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ARTICLES

Towards Universal Logic: Gaggle Logics

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Abstract

A class of non–classical logics called gaggle logics is introduced, based on a Kripke–style relational semantics and inspired by Dunn's gaggle theory. These logics deal with connectives of arbitrary arity and we show that they capture a wide range of non–classical logics. In particular, we list the 96 binary connectives and 16 unary connectives of basic gaggle logic and relate their truth conditions to the non-classical logics of the literature. We establish connections between gaggle theory and group theory. We show that Dunn's abstract law of residuation corresponds to an action of transpositions of the symmetric group on the set of connectives of gaggle logics and that Dunn's families of connectives are orbits of the same action. Other operations on connectives, such as dual and Boolean negation, are also reformulated in terms of actions of groups and their combination is defined by means of free groups and free products. We show how notions of groups arise naturally from our gaggle logics and how gaggle logics can be canonically defined from given groups. Our other main contribution deals with the proof theory of gaggle logics. We show how sound and complete calculi can be systematically computed from any basic gaggle logic with or without Boolean connectives. These calculi are display calculi and we prove that the cut rule can be systematically eliminated from proofs. This allows us to prove that basic gaggle logics are decidable.

Keywords: substructural logics, residuation, gaggle theory, display calculus, group theory, action of group, free group and free product.

1 Introduction

A wide variety of non–classical logics have been introduced over the past decades, such as relevant logics, linear logics and Lambek calculi, to name just a few. On the one hand, this diversity is an asset since each logic has an interest for a specific purpose, and one can select, and resort to, some of them for reasoning about a given applicative issue [38]. In fact, many of these non–classical logics have been developed for solving concrete problems in computer science: for example, dynamic logics [24], Hoare and separation logics [25, 43] for reasoning about computer programs, and description logics [3] for formalizing ontologies of the semantic web. Acknowledging and dealing with this plurality and diversity of logics is in a sense at the origin of the development of a philosophical stance in logic called "logical pluralism" [5]. On the other hand, and from a theoretical point of view, this plurality can be felt as problematic because it threatens the unity and the unifying power of logic. Indeed, all logics already have in common the same terminology and notions, such as truth, validity, conservativity and interpolation, and this is also an asset. Nevertheless, one can argue that non–classical logics are still disorganized and scattered and somehow miss a common formal ground. As Gabbay summarised the state of play (vis-à-vis non-monotonic logics) in the early 1980s, "we have had a multitude of systems generally accepted as 'logics' without a unifying underlying theory and many had semantics without proof theory. Many had proof theory without semantics, though almost all of them were based on some sound intuitions of one form or another. Clearly there was the need for a general unifying framework." [15, p. 184].

In response to that situation, a number of efforts have been made by some logicians to provide a genuine unity to logic as witnessed for example by the development of abstract model theory and "institutions" [4, 33, 19], the introduction of "labelled deductive systems" by Gabbay [17] or the "basic logic" of Sambin & al. [45] (see [16] for details and more examples). This led to the rise of a research thread sometimes referred to (nowadays) as "Universal Logic". Many kinds of semantics, such as algebraic, categorial, topological, phase or relational semantics, have been introduced and developed, sometimes for the express purpose of tackling this issue [46]. Within that line of research, Dunn's gaggle theory [10, 11, 7] is one of the most well–known frameworks based on the relational Kripke-style semantics which itself deals with the aforementioned problem. Dunn's gaggle theory is an attempt to understand the Kripke semantics of non-classical logics in a disciplined, systematic way.[1]

We share the ideal and the objective of "Universal Logic", but, in our view, gaggle theory is only a first step. Indeed, this theory does not really introduce an actual logic or logical framework that can serve as a foundation for non–classical logics, in the same way as the Lambek calculus is sometimes presented as the foundational logic of the varied substructural logics [42]. However, as we will show, gaggle theory provides formal methods to define a generic logic. In fact, it allows us to define a

[1]Dunn "owe[s] the name "gaggle" to [his] colleague Paul Eisenberg (a historian of philosophy, not a logician), who supplied it at [his] request for a name like a "group", but which suggested a certain amount of complexity and disorder." [10, p. 31]

class of logics that can handle connectives of arbitrary arity. Building on (partial) gaggle theory, we will define a class of non-classical logics that we call gaggle logics and which generalize the Lambek calculus and other substructural logics in many directions.

In doing so, we will establish connections between gaggle theory and group theory. We will show that Dunn's abstract law of residuation corresponds to an action of transpositions of the symmetric group (the group of permutations) on the set of connectives of gaggle logics and that Dunn's families of connectives are orbits of the same action. Other operations on connectives, such as dual and Boolean negation, will also be reformulated in terms of actions of groups, and their combination will be defined by means of free groups and free products. We will also show how notions of groups arise naturally from our gaggle logics and how gaggle logics can be canonically defined from given groups.

Our other main contribution will deal with the proof theory of gaggle logics. We will show how sound and complete calculi can be systematically computed and defined for any basic gaggle logic given by its set of connectives. This generic result is in line with our 'universal' approach explained above and constitutes the main technical advance of the article. We will use a specific Henkin construction method to prove the strong completeness of our calculi. Our main objective is to obtain sound and complete proof calculi for basic gaggle logics without the Boolean connectives. However, we will need to add them anyway and proceed in two steps. Firstly, we will consider a language with the Boolean connectives and prove completeness with them (Section 7). Secondly, after proving the cut elimination (via the proof of conditions $(C1)-(C8)$), we will obtain sound and complete calculi for basic gaggle logics without the Boolean connectives thanks to a proof-theoretical analysis of the calculi obtained (Sections 8 and 9, proof of Theorem 53). The cut elimination will also entail that basic gaggle logics are conservative extensions of each other and are decidable.

Organization of the article. In Section 2, we recall the basic results of (partial) gaggle theory. In Section 3, we recall the basics of group theory including the symmetric group (the group of permutations), free groups, free products and actions of groups. In Section 4, we introduce our gaggle logics and define our actions of groups on the gaggle connectives, in particular the residuation and the Boolean negation. In Section 5, we prove that Dunn's abstract laws of residuation are actions of transpositions of the symmetric group on the set of connectives and that Dunn's families of connectives are orbits of the action of the symmetric group. In Section 6, we relate our gaggle logics with the literature by listing the 96 binary connectives and the 16 unary connectives of basic gaggle logic while mentioning which connectives have already been introduced in a publication. We also mention two logics which cannot be embedded in gaggle logics. In Section 7, we introduce our display calculi. In Section 8 we prove that our calculi satisfy the display property and that the cut rule can be eliminated from any proof. Then, in Section 9, thanks to cut–elimination, we provide sound and strongly complete display calculi for gaggle logics without Boolean connectives. We also prove that basic gaggle logics are decidable. In Section 10, we show how notions of groups arise naturally from our gaggle logics and how gaggle logics can be canonically defined from given groups. We conclude in Section 11. Long proofs are in the Appendix.

2 The core of gaggle theory

We present the core ideas of (partial) gaggle theory [10, 11]. Partial gaggle first appeared in Dunn [11] as a generalization of a gaggle that has just an underlying poset, not necessarily a distributive lattice as required for a gaggle in Dunn [10]. For our purpose, the presentation of (partial) gaggle theory is slightly different from the usual presentation of this theory. The definitions are the same (although they are sometimes instantiated) but the results of this theory are differently presented. Our results can nevertheless easily be obtained from the original presentation [11].

In this section, we consider given an integer $n \in \mathbb{N}$ and a non-empty set W. $\mathcal{P}(W)$ is the set of subsets of W and if S is a set, S^n is the Cartesian product $S \times \ldots \times S$, n times. A *n–ary function f on $\mathcal{P}(W)$* is a function $f : \mathcal{P}(W)^n \to \mathcal{P}(W)$ and a *n–ary relation R over W* is a subset of W^n. We write $Rw_1 \ldots w_n$ for $(w_1, \ldots, w_n) \in R$. For all $m, n \in \mathbb{N}$, the expression $[\![m; n]\!]$ denotes the set $\{m, \ldots, n\}$ if $m \leq n$, and the empty set $\varnothing$ otherwise. In the sequel, we will resort to polarity groups, in particular to the *negation group* $P_{(+,-)}$ and later to the *anti-group* $P_{(+,\sim)}$.

Definition 1 (Polarity groups)**.** Let (x, y) be an ordered pair. The *polarity group associated to* (x, y) is $P_{(x,y)} \triangleq (\{x, y\}, \cdot)$ where the operation $\cdot : P_{(x,y)} \times P_{(x,y)} \to P_{(x,y)}$ is defined by $x \cdot y = y \cdot x = y$ and $x \cdot x = y \cdot y = x$. For all $\pm, \pm' \in \{x, y\}$, we write $\pm\pm'$ for $\pm \cdot \pm'$. □

Note that x is the neutral element of a polarity group.

Definition 2 (Trace, contrapositive trace)**.** A *(n–ary) trace* is a tuple $t = (\pm_1, \ldots, \pm_n, \pm) \in \{+, -\}^{n+1}$, often denoted $t = (\pm_1, \ldots, \pm_n) \mapsto \pm$. If $j \in [\![1; n]\!]$, then the *contrapositive trace of t with respect to its j^{th} argument* is the trace $t^j \triangleq (\pm_1, \ldots, -\pm, \ldots, \pm_n) \mapsto -\pm_j$. □

Note that the contrapositive operation on traces is symmetric: $\left(t^j\right)^j = t$.

Example 3. The 2–ary traces $(-,-) \mapsto -$ and $(-,+) \mapsto +$ are contrapositive with respect to (w.r.t.) their first argument.

Definition 4 (Relation negation and permutation)**.** Let R be an arbitrary $n+1$–ary relation over W. Then, for all $j \in \{1,\dots,n\}$, we define the $n+1$–ary relation $-R$ as follows: for all $w_1,\dots,w_n,w \in W$,

$$-Rw_1\dots w_n w \text{ iff } (w_1,\dots,w_n,w) \notin R$$

$\mathfrak{S}_{n+1}$ denotes the set of permutations of the set $[\![1;n+1]\!]$ (see Section 3 for details). If $\sigma \in \mathfrak{S}_{n+1}$ is a permutation then its inverse permutation is denoted σ^-. We define the $n+1$–ary relation R^σ as follows: for all $w_1,\dots,w_{n+1} \in W$,

$$R^\sigma w_1\dots w_{n+1} \text{ iff } Rw_{\sigma^-(1)}\dots w_{\sigma^-(n+1)}$$

We also define $+R \triangleq R$ and if $\pm \in \{+,-\}$ then $R^{\pm\sigma}$ denotes $\pm R^\sigma$. □

Definition 5 (Logical functions associated to a trace and a relation)**.** Let $t = (\pm_1,\dots,\pm_n) \mapsto \pm$ be a n–ary trace and let R be a $n+1$–ary relation on W. The *n–ary function f on $\mathcal{P}(W)$ associated to t and R*, denoted f_R^t, is defined as follows:

- If $n=0$, $f_R^t \triangleq R$;
- If $n>0$, then for all $W_1,\dots,W_n \in \mathcal{P}(W)$,

$$f_R^t(W_1,\dots,W_n) \triangleq \left\{w \in W \mid \mathcal{C}_R^t(W_1,\dots,W_n,w)\right\}$$

where $\mathcal{C}_R^t(W_1,\dots,W_n,w)$ is called the *truth condition* of the function f_R^t and is defined as follows:

- if $\pm = +$: "for all $w_1,\dots,w_n \in W$, we have $w_1 \pitchfork W_1$ or $\dots$ or $w_n \pitchfork W_n$ or $Rw_1\dots w_n w$";
- if $\pm = -$: "there are $w_1,\dots,w_n \in W$ such that $w_1 \pitchfork W_1$ and $\dots$ and $w_n \pitchfork W_n$ and $Rw_1\dots w_n w$";

where, for all $j \in [\![1;n]\!]$, $w_j \pitchfork W_j \triangleq \begin{cases} w_j \in W_j & \text{if } \pm_j\pm = +; \\ w_j \notin W_j & \text{if } \pm_j\pm = -. \end{cases}$ □

Example 6. Let R be a 3–ary relation on W and let σ be the permutation $(2,3,1)$ on the set $[\![1;3]\!]$ (see Section 3 for details). Then, we have that $R^\sigma uvw$ if, and only if, $Rwuv$.

- If $t = (-,-) \mapsto -$ then the function $f_R^t : \mathcal{P}(W) \times \mathcal{P}(W) \to \mathcal{P}(W)$, whose truth condition is $\mathcal{C}_R^t(W_1, W_2, w) = \exists uv\,(u \in W_1 \wedge v \in W_2 \wedge Ruvw)$, defines the semantics of a connective, that we denote $\circ$, as follows: for all $w \in W$,

$$\begin{aligned} w \in [\![\varphi \circ \psi]\!] &\text{ iff } w \in f_R^t([\![\varphi]\!], [\![\psi]\!]) \\ &\text{ iff } \exists uv\,(u \in [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge Ruvw) \end{aligned}$$

- If $t = (-,+) \mapsto +$ then the function $f_{-R^\sigma}^t : \mathcal{P}(W) \times \mathcal{P}(W) \to \mathcal{P}(W)$, whose truth condition is $\mathcal{C}_{-R^\sigma}^t(W_1, W_2, w) = \forall vu\,(v \in W_1 \vee u \in W_2 \vee -R^\sigma vuw)$, defines the semantics of a connective that we denote $\backslash$, as follows: for all $w \in W$,

$$\begin{aligned} w \in [\![\varphi \backslash \psi]\!] &\text{ iff } w \in f_{-R^\sigma}^{t'}([\![\varphi]\!], [\![\psi]\!]) \\ &\text{ iff } \forall vu\,(v \notin [\![\varphi]\!] \vee u \in [\![\psi]\!] \vee -R^\sigma uvw) \\ &\text{ iff } \forall vu\,((Rwuv \wedge u \in [\![\varphi]\!]) \to v \in [\![\psi]\!]). \end{aligned}$$

Definition 7 (Isotonic and antitonic functions)**.** Let f be a n–ary function on $\mathcal{P}(W)$. We say that f is *isotonic (resp. antitonic) with respect to the j^{th} argument*, written $tn(f,j) = +$ (resp. $tn(f,j) = -$), when for all $W_1, \ldots, W_{j-1}, W_{j+1}, \ldots, W_n, X, Y \in \mathcal{P}(W)$,

$$\begin{aligned} &\text{if } X \subseteq Y \\ &\text{then } f(W_1, \ldots, W_{j-1}, X, W_{j+1}, \ldots, W_n) \subseteq f(W_1, \ldots, W_{j-1}, Y, W_{j+1}, \ldots, W_n) \\ &(\text{resp. } f(W_1, \ldots, W_{j-1}, Y, W_{j+1}, \ldots, W_n) \subseteq f(W_1, \ldots, W_{j-1}, X, W_{j+1}, \ldots, W_n)). \end{aligned}$$

□

Example 8. If $[\![\varphi]\!] \subseteq [\![\varphi']\!]$ then $[\![\varphi' \backslash \psi]\!] \subseteq [\![\varphi \backslash \psi]\!]$ because $tn(f_{-R^\sigma}^{t'}, 1) = -$, and $[\![\varphi \circ \psi]\!] \subseteq [\![\varphi' \circ \psi]\!]$ because $tn(f_R^t, 1) = +$.

Definition 9 (Relation transformations)**.** Let R be an arbitrary $n+1$–ary relation over W. Then, for all $j \in \{1, \ldots, n\}$, we define the $n+1$–ary relation R^j as follows: for all $w_1, \ldots, w_n, w \in W$,

$$R^j w_1 \ldots w_n w \text{ iff } Rw_1 \ldots w \ldots w_n w_j$$

If $t = (\pm_1, \ldots, \pm_n) \mapsto \pm$ and $t' = (\pm_1', \ldots, \pm_n') \mapsto \pm'$ are two n–ary traces which are contrapositive w.r.t. their j^{th} argument, we define the $n+1$–ary relation $(t', t)(R)$ over W as follows:

$$(t', t)(R) \triangleq \begin{cases} R^j & \text{if } \pm = \pm'; \\ -R^j & \text{otherwise.} \end{cases}$$

□

Theorem 10. *Let R be a $n+1$–ary relation over W. Let $t = (\pm_1, \dots, \pm_n) \mapsto \pm$ and $t' = (\pm'_1, \dots, \pm'_n) \mapsto \pm'$ be two contrapositive n–ary traces w.r.t. their j^{th} argument. Let f (resp. f') be the n–ary function on $\mathcal{P}(W)$ associated to t and R (resp. associated to t' and $(t', t)(R)$). Then, if $n > 0$:*

1. *for all $j \in [\![1; n]\!]$, $tn(f, j) = \pm_j \pm$ (and thus $tn(f', j) = \pm'_j \pm'$ too);*

2. *f and f' satisfy the* abstract law of residuation w.r.t. their j^{th} argument*: for all $W_1, \dots, W_n, X \in \mathcal{P}(W)$,*

$$S(f, W_1, \dots, W_j, \dots, W_n, X) \quad \textit{iff} \quad S(f', W_1, \dots, X, \dots, W_n, W_j).$$

where $S(f, W_1, \dots, W_n, X) \triangleq \begin{cases} f(W_1, \dots, W_n) \subseteq X & \textit{if } \pm = - \\ X \subseteq f(W_1, \dots, W_n) & \textit{if } \pm = +. \end{cases}$

Example 11. Let us define $\varphi \Vdash \psi$ by for all $w \in W$, $w \in [\![\varphi]\!]$ implies that $w \in [\![\psi]\!]$. Then, the following holds:

- if $\psi \Vdash \psi'$ then $\varphi \circ \psi \Vdash \varphi \circ \psi'$ because $tn(f^t_R, 2) = +$, and if $\varphi \Vdash \varphi'$ then $\varphi' \backslash \psi \Vdash \varphi \backslash \psi$ because $tn(f^{t'}_{-R^\sigma}, 1) = -$. In other words, f^t_R is isotonic w.r.t. its second argument and $f^{t'}_{-R^\sigma}$ is antitonic w.r.t. its first argument.

- $\varphi \circ \psi \Vdash \chi$ iff $\varphi \Vdash \psi \backslash \chi$, because t and t' are contrapositive w.r.t. their first argument.

3 Group theory

We first recall some basics of group theory (see for instance [44] for more details).

Permutations and cycles. If X is a non-empty set, a *permutation* is a bijection $\sigma : X \to X$. We denote the set of all permutations of X by $\mathfrak{S}_X$. In the important special case when $X = \{1, \dots, n\}$, we write $\mathfrak{S}_n$ instead of $\mathfrak{S}_X$. Note that $|\mathfrak{S}_n| = n!$, where $|Y|$ denotes the number of elements in a set Y. A permutation σ on the set $\{1, \dots, n\}$ such that $\sigma(1) = x_1, \sigma(2) = x_2, \dots, \sigma(n) = x_n$ is denoted $(x_1, x_2, \dots, x_n)$. For example, $(1, 3, 2)$ is the permutation σ such that $\sigma(1) = 1, \sigma(2) = 3$ and $\sigma(3) = 2$.

If $x \in X$ and $\sigma \in \mathfrak{S}_X$, then σ *fixes* x if $\sigma(x) = x$ and σ *moves* x if $\sigma(x) \neq x$. Let $j_1, \dots, j_r$ be distincts integers between 1 and n. If $\sigma \in \mathfrak{S}_n$ fixes the remaining $n - r$ integers and if $\sigma(j_1) = j_2, \sigma(j_2) = j_3, \dots, \sigma(j_{r-1}) = j_r, \sigma(j_r) = j_1$ then σ is an r*–cycle*; one also says that σ is a cycle of *length* r. Denote σ by $(j_1\ j_2\ \dots\ j_r)$. A 2–cycle which merely interchanges a pair of elements is called a *transposition*.

Two permutations $\sigma, \tau \in \mathfrak{S}_X$ are *disjoint* if every x moved by one is fixed by the other. A family of permutations $\sigma_1, \sigma_2, \dots, \sigma_n$ is *disjoint* if each pair of them is disjoint. Every permutation $\sigma \in \mathfrak{S}_n$ is either a cycle or a product of disjoint cycles. Moreover, this factorization is unique except for the order in which the factors occur.

Groups. A *group* $(G, \circ)$ is a non–empty set G equipped with an associative operation $\circ : G \times G \to G$ and containing an element denoted 1_G called the *neutral element* such that:

- $1_G \circ a = a = a \circ 1_G$ for all $a \in G$;
- for every $a \in G$, there is an element $b \in G$ such that $a \circ b = 1_G = b \circ a$.

This element b is unique and called the *inverse* of a, denoted a^{-1}. The set $\mathfrak{S}_n$ with the composition operation is a group called the *symmetric group on n letters*.

A non–empty subset S of a group G is a *subgroup* of G if $s \in S$ implies $s^{-1} \in S$ and $s, t \in S$ imply $s \circ t \in S$. In that case, S is also a group in its own right.

If X is a subset of a group G, then the smallest subgroup of G containing X, denoted by $\langle X \rangle$, is called the *subgroup generated by X*. For example, $\mathfrak{S}_n = \langle (1\,2), (2\,3), \dots, (i\,i+1), \dots, (n-1\,n) \rangle = \langle (n\,1), (n\,2), \dots, (n\,n-1) \rangle = \langle (n-1\,n), (1\,2\,\dots\,n) \rangle$. $\mathfrak{S}_n$ is also generated by $(1\ 2)$ and 3–cycles. For $n \geq 3$, the *alternating group* $\mathfrak{A}_n$ is the subgroup of $\mathfrak{S}_n$ generated by the n–cycles of $\mathfrak{S}_n$.

In fact, if X is non–empty, then $\langle X \rangle$ is the set of all the words on X, that is, elements of G of the form $x_1^{\pm_1} x_2^{\pm_2} \dots x_n^{\pm_n}$ where $x_1, \dots, x_n \in X$ and $\pm_1, \dots, \pm_n$ are either -1 or empty.

Free groups and free products. If X is a subset of a group F, then F is a *free group* with *basis* X if, for every group G and every function $f : X \to G$, there exists a unique homomorphism $\varphi : F \to G$ extending f. One can prove that a free group with basis X always exists and that X generates F. We therefore use the notation $F = \langle X \rangle$ also for free groups.

If G and H are groups, the *free product* of G and H is a group P and homomorphisms j_G and j_H such that, for every group Q and all homomorphisms $f_G : G \to Q$ and $f_H : H \to Q$, there exists a unique homomorphism $\varphi : P \to Q$ with $\varphi j_G = f_G$ and $\varphi j_H = f_H$. Such a group always exists and it is unique modulo isomorphism, we denote it $G * H$. This definition can be generalized canonically to the case of a finite number of groups $G_1, \dots, G_n$, yielding the free product $G_1 * \dots * G_n$.

Group actions. If X is a set and G a group, an *action of G on X* is a function $\alpha : G \times X \to X$ given by $(g, x) \mapsto gx$ such that:

- $1x = x$ for all $x \in X$;
- $(g_1 g_2)x = g_1(g_2 x)$ for all $x \in X$ and all $g_1, g_2 \in G$.

An action of G on X is *transitive* if for every $x, y \in X$, there exists $g \in G$ such that $y = gx$; it is *faithful* if for $gx = x$ for all $x \in X$ implies that $g = 1$.

If $x \in X$ and α an action of a group G on X, then the *orbit* of x under α is $\mathcal{O}_\alpha(x) \triangleq \{\alpha(g,x) \mid g \in G\}$. The orbits form a partition of X. The *stabilizer* of x, denoted by G_x, is the subgroup $G_x \triangleq \{g \in G \mid gx = x\}$ of G. If G is finite, then we have that $|\mathcal{O}_\alpha(x)| = \frac{|G|}{|G_x|}$. Moreover, if X and G are finite then the number N of orbits of X is $N = \frac{1}{|G|}\sum_{\tau \in G} F(\tau)$ where, for $\tau \in G$, $F(\tau)$ is the number of $x \in X$ fixed by τ (Burnside's lemma). Finally, if $X' \subseteq X$ then $\mathcal{O}_\alpha(X')$ denotes $\bigcup\limits_{x' \in X'} \mathcal{O}_\alpha(x')$.

Fact 12. *If α is an action of G on a set X and H is a subgroup of G, then the restriction of α to H, denoted α_H, is also an action of H on the set X.*

Definition 13. Let G and H be two groups. If α and β are actions of G and H on a set X, then the *free action* $\alpha * \beta$ is the mapping $\alpha * \beta : G * H \times X \to X$ given by $\alpha * \beta(g,x) \triangleq \alpha(g_1, \beta(h_1, \ldots, \alpha(g_n, \beta(h_n, x))))$, where $g = g_1 h_1 \ldots g_n h_n$ is the factorization of g in the free group $G * H$. □

This definition can be generalized canonically to the case of a finite number of actions $\alpha_1, \ldots, \alpha_n$, yielding the mapping $\alpha_1 * \ldots * \alpha_n$.

Proposition 14. *If $\alpha_1, \ldots, \alpha_n$ are actions of $G_1, \ldots, G_n$ on a set X respectively, then the mapping $\alpha_1 * \ldots * \alpha_n$ is an action of the (free) group $G_1 * \ldots * G_n$ on X.*

4 From gaggle theory to gaggle logics

The introduction of the formal concepts of gaggle theory are motivated by some heuristic and logical reasons (see for example [41] for informal explanations). We are going to reformulate these formal concepts of gaggle theory because we want to make more clear the connection between traces and the relational Kripke–style semantics that they induce. Thereby, we replace the notion of trace by our notion of 'signature' which highlights and distinguishes in a more immediate way the different semantic ingredients that compose gaggle theory. More specifically, the output of a trace (+ or −) is replaced by a quantification signature ($\forall$ or $\exists$). Doing so, our reformulation will capture and represent more directly and faithfully the tonicity of the connective defined by a given trace/signature and the formulation of its truth condition (even if, as we said, the notion of trace output was introduced for different heuristic reasons [41]).

In this section, we show how gaggle theory, and in particular Definition 5, leads to the definition of finite families of connectives of arbitrary arities which are related to each other by the abstract law of residuation of Theorem 10.

4.1 From traces to gaggle connectives

Informally, $\forall$ is associated with $+$ and $\exists$ is associated with $-$. We formalize this association with the function $\pm : \{\forall, \exists\} \to \{+, -\}$ defined by $\pm(\forall) \triangleq +, \pm(\exists) \triangleq -$ and the inverse function $Æ : \{+, -\} \to \{\forall, \exists\}$ defined by $Æ(+) \triangleq \forall, Æ(-) \triangleq \exists$. Also, we define the function $+ : \{\forall, \exists\} \to \{\forall, \exists\}$ by $+(\forall) \triangleq \forall$ and $+(\exists) \triangleq \exists$ and the function $- : \{\forall, \exists\} \to \{\forall, \exists\}$ by $-(\forall) \triangleq \exists$ and $-(\exists) \triangleq (\forall)$. For better readability, we write $+\forall, +\exists, -\forall, -\exists$ instead of $-(\forall), +(\exists), -(\forall), -(\exists)$.

Definition 15 (Signatures versus traces)**.** A *(n–ary) signature* s is a tuple $s = (Æ, (\pm_1, \ldots, \pm_n)) \in \{\forall, \exists\} \times \{+, -\}^n$. If $s = (Æ, (\pm_1, \ldots, \pm_n))$ is a n–ary signature and $t = (\pm_1, \ldots, \pm_n, \pm)$ a n–ary trace, then

- The *trace $T(s)$ equivalent to s* is the trace $(\pm'_1, \ldots, \pm'_n) \mapsto \pm$ where $\pm \triangleq \pm(Æ)$ and $\pm'_j \triangleq \pm\pm_j$ for all $j \in [\![1; n]\!]$.
- The *signature $S(t)$ equivalent to t* is the signature $(Æ, (\pm'_1, \ldots, \pm'_n))$ where $Æ \triangleq Æ(\pm)$ and $\pm'_j \triangleq \pm\pm_j$ for all $j \in [\![1; n]\!]$. $\square$

Note that the derived notion of tonicity $tn(f, j)$ determined in Theorem 10 is now taken as primitive with our notion of signature. Then, we can easily prove the following:

$$s = S(T(s)) \qquad\qquad t = T(S(t))$$

We also reformulate the definition of contrapositive trace in terms of signature as follows. If $s = (Æ, (\pm_1, \ldots, \pm_n))$ is a n–ary signature and $r_j = (n{+}1\ j)$ a transposition with $j \in [\![1; n]\!]$, then we define

$$r_j s \triangleq (-\pm_j Æ, (-\pm_j \pm_1, \ldots, \pm_j, \ldots, -\pm_j \pm_n)). \tag{1}$$

Then, we can easily prove the following: for all n–ary traces t and n–ary signatures s,

$$r_j s = S\left(T(s)^j\right) \qquad\qquad t^j = T\left(r_j S(t)\right)$$

Moreover, for every cycle c fixing $n+1$, we define

$$cs \triangleq \left(Æ, \left(\pm_{c(1)}, \pm_{c(2)}, \ldots, \pm_{c(n)}\right)\right). \tag{2}$$

This definition is coherent with Expression (1). Indeed, the transpositions $(n+1\ 1), (n+1\ 2), \ldots, (n+1\ n)$ generate $\mathfrak{S}_{n+1}$ and every cycle fixing $n+1$ can be factorized into a sequence of transpositions of the form $(n+1\ j)$ so that, applying iteratively Expression (1), we obtain Expression (2).

Definition 16 (Gaggle connectives)**.** The set of *atoms* $\mathbb{P}$ and *connectives* $\mathbb{C}$ are:

$$\mathbb{P} \triangleq \mathfrak{S}_1 \times \{+,-\} \times \{\forall, \exists\} \qquad \mathbb{C} \triangleq \mathbb{P} \cup \bigcup_{n \in \mathbb{N}^*} \mathfrak{S}_{n+1} \times \{+,-\} \times \{\{\forall, \exists\} \times \{+,-\}^n\}.$$

Both atoms and connectives can be represented by triples $p = (1, \pm, Æ)$ (for atoms) and $\otimes = (\sigma, \pm, (Æ, (\pm_1, \ldots, \pm_n)))$ (for connectives) where $\sigma \in \mathfrak{S}_{n+1}$, $\pm \in \{+,-\}$ and $(Æ, (\pm_1, \ldots, \pm_n)) \in \{\forall, \exists\} \times \{+,-\}^n$. The *arity* of an atom is 0, the *arity* of a connective $\otimes = (\sigma, \pm, (Æ, (\pm_1, \ldots, \pm_n))) \in \mathbb{C}$, denoted $a(\otimes)$, is n, its *signature* is $(Æ, (\pm_1, \ldots, \pm_n))$, its *quantification signature* is Æ and its *tonicity signature* is $(\pm_1, \ldots, \pm_n)$. For all $j \in [\![1; n]\!]$, $tn(\otimes, j)$ denotes $\pm_j$. Atoms are denoted p, p_1, p_2, *etc.* and connectives are denoted $\otimes, \otimes_1, \otimes_2$, *etc.* The set of n–ary connectives, for $n > 0$, is denoted $\mathbb{C}_n$. □

Fact 17. *The number of n–ary gaggle connectives is* $(n+1)! \cdot 2^{n+2}$.

Proof: It follows from the very definition of connectives. □

4.2 Actions of groups on gaggle connectives

In this section, we introduce actions on the set of gaggle connectives. In the next sections, we will show that they generalize standard notions of residuations, duals and Boolean negation.

Definition 18 (Action of the symmetric group)**.** Let $n \in \mathbb{N}^*$. We define the function $\alpha_n : \mathfrak{S}_{n+1} \times \mathbb{C}_n \to \mathbb{C}_n, (\tau, \otimes) \mapsto \tau\otimes$ inductively as follows. Let $\otimes = (\sigma, \pm, (Æ, (\pm_1, \ldots, \pm_n))) \in \mathbb{C}_n$ and let $c \in \mathfrak{S}_{n+1}$.

- If c is the transposition $r_j = (j\ n+1)$, then $r_j\otimes \triangleq (r_j \circ \sigma, -\pm_j \pm, r_j s)$, *i.e.*:

  $$r_j\otimes \triangleq ((j\ n+1) \circ \sigma, -\pm_j \pm, (-\pm_j Æ, (-\pm_j \pm_1, \ldots, \pm_j, \ldots, -\pm_j \pm_n)).$$

  The connective r_j is called the *residual of $\otimes$ w.r.t. its j^{th} argument.*

Permutations of $\mathfrak{S}_2$	1–ary signatures
$\tau_1 = (1,2)$ $\tau_2 = (2,1)$	$t_1 = (\exists, +)$ $t_2 = (\forall, +)$
	$t_3 = (\forall, -)$
	$t_4 = (\exists, -)$
Permutations of $\mathfrak{S}_3$	**2–ary signatures**
$\sigma_1 = (1,2,3)$ $\sigma_2 = (3,2,1)$ $\sigma_3 = (3,1,2)$ $\sigma_4 = (2,1,3)$ $\sigma_5 = (2,3,1)$ $\sigma_6 = (1,3,2)$	$s_1 = (\exists, (+,+))$ $s_2 = (\forall, (+,-))$ $s_3 = (\forall, (-,+))$
	$s_4 = (\forall, (+,+))$ $s_5 = (\exists, (+,-))$ $s_6 = (\exists, (-,+))$
	$s_7 = (\exists, (-,-))$
	$s_8 = (\forall, (-,-))$

Figure 1: Permutations of $\mathfrak{S}_2$ and $\mathfrak{S}_3$ and 'families' of 1–ary and 2–ary signatures

- If c is the cycle $(j_1 \ j_2 \ \dots \ j_k \ n+1)$, then $c\circledast \triangleq r_{j_1}(r_{j_2}\dots(r_{j_k}\circledast))$, where $r_j \triangleq (j \ n+1)$ for all j.
- If c is a cycle fixing $n+1$, then $c\circledast \triangleq (c\circ\sigma, \pm, cs)$, *i.e.*:

$$c\circledast \triangleq \left(c\circ\sigma, \pm, \left(\text{Æ}, \left(\pm_{c(1)}, \pm_{c(2)}, \dots, \pm_{c(n)}\right)\right)\right).$$

Finally, if τ is an arbitrary permutation of $\mathfrak{S}_{n+1}$, it can be factorized into a product of disjoint cycles $\tau = c_1c_2\dots c_k$ and this factorization is unique (modulo its order) [44]. So, we define $\tau\circledast \triangleq c_1(c_2\dots(c_k\circledast))$. □

The mapping α_n is well-defined because one can easily prove that any other ordering of the disjoint cycles $c_1,\dots,c_k$ of τ yields the same outcome for $\tau\circledast$. Our definition is based on cycles and not on transpositions because the decomposition of any permutation into disjoint cycles is unique (modulo its order), unlike its decomposition into transpositions.

Proposition 19. *For all $n \in \mathbb{N}^*$, the mapping $\alpha_n : \mathfrak{S}_{n+1} \times \mathbb{C}_n \to \mathbb{C}_n$ is a group action of $\mathfrak{S}_{n+1}$ on $\mathbb{C}_n$. For all $n \in \mathbb{N}^*$, the group actions α_n (and all their restrictions to subgroups G) are not transitive, the cardinality of each orbit is $|\mathfrak{S}_{n+1}|$ (resp. $|G|$) and the number of orbits is $4\cdot 2^n$ (resp. $\frac{|\mathbb{C}_n|}{|G|}$).*

Proof: (sketch) The condition $(\tau_1 \circ \tau_2)\circledast = \tau_1(\tau_2 \circledast)$ of the definition of group actions is proved by induction on τ_1. The other results follow from group theory because for all $x \in \mathbb{C}_n$, $G_x = \{1\}$. □

Definition 20 (Actions of the negation group and the anti-group)**.** Let $n \in \mathbb{N}^*$. We define the functions $\beta_n : P_{(+,-)} \times \mathbb{C}_n \to \mathbb{C}_n, (\pm, \circledast) \mapsto \pm\circledast$ and $\gamma_n : P_{(+,\sim)} \times \mathbb{C}_n \to \mathbb{C}_n, (\pm, \circledast) \mapsto \pm\circledast$ as follows: if $\circledast = (\sigma, \pm, (\text{Æ}, (\pm_1, \ldots, \pm_n))) \in \mathbb{C}_n$, then

- $+\circledast \triangleq \circledast$
- $\sim \circledast \triangleq (\sigma, -\pm, (\text{Æ}, (\pm_1, \ldots, \pm_n)))$
- $-\circledast \triangleq (\sigma, -\pm, (-\text{Æ}, (-\pm_1, \ldots, -\pm_n)))$.

$-\circledast$ and $\sim \circledast$ are called the *Boolean negation* and the *symmetry* of $\circledast$ respectively. Moreover, if $\circledast$ is an atom $p = (1, \pm, \text{Æ})$, then we also define $-p \triangleq (1, -\pm, -\text{Æ})$. □

As we will see in Proposition 29, our definition of Boolean negation does correspond to the intended (Boolean) negation.

Proposition 21. *For all $n \in \mathbb{N}^*$, the functions β_n and γ_n are non–transitive actions. For both actions, the cardinality of each orbit is* 2 *and the number of orbits is* $\frac{|\mathbb{C}_n|}{2}$.

Proof: It follows from the application of Burnside Lemma. Only + fixes connectives of $\mathbb{C}_n$ and it fixes all of them. − and ~ do not fix any element of $\mathbb{C}_n$. □

4.3 Gaggle logics

Our introduction of 'gaggle logics', like many semantic-based logics, is made in three parts: first, we define their language (Definition 22), then their class of models (Definition 24) and finally their satisfaction relation (Definition 25).

Definition 22 ((Boolean) gaggle language)**.** The *gaggle language* $\mathcal{L}^0$ is the smallest set that contains the propositional letters and that is closed under the gaggle connectives. That is,

- $\mathbb{P} \subseteq \mathcal{L}^0$;
- for all $\circledast \in \mathbb{C}$ of arity $n > 0$ and for all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}^0$, we have $\circledast(\varphi_1, \ldots, \varphi_n) \in \mathcal{L}^0$.

The *Boolean gaggle language* $\mathcal{L}$ is the smallest set that contains the propositional letters and that is closed under the gaggle connectives as well as the Boolean connectives $\wedge, \vee$ and $\neg$.

Elements of $\mathcal{L}$ are called *formulas* and are denoted $\varphi, \psi, \alpha, \ldots$ For all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}$, $\varphi_1 \wedge \ldots \wedge \varphi_n$ and $\varphi_1 \vee \ldots \vee \varphi_n$ stand for $((\varphi_1 \wedge \varphi_2) \wedge \ldots \wedge \varphi_n)$ and $((\varphi_1 \vee \varphi_2) \vee \ldots \vee \varphi_n)$ respectively.

If $\mathsf{C} \subseteq \mathbb{C} \cup \{\wedge, \vee, \neg\}$ is such that $\mathsf{C} \cap \mathbb{P} \neq \varnothing$, then an element of $\mathcal{L}_{\mathsf{C}}$ is an element of $\mathcal{L}$ that contains only connectives and atoms of C. *In the sequel, we assume that all the sets of atoms and connectives $\mathsf{C} \subseteq \mathbb{C} \cup \{\wedge, \vee, \neg\}$ are such that $\mathsf{C} \cap \mathbb{P} \neq \varnothing$.* $\square$

Remark 23. We could consider a countable number of copies of the atoms and connectives: $\mathbb{P}' \triangleq \bigcup_{i \in \mathbb{N}} \{\circledast_i \mid \circledast \in \mathbb{P}\}$, $\mathbb{C}' \triangleq \bigcup_{i \in \mathbb{N}} \{\circledast_i \mid \circledast \in \mathbb{C}\}$. Indeed, in general we need a countable number of atoms or, like in some modal logics, we need multiple modalities of the same (similarity) type. All the results that follow would still hold in this extended language.

Definition 24 (C–models and C–frames)**.** Let $\mathsf{C} \subseteq \mathbb{C}$. A *C–model* is a tuple $M = (W, \mathcal{R})$ where W is a non-empty set and $\mathcal{R}$ is a set of relations over W. Each n–ary connective $\circledast \in \mathsf{C}$ is associated to a $n+1$–ary relation $R_\circledast$ such that for all connectives $\circledast_1, \circledast_2 \in \mathsf{C}$, we have that $R_{\circledast_1} = R_{\circledast_2}$ iff $\mathcal{O}_{\alpha_n * \beta_n}(\circledast_1) = \mathcal{O}_{\alpha_n * \beta_n}(\circledast_2)$.

We abusively write $w \in M$ for $w \in W$. A *pointed C–model* (M, w) is a C–model M together with a state $w \in M$. The class of all pointed C–models is denoted $\mathcal{M}_{\mathsf{C}}$ and simply $\mathcal{M}$ when $\mathsf{C} = \mathbb{C}$. A *C–frame* is a $\mathsf{C} \backslash \mathbb{P}$–model. The class of all pointed C–frames is denoted $\mathcal{F}_{\mathsf{C}}$ and simply $\mathcal{F}$ when $\mathsf{C} = \mathbb{C}$. $\square$

Definition 25 (Gaggle logics)**.** Let $\mathsf{C} \subseteq \mathbb{C}$ and let $M = (W, \mathcal{R})$ be a C–model. We define the *interpretation function of $\mathcal{L}_C$ in M*, denoted $[\![\cdot]\!]^M : \mathcal{L}_{\mathsf{C}} \to \mathcal{P}(W)$, inductively as follows: for all $p \in \mathsf{C} \cap \mathbb{P}$ and all $\circledast \in \mathsf{C}$ of arity $n > 0$ and signature denoted $(\sigma, \pm, s)$, for all $\varphi, \psi, \varphi_1, \ldots, \varphi_n \in \mathcal{L}_{\mathsf{C}}$,

$$\begin{array}{rcl}
[\![p]\!]^M & \triangleq & \pm R_p \\
[\![\neg\varphi]\!]^M & \triangleq & W - [\![\varphi]\!]^M \\
[\![(\varphi \wedge \psi)]\!]^M & \triangleq & [\![\varphi]\!]^M \cap [\![\psi]\!]^M \\
[\![(\varphi \vee \psi)]\!]^M & \triangleq & [\![\varphi]\!]^M \cup [\![\psi]\!]^M \\
[\![\circledast(\varphi_1, \ldots, \varphi_n)]\!]^M & \triangleq & f_\circledast([\![\varphi_1]\!]^M, \ldots, [\![\varphi_n]\!]^M)
\end{array}$$

where the function $f_\circledast = f^t_{R_\circledast^{\pm\sigma}}$ with $t = T(s)$ defined in Section 4.1 and $f^t_{R_\circledast^{\pm\sigma}}$ in Definition 5. That is, $f_\circledast$ is defined as follows: for all $W_1, \ldots, W_n \in \mathcal{P}(W)$, $f_\circledast(W_1, \ldots, W_n) \triangleq \{w \in W \mid \mathcal{C}^\circledast(W_1, \ldots, W_n, w)\}$ where $\mathcal{C}^\circledast(W_1, \ldots, W_n, w)$ is called the *truth condition* of $\circledast$ and is:

- if Æ = $\forall$: "$\forall w_1, \ldots, w_n \in W \, (w_1 \pitchfork W_1 \vee \ldots \vee w_n \pitchfork W_n \vee R_{\circledast}^{\pm\sigma} w_1 \ldots w_n w)$";
- if Æ = $\exists$: "$\exists w_1, \ldots, w_n \in W \, (w_1 \pitchfork W_1 \wedge \ldots \wedge w_n \pitchfork W_n \wedge R_{\circledast}^{\pm\sigma} w_1 \ldots w_n w)$";

where, for all $j \in [\![1; n]\!]$, $w_j \pitchfork W_j \triangleq w_j \in W_j$ if $\pm_j = +$ and $w_j \pitchfork W_j \triangleq w_j \notin W_j$ if $\pm_j = -$ and $R_{\circledast}^{\pm\sigma} w_1 \ldots w_{n+1}$ iff $\pm R_{\circledast} w_{\sigma^-(1)} \ldots w_{\sigma^-(n+1)}$ (we recall that $+R_{\circledast} \triangleq R_{\circledast}$ and $-R_{\circledast} \triangleq W^{n+1} - R_{\circledast}$).

We extend the definition of the interpretation function $[\![\cdot]\!]^M$ to C–frames as follows: for all $\varphi \in \mathcal{L}_{\mathsf{C}}$ and all C–frames F,

$$[\![\varphi]\!]^F \triangleq \bigcap \left\{ [\![\varphi]\!]^{(F,\mathcal{P})} \mid \mathcal{P} \text{ a set of } n\text{–ary relations over } W \text{such that } (F, \mathcal{P}) \text{ is a } \mathsf{C}\text{–model} \right\}$$

If $\mathcal{E}_{\mathsf{C}}$ is a class of pointed C–models or C–frames, the *satisfaction relation* $\Vdash \subseteq \mathcal{E}_{\mathsf{C}} \times \mathcal{L}_{\mathsf{C}}$ is defined as follows: for all $\varphi \in \mathcal{L}_{\mathsf{C}}$ and all $(M, w) \in \mathcal{E}_{\mathsf{C}}$, $((M, w), \varphi) \in \Vdash$ iff $w \in [\![\varphi]\!]^M$. We usually write $(M, w) \Vdash \varphi$ instead of $((M, w), \varphi) \in \Vdash$. The triple $(\mathcal{L}_{\mathsf{C}}, \mathcal{E}_{\mathsf{C}}, \Vdash)$ is a logic called the *gaggle logic associated to $\mathcal{E}_{\mathsf{C}}$ and C.* The logics of the form $(\mathcal{L}_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$ are called *basic gaggle logics.* We call them *Boolean (basic) gaggle logics* when their language includes the Boolean connectives $\wedge, \vee, \neg$. $\square$

The truth conditions of the above definitions have been introduced in a different formal approach by Bimbó & Dunn [7] and for some particular cases by Dunn [10] and Dunn & Hardegree [13]. However, it is the first time that they are spelled out systematically and in a comprehensive manner.

Example 26 (Lambek calculus, modal logic)**.** The Lambek calculus $(\mathcal{L}_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$ where $\mathsf{C} = \{p, \circ, \backslash, /\}$ defined in Section 2 is an example of basic gaggle logic. Here $\circ, \backslash, /$ are the connectives $(\sigma_1, +, s_1), (\sigma_5, -, s_3), (\sigma_3, -, s_2)$. Another example of gaggle logic is modal logic $(\mathcal{L}_{\mathsf{C}}, \mathcal{E}_{\mathsf{C}}, \Vdash)$ where $\mathsf{C} = \{p, \top, \bot, \wedge, \vee, \Diamond, \Box\}$ is such that

- $\top, \bot$ are the connectives $(1, +, \exists)$ and $(1, -, \forall)$ respectively;
- $\wedge, \vee, \Diamond, \Box$ are the connectives $(\sigma_1, +, s_1), (\sigma_1, -, s_4), (\tau_2, +, s_1), (\tau_2, -, s_2)$ respectively;
- the C-models $M = (W, \mathcal{R}) \in \mathcal{E}_{\mathsf{C}}$ are such that $R_\wedge = R_\vee = \{(w, w, w) \mid w \in W\}$, $R_\Diamond = R_\Box$, $R_\top = R_\bot = W$.

Indeed, one can easily show that, with these conditions on the C–models of $\mathcal{E}_{\mathsf{C}}$, we have that for all $(M, w) \in \mathcal{E}_{\mathsf{C}}$, $(M, w) \Vdash (\sigma_1, +, s_1)(\varphi, \psi)$ iff $(M, w) \Vdash \varphi$ and $(M, w) \Vdash \psi$, and $(M, w) \Vdash (\sigma_1, -, s_4)(\varphi, \psi)$ iff $(M, w) \Vdash \varphi$ or $(M, w) \Vdash \psi$. Note that the Boolean conjunction and disjunction $\wedge$ and $\vee$ are defined using the connectives of $\mathbb{C}$ by means of special relations $R_\wedge$ and $R_\vee$. They could obviously be defined directly. Many more examples will be given in Section 6.

5 Residual, Boolean negation, dual and switch

The action of specific permutations on the set of connectives corresponds to well–known operations used in proof theory. For example, the action of a transposition $(j\ n+1)$ corresponds to the abstract law of residuation for the j^{th} argument. This operation of residuation turns out to be central since every permutation can be decomposed into a composition of transpositions. Yet, we argue that the actions of cycles is more central because every permutation can be decomposed *uniquely* into disjoint cycles. Moreover, the symmetric group $\mathfrak{S}_{n+1}$ is also generated by the cycles $(1\ \dots\ n+1)$ and $(n\ n+1)$ and the alternation group is generated by the $n+1$-cycles of $\mathfrak{S}_{n+1}$. This confirms an observation already made in [2] which highlighted the role of 3-cycles for substructural and update logics in the formal connections that exist between connectives.

Proposition 27. *Let t be a n-ary trace, R a $n+1$-ary relation over W and $\sigma \in \mathfrak{S}_{n+1}$. Then, $f^t_{R^{\pm\sigma}} = f_{\circledast}$ where $\circledast = (\sigma, \pm, S(t)) = (\sigma, \pm, (Æ, (\pm_1, \dots, \pm_n)))$. Moreover, if $j \in [\![1; n]\!]$, then the n-ary function associated to t^j and $(t^j, t)(R)$ of Definition 5 is $f_{r_j\circledast}$ where $r_j\circledast$, the* residual of $\circledast$ w.r.t. its j^{th} argument*, was defined in Definition 18:*

$$r_j\circledast \triangleq ((j\ n+1) \circ \sigma, -\pm_j \pm, (-\pm_j Æ, (-\pm_j \pm_1, \dots, \pm_j, \dots, -\pm_j \pm_n))).$$

Therefore, we have the following property: for all $\varphi_1, \dots, \varphi_j, \dots, \varphi_n, \varphi \in \mathcal{L}$,

$$S[\circledast, \varphi_1, \dots, \varphi_j, \dots, \varphi_n, \varphi] \qquad \textit{iff} \qquad S[r_j\circledast, \varphi_1, \dots, \varphi, \dots, \varphi_n, \varphi_j] \tag{3}$$

where $S[\circledast, \varphi_1, \dots, \varphi_n, \varphi] \triangleq \begin{cases} \circledast(\varphi_1, \dots, \varphi_n) \Vdash \varphi & \text{if } Æ = \exists \\ \varphi \Vdash \circledast(\varphi_1, \dots, \varphi_n) & \text{if } Æ = \forall \end{cases}.$

Proof: It follows straightforwardly from our definitions. Expression (3) follows from Theorem 10 (item 2). □

Hence, $r_j\circledast$ does correspond to the residual connective of $\circledast$ w.r.t. its j^{th} argument as it is usually defined in Dunn's theory.

Definition 28 (Dual and switch operations)**.** Let $\circledast = (\sigma, \pm, (Æ, (\pm_1, \dots, \pm_n))) \in \mathbb{C}$ be a n–ary connective and let $j \in [\![1; n]\!]$.

- The *switch of $\circledast$ w.r.t. its j^{th} argument* is the n-ary connective

$$s_j\circledast \triangleq (\sigma, \pm, (Æ, (\pm_1, \dots, -\pm_j, \dots, \pm_n))).$$

- The *dual of* $\circledast$ *w.r.t. its* j^{th} *argument* is the n–ary connective

$$d_j\circledast \triangleq (\sigma, -\pm, (-\text{Æ}, (-\pm_1, \ldots, \pm_j, \ldots, -\pm_n))).$$

- The *dual of* $\circledast$ is the n-ary connective

$$d\circledast \triangleq (\sigma, -\pm, (-\text{Æ}, (\pm_1, \ldots, \pm_n))). \qquad \square$$

The following proposition shows that our terminology for "Boolean negation" and "dual" is appropriate and does correspond to the standard intuitive meaning (see Blackburn & Al. [8, Def 1.13] for example).

Proposition 29. *Let* $\circledast \in \mathbb{C}$ *be a* n*–ary connective and let* $\varphi_1, \ldots, \varphi_n \in \mathcal{L}$*. Then, for all (appropriate) pointed models* (M, w)*,*

$$\begin{array}{lll}
(M,w) \Vdash -\circledast(\varphi_1, \ldots, \varphi_n) & \textit{iff} & (M,w) \Vdash \circledast(\varphi_1, \ldots, \varphi_n) \textit{ does not hold} \\
(M,w) \Vdash s_j \circledast(\varphi_1, \ldots, \varphi_n) & \textit{iff} & (M,w) \Vdash \circledast(\varphi_1, \ldots, \neg\varphi_j, \ldots, \varphi_n) \\
(M,w) \Vdash d_j \circledast(\varphi_1, \ldots, \varphi_n) & \textit{iff} & (M,w) \Vdash -\circledast(\varphi_1, \ldots, \neg\varphi_j, \ldots, \varphi_n) \\
(M,w) \Vdash d \circledast(\varphi_1, \ldots, \varphi_n) & \textit{iff} & (M,w) \Vdash -\circledast(\neg\varphi_1, \ldots, \neg\varphi_n)
\end{array}$$

The following proposition shows that the switch as well as the dual operations are definable in terms of residuations and Boolean negation.

Proposition 30. *If* $\circledast \in \mathbb{C}_n$ *is a* n*–ary connective, then for all* $j \in [\![1; n]\!]$*,*

- $s_j\circledast = r_j - r_j\circledast$
- $d_j\circledast = r_j - r_j - \circledast$
- $d\circledast = s_1 \ldots s_n - \circledast$.

Proof: See the Appendix, Section A. $\square$

Proposition 31. *Dunn's (complete) families of* n*–ary connectives are orbits* $\mathcal{O}_{\alpha_n}(\circledast)$ *of the group action* α_n*. These families/orbits form a partition of the set of* n*–ary connectives.*

Proof: It follows easily from Dunn's and our definitions. $\square$

Dunn's families of n–ary connectives are called "complete families" of operations by Bimbó & Dunn [7]. Likewise, two n–ary connectives $\circledast, \circledast \in \mathbb{C}_n$ are "colligated" in the sense of Bimbó & Dunn [7] when they belong to the same orbit $\mathcal{O}_{\alpha_n}(\circledast)$.

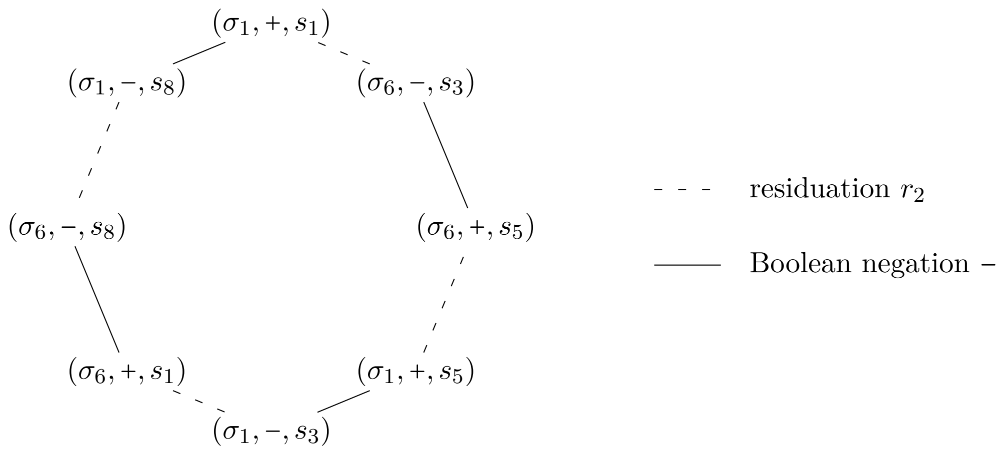

Figure 2: The 8 connectives of the orbit $\mathcal{O}_{\alpha_{G_2}}((\sigma_1,+,s_1))$

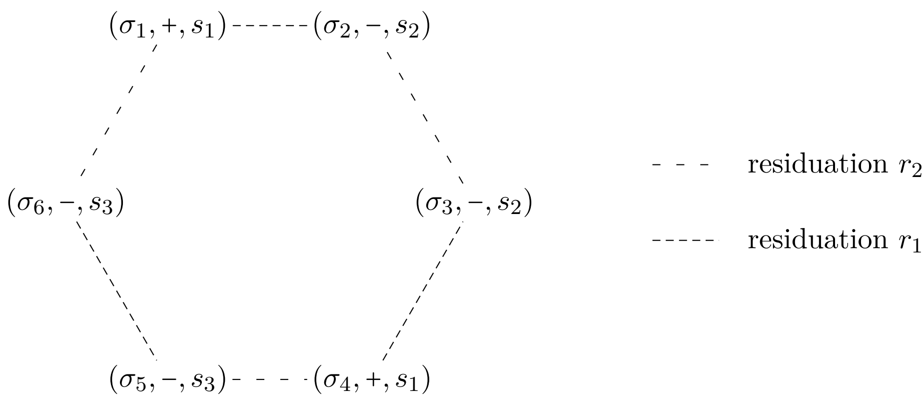

Figure 3: The 6 connectives of the orbit $\mathcal{O}_{\alpha_2}((\sigma_1,+,s_1))$

Proposition 32. *Let $n \in \mathbb{N}^*$, $j \in [\![1;n]\!]$ and let us define $G_j = \langle r_j \rangle * P_{(+,-)}$. Since G_j is a subgroup of $\mathfrak{S}_{n+1} * P_{(+,-)}$, let us denote by α_{G_j} the action of G_j on $\mathbb{C}_n$ induced by the free action $\alpha_n * \beta_n$. Then, for all connectives $\circledast$ of arity n,*

1. *$\mathcal{O}_{\alpha_{G_j}}(\circledast)$ is isomorphic to a cyclic group of order 8.*

2. *$\{\mathcal{O}_{\alpha_n * \beta_n}(\circledast), \mathcal{O}_{\alpha_n * \beta_n}(\sim \circledast)\}$ forms a partition of the set $\mathbb{C}_n$ of connectives of arity n. Moreover, the mapping $\tilde{\cdot} : \mathcal{O}_{\alpha_n * \beta_n}(\circledast) \to \mathcal{O}_{\alpha_n * \beta_n}(\sim \circledast)$, $x \mapsto \sim x$ is involutive.*

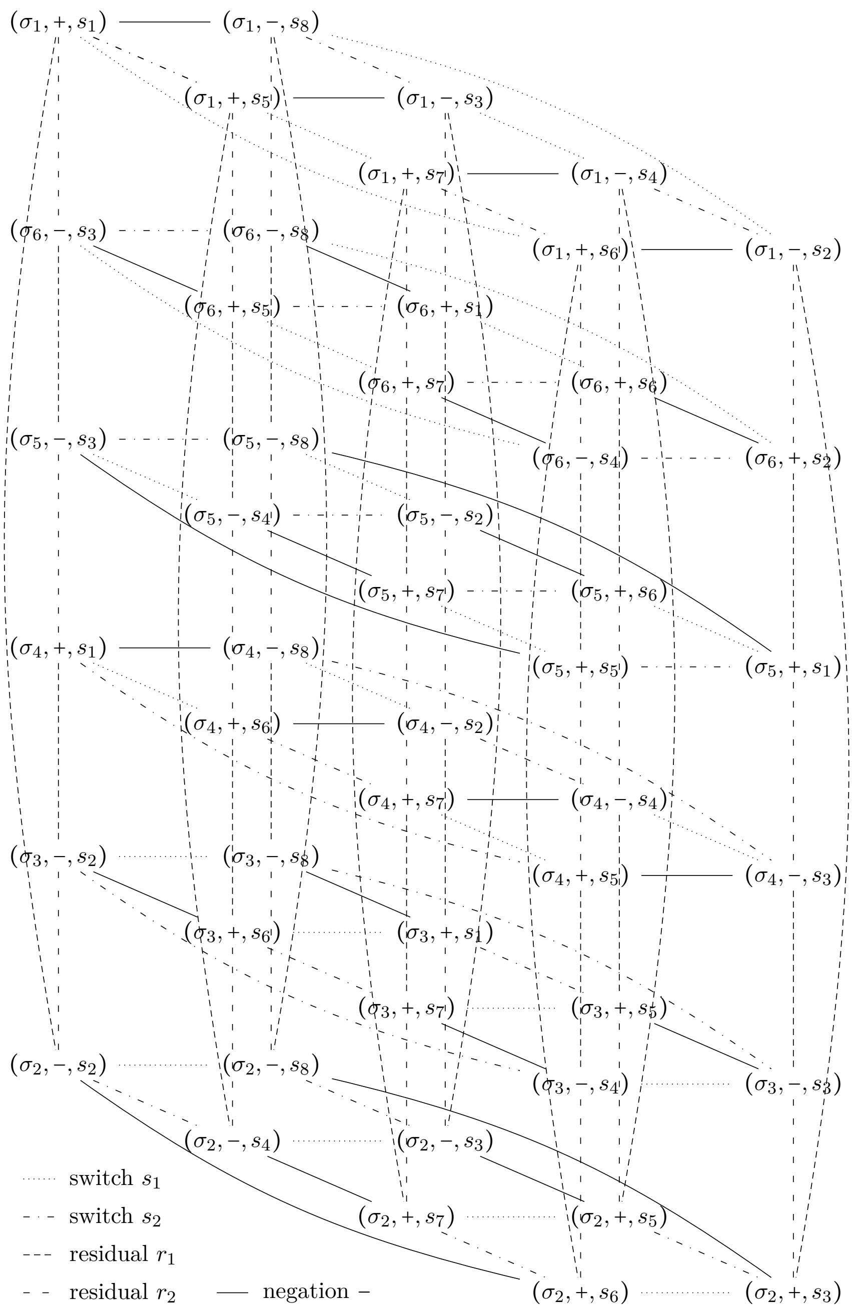

Figure 4: The 48 connectives of the orbit $\mathcal{O}_{\alpha_2 * \beta_2}((\sigma_1,+,s_1))$ related to each other by residual, negation and switch operations.

3. *For all $n \in \mathbb{N}^*$, the free action $\alpha_n * \beta_n * \gamma_n$ on the set of connectives $\mathbb{C}_n$ is transitive.*

Proof: See the Appendix, Section A. □

So, for every pair of connectives $(\circledast, \circledast')$, there exists a sequence of residuation(s), negation(s) and symmetry which transforms $\circledast$ into $\circledast'$. In other words, every gaggle connective $\circledast \in \mathbb{C}_n$ can be obtained from another connective $\circledast' \in \mathbb{C}_n$ with a suitable choice of element in the free groups $\mathfrak{S}_{n+1} * P_{(+,-)} * P_{(+,\sim)}$: for all $\circledast, \circledast' \in \mathbb{C}_n$, there is $g \in \mathfrak{S}_{n+1} * P_{(+,-)} * P_{(+,\sim)}$ such that $\circledast' = \alpha_n * \beta_n * \gamma_n(g, \circledast)$.

Example 33. In Figure 2, we represent the orbit $\mathcal{O}_{\alpha_{G_2}}((\sigma_1, +, s_1))$. It is isomorphic to a group of order 8 according to the first item of Proposition 32. In Figure 4, we represent the orbit $\mathcal{O}_{\alpha_2 * \beta_2}((\sigma_1, +, s_1))$ where the 48 binary connectives are related to each other by means of residuation, switch or Boolean negation. The other 48 binary connectives of the orbit $\mathcal{O}_{\alpha_2 * \beta_2}(\sim(\sigma_1, +, s_1))$ are obtained symmetrically by switching everywhere $-$ to $+$ and $+$ to $-$. These two orbits form a partition of $\mathbb{C}_2$ according to the second item of Proposition 32. The orbits $\mathcal{O}_{\alpha_2}(\circledast)$ of the binary connectives $\circledast$ of $\mathbb{C}_2$ are given in Figures 7, 8, 9, 10, 11 and 12. Every orbit $\mathcal{O}_{\alpha_2}(\circledast)$ is of cardinality $6 = |\mathfrak{S}_3|$. In order to follow common notations, binary connectives are denoted $\varphi \circledast \psi$ instead of $\circledast(\varphi, \psi)$. Finally, the orbit of $\mathcal{O}_{\alpha_2}((\sigma_1, +, s_1))$ is represented graphically in Figure 3, it corresponds to the outermost left vertical line of Figure 4.

6 Gaggle logics in the literature

In this section, we provide formal connections between our gaggle logics and substructural and non-classical logics. The last columns of our tables indicate the relevant publication where the gaggle logic connective was introduced for the first time. A logic close to our approach with connectives of arbitrary arity is the Generalized Lambek Calculus of Kolowska-Gawiejnovicz [26]. It is in fact the basic gaggle logic $(\mathcal{L}_\mathsf{C}, \mathcal{M}_\mathsf{C}, \Vdash)$ where $\mathsf{C} = \bigcup_{n \in \mathbb{N}^*} \{\circledast_n, r_i \circledast_n \mid, i = 1, \dots, n\}$ with $\circledast_n$ the n-ary connnective $(1, +, (\exists, (+, \dots, +)))$. ($\circledast_n$ and $r_i \circledast_n$ are denoted f and f/i in [26].)

6.1 Binary and unary connectives of basic gaggle logic

The truth conditions of the 16 unary gaggle connectives of gaggle logic are given in Figure 6 and those of the 96 binary gaggle connectives of gaggle logic in Figures 7, 8, 9, 10, 11 and 12. Many of these unary and binary connectives have already been introduced in the literature [30, 23, 28, 29, 40, 31, 22, 42, 2]. For example, the binary connectives $(\sigma_1, +, s_1)$, $(\sigma_5, s_3, -)$ and $(\sigma_3, s_2, -)$ are the fusion $\circ$, implication $\backslash$ and co-implication $/$ connectives of the Lambek calculus [30] used to illustrate our examples in Section 2. They are also denoted $\otimes_3$, $\supset_1$ and $\subset_2$ in update logic [2].[2] In the third column of the tables, we provide the bibliographical references where the connectives were first introduced. Note that each binary connective $\circledast$ has a commutative version $\circledast'$ which belongs to the same orbit/family so that for all formulas φ, ψ we have that $\varphi \circledast \psi = \psi \circledast' \varphi$. So, instead of 6 different connectives for each 2–ary orbit, we genuinely have 3 different connectives. This is in line with a result about colligated operations of Bimbó & Dunn [7]. For each orbit, one goes from one connective to the next by alternating residuations w.r.t. the first or the second argument, like in Figure 3. For example, $(\sigma_1, +, s_1) = r_1\ (\sigma_2, -, s_2) = r_1 r_2\ (\sigma_3, -, s_2) = r_1 r_2 r_1\ (\sigma_4, +, s_1) = r_1 r_2 r_1 r_2\ (\sigma_5, -, s_3) = r_1 r_2 r_1 r_2 r_1\ (\sigma_6, -, s_3)$.

To each family/orbit of connectives corresponds a series of laws of residuation. These laws are all instances of the same abstract law of residuation of Definition 10 and correspond to the action of transpositions of the form $(j\ n+1)$ on the set of connectives. They are of different types depending on the family/orbit to which they belong. These types were denoted in the literature: residuation connection, dual residuation connection, Galois connection and dual Galois connection (denoted *rp*, *drp*, *gc* and *dgc* by Goré [22]). These different 'types' of instance of the same abstract law of residuation for binary and unary connectives are given in Figure 5. In particular, note that the notion of dual residuation is the same as our definition of dual w.r.t. the j^{th} argument (Definition 28 and Proposition 29).

6.2 Non gaggle logics

Some connectives of non–classical logics are not connectives of gaggle logics. We mention two of them here. First, the standard modal connective interpreted over a neighborhood semantics [34, 35, 47]. It cannot be expressed by a combination of gaggle logic connectives, because its reformulation with a ternary relation contains an alternation of quantifiers that cannot occur in any function of Definition 5:

$$w \in [\![\Box\varphi]\!] \text{ iff } \exists u \forall v \,(Rwuv \leftrightarrow v \in [\![\varphi]\!])\,.$$

[2]There is a number of important typographical mistakes about *dual* update logic in [2]. In particular, in Definition 20 (dual update logic) of [2], y and z should be swapped in the truth conditions of $\prec_i$ and $\succ_i$. There are also some errors in the case study of Section 8 about bi-intuitionistic logic. A fully corrected version of [2] is available at `https://hal.inria.fr/hal-01476234v2/document`.

‘Type’ of the abstract law	Binary connectives	Unary connectives
Residuation	$\begin{array}{c} \varphi \otimes_{\mathrm{i}} \psi \Vdash \chi \\ \hline\hline \varphi \Vdash \psi \supset_{\mathrm{j}} \chi \\ \hline\hline \psi \Vdash \chi \subset_{\mathrm{k}} \varphi \end{array} \begin{array}{l} 1 \\ 2 \end{array}$	$\begin{array}{c} \Diamond^{-}\varphi \Vdash \psi \\ \hline\hline \varphi \Vdash \Box\psi \end{array} \qquad \begin{array}{c} \Diamond\varphi \Vdash \psi \\ \hline\hline \varphi \Vdash \Box^{-}\psi \end{array}$
Dual residuation	$\begin{array}{c} \chi \Vdash \varphi \oplus_{\mathrm{i}} \psi \\ \hline\hline \psi \succ_{\mathrm{j}} \chi \Vdash \varphi \\ \hline\hline \chi \prec_{\mathrm{k}} \varphi \Vdash \psi \end{array}$	
Galois	$\begin{array}{c} \varphi \mid_{\mathrm{i}} \psi \Vdash \chi \\ \hline\hline \varphi \mid_{\mathrm{j}} \chi \Vdash \psi \\ \hline\hline \psi \mid_{\mathrm{k}} \chi \Vdash \varphi \end{array}$	$\begin{array}{c} \varphi^{\mathbf{1}} \Vdash \psi \\ \hline\hline {}^{\mathbf{1}}\psi \Vdash \varphi \end{array}$
Dual Galois	$\begin{array}{c} \chi \Vdash \varphi \downarrow_{\mathrm{i}} \psi \\ \hline\hline \psi \Vdash \varphi \downarrow_{\mathrm{j}} \chi \\ \hline\hline \varphi \Vdash \psi \downarrow_{\mathrm{k}} \chi \end{array}$	$\begin{array}{c} \psi \Vdash \varphi^{\mathbf{0}} \\ \hline\hline \varphi \Vdash {}^{\mathbf{0}}\psi \end{array}$

Figure 5: Instances of the abstract law of residuation
$(i, j, k) \in \{(3, 1, 2), (2, 3, 1), (1, 2, 3)\}$

Second, the disjunction of connexive logics interpreted over the ternary semantics of relevant logics [37]. It cannot be expressed in basic gaggle logic either, because its formulation contains a pattern of Boolean connectives absent from the functions of Definition 5:

$$w \in [\![\varphi \vee \psi]\!] \text{ iff } \exists uv \left(Rwuv \wedge \left(u \in [\![\varphi]\!] \vee v \in [\![\psi]\!]\right)\right).$$

7 Calculi for Boolean gaggle logics

After some general definitions in Section 7.1 and definitions of structures and consecutions for gaggle logics in Definition 40, we introduce in Section 7.3 our calculus for *Boolean* basic gaggle logics. The calculus is a display calculus.

7.1 Preliminary definitions

These definitions are very general and apply to any kind of formalism.

Gaggle connective	Truth condition	Substructural connective
The existentially positive orbit: residuations		
$(\tau_1, +, t_1)\ \varphi$	$\exists v\,(v \in [\![\varphi]\!] \wedge Rvw)$	$\Diamond^{-}\varphi$ [40] $\Diamond_{\downarrow}$ [10]
$(\tau_2, -, t_2)\ \varphi$	$\forall v\,(v \in [\![\varphi]\!] \vee -Rwv)$	$\Box\varphi$ [28]
The universally positive orbit: residuations		
$(\tau_1, +, t_2)\ \varphi$	$\forall v\,(v \in [\![\varphi]\!] \vee Rvw)$	$+_{\downarrow}\varphi$ [10] [13, p. 401]
$(\tau_2, -, t_1)\ \varphi$	$\exists v\,(v \in [\![\varphi]\!] \wedge -Rwv)$	[10]
The existentially negative orbit: Galois connections		
$(\tau_1, +, t_4)\ \varphi$	$\exists v\,(v \notin [\![\varphi]\!] \wedge Rvw)$	$?\varphi$ [10][13, p. 402] $\boxminus_1\varphi$ [10][7, Def. 10.7.7]
$(\tau_2, +, t_4)\ \varphi$	$\exists v\,(v \notin [\![\varphi]\!] \wedge Rwv)$	$?_{\downarrow}\varphi$ [10][14] [13, p. 402] $\boxminus_2\varphi$ [7, Def. 10.7.7]
The universally negative orbit: dual Galois connections		
$(\tau_1, +, t_3)\ \varphi$	$\forall v\,(v \notin [\![\varphi]\!] \vee Rvw)$	$\varphi^{\perp}$ [10, 12] $\varphi^{\mathbf{o}}$ [22] $\ominus_1\varphi$ [7, Def. 10.7.2]
$(\tau_2, +, t_3)\ \varphi$	$\forall v\,(v \notin [\![\varphi]\!] \vee Rwv)$	$\sim\varphi$ [20] ${}^{\perp}\varphi$ [10, 12] ${}^{\mathbf{o}}\varphi$ [22] $\ominus_2\varphi$ [7, Def. 10.7.2]
The symmetrical existentially positive orbit: residuations		
$(\tau_1, -, t_1)\ \varphi$	$\exists v\,(v \in [\![\varphi]\!] \wedge -Rvw)$	[10]
$(\tau_2, +, t_2)\ \varphi$	$\forall v\,(v \in [\![\varphi]\!] \vee Rwv)$	$+\varphi$ [10] [13, p. 402] φ^{*} [7, Def. 7.1.19]
The symmetrical universally positive orbit: residuations		
$(\tau_1, -, t_2)\ \varphi$	$\forall v\,(v \in [\![\varphi]\!] \vee -Rvw)$	$\Box^{-}\varphi$ [40] $\Box_{\downarrow}$ [10]
$(\tau_2, +, t_1)\ \varphi$	$\exists v\,(v \in [\![\varphi]\!] \wedge Rwv)$	$\Diamond\varphi$ [28]
The symmetrical existentially negative orbit: Galois connections		
$(\tau_1, -, t_4)\ \varphi$	$\exists v\,(v \notin [\![\varphi]\!] \wedge -Rvw)$	$?\varphi$ [10][7, Ex. 1.4.5] $\varphi^{\mathbf{1}}$ [22]
$(\tau_2, -, t_4)\ \varphi$	$\exists v\,(v \notin [\![\varphi]\!] \wedge -Rwv)$	$?_{\downarrow}\varphi$ [10] [7, Ex. 1.4.5] ${}^{\mathbf{1}}\varphi$ [22]
The symmetrical universally negative orbit: dual Galois connections		
$(\tau_1, -, t_3)\ \varphi$	$\forall v\,(v \notin [\![\varphi]\!] \vee -Rvw)$	[10]
$(\tau_2, -, t_3)\ \varphi$	$\forall v\,(v \notin [\![\varphi]\!] \vee -Rwv)$	$\neg_h\varphi$ [29, 42] $\perp\varphi$ [14]

Figure 6: The 1–ary gaggle connectives

Gaggle connective	Truth condition	Substructural connective
	The conjunction orbit $\mathcal{O}_{\alpha_3}(\ (\sigma_1,+,s_1)\ )$: residuations	
$\varphi\ (\sigma_1,+,s_1)\ \psi$	$\exists uv\,(u \in \llbracket\varphi\rrbracket \wedge v \in \llbracket\psi\rrbracket \wedge Ruvw)$	$\varphi \circ \psi$ [30], $\varphi \otimes_3 \psi$ [2]
$\varphi\ (\sigma_2,-,s_2)\ \psi$	$\forall uv\,(u \in \llbracket\varphi\rrbracket \vee v \notin \llbracket\psi\rrbracket \vee -Rwvu)$	
$\varphi\ (\sigma_3,-,s_2)\ \psi$	$\forall uv\,(u \in \llbracket\varphi\rrbracket \vee v \notin \llbracket\psi\rrbracket \vee -Rvwu)$	$/$ [30], $\varphi \subset_2 \psi$ [2]
$\varphi\ (\sigma_4,+,s_1)\ \psi$ $= \psi\ (\sigma_1,+,s_1)\ \varphi$	$\exists uv\,(u \in \llbracket\varphi\rrbracket \wedge v \in \llbracket\psi\rrbracket \wedge Rvuw)$	
$\varphi\ (\sigma_5,-,s_3)\ \psi$ $= \psi\ (\sigma_2,-,s_2)\ \varphi$	$\forall uv\,(u \notin \llbracket\varphi\rrbracket \vee v \in \llbracket\psi\rrbracket \vee -Rwuv)$	$\backslash$ [30], $\varphi \supset_1 \psi$ [2]
$\varphi\ (\sigma_6,-,s_3)\ \psi$ $= \psi\ (\sigma_3,-,s_2)\ \varphi$	$\forall uv\,(u \notin \llbracket\varphi\rrbracket \vee v \in \llbracket\psi\rrbracket \vee -Ruwv)$	
	The not–but orbit $\mathcal{O}_{\alpha_3}(\ (\sigma_1,+,s_6)\ )$: residuations	
$\varphi\ (\sigma_1,+,s_6)\ \psi$	$\exists uv\,(u \notin \llbracket\varphi\rrbracket \wedge v \in \llbracket\psi\rrbracket \wedge Ruvw)$	$\varphi \succ_3 \psi$ [2]
$\varphi\ (\sigma_2,+,s_6)\ \psi$	$\exists uv\,(u \notin \llbracket\varphi\rrbracket \wedge v \in \llbracket\psi\rrbracket \wedge Rwvu)$	
$\varphi\ (\sigma_3,-,s_4)\ \psi$	$\forall uv\,(u \in \llbracket\varphi\rrbracket \vee v \in \llbracket\psi\rrbracket \vee -Rvwu)$	$\varphi \oplus_2 \psi$ [2]
$\varphi\ (\sigma_4,+,s_5)\ \psi$ $= \psi\ (\sigma_1,+,s_6)\ \varphi$	$\exists uv\,(u \in \llbracket\varphi\rrbracket \wedge v \notin \llbracket\psi\rrbracket \wedge Rvuw)$	
$\varphi\ (\sigma_5,+,s_5)\ \psi$ $= \psi\ (\sigma_2,+,s_6)\ \varphi$	$\exists uv\,(u \in \llbracket\varphi\rrbracket \wedge v \notin \llbracket\psi\rrbracket \wedge Rwuv)$	$\varphi \prec_1 \psi$ [2]
$\varphi\ (\sigma_6,-,s_4)\ \psi$ $= \psi\ (\sigma_3,-,s_4)\ \varphi$	$\forall uv\,(u \in \llbracket\varphi\rrbracket \vee v \in \llbracket\psi\rrbracket \vee -Ruwv)$	
	The but–not orbit $\mathcal{O}_{\alpha_3}(\ (\sigma_1,+,s_5)\ )$: residuations	
$\varphi\ (\sigma_1,+,s_5)\ \psi$	$\exists uv\,(u \in \llbracket\varphi\rrbracket \wedge v \notin \llbracket\psi\rrbracket \wedge Ruvw)$	$\varphi \prec_3 \psi$ [2]
$\varphi\ (\sigma_2,-,s_4)\ \psi$	$\forall uv\,(u \in \llbracket\varphi\rrbracket \vee v \in \llbracket\psi\rrbracket \vee -Rwvu)$	
$\varphi\ (\sigma_3,+,s_6)\ \psi$	$\exists uv\,(u \notin \llbracket\varphi\rrbracket \wedge v \in \llbracket\psi\rrbracket \wedge Rvwu)$	$\varphi \succ_2 \psi$ [2]
$\varphi\ (\sigma_4,+,s_6)\ \psi$ $= \psi\ (\sigma_1,+,s_5)\ \varphi$	$\exists uv\,(u \notin \llbracket\varphi\rrbracket \wedge v \in \llbracket\psi\rrbracket \wedge Rvuw)$	$\varphi \obslash \psi$ [23, 36]
$\varphi\ (\sigma_5,-,s_4)\ \psi$ $= \psi\ (\sigma_2,-,s_4)\ \varphi$	$\forall uv\,(u \in \llbracket\varphi\rrbracket \vee v \in \llbracket\psi\rrbracket \vee -Rwuv)$	$\varphi \oplus \psi$ [23, 36] $\varphi \oplus_1 \psi$ [2]
$\varphi\ (\sigma_6,+,s_5)\ \psi$ $= \psi\ (\sigma_3,+,s_6)\ \varphi$	$\exists uv\,(u \in \llbracket\varphi\rrbracket \wedge v \notin \llbracket\psi\rrbracket \wedge Ruwv)$	$\varphi \oslash \psi$ [23, 36]

Figure 7: The 2–ary gaggle connectives

Gaggle connective	Truth condition	Substructural connective
The symmetrical conjunction orbit $\mathcal{O}_{\alpha_3}(\ (\sigma_1, -, s_1)\ )$: residuations		
$\varphi\ (\sigma_1, -, s_1)\ \psi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge -Ruvw)$	$\varphi \circ \psi$ [7, Def. 5.2.3]
$\varphi\ (\sigma_2, +, s_2)\ \psi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee Rwvu)$	
$\varphi\ (\sigma_3, +, s_2)\ \psi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee Rvwu)$	
$\varphi\ (\sigma_4, -, s_1)\ \psi$ $= \psi\ (\sigma_1, -, s_1)\ \varphi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge -Rvuw)$	
$\varphi\ (\sigma_5, +, s_3)\ \psi$ $= \psi\ (\sigma_2, +, s_2)\ \varphi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee Rwuv)$	$\varphi \rightarrow \psi$ [7, Def. 5.2.3]
$\varphi\ (\sigma_6, +, s_3)\ \psi$ $= \psi\ (\sigma_3, +, s_2)\ \varphi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee Ruwv)$	
The symmetrical not–but orbit $\mathcal{O}_{\alpha_3}(\ (\sigma_1, -, s_6)\ )$: residuations		
$\varphi\ (\sigma_1, -, s_6)\ \psi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge -Ruvw)$	
$\varphi\ (\sigma_2, -, s_6)\ \psi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge -Rwvu)$	
$\varphi\ (\sigma_3, +, s_4)\ \psi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee Rvwu)$	
$\varphi\ (\sigma_4, -, s_5)\ \psi$ $= \psi\ (\sigma_1, -, s_6)\ \varphi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge -Rvuw)$	
$\varphi\ (\sigma_5, -, s_5)\ \psi$ $= \psi\ (\sigma_2, -, s_6)\ \varphi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge -Rwuv)$	
$\varphi\ (\sigma_6, +, s_4)\ \psi$ $= \psi\ (\sigma_3, +, s_4)\ \varphi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee Ruwv)$	
The symmetrical but–not orbit $\mathcal{O}_{\alpha_3}(\ (\sigma_1, -, s_5)\ )$: residuations		
$\varphi\ (\sigma_1, -, s_5)\ \psi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge -Ruvw)$	
$\varphi\ (\sigma_2, +, s_4)\ \psi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee Rwvu)$	
$\varphi\ (\sigma_3, -, s_6)\ \psi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge -Rvwu)$	
$\varphi\ (\sigma_4, -, s_6)\ \psi$ $= \psi\ (\sigma_1, -, s_5)\ \varphi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge -Rvuw)$	
$\varphi\ (\sigma_5, +, s_4)\ \psi$ $= \psi\ (\sigma_2, +, s_4)\ \varphi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee Rwuv)$	
$\varphi\ (\sigma_6, -, s_5)\ \psi$ $= \psi\ (\sigma_3, -, s_6)\ \varphi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge -Ruwv)$	

Figure 8: The 2–ary gaggle connectives

Gaggle connective	Truth condition	Substructural connective
The disjunction orbit $\mathcal{O}_{\alpha_3}(\ (\sigma_1, -, s_4)\ )$: dual residuations		
$\varphi\ (\sigma_1, -, s_4)\ \psi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee -Ruvw)$	$\varphi \oplus_3 \psi$ [2]
$\varphi\ (\sigma_2, +, s_5)\ \psi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge Rwvu)$	
$\varphi\ (\sigma_3, +, s_5)\ \psi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge Rvwu)$	$\varphi \prec_2 \psi$ [2]
$\varphi\ (\sigma_4, -, s_4)\ \psi$ $= \psi\ (\sigma_1, -, s_4)\ \varphi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee -Rvuw)$	
$\varphi\ (\sigma_5, +, s_6)\ \psi$ $= \psi\ (\sigma_2, +, s_5)\ \varphi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge Rwuv)$	$\varphi \succ_1 \psi$ [2]
$\varphi\ (\sigma_6, -, s_6)\ \psi$ $= \psi\ (\sigma_3, +, s_5)\ \varphi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge Ruwv)$	
The implication orbit $\mathcal{O}_{\alpha_3}(\ (\sigma_1, -, s_3)\ )$: dual residuations		
$\varphi\ (\sigma_1, -, s_3)\ \psi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee -Ruvw)$	$\varphi \supset_3 \psi$ [2]
$\varphi\ (\sigma_2, -, s_3)\ \psi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee -Rwvu)$	
$\varphi\ (\sigma_3, +, s_1)\ \psi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge Rvwu)$	$\varphi \otimes_2 \psi$ [2]
$\varphi\ (\sigma_4, -, s_2)\ \psi$ $= \psi\ (\sigma_1, +, s_3)\ \varphi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee -Rvuw)$	
$\varphi\ (\sigma_5, -, s_2)\ \psi$ $= \psi\ (\sigma_2, -, s_3)\ \varphi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee -Rwuv)$	$\varphi \subset_1 \psi$ [2]
$\varphi\ (\sigma_6, +, s_1)\ \psi$ $= \psi\ (\sigma_3, +, s_1)\ \varphi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge Ruwv)$	
The coimplication orbit $\mathcal{O}_{\alpha_3}(\ (\sigma_1, -, s_2)\ )$: dual residuations		
$\varphi\ (\sigma_1, -, s_2)\ \psi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee -Ruvw)$	$\varphi \subset_3 \psi$ [2]
$\varphi\ (\sigma_2, +, s_1)\ \psi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge Rwvu)$	
$\varphi\ (\sigma_3, -, s_3)\ \psi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee -Rvwu)$	$\varphi \supset_2 \psi$ [2]
$\varphi\ (\sigma_3, -, s_3)\ \psi$ $= \psi\ (\sigma_1, -, s_2)\ \varphi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee -Rvuw)$	
$\varphi\ (\sigma_5, +, s_1)\ \psi$ $= \psi\ (\sigma_2, +, s_1)\ \varphi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge Rwuv)$	$\varphi \otimes_1 \psi$ [2]
$\varphi\ (\sigma_6, -, s_2)\ \psi$ $= \psi\ (\sigma_3, -, s_3)\ \varphi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee -Ruwv)$	

Figure 9: The 2–ary gaggle connectives

Gaggle connective	Truth condition	Substructural connective
The symmetrical disjunction orbit $\mathcal{O}_{\alpha_3}(\ (\sigma_1, +, s_4)\ )$: dual residuations		
$\varphi\ (\sigma_1, +, s_4)\ \psi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee Ruvw)$	$\varphi \oplus \psi$ [22]
$\varphi\ (\sigma_2, -, s_5)\ \psi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge -Rwvu)$	$\varphi \prec \psi$ [22]
$\varphi\ (\sigma_3, -, s_5)\ \psi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge -Rvwu)$	
$\varphi\ (\sigma_4, +, s_4)\ \psi$ $= \psi\ (\sigma_1, +, s_4)\ \varphi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee Rvuw)$	
$\varphi\ (\sigma_5, -, s_6)\ \psi$ $= \psi\ (\sigma_2, -, s_5)\ \varphi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge -Rwuv)$	
$\varphi\ (\sigma_6, +, s_6)\ \psi$ $= \psi\ (\sigma_3, -, s_5)\ \varphi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge -Ruwv)$	$\varphi \succ \psi$ [22]
The symmetrical implication orbit $\mathcal{O}_{\alpha_3}(\ (\sigma_1, +, s_3)\ )$: dual residuations		
$\varphi\ (\sigma_1, +, s_3)\ \psi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee Ruvw)$	
$\varphi\ (\sigma_2, +, s_3)\ \psi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee Rwvu)$	
$\varphi\ (\sigma_3, -, s_1)\ \psi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge -Rvwu)$	
$\varphi\ (\sigma_4, +, s_2)\ \psi$ $= \psi\ (\sigma_1, +, s_3)\ \varphi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee Rvuw)$	
$\varphi\ (\sigma_5, +, s_2)\ \psi$ $= \psi\ (\sigma_2, +, s_3)\ \varphi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee Rwuv)$	
$\varphi\ (\sigma_6, -, s_1)\ \psi$ $= \psi\ (\sigma_3, -, s_1)\ \varphi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge -Ruwv)$	
The symmetrical coimplication orbit $\mathcal{O}_{\alpha_3}(\ (\sigma_1, +, s_2)\ )$: dual residuations		
$\varphi\ (\sigma_1, +, s_2)\ \psi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee Ruvw)$	
$\varphi\ (\sigma_2, -, s_1)\ \psi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge -Rwvu)$	
$\varphi\ (\sigma_3, +, s_3)\ \psi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee Rvwu)$	
$\varphi\ (\sigma_4, +, s_3)\ \psi$ $= \psi\ (\sigma_1, +, s_2)\ \varphi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee Rvuw)$	
$\varphi\ (\sigma_5, -, s_1)\ \psi$ $= \psi\ (\sigma_2, -, s_1)\ \varphi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge -Rwuv)$	
$\varphi\ (\sigma_6, +, s_2)\ \psi$ $= \psi\ (\sigma_3, +, s_3)\ \varphi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee Ruwv)$	

Figure 10: The 2–ary gaggle connectives

Gaggle connective	Truth condition	Substructural connective
The stroke orbit $\mathcal{O}_{\alpha_3}$ ($(\sigma_1,+,s_7)$): Galois connections		
$\varphi\ (\sigma_1,+,s_7)\ \psi$	$\exists uv\,(u\notin[\![\varphi]\!]\wedge v\notin[\![\psi]\!]\wedge Ruvw)$	$\varphi \mid_3 \psi$ [1, 22]
$\varphi\ (\sigma_2,+,s_7)\ \psi$	$\exists uv\,(u\notin[\![\varphi]\!]\wedge v\notin[\![\psi]\!]\wedge Rwvu)$	
$\varphi\ (\sigma_3,+,s_7)\ \psi$	$\exists uv\,(u\notin[\![\varphi]\!]\wedge v\notin[\![\psi]\!]\wedge Rvwu)$	
$\varphi\ (\sigma_4,+,s_7)\ \psi = \psi\ (\sigma_1,+,s_7)\ \varphi$	$\exists uv\,(u\notin[\![\varphi]\!]\wedge v\notin[\![\psi]\!]\wedge Rvuw)$	
$\varphi\ (\sigma_5,+,s_7)\ \psi = \psi\ (\sigma_2,+,s_7)\ \varphi$	$\exists uv\,(u\notin[\![\varphi]\!]\wedge v\notin[\![\psi]\!]\wedge Rwuv)$	$\varphi \mid_1 \psi$ [1, 22]
$\varphi\ (\sigma_6,+,s_7)\ \psi = \psi\ (\sigma_3,+,s_7)\ \varphi$	$\exists uv\,(u\notin[\![\varphi]\!]\wedge v\notin[\![\psi]\!]\wedge Ruwv)$	$\varphi \mid_2 \psi$ [1, 22]
The dagger orbit $\mathcal{O}_{\alpha_3}$ ($(\sigma_1,-,s_8)$): Galois connections		
$\varphi\ (\sigma_1,-,s_8)\ \psi$	$\forall uv\,(u\notin[\![\varphi]\!]\vee v\notin[\![\psi]\!]\vee -Ruvw)$	$\varphi \downarrow_3 \psi$ [1, 22]
$\varphi\ (\sigma_2,-,s_8)\ \psi$	$\forall uv\,(u\notin[\![\varphi]\!]\vee v\notin[\![\psi]\!]\vee -Rwvu)$	
$\varphi\ (\sigma_3,-,s_8)\ \psi$	$\forall uv\,(u\notin[\![\varphi]\!]\vee v\notin[\![\psi]\!]\vee -Rvwu)$	
$\varphi\ (\sigma_4,-,s_8)\ \psi = \psi\ (\sigma_1,-,s_8)\ \varphi$	$\forall uv\,(u\notin[\![\varphi]\!]\vee v\notin[\![\psi]\!]\vee -Rvuw)$	
$\varphi\ (\sigma_5,-,s_8)\ \psi = \psi\ (\sigma_2,-,s_8)\ \varphi$	$\forall uv\,(u\notin[\![\varphi]\!]\vee v\notin[\![\psi]\!]\vee -Rwuv)$	$\varphi \downarrow_1 \psi$ [1, 22]
$\varphi\ (\sigma_6,-,s_8)\ \psi = \psi\ (\sigma_3,-,s_8)\ \varphi$	$\forall uv\,(u\notin[\![\varphi]\!]\vee v\notin[\![\psi]\!]\vee -Ruwv)$	$\varphi \downarrow_2 \psi$ [1, 22]

Figure 11: The 2–ary gaggle connectives

Definition 34 (Logic)**.** A *logic* is a triple $\mathsf{L} = (\mathcal{L}, E, \models)$ where

- $\mathcal{L}$ is a *language* defined as a set of well-formed expressions built from a set of *connectives* C and a set of *atoms* $\mathbb{P}$;
- E is a *class of pointed models or frames*;
- $\models$ is a *satisfaction relation* which relates in a compositional manner elements of $\mathcal{L}$ to models of E by means of so-called *truth conditions*.

A $\mathcal{L}$*–consecution* is an expression of the form $\varphi \vdash \psi$, $\vdash \psi$ or $\varphi \vdash$, where $\varphi, \psi \in \mathcal{L}$. $\square$

Our definition of a calculus and of an inference rule is taken from [32].

Gaggle connective	Truth condition	Substructural connective
The symmetrical stroke orbit $\mathcal{O}_{\alpha_3}$ ($(\sigma_1,-,s_7)$): dual Galois connections		
$\varphi\ (\sigma_1,-,s_7)\ \psi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge -Ruvw)$	
$\varphi\ (\sigma_2,-,s_7)\ \psi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge -Rwvu)$	
$\varphi\ (\sigma_3,-,s_7)\ \psi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge -Rvwu)$	
$\varphi\ (\sigma_4,-,s_7)\ \psi = \psi\ (\sigma_1,-,s_7)\ \varphi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge -Rvuw)$	
$\varphi\ (\sigma_5,-,s_7)\ \psi = \psi\ (\sigma_2,-,s_7)\ \varphi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge -Rwuv)$	
$\varphi\ (\sigma_6,-,s_7)\ \psi = \psi\ (\sigma_3,-,s_7)\ \varphi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge -Ruwv)$	
The symmetrical dagger orbit $\mathcal{O}_{\alpha_3}$ ($(\sigma_1,+,s_8)$): dual Galois connections		
$\varphi\ (\sigma_1,+,s_8)\ \psi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee Ruvw)$	
$\varphi\ (\sigma_2,+,s_8)\ \psi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee Rwvu)$	
$\varphi\ (\sigma_3,+,s_8)\ \psi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee Rvwu)$	
$\varphi\ (\sigma_4,+,s_8)\ \psi = \psi\ (\sigma_1,+,s_8)\ \varphi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee Rvuw)$	
$\varphi\ (\sigma_5,+,s_8)\ \psi = \psi\ (\sigma_2,+,s_8)\ \varphi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee Rwuv)$	
$\varphi\ (\sigma_6,+,s_8)\ \psi = \psi\ (\sigma_3,+,s_8)\ \varphi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee Ruwv)$	

Figure 12: The 2–ary gaggle connectives

Definition 35 (Conservativity)**.** Let $\mathsf{L} = (\mathcal{L}, E, \models)$ and $\mathsf{L}' = (\mathcal{L}', E', \models')$ be two logics such that $\mathcal{L} \subseteq \mathcal{L}'$. We say that L' is a *conservative extension* of L when $\{\varphi \in \mathcal{L} \mid \models_{\mathsf{L}} \varphi\} = \mathcal{L} \cap \{\varphi' \in \mathcal{L}' \mid \models'_{\mathsf{L}'} \varphi'\}$. □

Definition 36 (Calculus and sequent calculus)**.** Let $\mathsf{L} = (\mathcal{L}, E, \models)$ be a logic. A *calculus* P for $\mathcal{L}$ is a set of elements of $\mathcal{L}$ called *axioms* and a set of *inference rules*. Most often, one can effectively decide whether a given element of $\mathcal{L}$ is an axiom. To be more precise, an *inference rule* R for $\mathcal{L}$ is a relation among elements of $\mathcal{L}$ such that there is a unique $l \in \mathbb{N}^*$ such that, for all $\varphi, \varphi_1, \ldots, \varphi_l \in \mathcal{L}$, one can effectively decide whether $(\varphi_1, \ldots, \varphi_l, \varphi) \in R$. The elements $\varphi_1, \ldots, \varphi_l$ are called the *premises* and φ is called the *conclusion* and we say that φ is a *direct consequence of* $\varphi_1, \ldots, \varphi_l$

by virtue of R. Let $\Gamma \subseteq \mathcal{L}$ and let $\varphi \in \mathcal{L}$. We say that φ is *provable* (from Γ) in P or a *theorem* of P, denoted $\vdash_{\mathsf{P}} \varphi$ (resp. $\Gamma \vdash_{\mathsf{P}} \varphi$), when there is a *proof* of φ (from Γ) in P, that is, a finite sequence of formulas ending in φ such that each of these formulas is:

1. either an instance of an axiom of P (or a formula of Γ);
2. or the direct consequence of preceding formulas by virtue of an inference rule R.

If $\mathcal{S}$ is a set of $\mathcal{L}$–consecutions, this set $\mathcal{S}$ can be viewed as a language. In that case, we call *sequent calculus for* $\mathcal{S}$ a calculus for $\mathcal{S}$.

Axioms and inference rules are often represented by means of *axiom schemas* and *inference rule schemas*, that is, expressions of the following form, depending on whether we deal with formulas of $\mathcal{L}$ or $\mathcal{L}$–consecutions:

Axiom schemas:

α $\qquad\qquad$ $A \vdash B$

Inference rule schemas:

$$\frac{\alpha_1 \quad \dots \quad \alpha_n}{\alpha} \qquad\qquad \frac{A_1 \vdash B_1 \quad \dots \quad A_n \vdash B_n}{A \vdash B}$$

where $\alpha_1, \dots, \alpha_n, \alpha$ are built up from *variables* often denoted $\varphi, \psi, \dots$ and the connectives of C and, likewise, $A_1, \dots, A_n, B_1, \dots, B_n, A, B$ are built up from *variables* often denoted $X, Y, \dots$ and the connectives of C. In this representation, inference rules and axioms schemas are closed by *uniform substitution*: each variable can be replaced uniformly by *any* well-formed expression of $\mathcal{L}$.

An inference rule R' is *derivable from an inference rule R in* P when there is a finite sequence of rules $R_1, \dots, R_n$ of P, with at least one of them equal to R, such that $R' = R_1 \circ \dots \circ R_n$. $\square$

Definition 37 (Truth, validity, logical consequence)**.** Let $\mathsf{L} = (\mathcal{L}, E, \models)$ be a logic. Let $M \in E$, $\varphi \in \mathcal{L}$, R be an inference rule for $\mathcal{L}$ and S, S' be either inference rules for $\mathcal{L}$ or formulas of $\mathcal{L}$. If Γ is a set of formulas or inference rules, we write $M \models \Gamma$ when for all $\varphi \in \Gamma$, we have $M \models \varphi$. Then, we say that

- φ is *true* (*satisfied*) at M or M is a *model* of φ when $M \models \varphi$;
- φ is *valid*, denoted $\models_{\mathsf{L}} \varphi$, when for all models $M \in E$, we have $M \models \varphi$;
- R is *true* (*satisfied*) at M or M is a *model* of R, denoted $M \models R$, when for all $(\varphi_1, \dots, \varphi_l, \varphi) \in R$, if $M \models \varphi_i$ for all $i \in \{1, \dots, l\}$, then $M \models \varphi$.

An inference rule R is *equivalent* to another inference rule R' iff for all $M \in E$, $M \models R$ iff $M \models R'$. □

Definition 38 (Soundness and completeness)**.** Let $\mathsf{L} = (\mathcal{L}, E, \models)$ be a logic. Let P be a calculus for $\mathcal{L}$. Then,

- P is *sound* for the logic L when for all $\varphi \in \mathcal{L}$, if $\vdash_{\mathsf{P}} \varphi$, then $\models_{\mathsf{L}} \varphi$.
- P is *(strongly) complete* for the logic L when for all $\varphi \in \mathcal{L}$ (and all $\Gamma \subseteq \mathcal{L}$), if $\models_{\mathsf{L}} \varphi$, then $\vdash_{\mathsf{P}} \varphi$ (resp. if $\Gamma \models_{\mathsf{L}} \varphi$, then $\Gamma \vdash_{\mathsf{P}} \varphi$). □

7.2 Structures and consecutions

In order to provide a sound and complete calculus for a gaggle logic based on a set of connectives $\mathsf{C} \subseteq \mathbb{C}$, we will need to resort to the connectives of C which are in the orbits of the free action $\alpha_n * \beta_n$ (for appropriate ns). We introduce these extra connectives in the language as *structural* connectives: they will appear in the proof derivations but not in the formulas proved by the calculus.

Definition 39 (Structural connectives)**.** *(Gaggle) structural connectives*, denoted $[\mathbb{C}]$, are a copy of the connectives: for all $\mathsf{C} \subseteq \mathbb{C}$,

$$[\mathsf{C}] \triangleq \{[\circledast] \mid \circledast \in \mathsf{C}\}\,.$$

Structural connectives are denoted $[p], [p_1], [p_2], \dots$ and $[\circledast], [\circledast_1], [\circledast_2], \dots$ For all $\circledast = (\sigma, \pm, s) \in \mathbb{C}$, the *arity*, *signature*, *tonicity signature*, *quantification signature* of $[\circledast]$ are the same as $\circledast$.

We also introduce the *(Boolean) structural connective* $\,,\,$. □

Definition 40 (Structural gaggle language and consecutions)**.** The *structural gaggle language* $[\mathcal{L}]$ is the smallest set that contains the gaggle language $\mathcal{L}$, the structures $*\varphi$ for all $\varphi \in \mathcal{L}$ as well as $[\mathbb{P}]$ and that is closed under the structural connectives of $[\mathbb{C}] \cup \{\,,\,\}$.

A $\mathcal{L}$*–consecution* (resp. $[\mathcal{L}]$*–consecution*) is an expression of the form $\varphi \vdash \psi$ (resp. $X \vdash Y$), where $\varphi, \psi \in \mathcal{L}$ (resp. $X, Y \in [\mathcal{L}]$). The set of all (Boolean) $\mathcal{L}$–consecutions (resp. $[\mathcal{L}]$–consecutions) is denoted $\mathcal{S}$ (resp. $[\mathcal{S}]$) and the set of all $\mathcal{L}^0$–consecutions is denoted $\mathcal{S}^0$. If $\mathsf{C} \subseteq \mathbb{C}$ then an element of $[\mathcal{L}]_{\mathsf{C}}$ (resp. $\mathcal{S}^0_{\mathsf{C}}$, $\mathcal{S}_{\mathsf{C}}$, $[\mathcal{S}]_{\mathsf{C}}$) is an element of $[\mathcal{L}]$ (resp. $\mathcal{S}^0$, $\mathcal{S}$, $[\mathcal{S}]$) which contains only connectives of $[\mathsf{C}]$.

Elements of $\mathcal{L}$ (resp. $[\mathcal{L}]$ and $[\mathcal{S}]$) are called *formulas* (resp. *structures* and *consecutions*); they are denoted $\varphi, \psi, \alpha, \dots$ (resp. $X, Y, A, B, \dots$ and $X \vdash Y, A \vdash B, \dots$). □

Definition 41 (Boolean negation)**.** Let $X \in [\mathcal{L}]$ be a structure. The *Boolean negation of* X, denoted $*X$, is defined inductively as follows:

$$*X \triangleq \begin{cases} [-\circledast](X_1, \ldots, X_n) & \text{if } X = [\circledast](X_1, \ldots, X_n) \\ (*X_1 \ , \ *X_2) & \text{if } X = (X_1 \ , \ X_2) \\ \varphi & \text{if } X = *\varphi \\ *\varphi & \text{if } X = \varphi \in \mathcal{L} \end{cases}$$

where $-\circledast$ was defined in Definition 20. □

Note that from that definition, for all structures $X \in [\mathcal{L}]$, it follows that $**X = X$.

Definition 42 (Formula associated to a structure)**.** We define inductively the function τ_0 and τ_1 from structures of $[\mathcal{L}]$ to formulas of $\mathcal{L}$ as follows: for all $i \in \{0, 1\}$, all $\circledast = (\sigma, \pm, (Æ, (\pm_1, \ldots, \pm_n)))$,

$$\begin{array}{rcl} \tau_i(\varphi) & \triangleq & \varphi \\ \tau_i(*\varphi) & \triangleq & \neg\varphi \\ \tau_0 \, (X \ , \ Y) & \triangleq & (\tau_0(X) \wedge \tau_0(Y)) \\ \tau_1 \, (X \ , \ Y) & \triangleq & (\tau_1(X) \vee \tau_1(Y)) \\ \tau_i \, ([\circledast] \, (X_1, \ldots, X_n)) & \triangleq & \circledast(\tau_{i_1}(X_1), \ldots, \tau_{i_n}(X_n)) \end{array}$$

where for all $j \in [\![1; n]\!]$, $\tau_{i_j}(X_j) \triangleq \begin{cases} \tau_i(X_j) & \text{if } \pm_j = + \\ \tau_{1-i}(X_j) & \text{if } \pm_j = - \end{cases}$.

Then, we define the function τ from $[\mathcal{L}]$–consecutions of $[\mathcal{S}]$ to $\mathcal{L}$–consecutions of $\mathcal{S}$ as follows:

$$\tau(X \models Y) \quad \triangleq \quad \tau_0(X) \models \tau_1(Y)$$ □

Instead of a single structural connective $\ ,\ $, we could introduce two Boolean structural connectives $[\wedge]$, $[\vee]$ as a copy of the Boolean connectives $\wedge, \vee$, like for the other gaggle connectives $\circledast$. This would not be usual but in line with our approach. This would greatly simplify the definition of the function τ since the interpretation of the structural connectives would then not be context-dependent as here. In particular one would not need two functions τ_0 and τ_1. We proceed as follows on the one hand in order to stay in line with current practice and on the other hand because it simplifies the subsequent calculus GGL_C of Figure 13: we use one structural connective (,) instead of two ($[\wedge]$ and $[\vee]$). This said, it would be easily possible to adapt and rewrite the calculus GGL_C with these two structural connectives $[\wedge]$ and $[\vee]$: the structural connective $\ ,\ $ would need to be replaced by $[\wedge]$ in the premise of (dr_2) and in $(\mathsf{B} \vdash), (\mathsf{Cl} \vdash), (\mathsf{K} \vdash), (\wedge \vdash)$ and by $[\vee]$ in the conclusion of (dr_2) and in $(\vdash \mathsf{B}), (\vdash \mathsf{Cl}), (\vdash \mathsf{K}), (\vdash \vee)$ (see below).

Definition 43 (Interpretation of gaggle structures and consecutions)**.** Let $\mathsf{C} \subseteq \mathbb{C}$ and let $M = (W, \mathcal{R})$ be a C–model. We extend the interpretation function $[\![\cdot]\!]^M$ of $\mathcal{L}_\mathsf{C}$ in M to $\mathcal{L}_\mathsf{C}$–consecutions of $\mathcal{S}_\mathsf{C}$ as follows: for all $\varphi, \psi \in \mathcal{L}_\mathsf{C}$ and all $w \in W$, we have that $w \in [\![\varphi \vdash \psi]\!]^M$ iff if $w \in [\![\varphi]\!]^M$ then $w \in [\![\psi]\!]^M$, we have that $w \in [\![\ \vdash \psi]\!]^M$ iff $w \in [\![\psi]\!]^M$ and we have that $w \in [\![\varphi \vdash\ ]\!]^M$ iff $w \notin [\![\varphi]\!]^M$. We then extend in a natural way the interpretation function $[\![\cdot]\!]^M$ of $\mathcal{L}_\mathsf{C}$ in M to $[\mathcal{L}]_\mathsf{C}$–consecutions of $[\mathcal{S}]_\mathsf{C}$ as follows: for all $X \in \mathcal{L}_\mathsf{C}$, all $X \vdash Y \in [\mathcal{S}]_\mathsf{C}$ and all $w \in W$, we have that $w \in [\![X \vdash Y]\!]^M$ if, and only if, $w \in [\![\tau(X \vdash Y)]\!]^M$. If $\mathcal{E}_\mathsf{C}$ is a class of C–models, then the satisfaction relation $\Vdash\ \subseteq \mathcal{E}_\mathsf{C} \times [\mathcal{S}]_\mathsf{C}$ is defined like for formulas of $\mathcal{L}$. □

7.3 Our display calculus

We introduce a calculus for *Boolean* basic gaggle logics. Our calculus is defined relatively to an orbit/family of connectives. This means that if we have a basic gaggle logic defined on the basis of some connectives C and if we want to obtain a sound and complete calculus for that logic, we need to consider in the proof system the following associated set of connectives:

$$\mathcal{O}(\mathsf{C}) \triangleq \bigcup_{\circledast \in \mathsf{C}} \left\{ \mathcal{O}_{\alpha_n * \beta_n}(\circledast) \mid a(\circledast) = n \right\} \tag{4}$$

This set of connectives $\mathcal{O}(\mathsf{C})$ is stable under the free action $\alpha_n * \beta_n$: for all $\circledast \in \mathcal{O}(\mathsf{C})$, we have that $\mathcal{O}_{\alpha_n * \beta_n}(\circledast) \subseteq \mathcal{O}(\mathsf{C})$. This is because in the completeness proof, we need to apply the abstract law of residuation for any arguments j (associated to the residuation operator of Definition 18) and consider the Boolean negation for each connective. This entails that we must consider the orbits of the connectives of C under the free action $\alpha_n * \beta_n$.

Definition 44. Let $\mathsf{C} \subseteq \mathbb{C}$. We denote by GGL_C the calculus of Figure 13 where the introduction rules $(\vdash \circledast)$ and $(\circledast \vdash)$ are defined for the connectives $\circledast$ of C and where the rule (dr_1) is defined for the elements τ of an arbitrary set of generators of $\mathfrak{S}_{n+1}$ (for each n ranging over the arities of the connectives of C). □

Theorem 45 (Soundness and strong completeness)**.** *Let $\mathsf{C} \subseteq \mathbb{C}$ be such that $\mathcal{O}(\mathsf{C}) = \mathsf{C}$. The calculus GGL_C is sound and strongly complete for the Boolean basic gaggle logic $(\mathcal{S}_\mathsf{C}, \mathcal{M}_\mathsf{C}, \Vdash)$.*

Proof: See the Appendix, Section B. □

Some comments about the rules of the calculus GGL_C are needed.

Structural rules:

$$\frac{((X\ ,\ Y)\ ,\ Z)\vdash U}{(X\ ,\ (Y\ ,\ Z))\vdash U}\ (\mathsf{B}\vdash) \qquad \frac{(X\ ,\ Y)\vdash U}{(Y\ ,\ X)\vdash U}\ (\mathsf{Cl}\vdash)$$

$$\frac{X\vdash U}{(X\ ,\ Y)\vdash U}\ (\mathsf{K}\vdash) \qquad \frac{(X\ ,\ X)\vdash U}{X\vdash U}\ (\mathsf{WI}\vdash)$$

$$\frac{U\vdash\varphi \quad \varphi\vdash V}{U\vdash V}\ cut$$

Display rules:

$$\frac{S([\circledast],X_1,\ldots,X_n,X_{n+1})}{S\big([\tau\circledast],X_{\tau(1)},\ldots,X_{\tau(n)},X_{\tau(n+1)}\big)}\ (\mathsf{dr}_1) \qquad \frac{(X\ ,\ Y)\vdash Z}{X\vdash(Z\ ,\ *Y)}\ (\mathsf{dr}_2)$$

Introduction rules:

$$\frac{U\vdash\ *\varphi}{U\vdash\neg\varphi}\ (\vdash\neg) \qquad \frac{*\varphi\vdash U}{\neg\varphi\vdash U}\ (\neg\vdash)$$

$$\frac{X\vdash\varphi \quad Y\vdash\psi}{(X\ ,\ Y)\vdash(\varphi\wedge\psi)}\ (\vdash\wedge) \qquad \frac{(\varphi\ ,\ \psi)\vdash U}{(\varphi\wedge\psi)\vdash U}\ (\wedge\vdash)$$

$$\frac{U\vdash(\varphi\ ,\ \psi)}{U\vdash(\varphi\vee\psi)}\ (\vdash\vee) \qquad \frac{\varphi\vdash X \quad \psi\vdash Y}{(\varphi\vee\psi)\vdash(X\ ,\ Y)}\ (\vee\vdash)$$

$$\frac{U_1\vdash V_1 \quad \ldots \quad U_n\vdash V_n}{S([\circledast],X_1,\ldots,X_n,\circledast(\varphi_1,\ldots,\varphi_n))}\ (\vdash\circledast) \qquad \frac{S([\circledast],\varphi_1,\ldots,\varphi_n,U)}{S(\circledast,\varphi_1,\ldots,\varphi_n,U)}\ (\circledast\vdash)$$

In rules $(\vdash\circledast)$ and $(\circledast\vdash)$, for all $\circledast=(\sigma,\pm,(Æ,(\pm_1,\ldots,\pm_n)))\in\mathsf{C}$:

- for all $j\in[\![1;n]\!]$, we set $U_j\vdash V_j \triangleq \begin{cases} X_j\vdash\varphi_j & \text{if } \pm_j\pm(Æ)=- \\ \varphi_j\vdash X_j & \text{if } \pm_j\pm(Æ)=+ \end{cases}$
  such that, in rule $(\vdash\circledast)$, for all j X_j is not empty and if φ_j is empty for some j then $\circledast(\varphi_1,\ldots,\varphi_n)$ is also empty.
- for all $\star\in\{\circledast,[\circledast]\}$, $S(\circledast,X_1,\ldots,X_n,X)\triangleq\begin{cases}\star(X_1,\ldots,X_n)\vdash X & \text{if } Æ=\exists \\ X\vdash\ \star(X_1,\ldots,X_n) & \text{if } Æ=\forall\end{cases}$.

If X is empty then $*X$ is empty and $(X\ ,\ Y)$ and $(Y\ ,\ X)$ are equal to Y.

Figure 13: Calculus $\mathsf{GGL_C}$

• The axioms and inference rules for atoms p are special instances of the rules $(\vdash \circledast)$ and $(\circledast \vdash)$ of Figure 13. With $\circledast = p$, we have that $n = 0$ and, replacing $\circledast$ with p in $(\vdash \circledast)$, we obtain the inference rules below. Note that $(\vdash p)$ is in fact an axiom.

$$\frac{}{S([p]\,,p)}\ (\vdash p) \qquad\qquad \frac{S\,([p]\,,X)}{S\,(p,X)}\ (p \vdash)$$

where, if $\circledast$ is p or $[p]$, then $S(\circledast, X) \triangleq \begin{cases} \circledast \models X & \text{if } Æ = \exists \\ X \models \circledast & \text{if } Æ = \forall \end{cases}$.

Hence, for all $p = (1, \pm, Æ)$, if $Æ = \exists$ then $(\vdash p)$ and $(p \vdash)$ rewrite as follows:

$$\frac{}{[p] \models p}\ (\vdash p) \qquad\qquad \frac{[p] \models X}{p \models X}\ (p \vdash) \tag{5}$$

and if $Æ = \forall$ then $(\vdash p)$ and $(p \vdash)$ rewrite as follows:

$$\frac{}{p \models [p]}\ (\vdash p) \qquad\qquad \frac{X \models [p]}{X \models p}\ (p \vdash) \tag{6}$$

Note that in both cases, the standard axiom $p \models p$ is derivable by applying $(p \vdash)$ once again to $[p] \models p$ or $p \models [p]$. If $[p]$ is replaced by $\mathbf{I}$ and p by $\top$ in the first pair and if $[p]$ is replaced by $\mathbf{I}$ and p by $\bot$ in the second pair then we obtain respectively the operational rules $(\models \top)$, $(\top \models)$, $(\bot \models)$ and $(\models \bot)$ of Kracht [27] and Belnap [6]. This is meaningful since truth constants can be seen as special atoms, those that are always true or always false. Then, one needs, like in the calculus **DLM** of Kracht [27], to impose some conditions on these atoms by means of particular structural inference rules so that these special atoms $\top$ and $\bot$ do behave as truth constants, as intended. Note that the reading of **I**, either as $\top$ or as $\bot$, is clearly separated here by means of two structural constants, whereas in the literature it is disambiguated depending on the context, whether it is in antecedent part or consequent part of a consecution. Alternatively, one can easily prove (by extending the proof of Section **??**) that adding the following axioms to our calculus $\mathsf{GGL_C}$ is enough to capture the standard truth constants $\top$ and $\bot$:

$$\frac{}{\bot \models}\ (\bot \vdash) \qquad\qquad \frac{}{\models \top}\ (\vdash \top)$$

• The Boolean operator $*$ transforms the structures on which it is applied. It does not function as an operator applied externally on structures, it modifies them internally. Hence, for example, for any structure $[\circledast]\,(X_1, \ldots, X_n)$, $*\,[\circledast]\,(X_1, \ldots, X_n)$ is equal to $[-\circledast]\,(X_1, \ldots, X_n)$. In that sense, it is formally different from the usual structural connective $*$ used in display logics, even if its semantic meaning is the same (it behaves as a Boolean negation). Moreover, because by Definition 41 $* * X = X$, the following rule is a reformulation of the display rule (dr_2) (premise and conclusion are turned upside down):

$$\frac{X \vdash (Y \;,\; Z)}{\overline{(X \;,\; *Z) \vdash Y}}$$

• Because of our convention that if X is empty then $(X \;,\; Y)$ and $(Y \;,\; X)$ are equal to Y, the following rules are specific instances of the display rule (dr_2):

$$\frac{(X \;,\; Y) \vdash}{\overline{X \vdash \; *Y}} \qquad\qquad \frac{\vdash (Z \;,\; Y)}{\overline{*Y \vdash Z}}$$

Likewise, if $\circledast = (\sigma, \pm, (Æ, (\pm_1, \dots, \pm_n)))$ is such that, for example, $Æ = \exists$ and $\pm_j = +$, then the following rule is an instance of the rule $(\vdash \circledast)$, because of our conventions about empty structures in the rule $(\vdash \circledast)$:

$$\frac{U_1 \vdash V_1 \quad \dots \quad X_j \vdash \quad \dots \quad U_n \vdash V_n}{[\circledast]\,(X_1, \dots, X_j, \dots, X_n) \vdash} \tag{7}$$

• The introduction rule $(\vdash \circledast)$ of our calculus is a direct translation in gaggle logics of the tonicity relations of Theorem 10. Likewise, the structural rule (dr_1) is a translation and a generalization of the abstract law of residuation of Theorem 10 (see Proposition 27).

• As shown in Example 26, $\wedge$ and $\vee$ can be formalized by the gaggle connectives $(\sigma_1, +, s_1)$ and $(\sigma_1, -, s_4)$ if these are interpreted on identity ternary relations (which can be obtained by imposing the validity of the classic structural rules involving these connectives). Hence, unsurprisingly, rules $(\vdash \vee)$ and $(\wedge \vdash)$ are instances of the (gaggle) rule $(\circledast \vdash)$ and rules $(\vdash \wedge)$ and $(\vee \vdash)$ are also instances of the (gaggle) rule $(\vdash \circledast)$.

This said, one could equivalently replace $(\vdash \wedge)$ and $(\vee \vdash)$ by their extensional/additive version $(\vdash \wedge)'$ and $(\vee \vdash)'$ of Proposition 46 and still obtain the completeness of the resulting calculus. In fact, completeness still holds if one also removes the contraction rule $(\mathsf{WI} \vdash)$ because a contraction is hidden in the extensional/additive version of the conjunction and disjunction rule. Yet, one needs the contraction rule $(\mathsf{WI} \vdash)$ explicitly to prove cut elimination, in particular for condition $(C8)$ with the conjunction case (see Theorem 49). So, we prefer to take in our calculus the intensional/multiplicative version $(\vdash \wedge)$ and $(\vee \vdash)$ of the conjunction and disjunction rules because they are instances of the general rules $(\circledast \vdash)$ and $(\vdash \circledast)$ for gaggle connectives.

• Our calculus has the subformula property, but not the substructure property: every formula appearing in a cut–free proof of a consecution is a subformula of a formula of the final consecution.

• In the calculus $\mathsf{GGL_C}$, we do not need to consider *all* permutations τ of the symmetric group $\mathfrak{S}_{n+1}$. In fact, it suffices to consider only a set of generators of $\mathfrak{S}_{n+1}$ because rules for any permutations are derivable from these rules for generators as the following proposition shows. One could naturally consider transpositions because they generate the symmetric group and correspond to residuation operations. One could consider as well other generators of the symmetric group $\mathfrak{S}_{n+1}$, such as the pair $\{(n\ n+1), (1\ 2\ \ldots\ n+1)\}$ or the set of generators $\{(1\ 2), (2\ 3), \ldots, (i\ i+1), \ldots, (n\ n+1)\}$ or $(1\ 2)$ together with the 3–cycles (see Section 3). Hence, one can reduce the number of inference rules (dr_1) from $(n+1)!$ to 2: it suffices to define the calculus $\mathsf{GGL_C}$ only with the rules (dr_1) where $\tau = (n\ n+1)$ and $\tau = (1\ 2\ \ldots\ n+1)$ for example. Indeed, the rules (dr_1) with $\tau \in \mathfrak{S}_{n+1}$ different from $(n\ n+1)$ and $(1\ 2\ \ldots\ n+1)$ are all derivable from these two rules since these two cycles generate $\mathfrak{S}_{n+1}$.

Proposition 46. *Let $C \subseteq \mathbb{C}$ and let $\circledast \in C$ be a n-ary connective. The following rules are all derivable in GGL_C.*

$$\frac{X \vdash Y}{\overline{\overline{*Y \vdash *X}}}\ (dr'_2) \qquad \frac{S([\circledast], X_1, \ldots, X_j, \ldots, X_n, X)}{S([s_j\circledast], X_1, \ldots, *X_j, \ldots, X_n, X)}\ (sw^j)$$

$$\frac{*X \vdash Y}{\overline{\overline{*Y \vdash X}}}\ (dr''_2) \qquad \frac{X \vdash *Y}{\overline{\overline{Y \vdash *X}}}\ (dr'''_2)$$

$$\frac{U \vdash ((X\ ,\ Y)\ ,\ Z)}{\overline{\overline{U \vdash (X\ ,\ (Y\ ,\ Z))}}}\ (\vdash B) \qquad \frac{U \vdash (X\ ,\ Y)}{U \vdash (Y\ ,\ X)}\ (\vdash CI)$$

$$\frac{U \vdash X}{U \vdash (X\ ,\ Y)}\ (\vdash K) \qquad \frac{U \vdash (X\ ,\ X)}{U \vdash X}\ (\vdash WI)$$

$$\frac{U \vdash \varphi \quad U \vdash \psi}{U \vdash (\varphi \wedge \psi)}\ (\vdash \wedge)' \qquad \frac{\varphi \vdash U \quad \psi \vdash U}{(\varphi \vee \psi) \vdash U}\ (\vee \vdash)'$$

The rule (dr'_2) is called the Boolean negation rule *and the rule (sw^j), for $j \in [\![1;n]\!]$, is called the* switch rule w.r.t. the j^{th} argument. *The rule (dr_1) is also derivable in GGL_C,* for all $\tau \in \mathfrak{S}_{n+1}$.

Proof: See the Appendix, Section A. □

8 Cut elimination and displayability

In this section, we prove that the cut rule can be eliminated from any proof of GGL_C. This result relies on the fact that our gaggle calculi are in fact display calculi and enjoy the display property: every substructure of a consecution provable in GGL_C can be displayed as the sole antecedent or consequent of a provably equivalent consecution. In display calculi [6], the antecedent or consequent position depends on the kind of position in which the given substructure appears in the consecution: either in "antecedent part" or in "consequent part". In standard display logics, these two related notions are defined on the basis of the parity of the number of structural connectives $*$ that occur in front of the given substructure (odd or even). Since our framework is more abstract, we reformulate these two notions in a more abstract form based on the tonicity of the connectives that occur in front of the substructure. This leads us to define the following notions of 'protoantecedant part' and 'protoconsequent part'. A similar notion was defined by Goré [21] without Boolean structural connectives.

Definition 47 (Protoantecedent and protoconsequent part)**.** Let $X, Y, Z \in [\mathcal{L}]$ be structures. If Z is a substructure of X, then $tn(X, Z)$ is defined inductively as follows:

- if $X = Z$ then $tn(X, Z) \triangleq +$;
- if $X = *Y$ and Z appears in Y then $tn(X, Z) \triangleq -tn(Y, Z)$;
- if $X = (X_1 \ , \ X_2)$ and Z appears in X_j then $tn(X, Z) \triangleq tn(X_j, Z)$;
- if $X = [\circledast](X_1, \ldots, X_n)$ and Z appears in X_j then $tn(X, Z) \triangleq tn(\circledast, j) tn(X_j, Z)$.

If $X \vdash Y$ is a $[\mathcal{L}]$–consecution, then X is called the *antecedent* and Y is called the *consequent* of $X \vdash Y$. If Z is a substructure of X or Y, Z is called a *protoantecedent part* (resp. *protoconsequent part*) of $X \vdash Y$ when $tn(X, Z) = +$ or $tn(Y, Z) = -$ (resp. $tn(X, Z) = -$ or $tn(Y, Z) = +$). □

Proposition 48 (Display property)**.** *Let $\mathsf{C} \subseteq \mathbb{C}$. For all $[\mathcal{L}]$–consecutions $X \vdash Y$ provable in GGL_C and for all substructure Z of $X \vdash Y$,*

- *if Z is protoantecedent part of $X \vdash Y$ then there exists a structure $W \in [\mathcal{L}]$ such that $Z \vdash W$ is provably equivalent to $X \vdash Y$ in $\mathsf{GGL_C}$;*
- *if Z is protoconsequent part of $X \vdash Y$ then there exists a structure $W \in [\mathcal{L}]$ such that $W \vdash Z$ is provably equivalent to $X \vdash Y$ in $\mathsf{GGL_C}$.*

Hence, $\mathsf{GGL_C}$ is a display calculus.

Proof: It follows from an inductive application of the display rules (dr_1) and (dr_2) on each substructure of X (or Y) containing Z, from the outermost one to the innermost one (Z itself). We use (dr_1) if we have to 'unfold' a structural gaggle connective $[\circledast]$ and (dr_2) (or one of its derived rules) if we have to 'unfold' the structural Boolean connective , . □

Theorem 49 (Cut–elimination)**.** *Let $\mathsf{C} \subseteq \mathbb{C}$. The calculus $\mathsf{GGL_C}$ is cut–eliminable: it is possible to eliminate all occurrences of the cut rule from a given proof in order to obtain a cut-free proof of the same consecution.*

Proof: See the Appendix, Section C. □

As usual in proof theory and ever since Gentzen [18], the fact that the cut rule can be eliminated from any proof is of practical and theoretical importance and we easily obtain a number of significant results about our logics. This also holds in our setting.

Theorem 50 (Conservativity)**.** *If $\mathsf{C} \subseteq \mathsf{C}' \subseteq \mathbb{C}$ then the logic $\left(\mathcal{S}_{\mathsf{C}'}, \mathcal{M}_{\mathsf{C}'}, \Vdash\right)$ is a conservative extension of the logic $\left(\mathcal{S}_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash\right)$.*

Proof: It is standard because our calculi have the subformula property. See for example [39] for details. □

Theorem 51 (Soundness and strong completeness)**.** *Let $\mathsf{C} \subseteq \mathbb{C}$. The calculus $\mathsf{GGL_C}$ is sound and strongly complete for the Boolean basic gaggle logic $(\mathcal{S}_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$.*

Proof: Since any proof of a consecution $\varphi \vdash \psi \in \mathcal{S}_{\mathsf{C}}$ can be cut–free and our calculus has the subformula property, it contains only the introduction rules $(\vdash \circledast)$ for the connectives of C. (The introduction rules for the other connectives of $\mathcal{O}(\mathsf{C}) - \mathsf{C}$ were needed in the initial completeness proof before the cut elimination theorem for Lemma 68.) □

The difference between the above theorem and Theorem 45 is that the set of connectives C considered is not assumed to be such that $\mathsf{C} = \mathcal{O}(\mathsf{C})$ (we recall that $\mathcal{O}(\mathsf{C})$ is defined by Expression (4)). Thanks to cut–elimination, the completeness result also holds if we do not have equality. This said, all connectives of $\mathcal{O}(\mathsf{C})$ do appear in the calculus, but only as structural connectives.

9 Calculi for gaggle logics

Until now, our calculi are sound and complete for logics including the Boolean connectives. However, we would like to obtain calculi for plain gaggle logics, without Boolean connectives. Indeed, we consider the latter to be more primitive than Boolean gaggle logics because even the Boolean connectives can be seen as particular gaggle connectives, interpreted over special relations (identity relations, see Example 26). These special relations are obtained at the proof-theroretical level by imposing the validity of Gentzen's structural rules. So, in this section, we are going to define sound and complete calculi for (plain) gaggle logics, without Boolean connectives.

Definition 52. Let $\mathsf{C} \subseteq \mathbb{C}$. We denote by $\mathsf{GGL}^0_\mathsf{C}$ the calculus of Figure 14 where the introduction rules $(\vdash \otimes)$ and $(\otimes \vdash)$ are defined for the connectives $\otimes$ of C and where the rule (dr_1) is defined for the elements τ of an arbitrary set of generators of $\mathfrak{S}_{n+1}$ (for each n ranging over the arities of the connectives of C). $\square$

Note that $\left(\mathsf{dr}'_2\right)$ (introduced in Proposition 46) is in $\mathsf{GGL}^0_\mathsf{C}$ instantiated with gaggle connectives. More precisely, in $\mathsf{GGL}^0_\mathsf{C}$, an application of $\left(\mathsf{dr}'_2\right)$ is of the following form:

$$\frac{[\otimes]\,(X_1,\ldots,X_m) \vdash [\otimes']\,(X'_1,\ldots,X'_n)}{[-\otimes']\,(X'_1,\ldots,X'_n) \vdash [-\otimes]\,(X_1,\ldots,X_m)} \qquad \frac{\otimes(\varphi_1,\ldots,\varphi_m) \vdash \otimes'\,(\varphi'_1,\ldots,\varphi'_n)}{* \otimes'\,(\varphi'_1,\ldots,\varphi'_n) \vdash * \otimes(\varphi_1,\ldots,\varphi_m)}$$

An equivalent axiomatization of $\mathsf{GGL}^0_\mathsf{C}$ is obtained if we replace rule $\left(\mathsf{dr}'_2\right)$ by the switch rule $\left(\mathsf{sw}^j\right)$ of Proposition 46, for each $j \in [\![1;n]\!]$:

$$\frac{S([\otimes]\,, X_1,\ldots,X_j,\ldots,X_n,X)}{S([s_j\otimes]\,, X_1,\ldots,*X_j,\ldots,X_n,X)}\ \left(\mathsf{sw}^j\right).$$

This is due to the fact that the switch rule is derivable in $\mathsf{GGL}^0_\mathsf{C}$ and, vice versa, $\left(\mathsf{dr}'_2\right)$ is derivable from the switch rule and (dr_1) thanks to Proposition 30.

The main difference between GGL_C and $\mathsf{GGL}^0_\mathsf{C}$ lies in the fact that the introduction rules for the Boolean connectives have been removed as well as the structural rules.

Display rules:

$$\frac{S\left([\circledast], X_1, \ldots, X_n, X_{n+1}\right)}{S\left([\tau\circledast], X_{\tau(1)}, \ldots, X_{\tau(n)}, X_{\tau(n+1)}\right)}\ (\mathsf{dr}_1) \qquad \frac{X \vdash Y}{*Y \vdash *X}\ (\mathsf{dr}_2')$$

(the rules (dr_1) and (dr_2') are printed with double lines)

Introduction rules:

$$\frac{U_1 \vdash V_1 \quad \ldots \quad U_n \vdash V_n}{S\left([\circledast], X_1, \ldots, X_n, \circledast\left(\varphi_1, \ldots, \varphi_n\right)\right)}\ (\vdash \circledast) \qquad \frac{S\left([\circledast], \varphi_1, \ldots, \varphi_n, U\right)}{S\left(\circledast, \varphi_1, \ldots, \varphi_n, U\right)}\ (\circledast \vdash)$$

In rules $(\vdash \circledast)$ and $(\circledast \vdash)$, for all $\circledast = (\sigma, \pm, (Æ, (\pm_1, \ldots, \pm_n))) \in \mathsf{C}$:

- for all $j \in [\![1;n]\!]$, we set $U_j \vdash V_j \triangleq \begin{cases} X_j \vdash \varphi_j & \text{if } \pm_j \pm (Æ) = - \\ \varphi_j \vdash X_j & \text{if } \pm_j \pm (Æ) = + \end{cases}$
  such that, in rule $(\vdash \circledast)$, for all j X_j is not empty and with the convention that if φ_j is empty for some j then $\circledast(\varphi_1, \ldots, \varphi_n)$ is also empty.
- for all $\star \in \{\circledast, [\circledast]\}$, $S(\star, X_1, \ldots, X_n, X) \triangleq \begin{cases} \star(X_1, \ldots, X_n) \vdash X & \text{if } Æ = \exists \\ X \vdash \star(X_1, \ldots, X_n) & \text{if } Æ = \forall. \end{cases}$

Figure 14: Calculus $\mathsf{GGL}^0_{\mathsf{C}}$

Theorem 53 (Soundness and strong completeness)**.** *Let $\mathsf{C} \subseteq \mathbb{C}$. The calculus $\mathsf{GGL}^0_{\mathsf{C}}$ is sound and strongly complete for the basic gaggle logic $(\mathcal{S}^0_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$.*

Proof: See the Appendix, Section C. □

Goré [21] introduces a calculus $\delta\mathbf{OP}$ which is basically our calculus $\mathsf{GGL}^0_{\mathsf{C}}$ without the rule (dr_2'). Restall [41] establishes connections between gaggle theory and display logics and sketches a similar calculus (without proving condition $(C8)$). This difference between our and their calculi is due to the fact that they do not deal with Boolean negation and do not consider it in their approach and framework. As one can notice, this complicates the proofs tremendously even if the addition in the calculi is minimal. This said, Goré [21] recognizes the dual character, in a proof–theoretical sense, of pairs of traces which are obtained from each other by multiplying every argument of the trace by $-$. This leads him to introduce the function/connective f^{Δ} of trace $-t$ associated to a function f of trace t. However, he does not make the connection between this function/connective f^{Δ} and the Boolean negation of f as we do (see Definition 20 and Proposition 29). Therefore, he proves the soundness and completeness of his calculus but with respect to two distinct yet dual semantics based on Dunn's tonoids. As such, he does not connect his algebraic semantics with the Kripke–style relational semantics (elicited by Dunn) explicitly as we do. A similar observation regarding the role of Boolean negation in his and our work was already made in [2].

Theorem 54 (Decidability)**.** *Let $\mathsf{C} \subseteq \mathbb{C}$ and let $\varphi, \psi \in \mathcal{L}^0_{\mathsf{C}}$. The problem of determining whether φ or $\varphi \vdash \psi$ are valid in the logics $(\mathcal{L}^0_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$ and $(\mathcal{S}^0_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$ (respectively) is decidable.*

Proof: It suffices to observe that the set of consecutions that can lead to a *cut-free* proof of $\varphi \vdash \psi$ in $\mathsf{GGL}^0_{\mathsf{C}}$ is finite. The problem of finding a proof of $\varphi \vdash \psi$ thus boils down to a graph reachability problem in a finite graph whose edges are labeled by the rules. This problem is decidable. We then obtain the result by the completeness of $\mathsf{GGL}^0_{\mathsf{C}}$ for $(\mathcal{L}^0_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$ and $(\mathcal{S}^0_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$ of Theorem 53. □

10 Logics defining groups and groups defining logics

In this section, we are going to show how notions of groups arise naturally from our gaggle logics and how gaggle logics can be canonically defined from groups thanks to our connections with group theory.

10.1 Groups defined from logics

One problem solved in this article is the following: given an arbitrary basic gaggle logic (Boolean or not) defined by a set C of (gaggle) connectives, how do we compute and define uniformly a sound and complete calculus for that logic ? Theorems 51 and 53 of the previous sections have solved it. However, we needed in our calculi to introduce *all* connectives of $\mathcal{O}(\mathsf{C})$ (defined by Expression (4)) either as logical connectives in Theorem 45 or as structural connectives in Theorems 51 and 53. In this section, we are going to show that we can in fact limit further the connectives considered and not take the full orbits $\mathcal{O}(\mathsf{C})$ of C under the action $\alpha_n * \beta_n$. For that, we need to explore a bit more the proof–theoretical aspects of our gaggle logics in light of our connections with group theory.

We have introduced actions on the set of gaggle connectives. Even if we know how a permutation, the Boolean negation and their combinations act on connectives,

$$\frac{S\left(\left[\circledast\right], X_1, \ldots, X_n, X_{n+1}\right)}{S\left(\left[\tau\circledast\right], \pm^1 X_{\overline{\tau}(1)}, \ldots, \pm^n X_{\overline{\tau}(n)}, \pm^{n+1} X_{\overline{\tau}(n+1)}\right)}\ (\mathsf{dr}_3)$$

where $\tau \in \mathfrak{S}_{n+1} * P_{(+,-)}$ and

if $\tau = \tau_0 - \tau_1 \ldots - \tau_m$ with $m \geq 1$ then $\overline{\tau} \triangleq \tau_0\tau_1 \ldots \tau_m$ and for all $j \in [\![1; n+1]\!]$, $\pm^j \triangleq \pm^j_1 \pm^j_2 \ldots \pm^j_m$ with, for all $i \in [\![1; m]\!]$, $\pm^j_i \triangleq \begin{cases} * & \text{if } j = \tau_i\tau_{i+1} \ldots \tau_m(n+1) \\ \text{empty} & \text{otherwise} \end{cases}$;
if $\tau = \tau_0 - \tau_1 \ldots - \tau_{m-1}-$ with $m \geq 1$ then replace τ with $\tau_0 - \tau_1 \ldots - \tau_{m-1} - 1$;
if $\tau = -\tau_1 \ldots - \tau_{m-1} - \tau_m$ with $m \geq 1$ then replace τ with $1 - \tau_1 \ldots - \tau_{m-1} - \tau_m$;
if $\tau \in \mathfrak{S}_{n+1}$ then $\overline{\tau} \triangleq \tau$ and $\pm^1, \ldots, \pm^{n+1}$ are empty;
if $\tau = -$ then $\overline{\tau} = 1$ and $\pm^1, \ldots, \pm^n$ are empty and $\pm^{n+1} = -$.

Figure 15: Rule (dr_3)

we still do not know how their combination and iteration operate at the proof–theoretical level. Indeed, we have a rule (dr_1) for permutations $\tau_1, \ldots, \tau_n$ and a rule (dr'_2) for Boolean negation $-$, yet we do not have a rule combining both, for elements $\tau_0 - \tau_1 \ldots - \tau_m$ of the free group $\mathfrak{S}_{n+1} * P_{(+,-)}$. Such a rule is defined in Figure 15. One can easily prove that rule (dr_3) is valid and derives from (dr_1) and (dr'_2) in $\mathsf{GGL}^0_{\mathsf{C}}$. Conversely, with $\tau \in \mathfrak{S}_{n+1}$, we recover rule (dr_1) and with $\tau = -$ we recover rule (dr'_2). (The term "empty" could be replaced by $**$.)

Now, let us be given a set of connectives $\mathsf{C} \subseteq \mathbb{C}$ and assume without loss of generality that all connectives of C belong to the same orbit $\mathcal{O}(\mathsf{C}) = \mathcal{O}_{\alpha_n * \beta_n}(\circledast)$ (for some $\circledast \in \mathsf{C}$). What we would want in (dr_1) is to be able to 'go' from one connective $\circledast$ of C to an arbitrary other connective $\circledast'$ of C. By transitivity of the action $\alpha_n * \beta_n$, this is possible in $\mathcal{O}(\mathsf{C})$: given any two connectives $\circledast, \circledast' \in \mathsf{C}$, there is an element of the group $g \in \mathfrak{S}_{n+1} * P_{(+,-)}$ such that $\circledast' = \alpha_n * \beta_n(g, \circledast)$. This leads us to define a special subset G of $\mathfrak{S}_{n+1} * P_{(+,-)}$ such that for all $\circledast, \circledast' \in \mathsf{C}$ there is $g \in G$ such that $\circledast' = \alpha_n * \beta_n(g, \circledast)$. We want this set G to be a group. Indeed, informally, its composition operation should be associative, because of the definition of an action group, and every element g of G should have an inverse: if $\circledast' = \alpha_n * \beta_n(g, \circledast)$ then there should be a g^{-1} such that $\circledast = \alpha_n * \beta_n(g^{-1}, \circledast')$. This leads us to the following definition:

Definition 55 (Group associated to a set of connectives)**.** Let $\mathsf{C} \subseteq \mathbb{C}$. A *group associated to* C is a group G such that for all $n \in \mathbb{N}^*$, all $\circledast, \circledast' \in \mathsf{C} \cap \mathbb{C}_n$, there is $g \in G$ such that $\circledast' = \alpha_n * \beta_n(g, \circledast)$. □

Implicitly, note that $G \subseteq \bigcup_{n \in \mathbb{N}} \left\{ \mathfrak{S}_{n+1} * P_{(+,-)} \mid a(\circledast) = n, \circledast \in \mathsf{C} \right\}$. A group associated to a set of connectives always exists because the free group $\left\langle \bigcup_{n \in \mathbb{N}^*} \left\{ g \in \mathfrak{S}_{n+1} * P_{(+,-)} \mid \circledast' = g \circledast' \text{ for some } \circledast, \circledast' \in \mathsf{C} \cap \mathbb{C}_n \right\} \right\rangle$ satisfies the required condition. It is not in general unique because the action $\alpha_n * \beta_n$ is not faithful: we proved in Proposition 32 (item 1) that $-r_j - r_j - r_j - r_j \circledast = \circledast$.

Definition 56. Let $\mathsf{C} \subseteq \mathbb{C}$ and let G be a group associated to C. We denote by $\mathsf{GL}^0_{\mathsf{C},G}$ (resp. $\mathsf{GL}_{\mathsf{C},G}$) the calculus of Figure 14 (resp. Figure 13) where the introduction rules $(\vdash \circledast)$ and $(\circledast \vdash)$ are defined for the connectives $\circledast$ of C and where rules (dr_1) and (dr'_2) (resp. only (dr_1)) are replaced by rule (dr_3) which is defined for elements τ belonging to a set of generators of the group G. □

Theorem 57 (Soundness and strong completeness)**.** *Let* $\mathsf{C} \subseteq \mathbb{C}$ *and let* G *be a group associated to* C*. The calculus* $\mathsf{GL}^0_{\mathsf{C},G}$ *(*$\mathsf{GL}_{\mathsf{C},G}$*) is sound and strongly complete for the (Boolean) basic gaggle logic* $(\mathcal{S}^0_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$ *(resp.* $(\mathcal{S}_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$*).*

Proof: See the Appendix, Section C. □

Example 58. The symmetric group $\mathfrak{S}_3$ is a group associated to the connectives of the Lambek calculus [30] and update logic [2]. However, there is a simpler and smaller group associated. Indeed, the alternating group $\mathfrak{A}_3$, generated by the 3–cycle (123) (or (132) = (123) ∘ (123), see Section 3) is another group associated to the connectives of the Lambek calculus and update logic. This confirms an observation already made in [2] about the central role played by ternary cycles in update logic and substructural logics in general. The free group $\mathfrak{A}_3 * P_{(+,-)}$ is a group associated to the connectives of *dual* update logic [2], because the dual connectives of dual update logic are definable from the connectives of update logic thanks to Boolean negation (see [2, Proposition 16]).

10.2 Logics defined from groups

According to Cayley's theorem, every finite group of cardinal $n+1$ is isomorphic to a subgroup of the symmetric group $\mathfrak{S}_{n+1}$. Now, the restriction of the action α_n to any subgroup G of $\mathfrak{S}_{n+1}$ is also an action of G on $\mathbb{C}_n$. Therefore, every finite group G of cardinal $n+1$ induces a canonical group action α of G on $\mathbb{C}_n$ defined for all

$g \in G$ and $\circledast \in \mathbb{C}_n$ by $\alpha(g, \circledast) = \alpha_n(\varphi(g), \circledast)$, where φ is an isomorphism between G and the subgroup of $\mathfrak{S}_{n+1}$. Every finite group therefore defines a set of connectives obtained by considering the orbit of an arbitrary connective $\circledast \in \mathbb{C}$ by this canonical group action α. In other words, every finite group defines a class of logics. These logics provide a certain perspective on the whole set of gaggle connectives.

11 Conclusion

In this article we have introduced a uniform method to automatically compute sound and strongly complete calculi for a wide class of non–classical logics, basic gaggle logics. These calculi are display calculi and enjoy the cut elimination. This allowed us to prove in particular that basic gaggle logics are decidable. We further restrained the structural connectives needed in our calculi by introducing the notion of group associated to a set of connectives. We also established connections between gaggle theory and group theory. We showed that Dunn's abstract law of residuation corresponds to an action of transpositions of the symmetric group on the set of gaggle connectives and that Dunn's families of connectives are orbits of the same action of the symmetric group. Other operations on connectives, such as dual and Boolean negation, were also reformulated in terms of actions of groups and their combination was defined by means of free groups and free products.

Based on our connection with group theory, we argued that there are more 'basic' operations on connectives than Dunn's abstract law of residuation, based on cycles of the symmetric group rather than transpositions (which are cycles anyway), because every permutation factorizes uniquely into disjoint cycles. Residuation is still central because it corresponds to the action of transpositions of the symmetric group and transpositions generate it as well. Yet, there are many other generators and ways to present and represent the symmetric groups and its subgroups. What really matters from a proof-theoretical perspective is the set of generators of the groups that we consider and how groups can be presented. That is why the results in group theory regarding the presentation and classification of finite groups have now become quite relevant for the study of various (gaggle) logics.

Our connections with the theory of groups enable to study the structure of gaggle connectives in a very modular and systematic way, using bridges from algebra such as Cayley's theorem. Thanks to this bridge, each finite group can be seen as a set of operations acting on the set of connectives. Hence, each group generates and defines gaggle logics. Thus, the structure of the gaggle connectives can be studied under a variety of different viewpoints by means of different logics that correspond to the wide range of finite groups that can act on the connectives. This is similar to what happens in mathematics where the structure of (vectorial, Euclidean, *etc.*) spaces can be studied by different geometries corresponding to different groups of transformation acting on it: Euclidean geometry with the isometric group, hyperbolic geometry with the Lorentz group, affine geometry with the affine group, *etc.*

References

[1] Gerard Allwein and J. Michael Dunn. Kripke models for linear logic. *The Journal of Symbolic Logic*, 58(2):514–545, June 1993.

[2] Guillaume Aucher. Displaying Updates in Logic. *Journal of Logic and Computation*, 26(6):1865–1912, March 2016.

[3] Franz Baader. *The description logic handbook: Theory, implementation and applications.* Cambridge university press, 2003.

[4] Jon Barwise. Axioms for abstract model theory. *Annals of Mathematical Logic*, 7(2):221 – 265, 1974.

[5] Jeffrey C Beall and Greg Restall. Logical pluralism. *Australasian Journal of Philosophy*, 78(4):475–493, 2000.

[6] Nuel D. Belnap Jr. Display logic. *Journal of Philosophical Logic*, 11(4):375–417, 1982.

[7] Katalin Bimbó and J. Michael Dunn. *Generalized Galois Logics: Relational Semantics of Nonclassical Logical Calculi.* Number 188. Center for the Study of Language and Information, 2008.

[8] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic*, volume 53 of *Cambridge Tracts in Computer Science.* Cambridge University Press, 2001.

[9] Agata Ciabattoni and Revantha Ramanayake. Power and limits of structural display rules. *ACM Trans. Comput. Log.*, 17(3):17:1–17:39, 2016.

[10] J Michael Dunn. Gaggle theory: an abstraction of galois connections and residuation, with applications to negation, implication, and various logical operators. In *European Workshop on Logics in Artificial Intelligence*, pages 31–51. Springer Berlin Heidelberg, 1990.

[11] J Michael Dunn. Partial-gaggles applied to logics with restricted structural rules. In Peter Schroeder-Heister and Kosta Dosen, editors, *Substructural Logics*, pages 63–108. Clarendon Press: Oxford, 1993.

[12] J. Michael Dunn. *Philosophy of Language and Logic*, volume 7 of *Philosophical Perspectives*, chapter Perp and star: Two treatments of negation, pages 331–357. Ridgeview Publishing Company, Atascaredo, California, USA, 1993.

[13] J. Michael Dunn and Gary M. Hardegree. *Algebraic Methods in Philosophical Logic*. Number 41 in Oxford Logic Guides. Clarendon Press: Oxford, 2001.

[14] J. Michael Dunn and Chunlai Zhou. Negation in the context of gaggle theory. *Studia Logica*, 80(2-3):235–264, 2005.

[15] Dov Gabbay. What is a logical system ? In Dov Gabbay, editor, *What is a Logical System ?*, Studies in Logic and Computation, pages 179–216. Oxford University Press, 1994.

[16] Dov Gabbay, editor. *What is a Logical System ?* Studies in Logic and Computation. Oxford University Press, 1994.

[17] Dov Gabbay. *Labelled Deductive Systems*, volume 1. Oxford University Press, 1996.

[18] Gerhard Gentzen. Untersuchungen über das logische schließen. i. *Mathematische zeitschrift*, 39(1):176–210, 1935.

[19] Joseph A Goguen and Rod M Burstall. Institutions: Abstract model theory for specification and programming. *Journal of the ACM (JACM)*, 39(1):95–146, 1992.

[20] Robert Goldblatt. Semantic analysis of orthologic. *Journal of Philosophical Logic*, 3:19–35, 1974.

[21] Rajeev Goré. Gaggles, Gentzen and Galois: How to display your favourite substructural logic. *Logic Journal of IGPL*, 6(5):669–694, 1998.

[22] Rajeev Goré. Substructural logics on display. *Logic Journal of IGPL*, 6(3):451–504, 1998.

[23] Viktor Grishin. On a generalization of the Ajdukiewicz-Lambek system. In A. I. Mikhailov, editor, *Studies in Nonclassical Logics and Formal Systems*, pages 315–334. Nauka, Moscow, 1983.

[24] David Harel, Dexter Kozen, and Jerzy Tiuryn. *Dynamic Logic*. MIT Press, 2000.

[25] Tony (C.A.R.) Hoare. An axiomatic basis for computer programming. *Communications of the ACM*, 12(10):567–580, 1969.

[26] Miroslawa Kolowska-Gawiejnowicz. Powerset residuated algebras and generalized Lambek calculus. *Mathematical Logic Quarterly*, 43(1):60–72, 1 1997.

[27] Marcus Kracht. Power and weakness of the modal display calculus. In *Proof theory of modal logic*, pages 93–121. Springer, 1996.

[28] Saul A. Kripke. Semantical analysis of modal logic, i: Normal propositional calculi. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 8:113–116, 1963.

[29] Saul A. Kripke. *Formal Systems and Recursive Functions*, chapter Semantical Analysis of Intuitionistic Logic, I, pages 92–130. North Holland, Amsterdam, 1965.

[30] Joachim Lambek. The mathematics of sentence structure. *American mathematical monthly*, 65:154–170, 1958.

[31] Joachim Lambek. *Substructural Logics*, chapter From categorial grammar to bilinear logic, pages 207–237. Studies in Logic and Computation. Oxford University Press, 1993.

[32] Elliott Mendelson. *Introduction to mathematical logic*. CRC press, 1997.

[33] José Meseguer. General logics. In H.-D. Ebbinghaus, J. Fernandez-Prida, M. Garrido, D. Lascar, and M. Rodriquez Artalejo, editors, *Logic Colloquium'87*, volume 129 of *Studies in Logic and the Foundations of Mathematics*, pages 275 – 329. Elsevier, 1989.

[34] Richard Montague. *Contemporary Philosophy: a Survey*, chapter Pragmatics, pages 102–122. La Nuova Italia Editrice, Florence, 1968.

[35] Richard Montague. Universal grammar. *Theoria*, 36:373–398, 1970.

[36] Michael Moortgat. Symmetries in natural language syntax and semantics: the lambek-grishin calculus. In *Logic, Language, Information and Computation*, pages 264–284. Springer, 2007.

[37] Chris Mortensen. Aristotle's thesis in consistent and inconsistent logics. *Studia Logica*, 43:107–116, 1984.

[38] Lawrence S Moss. Applied logic: A manifesto. In *Mathematical problems from applied logic I*, pages 317–343. Springer, 2006.

[39] Hiroakira Ono. Proof-theoretic methods in nonclassical logic –an introduction. In Masako Takahashi, Mitsuhiro Okada, and Mariangiola Dezani-Ciancaglini, editors, *Theories of Types and Proofs*, volume Volume 2 of *MSJ Memoirs*, pages 207–254. The Mathematical Society of Japan, Tokyo, Japan, 1998.

[40] Arthur Prior. *Past, Present and Future.* Oxford: Clarendon Press, 1967.

[41] Greg Restall. Display logic and gaggle theory. *Reports on Mathematical Logic*, 29:133–146, 1995.

[42] Greg Restall. *An Introduction to Substructural Logics.* Routledge, 2000.

[43] John C. Reynolds. Separation logic: A logic for shared mutable data structures. In *17th IEEE Symposium on Logic in Computer Science (LICS 2002), Copenhagen, Denmark, Proceedings*, pages 55–74. IEEE Computer Society, 2002.

[44] Joseph J. Rotman. *An Introduction to the Theory of Groups*, volume 148 of *Graduate texts in mathematics.* Springer, 1995.

[45] Giovanni Sambin, Giulia Battilotti, and Claudia Faggian. Basic logic: reflection, symmetry, visibility. *The Journal of Symbolic Logic*, 65(03):979–1013, 2000.

[46] Peter Schroeder-Heister and Kosta Dosen, editors. *Substructural Logics*, volume 2 of *Studies in Logic and Computation.* Oxford Science Publication, 1993.

[47] Dana S. Scott. *Philosophical Problems in Logic*, chapter Advice on modal logic, pages 143–173. Reidel, 1970.

A Proofs of propositions 30, 32 and 46

Proposition 30. *If* $\circledast \in \mathbb{C}_n$ *is a* n*–ary connective, then for all* $j \in [\![1;n]\!]$*,*

- $s_j\circledast = r_j - r_j\circledast$
- $d_j\circledast = r_j - r_j - \circledast$
- $d\circledast = s_1 \dots s_n - \circledast.$

Proof: Let $\circledast = (\sigma, \pm, (Æ, (\pm_1, \dots, \pm_n))) \in \mathbb{C}_n$. Then,

$$\begin{aligned} r_j\circledast =& (\sigma, - \pm_j \pm, (- \pm_j Æ, (- \pm_j \pm_1, \dots, \pm_j, \dots, - \pm_j \pm_n))) \\ -r_j\circledast =& (\sigma, \pm_j\pm, (\pm_j Æ, (\pm_j\pm_1, \dots, -\pm_j, \dots, \pm_j\pm_n))) \\ r_j - r_j\circledast =& (\sigma, \pm, (Æ, (\pm_1, \dots, -\pm_j, \dots, \pm_n))) \end{aligned}$$

Moreover,

$$\begin{aligned} -\circledast =& (\sigma, -\pm, (-Æ, (-\pm_1, \dots, -\pm_n))) \\ r_j - \circledast =& ((j\ n+1) \circ \sigma, - \pm_j \pm, (- \pm_j Æ, (- \pm_j \pm_1, \dots, -\pm_j, \dots, - \pm_j \pm_n))) \\ -r_j - \circledast =& ((j\ n+1) \circ \sigma, (\pm_j\pm, (\pm_j Æ, (\pm_j\pm_1, \dots, \pm_j, \dots, \pm_j\pm_n)))) \\ r_j - r_j - \circledast =& (\sigma, -\pm, (-Æ, (-\pm_1, \dots, \pm_j, \dots, -\pm_n))) \end{aligned}$$

□

Proposition 32. *Let* $n \in \mathbb{N}^*$*,* $j \in [\![1;n]\!]$ *and let us define* $G_j = \langle r_j \rangle * P_{(+,-)}$*. Since* G_j *is a subgroup of* $\mathfrak{S}_{n+1} * P_{(+,-)}$*, let us denote by* α_{G_j} *the action of* G_j *on* $\mathbb{C}_n$ *induced by the free action* $\alpha_n * \beta_n$*. Then, for all connectives* $\circledast$ *of arity* n*,*

1. $\mathcal{O}_{\alpha_{G_j}}(\circledast)$ *is isomorphic to a cyclic group of order 8.*
2. $\left\{\mathcal{O}_{\alpha_n * \beta_n}(\circledast), \mathcal{O}_{\alpha_n * \beta_n}(\sim \circledast)\right\}$ *forms a partition of the set* $\mathbb{C}_n$ *of connectives of arity* n*. Moreover, the mapping* $\tilde{\cdot} : \mathcal{O}_{\alpha_n*\beta_n}(\circledast) \to \mathcal{O}_{\alpha_n*\beta_n}(\sim \circledast)$*,* $x \mapsto \sim x$ *is involutive.*
3. *For all* $n \in \mathbb{N}^*$*, the free action* $\alpha_n * \beta_n * \gamma_n$ *on the set of connectives* $\mathbb{C}_n$ *is transitive.*

Proof: For the first item, it suffices to prove that for all connectives $\circledast$ of arity n and all $j \in [\![1;n]\!]$, $-r_j - r_j - r_j - r_j \circledast = \circledast$. Let $\circledast = (\sigma, \pm, (Æ, (\pm_1, \dots, \pm_n)))$ and let r_j be the transposition $(j\ n+1)$. (See also Figure 2 for an example.)

$$\begin{aligned}
\circledast &= (\sigma, \pm, (Æ, (\pm_1, \dots, \pm_j, \dots, \pm_n)))\\
r_j \circledast &= (r_j \circ \sigma, - \pm_j \pm, (- \pm_j Æ, (- \pm_j \pm_1, \dots, \pm_j, \dots, - \pm_j \pm_n)))\\
-r_j \circledast &= (r_j \circ \sigma, \pm_j \pm, (\pm_j Æ, (\pm_j \pm_1, \dots, -\pm_j, \dots, \pm_j \pm_n)))\\
r_j - r_j \circledast &= (\sigma, \pm, (Æ, (\pm_1, \dots, -\pm_j, \dots, \pm_n)))\\
-r_j - r_j \circledast &= (\sigma, -\pm, (-Æ, (-\pm_1, \dots, \pm_j, \dots, -\pm_n)))\\
r_j - r_j - r_j \circledast &= (r_j \circ \sigma, \pm_j \pm, (\pm_j Æ, (\pm_j \pm_1, \dots, \pm_j, \dots, \pm_j \pm_n)))\\
-r_j - r_j - r_j \circledast &= (r_j \circ \sigma, - \pm_j \pm, (- \pm_j Æ, (- \pm_j \pm_1, \dots, -\pm_j, \dots, - \pm_j \pm_n)))\\
r_j - r_j - r_j - r_j \circledast &= (\sigma, -\pm, (-Æ, (-\pm_1, \dots, -\pm_j, \dots, -\pm_n)))\\
-r_j - r_j - r_j - r_j \circledast &= (\sigma, \pm, (Æ, (\pm_1, \dots, \pm_j, \dots, \pm_n))) = \circledast.
\end{aligned}$$

For the second item, one should first observe that $\mathcal{O}_{\alpha_n * \beta_n}(\circledast) \cap \mathcal{O}_{\alpha_n * \beta_n}(\sim \circledast) = \varnothing$ $(*)$. Indeed, for all $\circledast' = (\sigma', \pm', (Æ', (\pm'_1, \dots, \pm'_n))) \in \mathcal{O}_{\alpha_n * \beta_n}(\circledast)$, we have that $\pm' \pm (Æ') = \pm \pm (Æ)$ but at the same time, for all $\circledast' = (\sigma', \pm', (Æ', (\pm'_1, \dots, \pm'_n))) \in \mathcal{O}_{\alpha_n * \beta_n}(\sim \circledast)$, we also have that $\pm' \pm (Æ') = - \pm \pm (Æ)$. Now, we prove that for all $\circledast' = (\sigma', \pm', (Æ', (\pm'_1, \dots, \pm'_n)))$, if $\pm' \pm (Æ') = \pm \pm (Æ)$ then $\circledast' \in \mathcal{O}_{\alpha_n * \beta_n}(\circledast)$, and $\circledast' \in \mathcal{O}_{\alpha_n * \beta_n}(\sim \circledast)$ otherwise. First, assume that $\pm' \pm (Æ') = \pm \pm (Æ)$. Then, we define $\circledast'' = \sigma \sigma'^{-} \circledast'$. So, $\circledast'' = (\sigma, \pm'', (Æ'', (\pm''_1, \dots, \pm''_n)))$ and we still have that $\pm'' \pm (Æ'') = \pm' \pm (Æ') = \pm \pm (Æ)$. If $\pm'' = \pm$, then it only suffices to switch the tonicity of the arguments $j_1, \dots, j_k$ of $\circledast''$ such that $\pm''_j \neq \pm_j$. This can be done by applying the switch operation for the arguments $j_1, \dots, j_k$ to $\circledast''$. We then obtain that $s_{j_1} s_{j_2} \dots s_{j_k} \circledast'' = \circledast$. Thus, $s_{j_1} s_{j_2} \dots s_{j_k} \sigma \sigma'^{-} \circledast' = \circledast$. Second, assume that $\pm' \pm (Æ') = - \pm \pm (Æ)$. Then, we define $\circledast'' = \sim \circledast = (\sigma', \pm', (Æ'', (\pm'_1, \dots, \pm'_n)))$ and we have that $\pm' \pm (Æ'') = \pm'(-\pm(Æ')) = \pm \pm (Æ)$. So, we proceed like in the first case. We then obtain that there are $i_1, \dots, i_l \in [\![1;n]\!]$ such that $s_{i_1} s_{i_2} \dots s_{i_l} \sigma \sigma'^{-} \sim \circledast' = \circledast$. So, we have proved that for all $\circledast' = (\sigma', \pm', (Æ', (\pm'_1, \dots, \pm'_n)))$, it holds that $\circledast' \in \mathcal{O}_{\alpha_n * \beta_n}(\circledast)$ iff $\pm' \pm (Æ') = \pm \pm (Æ)$. This entails that $|\mathcal{O}_{\alpha_n * \beta_n}(\circledast)| = |\mathcal{O}_{\alpha_n * \beta_n}(\sim \circledast)| = (n+1)! \cdot 2^{n+1} = \frac{|\mathbb{C}_n|}{2}$. Therefore, $\mathcal{O}_{\alpha_n * \beta_n}(\circledast) \cup \mathcal{O}_{\alpha_n * \beta_n}(\sim \circledast) = \mathbb{C}_n$ and together with (*), we have that $\{\mathcal{O}_{\alpha_n * \beta_n}(\circledast), \mathcal{O}_{\alpha_n * \beta_n}(\sim \circledast)\}$ forms a partition of $\mathbb{C}_n$.

The third item follows easily from the second item. □

Proposition 46. *Let $C \subseteq \mathbb{C}$ and let $\circledast \in C$ be a n-ary connective. The following rules* *are all derivable in* $\mathsf{GGL_C}$.

$$\frac{X \vdash Y}{*Y \vdash *X}\ (\mathit{dr}_2') \qquad \frac{S([\circledast], X_1, \dots, X_j, \dots, X_n, X)}{S([s_j \circledast], X_1, \dots, *X_j, \dots, X_n, X)}\ (\mathit{sw}^j)$$

$$\frac{*X \vdash Y}{*Y \vdash X}\ (\mathit{dr}_2'') \qquad \frac{X \vdash *Y}{Y \vdash *X}\ (\mathit{dr}_2''')$$

$$\frac{U \vdash ((X \,,\, Y) \,,\, Z)}{U \vdash (X \,,\, (Y \,,\, Z))}\ (\vdash B) \qquad \frac{U \vdash (X \,,\, Y)}{U \vdash (Y \,,\, X)}\ (\vdash CI)$$

$$\frac{U \vdash X}{U \vdash (X \,,\, Y)}\ (\vdash K) \qquad \frac{U \vdash (X \,,\, X)}{U \vdash X}\ (\vdash WI)$$

$$\frac{U \vdash \varphi \quad U \vdash \psi}{U \vdash (\varphi \wedge \psi)}\ (\vdash \wedge)' \qquad \frac{\varphi \vdash U \quad \psi \vdash U}{(\varphi \vee \psi) \vdash U}\ (\vee \vdash)'$$

The rule (dr_2') *is called the* Boolean negation rule *and the rule* (sw^j), *for* $j \in [\![1; n]\!]$, *is called the* switch rule w.r.t. the j^{th} argument. *The rule* (dr_1) *is also derivable in* $\mathsf{GGL_C}$, for all $\tau \in \mathfrak{S}_{n+1}$.

Proof:

(dr_2') :

$$\begin{array}{cl} X \vdash Y & \\ \hline\hline (X \,,\, *Y) \vdash & (\mathsf{dr}_2) \\ \hline\hline (*Y \,,\, X) \vdash & (\mathsf{CI} \vdash) \\ \hline\hline *Y \vdash *X & (\mathsf{dr}_2) \end{array}$$

(dr_2'') :

$$\begin{array}{cl} *X \vdash Y & \\ \hline (*X \,,\, *Y) \vdash & (\mathsf{dr}_2) \\ \hline (*Y \,,\, *X) \vdash & (\mathsf{CI} \vdash) \\ \hline *Y \vdash X & (\mathsf{dr}_2) \end{array}$$

(dr_2''') :

$$\begin{array}{cl} X \vdash *Y & \\ \hline (X \,,\, Y) \vdash & (\mathsf{dr}_2) \\ \hline (Y \,,\, X) \vdash & (\mathsf{CI} \vdash) \\ \hline Y \vdash *X & (\mathsf{dr}_2) \end{array}$$

(sw^j) :

$$\begin{array}{cl} S(\circledast, X_1, \dots, X_j, \dots, X_n, X) & \\ \hline S(r_j \circledast, X_1, \dots, X, \dots, X_n, X_j) & (\mathsf{dr}_1) \\ \hline S(-r_j \circledast, X_1, \dots, X, \dots, X_n, *X_j) & (\mathsf{dr}_2') \\ \hline S(r_j - r_j \circledast, X_1, \dots, *X_j, \dots, X_n, X) & (\mathsf{dr}_1) \\ \hline S(s_j \circledast, X_1, \dots, *X_j, \dots, X_n, X) & \text{Rewrite} \end{array}$$

$(\vdash \mathsf{K})$:

$$\begin{array}{cl} U \vdash X & \\ \hline *X \vdash *U & (\mathsf{dr}_2') \\ \hline (*X \,,\, *Y) \vdash *U & (\mathsf{K} \vdash) \\ \hline U \vdash *(*X \,,\, *Y) & (\mathsf{dr}_2''') \\ \hline U \vdash (X \,,\, Y) & \text{Rewrite} \end{array}$$

$(\vdash \mathsf{CI}):$

$$\begin{array}{cl}
U \vdash (X \,,\, Y) & \\ \hline
(U \,,\, *(X \,,\, Y)) \vdash & (\mathsf{dr}_2) \\ \hline
(U \,,\, (*X \,,\, *Y)) \vdash & \text{Rewrite} \\ \hline
((*X \,,\, *Y) \,,\, U) \vdash & (\mathsf{CI} \vdash) \\ \hline
(*X \,,\, *Y) \vdash\ *U & (\mathsf{dr}_2) \\ \hline
(*Y \,,\, *X) \vdash\ *U & (\mathsf{CI} \vdash) \\ \hline
((*Y \,,\, *X) \,,\, U) \vdash & (\mathsf{dr}_2) \\ \hline
(U \,,\, (*Y \,,\, *X)) \vdash & (\mathsf{CI} \vdash) \\ \hline
U \vdash\ *(*Y \,,\, *X) & (\mathsf{dr}_2) \\ \hline
U \vdash (Y \,,\, X) & \text{Rewrite}
\end{array}$$

$(\vdash \mathsf{B}):$

$$\begin{array}{cl}
U \vdash ((X \,,\, Y) \,,\, Z) & \\ \hline
*((X \,,\, Y) \,,\, Z) \vdash\ *U & (\mathsf{dr}_2') \\ \hline
((*X \,,\, *Y) \,,\, *Z) \vdash\ *U & \text{Rewrite} \\ \hline
(*X \,,\, (*Y \,,\, *Z)) \vdash\ *U & (\mathsf{B} \vdash) \\ \hline
U \vdash\ *(*X \,,\, (*Y \,,\, *Z)) & (\mathsf{dr}_2') \\ \hline
U \vdash (X \,,\, (Y \,,\, Z)) & \text{Rewrite}
\end{array}$$

$(\vdash \mathsf{WI}):$

$$\begin{array}{cl}
U \vdash (X \,,\, X) & \\ \hline
(*X \,,\, *X) \vdash\ *U & (\mathsf{dr}_2') \\ \hline
*X \vdash\ *U & (\mathsf{WI} \vdash) \\ \hline
U \vdash X & (\mathsf{dr}_2')
\end{array}$$

$(\vdash \wedge)':$

$$\begin{array}{cl}
U \vdash \varphi \quad U \vdash \psi & \\ \hline
(U \,,\, U) \vdash (\varphi \wedge \psi) & (\vdash \wedge) \\ \hline
U \vdash (\varphi \wedge \psi) & (\mathsf{WI} \vdash)
\end{array}$$

$(\vee \vdash)':$

$$\begin{array}{cl}
\varphi \vdash U \quad \psi \vdash U & \\ \hline
(\varphi \vee \psi) \vdash (U \,,\, U) & (\vee \vdash) \\ \hline
(\varphi \vee \psi) \vdash U & (\vdash \mathsf{WI})
\end{array}$$

The last rewriting part in the proof of (sw^j) is due to Proposition 30. □

B Proof of theorem 45

Theorem 45 (Soundness and strong completeness)**.** *Let $\mathsf{C} \subseteq \mathbb{C}$ be such that $\mathcal{O}(\mathsf{C}) = \mathsf{C}$. The calculus GGL_C is sound and strongly complete for the Boolean basic gaggle logic $(\mathcal{S}_\mathsf{C}, \mathcal{M}_\mathsf{C}, \Vdash)$.*

In this section, $\mathsf{C} \subseteq \mathbb{C}$ is such that $\mathcal{O}(\mathsf{C}) = \mathsf{C}$. We provide the soundness and completeness proofs of Theorem 45. We adapt the proof methods introduced in [2], based on a Henkin construction, to our more abstract and general setting. We start by the soundness proof.

Lemma 59. *The calculus GGL_C is sound for the Boolean basic gaggle logic $(\mathcal{S}_\mathsf{C}, \mathcal{M}_\mathsf{C}, \Vdash)$.*

Proof: We only need to prove the soundness for the rules (dr_1) and $(\vdash \circledast)$, the soundness of the other rules being standard. The soundness of the inference rule $(\vdash \circledast)$ follows directly from item 1 of Theorem 10, the soundness of rule (dr_1) follows from an iterative application of item 2 of Theorem 10 (or Proposition 27) by the decomposition of permutations into cycles or transpositions. □

The completeness proof uses a canonical model built up from maximal GGL_C–consistent sets. First, we define the notions of GGL_C–consistent set and maximal GGL_C–consistent set. In the sequel, by abuse of notation and to ease the presentation, when we write $\varphi \models \psi$ we mean that $\varphi \models \psi$ is provable in the calculus GGL_C.

Definition 60 ((Maximal) GGL_C–consistent set)**.**

- A *GGL_C–consistent set* is a subset Γ of $\mathcal{L}_\mathsf{C}$ such that there are no $\varphi_1, \dots, \varphi_n \in \Gamma$ such that $\varphi_1 \;,\; \dots \;,\; \varphi_n \models \;.$ If $\varphi \in \mathcal{L}_\mathsf{C}$, we also say that φ is *GGL_C–consistent* when the set $\{\varphi\}$ is GGL_C–consistent.
- A *maximal GGL_C–consistent set* is a GGL_C–consistent set Γ of $\mathcal{L}_\mathsf{C}$ such that there is no $\varphi \in \mathcal{L}_\mathsf{C}$ satisfying both $\varphi \notin \Gamma$ and $\Gamma \cup \{\varphi\}$ is GGL_C–consistent. □

Lemma 61 (Cut lemma)**.** *Let Γ be a maximal GGL_C–consistent set. For all $\varphi_1, \dots, \varphi_n \in \Gamma$ and all $\varphi \in \mathcal{L}$, if $\varphi_1 \;,\; \dots \;,\; \varphi_n \models \varphi$ then $\varphi \in \Gamma$.*

Proof: First, we show that $\Gamma \cup \{\varphi\}$ is GGL_C–consistent. Assume towards a contradiction that it is not the case. Then, there are $\psi_1, \dots, \psi_m \in \Gamma$ such that $\psi_1 \;,\; \dots \;,\; \psi_m \;,\; \varphi \models \;.$ Then, by the rules (dr_2) and $(\mathsf{Cl} \vdash)$, we have that $\varphi \models \ast(\psi_1 \;,\; \dots \;,\; \psi_m)$. Now, by assumption, $\varphi_1 \;,\; \dots \;,\; \varphi_n \models \varphi$. Therefore, by the cut rule, we have that $\varphi_1 \;,\; \dots \;,\; \varphi_n \models \ast(\psi_1 \;,\; \dots \;,\; \psi_m)$. Then, by the rules (dr_2) and $(\mathsf{B} \vdash)$, we have that $\varphi_1 \;,\; \dots \;,\; \varphi_n \;,\; \psi_1 \;,\; \dots \;,\; \psi_m \models \;.$ However, $\varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_m \in \Gamma$. This entails that Γ is not GGL_C–consistent, which is impossible. Thus, $\Gamma \cup \{\varphi\}$ is GGL_C–consistent. Now, since Γ is a *maximal* GGL_C–consistent set, this implies that $\varphi \in \Gamma$. □

Lemma 62 (Lindenbaum lemma)**.** *Any GGL_C–consistent set can be extended into a maximal GGL_C–consistent set.*

Proof: Let $\varphi_1, \varphi_2, \dots, \varphi_n, \dots$ be an enumeration of $\mathcal{L}_\mathsf{C}$ (it exists because C is countable). We define the sets Γ_n inductively as follows:

$$\Gamma_0 \triangleq \Gamma$$

$$\Gamma_{n+1} \triangleq \begin{cases} \Gamma_n \cup \{\varphi_n\} & \text{if } \Gamma_n \cup \{\varphi_n\} \text{ is } \mathsf{GGL}_\mathsf{C}\text{–consistent} \\ \Gamma_n & \text{otherwise.} \end{cases}$$

Then, we define the subset Γ^+ of $\mathcal{L}$ as follows: $\Gamma^+ = \bigcup_{n \in \mathbb{N}} \Gamma_n$.

We show that Γ^+ is a maximal $\mathsf{GGL_C}$–consistent set. Clearly, for all $n \in \mathbb{N}$, Γ_n is $\mathsf{GGL_C}$–consistent by definition of Γ_n. So, if Γ^+ was not $\mathsf{GGL_C}$–consistent, there would be a $n_0 \in \mathbb{N}$ such that Γ_{n_0} is not $\mathsf{GGL_C}$–consistent, which is impossible. Now, assume towards a contradiction that Γ^+ is not a *maximal* $\mathsf{GGL_C}$–consistent set. Then, there is $\varphi \in \mathcal{L}_\mathsf{C}$ such that $\varphi \notin \Gamma^+$ and $\Gamma \cup \{\varphi\}$ is $\mathsf{GGL_C}$–consistent. But there is $n_0 \in \mathbb{N}$ such that $\varphi = \varphi_{n_0}$. Because $\varphi \notin \Gamma^+$, we also have that $\varphi_{n_0} \notin \Gamma_{n_0+1}$. So, $\Gamma_{n_0} \cup \{\varphi_{n_0}\}$ is not $\mathsf{GGL_C}$–consistent by definition of Γ^+. Therefore, $\Gamma^+ \cup \{\varphi\}$ is not $\mathsf{GGL_C}$–consistent either, which is impossible. □

Lemma 63. *The following consecutions are provable in GGL: for all $\varphi, \varphi' \in \mathcal{L}$, all $\circledast = (\sigma, \pm, (\exists, (\pm_1, \ldots, \pm_j, \ldots, \pm_n)))$,*

$$\varphi \vdash \varphi \tag{8}$$
$$((\varphi \vee \varphi') \wedge (\varphi \vee \neg\varphi')) \vdash \varphi \tag{9}$$
$$\varphi \vdash ((\varphi \wedge \neg\varphi') \vee (\varphi \wedge \varphi')) \tag{10}$$

if $\pm_j = +$ *then*

$$\circledast(\varphi_1, \ldots, \varphi_j \vee \varphi'_j, \ldots, \varphi_n) \vdash (\circledast(\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n) \vee \circledast(\varphi_1, \ldots, \varphi'_j, \ldots, \varphi_n)) \tag{11}$$

if $\pm_j = -$ *then*

$$\circledast(\varphi_1, \ldots, \varphi_j \wedge \varphi'_j, \ldots, \varphi_n) \vdash (\circledast(\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n) \vee \circledast(\varphi_1, \ldots, \varphi'_j, \ldots, \varphi_n)) \tag{12}$$

$$(\varphi\ ,\ \neg\psi) \vdash \quad \textit{iff}\ \varphi \vdash \psi \tag{13}$$

Proof: The proof of Expression (8) is by induction on φ. The proof of Expression

(9) is:

$$
\begin{prooftree}
\AxiomC{$\varphi \vdash \varphi$}
\RightLabel{$\mathsf{K}\,,$}
\UnaryInfC{$\varphi \vdash (\varphi \,,\, *(\varphi \vee \neg\varphi'))$}
\AxiomC{$\varphi \vdash \varphi$}
\RightLabel{$\mathsf{K}\,,$}
\UnaryInfC{$\varphi \vdash (\varphi \,,\, *\varphi')$}
\AxiomC{$\varphi' \vdash \varphi'$}
\RightLabel{$(\mathsf{dr}_2)\,,(\mathsf{dr}_2)$}
\UnaryInfC{$*\varphi' \vdash \, *\varphi'$}
\RightLabel{$(\neg \vdash)$}
\UnaryInfC{$\neg\varphi' \vdash \, *\varphi'$}
\RightLabel{$(\vdash \mathsf{Cl})\,,\mathsf{K}\,,$}
\UnaryInfC{$\neg\varphi' \vdash (\varphi \,,\, *\varphi')$}
\RightLabel{$(\vee \vdash)'$}
\BinaryInfC{$(\varphi \vee \neg\varphi') \vdash (\varphi \,,\, *\varphi')$}
\RightLabel{(dr_2)}
\UnaryInfC{$((\varphi \vee \neg\varphi') \,,\, \varphi') \vdash \varphi$}
\RightLabel{$(\mathsf{Cl} \vdash)$}
\UnaryInfC{$(\varphi' \,,\, (\varphi \vee \neg\varphi')) \vdash \varphi$}
\RightLabel{(dr_2)}
\UnaryInfC{$\varphi' \vdash (\varphi \,,\, *(\varphi \vee \neg\varphi'))$}
\RightLabel{$(\vee \vdash)'$}
\BinaryInfC{$(\varphi \vee \varphi') \vdash (\varphi \,,\, *(\varphi \vee \neg\varphi'))$}
\RightLabel{(dr_2)}
\UnaryInfC{$((\varphi \vee \varphi') \,,\, (\varphi \vee \neg\varphi')) \vdash \varphi$}
\RightLabel{$(\wedge \vdash)$}
\UnaryInfC{$((\varphi \vee \varphi') \wedge (\varphi \vee \neg\varphi')) \vdash \varphi$}
\end{prooftree}
$$

and the proof of Expression (10) is:

$$
\begin{prooftree}
\AxiomC{$\varphi \vdash \varphi$}
\RightLabel{$(\mathsf{K} \vdash)$}
\UnaryInfC{$(\varphi \,,\, *(\varphi \wedge \neg\varphi')) \vdash \varphi$}
\AxiomC{$(\varphi \vdash \varphi)$}
\RightLabel{$(\mathsf{K} \vdash)$}
\UnaryInfC{$(\varphi \,,\, *\varphi') \vdash \varphi$}
\AxiomC{$\varphi' \vdash \varphi'$}
\RightLabel{$(\mathsf{dr}_2)\,,(\mathsf{dr}_2)$}
\UnaryInfC{$*\varphi' \vdash \, *\varphi'$}
\RightLabel{$(\vdash \neg)$}
\UnaryInfC{$*\varphi' \vdash \neg\varphi'$}
\RightLabel{$(\mathsf{Cl} \vdash)\,,(\mathsf{K} \vdash)$}
\UnaryInfC{$(\varphi \,,\, *\varphi') \vdash \neg\varphi'$}
\RightLabel{$(\vdash \wedge)'$}
\BinaryInfC{$(\varphi \,,\, *\varphi') \vdash (\varphi \wedge \neg\varphi')$}
\RightLabel{(dr_2)}
\UnaryInfC{$(\varphi \,,\, *(\varphi \wedge \neg\varphi')) \vdash \varphi'$}
\RightLabel{$(\mathsf{K} \vdash)$}
\BinaryInfC{$(\varphi \,,\, *(\varphi \wedge \neg\varphi')) \vdash (\varphi \wedge \varphi')$}
\RightLabel{(dr_2)}
\UnaryInfC{$\varphi \vdash ((\varphi \wedge \neg\varphi') \,,\, (\varphi \wedge \varphi'))$}
\RightLabel{$(\vdash \vee)$}
\UnaryInfC{$\varphi \vdash ((\varphi \wedge \neg\varphi') \vee (\varphi \wedge \varphi'))$}
\end{prooftree}
$$

Proof of Expression (11). Assume that $\pm_j = +$. Then,

$$
\begin{prooftree}
\AxiomC{$[\circledast]\,(\varphi_1,\ldots,\varphi_n) \vdash \, \circledast(\varphi_1,\ldots,\varphi_n)$}
\RightLabel{$(\vdash \mathsf{K})$}
\UnaryInfC{$[\circledast]\,(\varphi_1,\ldots,\varphi_n) \vdash (\circledast(\varphi_1,\ldots,\varphi_j,\ldots,\varphi_n) \,,\, \circledast(\varphi_1,\ldots,\varphi_j',\ldots,\varphi_n))$}
\RightLabel{$(\vdash \vee)$}
\UnaryInfC{$[\circledast]\,(\varphi_1,\ldots,\varphi_n) \vdash (\circledast(\varphi_1,\ldots,\varphi_j,\ldots,\varphi_n) \vee \circledast(\varphi_1,\ldots,\varphi_j',\ldots,\varphi_n))$}
\RightLabel{(dr_1)}
\UnaryInfC{$\varphi_j \vdash \, [\tau_j\circledast]\,(\varphi_1,\ldots,(\circledast(\varphi_1,\ldots,\varphi_j,\ldots,\varphi_n) \vee \circledast(\varphi_1,\ldots,\varphi_j',\ldots,\varphi_n)),\ldots,\varphi_n)$}
\end{prooftree}
$$

Likewise, we prove that:

$$\varphi_j' \vdash [\tau_j \circledast](\varphi_1, \ldots, \circledast(\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n) \vee \circledast(\varphi_1, \ldots, \varphi_j', \ldots, \varphi_n), \ldots, \varphi_n).$$

So, by $(\vee \vdash)'$, we obtain that:

$$\varphi_j \vee \varphi_j' \vdash [\tau_j \circledast](\varphi_1, \ldots, \circledast(\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n) \vee \circledast(\varphi_1, \ldots, \varphi_j', \ldots, \varphi_n), \ldots, \varphi_n).$$

Thus, by (dr_1) and $(\circledast \vdash)$, we obtain that:

$$\circledast(\varphi_1, \ldots, \varphi_j \vee \varphi_j', \ldots, \varphi_n) \vdash \circledast(\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n) \vee \circledast(\varphi_1, \ldots, \varphi_j', \ldots, \varphi_n).$$

Proof of Expression (12). Assume that $\pm_j = -$. Then,

$$\dfrac{\dfrac{\dfrac{[\circledast](\varphi_1, \ldots, \varphi_n) \vdash \circledast(\varphi_1, \ldots, \varphi_n)}{[\circledast](\varphi_1, \ldots, \varphi_n) \vdash (\circledast(\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n)\ ,\ \circledast(\varphi_1, \ldots, \varphi_j', \ldots, \varphi_n))}\ (\vdash \mathsf{K})}{[\circledast](\varphi_1, \ldots, \varphi_n) \vdash (\circledast(\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n) \vee \circledast(\varphi_1, \ldots, \varphi_j', \ldots, \varphi_n))}\ (\vdash \vee)}{[\tau_j \circledast](\varphi_1, \ldots, (\circledast(\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n) \vee \circledast(\varphi_1, \ldots, \varphi_j', \ldots, \varphi_n)), \ldots, \varphi_n) \vdash \varphi_j}\ (\mathsf{dr}_1)$$

Likewise, we prove that:

$$[\tau_j \circledast](\varphi_1, \ldots, (\circledast(\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n) \vee \circledast(\varphi_1, \ldots, \varphi_j', \ldots, \varphi_n)), \ldots, \varphi_n) \vdash \varphi_j'.$$

So, by $(\vdash \wedge)'$, we obtain that:

$$[\tau_j \circledast](\varphi_1, \ldots, (\circledast(\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n) \vee \circledast(\varphi_1, \ldots, \varphi_j', \ldots, \varphi_n)), \ldots, \varphi_n) \vdash \varphi_j \wedge \varphi_j'.$$

Thus, by (dr_1) and $(\circledast \vdash)$, we obtain that:

$$\circledast(\varphi_1, \ldots, \varphi_j \wedge \varphi_j', \ldots, \varphi_n) \vdash (\circledast(\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n) \vee \circledast(\varphi_1, \ldots, \varphi_j', \ldots, \varphi_n)).$$

Proof of Expression (13):

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \vdash \psi}{*\psi \vdash *\varphi}\,(\mathsf{dr}_2')}{\neg\psi \vdash *\varphi}\,(\neg\vdash)}{(\neg\psi\ ,\ \varphi) \vdash}\,(\mathsf{dr}_2)}{(\varphi\ ,\ \neg\psi) \vdash}\,(\mathsf{Cl}\vdash) \qquad \dfrac{\dfrac{\dfrac{\varphi\ ,\ \neg\psi \vdash}{\varphi \vdash *\neg\psi}\,(\mathsf{dr}_2)}{\varphi \vdash \neg\neg\psi}\,(\neg\vdash) \quad \dfrac{\dfrac{\dfrac{\dfrac{\psi \vdash \psi}{*\psi \vdash *\psi}\,(\mathsf{dr}_2')}{*\psi \vdash \neg\psi}\,(\vdash\neg)}{*\neg\psi \vdash \psi}\,(\mathsf{dr}_2'')}{\neg\neg\psi \vdash \psi}\,(\neg\vdash)}{\varphi \vdash \psi}\,cut$$

□

Lemma 64. *Let $\circledast(\varphi_1,\dots,\varphi_n) \in \mathcal{L}$ with $\circledast = (\sigma,\pm,(\exists,(\pm_1,\dots,\pm_n)))$. If $\circledast(\varphi_1,\dots,\varphi_n)$ is $\mathsf{GGL_C}$–consistent then $\pm_1\varphi_1,\dots,\pm_n\varphi_n$ are $\mathsf{GGL_C}$–consistent, where* $\pm_j\varphi_j \triangleq \begin{cases}\varphi_j & \text{if } \pm_j = + \\ \neg\varphi_j & \text{if } \pm_j = -\end{cases}$.

Proof: We prove it by contraposition. If $\pm_j\varphi_j$ is $\mathsf{GGL_C}$–inconsistent then $\pm_j\varphi_j \vdash$. If $\pm_j = +$ then $\varphi_j \vdash$. If $\pm_j = -$ then $\neg\varphi_j \vdash$ and therefore $\vdash \varphi_j$ by the cut rule because $\neg\neg\varphi_j \vdash \varphi_j$ is provable. So, in both cases, applying Rule ($\vdash \circledast$), we obtain that $\circledast(\varphi_1,\dots,\varphi_n) \vdash$ and thus $\circledast(\varphi_1,\dots,\varphi_n)$ is $\mathsf{GGL_C}$–inconsistent. □

Definition 65 (Canonical model)**.** The *canonical model* is the tuple $(W^c,\mathcal{R}^c)$ where W^c is the set of all maximal $\mathsf{GGL_C}$–consistent sets of $\mathcal{L}_\mathsf{C}$ and $\mathcal{R}^c$ is a set of relations $R_\circledast$ over W^c, associated to the connectives $\circledast \in \mathsf{C}$ and defined by:

- if $\circledast = p$ then $\Gamma \in R_p^\pm$ iff $p \in \Gamma$ (where $p = (1,\pm,Æ)$);
- if $\circledast = (\sigma,\pm,(\exists,(\pm_1,\dots,\pm_n)))$ then $(\Gamma_1,\dots,\Gamma_{n+1}) \in R_\circledast^{\pm\sigma}$ iff for all $\varphi_1,\dots,\varphi_n \in \mathcal{L}_\mathsf{C}$, if $\varphi_1 \pitchfork \Gamma_1$ and … and $\varphi_n \pitchfork \Gamma_n$ then $\circledast(\varphi_1,\dots,\varphi_n) \in \Gamma_{n+1}$;
- if $\circledast = (\sigma,\pm,(\forall,(\pm_1,\dots,\pm_n)))$ then $(\Gamma_1,\dots,\Gamma_{n+1}) \notin R_\circledast^{\pm\sigma}$ iff for all $\varphi_1,\dots,\varphi_n \in \mathcal{L}_\mathsf{C}$, if $\circledast(\varphi_1,\dots,\varphi_n) \in \Gamma_{n+1}$ then $\varphi_1 \pitchfork \Gamma_1$ or … or $\varphi_n \pitchfork \Gamma_n$;

where for all $j \in [\![1;n]\!]$, $\varphi_j \pitchfork \Gamma_j \triangleq \begin{cases}\varphi_j \in \Gamma_j & \text{if } \pm_j = + \\ \varphi_j \notin \Gamma_j & \text{if } \pm_j = -\end{cases}$. □

Lemma 66 (Truth lemma)**.** *For all $\varphi \in \mathcal{L}$, for all maximal $\mathsf{GGL_C}$–consistent sets Γ, we have that $M^c,\Gamma \Vdash \varphi$ iff $\varphi \in \Gamma$.*

Proof: By induction on φ. The base case $\varphi = p \in \mathbb{P}$ holds trivially by definition of M^c.

- Case $\neg\varphi$.

Assume that $\neg\varphi \in \Gamma$ and assume towards a contradiction that it is not the case that $M^c, \Gamma \Vdash \neg\varphi$. Then, $M^c, \Gamma \Vdash \varphi$. So, by Induction Hypothesis, $\varphi \in \Gamma$. Now, $\varphi \ , \ \neg\varphi \vdash$ and $\neg\varphi \in \Gamma$ by assumption. Thus, Γ is not $\mathsf{GGL_C}$–consistent, which is impossible. Therefore, $M^c, \Gamma \Vdash \neg\varphi$.

Conversely, assume that $M^c, \Gamma \Vdash \neg\varphi$. Then, it is not the case that $M^c, \Gamma \Vdash \varphi$, so, by Induction Hypothesis, $\varphi \notin \Gamma$. Since Γ is a maximal $\mathsf{GGL_C}$–consistent set, this implies that $\Gamma \cup \{\varphi\}$ is not $\mathsf{GGL_C}$–consistent. So, there are $\varphi_1, \dots, \varphi_n \in \Gamma$ such that $\varphi_1 \ , \ \dots \ , \ \varphi_n \ , \ \varphi \vdash$. Thus, $\varphi_1 \ , \ \dots \ , \ \varphi_n \vdash * \varphi$ and also by $(\vdash \neg)$, $\varphi_1 \ , \ \dots \ , \ \varphi_n \vdash \neg\varphi$. Therefore, $\neg\varphi \in \Gamma$ by the cut lemma.

- Case $(\varphi \vee \psi)$.

We prove the following fact. It will prove the induction step because $M^c, \Gamma \Vdash \varphi \vee \psi$ iff $M^c, \Gamma \Vdash \varphi$ or $M^c, \Gamma \Vdash \psi$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$ by induction hypothesis.

Fact 67. *For all maximal $\mathsf{GGL_C}$–consistent sets Γ, $(\varphi \vee \psi) \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$.*

Without loss of generality, assume that $\varphi \in \Gamma$. Then, $\varphi \vdash \varphi$ implies $\varphi \vdash \varphi \vee \psi$ by $\mathsf{K}_,$ and $(\vdash \vee)$. So, by the cut lemma, $(\varphi \vee \psi) \in \Gamma$ since $\varphi \in \Gamma$. Conversely, we prove that $(\varphi \vee \psi) \in \Gamma$ implies that $\varphi \in \Gamma$ or $\psi \in \Gamma$. Assume that $(\varphi \vee \psi) \in \Gamma$ and assume towards a contradiction that $\varphi \notin \Gamma$ and $\psi \notin \Gamma$. Then, because Γ is a maximal $\mathsf{GGL_C}$–consistent set, there are $\varphi_1, \dots, \varphi_m \in \Gamma$ and $\psi_1, \dots, \psi_n \in \Gamma$ such that $\varphi_1 \ , \ \dots \ , \ \varphi_m \ , \ \varphi \vdash$ and $\psi_1 \ , \ \dots \ , \ \psi_n \ , \ \psi \vdash$. Thus, by $(\mathsf{K} \vdash)$, $(\mathsf{B} \vdash)$ and $(\mathsf{CI} \vdash)$, we have that $\varphi_1 \ , \ \dots \ , \ \varphi_m \ , \ \psi_1 \ , \ \dots \ , \ \psi_n \ , \ \varphi \vdash$ and $\varphi_1 \ , \ \dots \ , \ \varphi_m \ , \ \psi_1 \ , \ \dots \ , \ \psi_n \ , \ \psi \vdash$. Then, by rule (dr_2), we have that $\varphi \vdash * (\varphi_1 \ , \ \dots \ , \ \varphi_m \ , \ \psi_1 \ , \ \dots \ , \ \psi_n)$ and $\psi \vdash * (\varphi_1 \ , \ \dots \ , \ \varphi_m \ , \ \psi_1 \ , \ \dots \ , \ \psi_n)$. So, by rule $(\vee \vdash)'$, $(\varphi \vee \psi) \vdash * (\varphi_1 \ , \ \dots \ , \ \varphi_m \ , \ \psi_1 \ , \ \dots \ , \ \psi_n)$ and by rule (dr_2) and $(\mathsf{B} \vdash)$, $(\varphi \vee \psi) \ , \ \varphi_1 \ , \ \dots \ , \ \varphi_m \ , \ \psi_1 \ , \ \dots \ , \ \psi_n \vdash$. However, $(\varphi \vee \psi), \varphi_1, \dots, \varphi_m, \psi_1, \dots, \psi_n \in \Gamma$. Therefore, Γ is not $\mathsf{GGL_C}$–consistent, which is impossible. Thus, $\varphi \in \Gamma$ or $\psi \in \Gamma$.

- Case $(\varphi \wedge \psi)$.

We prove that $\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$. This will prove this induction step because $M^c, \Gamma \Vdash \varphi \wedge \psi$ iff $M^c, \Gamma \Vdash \varphi$ and $M^c, \Gamma \Vdash \psi$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$ by induction hypothesis. Assume that $\varphi \in \Gamma$ and $\psi \in \Gamma$. Then, since $\varphi \ , \ \psi \vdash \varphi \wedge \psi$ is provable, we have by the cut lemma that $\varphi \wedge \psi \in \Gamma$. Conversely, assume that $\varphi \wedge \psi \in \Gamma$ and assume towards a contradiction that $\varphi \notin \Gamma$. Since Γ is a maximal $\mathsf{GGL_C}$–consistent set, there is $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\varphi_1 \ , \ \ldots \ , \ \varphi_n \ , \ \varphi \models$. Now, by rule $(\mathsf{K} \vdash)$, we have that $\varphi_1 \ , \ \ldots \ , \ \varphi_n \ , \ \varphi \ , \ \psi \models$. Therefore, by rule $\mathsf{B}_\vee$, $\varphi_1 \ , \ \ldots \ , \ \varphi_n \ , \ (\varphi \ , \ \psi) \models$. Then, by rules $(\mathsf{Cl} \vdash)$ (dr_2), we have that $(\varphi \ , \ \psi) \models \ * (\varphi_1 \ , \ \ldots \ , \ \varphi_n)$. So, by rule $(\wedge \vdash)$, we have that $(\varphi \wedge \psi) \models \ * (\varphi_1 \ , \ \ldots \ , \ \varphi_n)$. Then, again by rules $(\mathsf{Cl} \vdash)$ and (dr_2), we obtain $\varphi_1 \ , \ \ldots \ , \ \varphi_n \ , \ (\varphi \ , \ \psi) \models$. Since $(\varphi \wedge \psi) \in \Gamma$ and $\varphi_1, \ldots, \varphi_n \in \Gamma$, this entails that Γ is not $\mathsf{GGL_C}$–consistent, which is impossible. Therefore, $\varphi \in \Gamma$. Likewise, we prove that $\psi \in \Gamma$.

- Case $\circledast(\varphi_1, \ldots, \varphi_n)$ with $\circledast = (\sigma, \pm, (\text{Æ}, (\pm_1, \ldots, \pm_n)))$.

First, we deal with the subcase Æ $= \exists$.

Assume that $\circledast(\varphi_1, \ldots, \varphi_n) \in \Gamma$. We have to show that $M^c, \Gamma \Vdash \circledast (\varphi_1, \ldots, \varphi_n)$, *i.e.*, there are $\Gamma_1, \ldots, \Gamma_n \in M^c$ such that $R_{\circledast}^{\pm\sigma} \Gamma_1 \ldots \Gamma_n \Gamma$ and $\Gamma_1 \pitchfork [\![\varphi_1]\!]$ and $\ldots$ and $\Gamma_n \pitchfork [\![\varphi_n]\!]$. We build these maximal $\mathsf{GGL_C}$–consistent sets $\Gamma_1, \ldots, \Gamma_n$ thanks to (pseudo) Algorithm 1 (because it does not terminate). This algorithm is such that if $\circledast (\ltimes_1 \pm_1 \Gamma_1, \ldots, \ltimes_n \pm_n \Gamma_n) \in \Gamma$ then for all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}$, there are $(\pm'_1, \ldots, \pm'_n) \in \{+, -\}^n$ such that $\circledast ((\ltimes_1 \pm_1 \Gamma_1^m) \times_1 (\pm'_1 \varphi_1^m), \ldots, (\ltimes_n \pm_n \Gamma_n^m) \times_n (\pm'_n \varphi_n^m)) \in \Gamma$. This is due to Expressions (9), (10) and Expressions (11), (12) of Lemma 63. What happens is that each $\ltimes_j \pm_j \Gamma_j$ is decomposed into disjunctions $((\ltimes_j \pm_j \Gamma_j) \wedge \varphi_n) \vee ((\ltimes_j \pm_j \Gamma_n) \wedge \neg\varphi_n)$ and conjunctions $((\ltimes_j \pm_j \Gamma_j) \vee \varphi_n) \wedge ((\ltimes_j \pm_j \Gamma_j) \vee \neg\varphi_n)$ depending on whether $\pm_j = +$ or $\pm_j = -$. Then, each decomposition of $\ltimes_j \pm_j \Gamma_n$ is replaced in Expression $\circledast (\ltimes_1 \pm_1 \Gamma_1, \ldots, \ltimes_n \pm_n \Gamma_n)$. This is possible thanks to rule $(\vdash \circledast)$ and this yields a new expression $(*)$. This new expression $(*)$ belongs to Γ because Γ is a maximal $\mathsf{GGL_C}$–consistent set, by the cut lemma. Then, we decompose again $(*)$ iteratively by applying Expressions (11) or (12). For each decomposition, at least one disjunct belongs to Γ because $\varphi \vee \psi \in \Gamma$ implies that either $\varphi \in \Gamma$ or $\psi \in \Gamma$ by Fact 67. Finally, after having decomposed each argument of $\circledast$, we obtain that there is $(\pm'_1, \ldots, \pm'_n) \in \{+, -\}^n$ such that $\circledast ((\ltimes_1 \pm_1 \Gamma_1^m) \times_1 (\pm'_1 \varphi_1^m), \ldots, (\ltimes_n \pm_n \Gamma_n^m) \times_n (\pm'_n \varphi_n^m)) \in \Gamma$.

Now, let $m \geq 0$ be fixed and assume that Γ_j^m is $\mathsf{GGL_C}$–consistent. Then, $\circledast ((\ltimes_1 \pm_1 \Gamma_1^m) \times_1 (\pm'_1 \varphi_1^m), \ldots, (\ltimes_n \pm_n \Gamma_n^m) \times_n (\pm'_n \varphi_n^m))$ is $\mathsf{GGL_C}$–consistent because it belongs to the $\mathsf{GGL_C}$–consistent set Γ_j^m. Thus, by Lemma 64, for all $j \in [\![1; n]\!]$, if $\pm_j = +$ then $\bigwedge \Gamma_j^m \wedge \pm'_j \varphi_j^m$ is $\mathsf{GGL_C}$–consistent and if $\pm_j = -$ then $\bigwedge \Gamma_j^m \wedge (-\pm'_j) \varphi_j^m$ is $\mathsf{GGL_C}$–consistent. That is, in both cases, Γ_j^{m+1} is $\mathsf{GGL_C}$–consistent. We have proved by induction that for all $m \geq 0$, Γ_j^m is $\mathsf{GGL_C}$–consistent. Thus, $\Gamma_1, \ldots, \Gamma_n$ are $\mathsf{GGL_C}$–

Algorithm 1

Require: $(\varphi_1, \ldots, \varphi_n) \in \mathcal{L}^n_{\mathsf{C}}$ and a maximal $\mathsf{GGL_C}$–consistent set Γ such that $\circledast(\varphi_1, \ldots, \varphi_n) \in \Gamma$ with $\circledast = (\sigma, \pm, (\exists, (\pm_1, \ldots, \pm_n)))$.

Ensure: A n–tuple of maximal $\mathsf{GGL_C}$–consistent sets $(\Gamma_1, \ldots, \Gamma_n)$ such that $R^{\pm\sigma}_{\circledast}\Gamma_1 \ldots \Gamma_n \Gamma$ and $\pm_1\varphi_1 \in \Gamma_1, \ldots, \pm_n\varphi_n \in \Gamma_n$.

Let $(\varphi^0_1, \ldots, \varphi^0_n), \ldots, (\varphi^m_1, \ldots, \varphi^m_n), \ldots$ be an enumeration of $\mathcal{L}^n_{\mathsf{C}}$;

$\Gamma^0_1 := \{\pm_1\varphi_1\}; \ldots; \Gamma^0_n := \{\pm_n\varphi_n\}$;

for all $m \geq 0$ **do**
  for all $(\pm'_1, \ldots, \pm'_n) \in \{+, -\}^n$ **do**
    if $\circledast\left(\left(\bowtie_1 \pm_1 \Gamma^m_1\right) \times_1 \left(\pm'_1\varphi^m_1\right), \ldots, \left(\bowtie_n \pm_n \Gamma^m_n\right) \times_n \left(\pm'_n\varphi^m_n\right)\right) \in \Gamma$ **then**
      $\Gamma^{m+1}_1 := \Gamma^m_1 \cup \{(\pm_1\pm'_1)\varphi^m_1\}$;
      $\vdots$
      $\Gamma^{m+1}_n := \Gamma^m_n \cup \{(\pm_n\pm'_n)\varphi^m_n\}$;
    end if
  end for
end for

$\Gamma_1 := \bigcup_{m \geq 0} \Gamma^m_1; \ldots; \Gamma_n := \bigcup_{m \geq 0} \Gamma^m_n$;

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where for all $\varphi \in \mathcal{L}$, $\pm\varphi \triangleq \begin{cases} \varphi & \text{if } \pm = + \\ \neg\varphi & \text{if } \pm = - \end{cases}$; for all $j \in [\![1; n]\!]$, $\times_j \triangleq \begin{cases} \wedge & \text{if } \pm_j = + \\ \vee & \text{if } \pm_j = - \end{cases}$ and
$\bowtie_j \pm_j \Gamma^m_j \triangleq \begin{cases} \bigwedge \left\{\varphi \mid \varphi \in \Gamma^m_j\right\} & \text{if } \pm_j = + \\ \bigvee \left\{\neg\varphi \mid \varphi \in \Gamma^m_j\right\} & \text{if } \pm_j = - \end{cases}$.

consistent. Moreover, for all $j \in [\![1; n]\!]$, Γ_j are *maximally* $\mathsf{GGL_C}$–consistent because by construction for all $\varphi \in \mathcal{L}$ either $\varphi \in \Gamma_j$ or $\neg\varphi \in \Gamma_j$.

Finally, we prove that $R^{\pm\sigma}_{\circledast}\Gamma_1 \ldots \Gamma_n \Gamma$, that is, we prove that for all $\psi_1, \ldots, \psi_n \in \mathcal{L}$ if $\psi_1 \pitchfork \Gamma_1$ and $\ldots$ and $\psi_n \pitchfork \Gamma_n$ then $\circledast(\psi_1, \ldots, \psi_n) \in \Gamma$, that is, since $\Gamma_1, \ldots, \Gamma_n$ are maximally $\mathsf{GGL_C}$–consistent sets, if $\pm_1\psi_1 \in \Gamma_1$ and $\ldots$ and $\pm_n\psi_n \in \Gamma_n$ then $\circledast(\psi_1, \ldots, \psi_n) \in \Gamma$. Assume that $\pm_1\psi_1 \in \Gamma_1$ and $\ldots$ and $\pm_n\psi_n \in \Gamma_n$, we are going to prove that $\circledast(\psi_1, \ldots, \psi_n) \in \Gamma$. Now $(\psi_1, \ldots, \psi_n) \in \mathcal{L}^n$, so there is $m_0 \geq 0$ such that $(\varphi_1^{m_0},\ldots,\varphi_n^{m_0}) = (\psi_1,\ldots,\psi_n)$. Since $\Gamma_1^{m_0+1} \subseteq \Gamma_1$ and $\ldots$ and $\Gamma_n^{m_0+1} \subseteq \Gamma_n$, we have that the tuple $(\pm_1',\ldots,\pm_n')$ satisfying the condition of line 8 of Algorithm 1 is $(+,\ldots,+)$, because of the way $\Gamma_1^{m_0+1}$,..., $\Gamma_n^{m_0+1}$ are defined. So, the condition of line 8, which is fulfilled, is $\circledast\left(\left(\bowtie_1 \pm_1 \Gamma_1^{m_0}\right) \times_1 \varphi_1^{m_0},\ldots,\left(\bowtie_n \pm_n \Gamma_n^{m_0}\right) \times_n \varphi_n^{m_0}\right) \in \Gamma$. Then, for all $j \in \llbracket 1;n \rrbracket$, if $\pm_j = +$ then $\left(\bowtie_j \pm_j \Gamma_j^{m_0}\right) \times_j \varphi_j^{m_0} \vdash \varphi_j^{m_0}$ and if $\pm_j = -$ then $\varphi_j^{m_0} \vdash \left(\bowtie_j \pm_j \Gamma_j^{m_0}\right) \times_j \varphi_j^{m_0}$. Therefore, applying rule $(\vdash \circledast)$, we obtain that $\circledast\left(\left(\bowtie_1 \pm_1 \Gamma_1^{m_0}\right) \times_1 \varphi_1^{m_0},\ldots,\left(\bowtie_n \pm_n \Gamma_n^{m_0}\right) \times_n \varphi_n^{m_0}\right) \vdash \circledast\left(\varphi_1^{m_0},\ldots,\varphi_n^{m_0}\right)$ is provable. Since we have proved that $\circledast\left(\left(\bowtie_1 \pm_1 \Gamma_1^{m_0}\right) \times_1 \varphi_1^{m},\ldots,\left(\bowtie_n \pm_n \Gamma_n^{m_0}\right) \times_n \varphi_n^{m_0}\right) \in \Gamma$, we obtain by the cut lemma that $\circledast\left(\varphi_1^{m_0},\ldots,\varphi_n^{m_0}\right) \in \Gamma$ as well, that is $\circledast\left(\psi_1,\ldots,\psi_n\right) \in \Gamma$.

Conversely, assume that $M^c, \Gamma \Vdash \circledast(\varphi_1,\ldots,\varphi_n)$, we are going to show that $\circledast(\varphi_1,\ldots,\varphi_n) \in \Gamma$. By definition, we have that there are $\Gamma_1,\ldots,\Gamma_n \in M^c$ such that $R_\circledast^{\pm\sigma}\Gamma_1\ldots\Gamma_n\Gamma$ and $\Gamma_1 \pitchfork \llbracket\varphi_1\rrbracket$ and $\ldots$ and $\Gamma_n \pitchfork \llbracket\varphi_n\rrbracket$. By Induction Hypothesis, we have that $\varphi_1 \pitchfork \Gamma_1$ and $\ldots$ and $\varphi_n \pitchfork \Gamma_n$. Then, by definition of $R_\circledast^{\pm\sigma}$ in Definition 65, we have that $\circledast(\varphi_1,\ldots,\varphi_n) \in \Gamma$.

Second, we deal with the subcase Æ = $\forall$.

Assume that $\circledast(\varphi_1,\ldots,\varphi_n) \in \Gamma$. We have to show that $M^c, \Gamma \Vdash \circledast(\varphi_1,\ldots,\varphi_n)$, *i.e.* for all $\Gamma_1,\ldots,\Gamma_n \in M^c$, $(\Gamma_1,\ldots,\Gamma_n,\Gamma) \in R_\circledast^{\pm\sigma}$ or $\Gamma_1 \pitchfork \llbracket\varphi_1\rrbracket$ or $\ldots$ or $\Gamma_n \pitchfork \llbracket\varphi_n\rrbracket$. Assume that $(\Gamma_1,\ldots,\Gamma_n,\Gamma) \notin R_\circledast^{\pm\sigma}$. Then, since $\circledast(\varphi_1,\ldots,\varphi_n) \in \Gamma$, we have by Definition 65 that $\varphi_1 \pitchfork \Gamma_1$ or $\ldots$ or $\varphi_n \pitchfork \Gamma_n$. So, by Induction Hypothesis, we have that $\Gamma_1 \pitchfork \llbracket\varphi_1\rrbracket$ or $\ldots$ or $\Gamma_n \pitchfork \llbracket\varphi_n\rrbracket$.

Conversely, we reason by contraposition and we assume that $\circledast(\varphi_1,\ldots,\varphi_n) \notin \Gamma$. We are going to show that $M^c, \Gamma \Vdash -\circledast(\varphi_1,\ldots,\varphi_n)$ (we recall that $-\circledast$ is a connective of $\mathbb{C}$), which will prove that it is not the case that $M^c, \Gamma \Vdash \circledast(\varphi_1,\ldots,\varphi_n)$ by Proposition 29. First, we prove that $\neg\circledast(\varphi_1,\ldots,\varphi_n) \vdash -\circledast(\varphi_1,\ldots,\varphi_n)$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi_1 \vdash \varphi_1 \quad \ldots \quad \varphi_n \vdash \varphi_n}{[-\circledast](\varphi_1,\ldots,\varphi_n) \vdash -\circledast(\varphi_1,\ldots,\varphi_n)}(\vdash\circledast)}{*[\circledast](\varphi_1,\ldots,\varphi_n) \vdash -\circledast(\varphi_1,\ldots,\varphi_n)}\text{Rewrite}}{*-\circledast(\varphi_1,\ldots,\varphi_n) \vdash [\circledast](\varphi_1,\ldots,\varphi_n)}(\mathsf{dr}_2'')}{*-\circledast(\varphi_1,\ldots,\varphi_n) \vdash \circledast(\varphi_1,\ldots,\varphi_n)}(\circledast\vdash)}{*\circledast(\varphi_1,\ldots,\varphi_n) \vdash -\circledast(\varphi_1,\ldots,\varphi_n)}(\mathsf{dr}_2'')}{\neg\circledast(\varphi_1,\ldots,\varphi_n) \vdash -\circledast(\varphi_1,\ldots,\varphi_n)}(\neg\vdash)$$

Then, by Fact 67 and because $\vdash (\varphi \vee \neg\varphi)$ is provable, we have that

$\neg\circledast(\varphi_1,\dots,\varphi_n) \in \Gamma$ or $\circledast(\varphi_1,\dots,\varphi_n) \in \Gamma$. So, by assumption, $\neg\circledast(\varphi_1,\dots,\varphi_n) \in \Gamma$. Therefore, by the cut lemma, since $\neg\circledast(\varphi_1,\dots,\varphi_n) \vdash -\circledast(\varphi_1,\dots,\varphi_n)$ we have that $-\circledast(\varphi_1,\dots,\varphi_n) \in \Gamma$. Hence, this case boils down to the case Æ $= \exists$ because $-\circledast = (\sigma, -\pm, (\exists, (-\pm_1,\dots,-\pm_n)))$. This case has been proved in the previous item and we thus have that $M^c, \Gamma \Vdash -\circledast(\varphi_1,\dots,\varphi_n)$. $\square$

We finally prove that the canonical model is indeed a C–model. For that, we need to prove the following lemma:

Lemma 68. *Let $\circledast \in C$ be a connective of arity $n \in \mathbb{N}$. Then, for all $\circledast' \in \mathcal{O}_{\alpha_n * \beta_n}(\circledast)$, we have that $R_\circledast = R_{\circledast'}$.*

Proof: We prove this lemma using the following two facts: for all $\circledast \in \mathbb{C}$, all transpositions $\tau_j = (j\ n+1)$,

$$\text{if } \circledast = (\sigma, \pm, (\exists, (\pm_1,\dots,\pm_n))) \text{ then } \circledast(\varphi_1,\dots,\tau_j\circledast(\varphi_1,\dots,\varphi_n),\dots,\varphi_n) \vdash \varphi_j \quad (14)$$

$$\text{if } \circledast = (\sigma, \pm, (\forall, (\pm_1,\dots,\pm_n)))\text{then } \varphi_j \vdash \circledast(\varphi_1,\dots,\tau_j\circledast(\varphi_1,\dots,\varphi_n),\dots,\varphi_n) \quad (15)$$

Expressions (14) and (15) are proved by a direct application of (dr_1) with τ_j and then $(\circledast\vdash)$ to the provable consecution $[\tau_j\circledast](\varphi_1,\dots,\varphi_n) \vdash \tau_j\circledast(\varphi_1,\dots,\varphi_n)$ if Æ$(\tau_j\circledast) = \exists$ and $\tau_j\circledast(\varphi_1,\dots,\varphi_n) \vdash [\tau_j\circledast](\varphi_1,\dots,\varphi_n)$ if Æ$(\tau_j\circledast) = \forall$.

First, we prove that for all $\circledast' \in \mathcal{O}_{\alpha_n}(\circledast)$, we have that $R_\circledast = R_{\circledast'}$. For that, it suffices to prove that for all transpositions $\tau_j = (j\ n+1)$, we have that $R_{\tau_j\circledast} = R_\circledast$ because the transpositions generate the symmetric group. Proving $R_\circledast \subseteq R_{\tau_j\circledast}$ or $R_{\tau_j\circledast} \subseteq R_\circledast$ for all $\tau_j = (j\ n+1)$ is enough, because by double inclusion we then have that $R_\circledast \subseteq R_{\tau_j\circledast} \subseteq R_{\tau_j\tau_j\circledast} = R_\circledast$ and thus $R_\circledast = R_{\tau_j\circledast}$.

- Case $\circledast = (\sigma, \pm, (\exists, (\pm_1,\dots,\pm_{j-1},+,\pm_{j+1},\dots,\pm_n)))$. Then, $\tau_j\circledast = (\tau_j\sigma, -\pm, (\forall, (-\pm_1,\dots,-\pm_{j-1},+,-\pm_{j+1},\dots,-\pm_n)))$.

Assume that $(\Gamma_1,\dots,\Gamma_{n+1}) \in R_\circledast^{\pm\sigma}$. We are going to show that $(\Gamma_1,\dots,\Gamma_{n+1}) \in R_{\tau_j\circledast}^{\pm\sigma}$, *i.e.* $(\Gamma_1,\dots,\Gamma_{n+1}) \notin R_{\tau_j\circledast}^{-\pm\sigma}$, *i.e.* $(\Gamma_1,\dots,\Gamma_{j-1},\Gamma_{n+1},\Gamma_{j+1},\dots,\Gamma_n,\Gamma_j) \notin R_{\tau_j\circledast}^{-\pm\tau_j\sigma}$. Let $\varphi_1,\dots,\varphi_n \in \mathcal{L}_\mathsf{C}$ and assume that $\tau_j\circledast(\varphi_1,\dots,\varphi_n) \in \Gamma_j$ and $\varphi_1 \pitchfork \Gamma_1$ and … and $\varphi_n \pitchfork \Gamma_n$ where $\varphi_i \pitchfork \Gamma_i \triangleq \begin{cases} \varphi_i \in \Gamma_i & \text{if } \pm_i = + \\ \varphi_i \notin \Gamma_i & \text{if } \pm_i = - \end{cases}$. We want to prove that $\varphi_j \in \Gamma_{n+1}$.

Since $(\Gamma_1,\dots,\Gamma_n,\Gamma_{n+1}) \in R_\circledast^{\pm\sigma}$ and $\varphi_1 \pitchfork \Gamma_1$ and … and $\tau_j\circledast(\varphi_1,\dots\varphi_n) \in \Gamma_j$ and … and $\varphi_n \pitchfork \Gamma_n$, we have that $M^c, \Gamma_{n+1} \Vdash \circledast(\varphi_1,\dots,\tau_j(\varphi_1,\dots,\varphi_n),\dots,\varphi_n)$. So, by the truth lemma, $\circledast(\varphi_1,\dots,\tau_j\circledast(\varphi_1,\dots,\varphi_n),\dots,\varphi_n) \in \Gamma_{n+1}$. Now, by Expression (14), $\circledast(\varphi_1,\dots,\tau_j\circledast(\varphi_1,\dots,\varphi_n),\dots,\varphi_n) \vdash \varphi_j$. Therefore, $\varphi_j \in \Gamma_{n+1}$ by the cut lemma.

• Case $\circledast = (\sigma, \pm, (\exists, (-, \dots, -)))$. Then, $\tau_j\circledast = (\tau_j\sigma, \pm, (\exists, (-, \dots, -)))$.

Assume that $(\Gamma_1, \dots, \Gamma_{n+1}) \in R^{\pm\sigma}_{\circledast}$, *i.e.* for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_{\mathsf{C}}$, if $\varphi_1 \notin \Gamma_1$ and … and $\varphi_n \notin \Gamma_n$ then $\circledast(\varphi_1, \dots, \varphi_n) \in \Gamma_{n+1}$ (1). We are going to show that $(\Gamma_1, \dots, \Gamma_{n+1}) \in R^{\pm\sigma}_{\tau_j\circledast}$, *i.e.* $(\Gamma_1, \dots, \Gamma_{n+1}, \dots \Gamma_n, \Gamma_j) \in R^{\pm\tau_j\sigma}_{\tau_j\circledast}$, *i.e.* for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_{\mathsf{C}}$, if $\varphi_1 \notin \Gamma_1$ and … and $\varphi_j \notin \Gamma_{n+1}$ and … and $\varphi_n \notin \Gamma$, then $\tau_j \circledast (\varphi_1, \dots, \varphi_n) \in \Gamma_j$. Assume that $\varphi_1 \notin \Gamma_1$ and … and $\varphi_j \notin \Gamma_{n+1}$ and … and $\varphi_n \notin \Gamma_n$. We want to prove that $\tau_j \circledast (\varphi_1, \dots, \varphi_n) \in \Gamma_j$.

Since $\varphi_j \notin \Gamma_{n+1}$, we have that $\circledast(\varphi_1, \dots, \tau_j \circledast (\varphi_1, \dots, \varphi_n), \dots, \varphi_n) \notin \Gamma_{n+1}$ because of the cut lemma since $\circledast(\varphi_1, \dots, \tau_j \circledast (\varphi_1, \dots, \varphi_n), \dots, \varphi_n) \vdash \varphi_j$ by Expression (14). Then, either $\varphi_1 \in \Gamma_1$ or $\varphi_2 \in \Gamma_2$ or … or $\tau_j \circledast (\varphi_1, \dots, \varphi_n) \in \Gamma_j$ or $\varphi_{j+1} \in \Gamma_{j+1}$ or … or $\varphi_n \in \Gamma_n$, because of (1). However, $\varphi_1 \notin \Gamma_1$,…, $\varphi_{j-1} \notin \Gamma_{j-1}$, $\varphi_{j+1} \notin \Gamma_{j+1}$, …, $\varphi_n \notin \Gamma_n$. Therefore, $\tau_j \circledast (\varphi_1, \dots, \varphi_n) \in \Gamma_j$.

• Case $\circledast = (\sigma, \pm, (\forall, (\pm_1, \dots \pm_{j-1}, +, \pm_{j+1}, \dots, \pm_n)))$. Then, $\tau_j\circledast = (\tau_j\sigma, -\pm, (\exists, (-\pm_1, \dots, -\pm_{j-1}, +, -\pm_{j+1}, \dots, -\pm_n)))$.

Assume that $(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}) \notin R^{\pm\sigma}_{\circledast}$. We are going to show that $(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}) \notin R^{\pm\sigma}_{\tau_j\circledast}$, *i.e.* $(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}) \in R^{-\pm\sigma}_{\tau_j\circledast}$ *i.e.* $(\Gamma_1, \dots, \Gamma_{n+1}, \Gamma_{j+1}, \dots, \Gamma_n, \Gamma_j) \in R^{-\pm\tau_j\sigma}_{\tau_j\circledast}$ *i.e.* for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_{\mathsf{C}}$, if $\varphi_1 \pitchfork \Gamma_1$ and … and $\varphi_j \in \Gamma_{n+1}$ and $\varphi_{j+1} \pitchfork \Gamma_{j+1}$ and … and $\varphi_n \pitchfork \Gamma_n$ then $\tau_j \circledast (\varphi_1, \dots, \varphi_n) \in \Gamma_j$ where $\varphi_i \pitchfork \Gamma_i \triangleq \begin{cases} \varphi_i \in \Gamma_i & \text{if } -\pm_i = + \\ \varphi_i \notin \Gamma_i & \text{if } -\pm_i = - \end{cases}$. Assume that $\varphi_1 \pitchfork \Gamma_1$ and … and $\varphi_j \in \Gamma_{n+1}$ and $\varphi_{j+1} \pitchfork \Gamma_{j+1}$ and … and $\varphi_n \pitchfork \Gamma_n$. We want to show that $\tau_j \circledast (\varphi_1, \dots, \varphi_n) \in \Gamma_j$.

Since $\varphi_j \in \Gamma_{n+1}$ and $\varphi_j \vdash \circledast(\varphi_1, \dots, \tau_j \circledast (\varphi_1, \dots, \varphi_n), \dots, \varphi_n)$ by Expression (15), we have by the cut lemma that $\circledast(\varphi_1, \dots, \tau_j \circledast (\varphi_1, \dots, \varphi_n), \dots \varphi_n) \in \Gamma_{n+1}$. So, $M^c, \Gamma_{n+1} \Vdash \circledast(\varphi_1, \dots, \tau_j \circledast (\varphi_1, \dots, \varphi_n), \dots, \varphi_n)$ by the truth lemma. That is, for all $\Gamma'_1, \dots, \Gamma'_n \in M^c$, either $(\Gamma'_1, \dots, \Gamma'_n, \Gamma_{n+1}) \in R^{\pm\sigma}_{\circledast}$ or not $\varphi_1 \pitchfork \Gamma_1$ or … or $\tau_j \circledast (\varphi_1, \dots, \varphi_n) \in \Gamma_j$ or … or not $\varphi_n \pitchfork \Gamma_n$ ($\varphi_i \pitchfork \Gamma_i$ is defined above). Take $(\Gamma'_1, \dots, \Gamma'_n) = (\Gamma_1, \dots, \Gamma_n)$. Then, by assumption, $(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}) \notin R^{\pm\sigma}_{\circledast}$ and $\varphi_1 \pitchfork \Gamma_1$ and … and $\varphi_{j-1} \pitchfork \Gamma_{j-1}$ and $\varphi_{j+1} \pitchfork \Gamma_{j+1}$ and … and $\varphi_n \pitchfork \Gamma_n$. Therefore, $\tau_j \circledast (\varphi_1, \dots, \varphi_n) \in \Gamma_j$.

• Case $\circledast = (\sigma, \pm, (\forall, (-, \dots, -)))$. Then, $\tau_j\circledast = (\tau_j\sigma, \pm, (\forall, (-, \dots, -)))$.

Assume that $(\Gamma_1, \dots, \Gamma_{n+1}) \notin R^{\pm\sigma}_{\circledast}$, *i.e.* for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_{\mathsf{C}}$, if $\circledast(\varphi_1, \dots, \varphi_n) \in \Gamma_{n+1}$ and $\varphi_1 \in \Gamma_1$ and …and $\varphi_n \in \Gamma_n$ then $\varphi_j \notin \Gamma_j$ (2). We are going to show that $(\Gamma_1, \dots, \Gamma_{n+1}) \notin R^{\pm\sigma}_{\tau_j\circledast}$, *i.e.* $(\Gamma_1, \dots, \Gamma_{n+1}, \dots, \Gamma_n, \Gamma_j) \notin R^{\pm\tau_j\sigma}_{\tau_j\circledast}$ *i.e.* for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_{\mathsf{C}}$ if $\tau_j \circledast (\varphi_1, \dots, \varphi_n) \in \Gamma_j$ and $\varphi_1 \in \Gamma_1$ and … and $\varphi_n \in \Gamma_n$ then $\varphi_j \notin \Gamma_{n+1}$. Assume that $\tau_j \circledast (\varphi_1, \dots, \varphi_n) \in \Gamma_j$ (3) and $\varphi_1 \in \Gamma_1$ and … and $\varphi_n \in \Gamma_n$. We want to prove that $\varphi_j \notin \Gamma_{n+1}$.

Assume towards a contradiction that $\varphi_j \in \Gamma_{n+1}$. Then, by Expression (15) and the cut lemma, $\circledast(\varphi_1, \ldots, \tau_j \circledast (\varphi_1, \ldots, \varphi_n), \ldots, \varphi_n) \in \Gamma_{n+1}$. Now, $\varphi_1 \in \Gamma_1$ and $\ldots$ and $\varphi_{j-1} \in \Gamma_{j-1}$ and $\varphi_{j+1} \in \Gamma_{j+1}$ and $\ldots$ and $\varphi_n \in \Gamma_n$. So, by (2), because $(\Gamma_1, \ldots, \Gamma_{n+1}) \notin R^{\pm\sigma}_{\circledast}$, we have that $\tau_j \circledast (\varphi_1, \ldots, \varphi_n) \notin \Gamma_j$. This contradicts (3).

Second, we prove that $R_{\circledast} = R_{-\circledast}$. Again, it suffices to prove that $R_{\circledast} \subseteq R_{-\circledast}$.

- Case $\circledast = (\sigma, \pm, (\exists, (\pm_1, \ldots, \pm_n)))$. Then, $-\circledast = (\sigma, -\pm, (\forall, (-\pm_1, \ldots, -\pm_n)))$.

$(\Gamma_1, \ldots, \Gamma_{n+1}) \in R^{\pm\sigma}_{\circledast}$ iff for all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}_{\mathsf{C}}$, if $\varphi_1 \pitchfork \Gamma_1$ and $\ldots$ and $\varphi_n \pitchfork \Gamma_n$ then $\circledast(\varphi_1, \ldots, \varphi_n) \in \Gamma_{n+1}$ where $\varphi_j \pitchfork \Gamma_j = \begin{cases} \varphi_j \in \Gamma_j & \text{if } \pm_j = + \\ \varphi_j \notin \Gamma_j & \text{if } \pm_j = - \end{cases}$. We are going to show that $(\Gamma_1, \ldots, \Gamma_{n+1}) \in R^{\pm\sigma}_{-\circledast}$, *i.e.* $(\Gamma_1, \ldots, \Gamma_{n+1}) \notin R^{-\pm\sigma}_{-\circledast}$ *i.e.* for all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}_{\mathsf{C}}$, if $-\circledast(\varphi_1, \ldots, \varphi_n) \in \Gamma_{n+1}$ then $\varphi_1 \pitchfork' \Gamma_1$ or $\ldots$ or $\varphi_n \pitchfork' \Gamma_n$ (1) where $\varphi_j \pitchfork' \Gamma_j = \begin{cases} \varphi_j \in \Gamma_j & \text{if } -\pm_j = + \\ \varphi_j \notin \Gamma_j & \text{if } -\pm_j = - \end{cases}$. So, for all j, $\varphi_j \pitchfork' \Gamma_j$ is (not $\varphi_j \pitchfork \Gamma_j$). Therefore, (1) holds iff if $\circledast(\varphi_1, \ldots, \varphi_n) \notin \Gamma_{n+1}$ and $\varphi_1 \pitchfork \Gamma_1$ and $\ldots$ and $\varphi_n \pitchfork \Gamma_n$ then not $\varphi_j \pitchfork \Gamma_j$

iff if $\varphi_1 \pitchfork \Gamma_1$ and $\ldots$ and $\varphi_n \pitchfork \Gamma_n$ then $\circledast(\varphi_1, \ldots, \varphi_n) \in \Gamma_{n+1}$

iff $(\Gamma_1, \ldots, \Gamma_{n+1}) \in R^{\pm\sigma}_{\circledast}$ which holds by assumption.

- Case $\circledast = (\sigma, \pm, (\forall, (\pm_1, \ldots, \pm_n)))$. It is proved like the previous case. □

Proof: (Completeness proof) We prove that for all sets $\Gamma \subseteq \mathcal{S}_{\mathsf{C}}$ and all $S = \varphi \vdash \psi \in \mathcal{S}_{\mathsf{C}}$, if $\Gamma \Vdash S$ holds then S is provable from Γ in $\mathsf{GGL}_{\mathsf{C}}$. We reason by contraposition. Assume that S is not provable from Γ in $\mathsf{GGL}_{\mathsf{C}}$. That is, there is no proof of $\varphi \vdash \psi$ in $\mathsf{GGL}_{\mathsf{C}}$ from Γ. Thus, it is not the case that $(\varphi\ ,\ \neg\psi) \vdash$ is provable in $\mathsf{GGL}_{\mathsf{C}} \cup \Gamma$ by Expression (13). Hence, $\{\varphi, \neg\psi\}$ is $\mathsf{GGL}_{\mathsf{C}} \cup \Gamma$–consistent (we can naturally adapt the definition of $\mathsf{GGL}_{\mathsf{C}}$–consistency to define the notion of $\mathsf{GGL}_{\mathsf{C}} \cup \Gamma$–consistency). So, by Lemma 62 (where $\mathsf{GGL}_{\mathsf{C}}$–consistency is replaced by $\mathsf{GGL}_{\mathsf{C}} \cup \Gamma$–consistency), it can be extended into a maximal $\mathsf{GGL}_{\mathsf{C}} \cup \Gamma$–consistent set Γ' such that $\{\varphi, \neg\psi\} \subseteq \Gamma'$. Now, Γ' is also $\mathsf{GGL}_{\mathsf{C}}$–consistent, so it is a state of the canonical model M^c. Then, by the truth Lemma 66, we have that $(M^c, \Gamma') \Vdash \varphi$ and $(M^c, \Gamma') \Vdash \neg\psi$, so it is not the case that $(M^c, \Gamma') \Vdash S$. Moreover, by the cut Lemma 61 and because Γ' is also $\mathsf{GGL}_{\mathsf{C}} \cup \Gamma$–consistent, we also have that $(M^c, \Gamma') \Vdash \Gamma$. Hence, we have found a pointed model (M^c, Γ'), which is indeed a C–model according to Lemma 68, such that $(M^c, \Gamma') \Vdash \Gamma$ but not $(M^c, \Gamma') \Vdash S$. That is, it is not the case that $\Gamma \Vdash S$. □

C Proofs of theorems 49, 53 and 57

Theorem 49 (Cut–elimination)**.** *Let* $\mathsf{C} \subseteq \mathbb{C}$*. The calculus* GGL_C *is cut–eliminable: it is possible to eliminate all occurrences of the cut rule from a given proof in order to obtain a cut-free proof of the same consecution.*

Proof: Since GGL_C is a display calculus in the general sense of Ciabattoni & Ramanayake [9], we only need to prove that it satisfies the conditions $(C2)$–$(C8)$ spelled out in [9] as proved by Belnap [6]. Note that condition $(C1)$ is not needed in Belnap's proof [6]. The conditions $(C2)$–$(C7)$ are easily checked on each rule of GGL_C. It remains to prove condition $(C8)$. It has already been proved in the literature for the Boolean connectives so we only prove it for the gaggle connectives. Instead of proving it in the general case, we prove it for $n = 2$ with $\circledast = (\sigma, \pm, (\exists, (+, -)))$. This should provide the reader with the main ideas underlying the proof in the general case. Basically, we display each subformula of the cut formula using the display rule (dr_1) and we apply the cut rule on each subformula.

$$\dfrac{\dfrac{X_1 \vdash \varphi_1 \quad \varphi_2 \vdash X_2}{[\circledast]\,(X_1, X_2) \vdash \circledast\,(\varphi_1, \varphi_2)}\,(\vdash \circledast) \quad \dfrac{[\circledast]\,(\varphi_1, \varphi_2) \vdash U}{\circledast(\varphi_1, \varphi_2) \vdash U}\,(\circledast \vdash)}{[\circledast]\,(X_1, X_2) \vdash U}\,\mathit{cut}\,(\circledast(\varphi_1, \varphi_2))$$

is transformed into

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{X_1 \vdash \varphi_1 \quad \dfrac{[\circledast]\,(\varphi_1, \varphi_2) \vdash U}{\varphi_1 \vdash [r_1\circledast]\,(U, \varphi_2)}\,(\mathsf{dr}_1)}{X_1 \vdash [r_1\circledast]\,(U, \varphi_2)}\,\mathit{cut}\,(\varphi_1)}{[\circledast]\,(X_1, \varphi_2) \vdash U}\,(\mathsf{dr}_1)}{[r_2\circledast]\,(X_1, U) \vdash \varphi_2}\,(\mathsf{dr}_1) \quad \varphi_2 \vdash X_2}{[r_2\circledast]\,(X_1, U) \vdash X_2}\,\mathit{cut}\,(\varphi_2)}{[\circledast]\,(X_1, X_2) \vdash U}\,(\mathsf{dr}_1)$$

We proceed similarly for the rules concerning the Boolean connectives $\neg, \wedge, \vee$ using the Boolean display rule (dr_2). □

Theorem 53 (Soundness and strong completeness)**.** *Let* $\mathsf{C} \subseteq \mathbb{C}$*. The calculus* $\mathsf{GGL}^0_\mathsf{C}$ *is sound and strongly complete for the basic gaggle logic* $(\mathcal{S}^0_\mathsf{C}, \mathcal{M}_\mathsf{C}, \Vdash)$*.*

Proof: We are going to perform a backward proof search and analyze the structure of a cut-free proof in GGL_C which ends up in a consecution of the following form, where $\varphi_1, \ldots, \varphi_k, \varphi'_1, \ldots, \varphi'_l \in \mathcal{L}^0_\mathsf{C}$ do not contain Boolean connectives:

$$\circledast(\varphi_1, \ldots, \varphi_k) \vdash \circledast'\,(\varphi'_1, \ldots, \varphi'_l).$$

Our aim is, via that analysis, to transfom the proof in $\mathsf{GGL_C}$ of the above consecution into a proof in $\mathsf{GGL^0_C}$ of the same consecution. This will prove the theorem.

Before proceeding further, note that the following rules are particular instances of $(\mathsf{K}\vdash)$ and $(\vdash\mathsf{K})$ (with X empty):

$$\frac{\vdash U}{Y \vdash U}\ (\mathsf{K}\vdash)' \qquad\qquad \frac{U \vdash}{U \vdash Y}\ (\vdash\mathsf{K})'$$

Since the proof is cut-free and the final consecution does not contain Boolean connectives, the Boolean rules $(\wedge\vdash)$, $(\vdash\wedge)$, $(\vee\vdash)$, $(\vdash\vee)$, $(\vdash\neg)$ and $(\neg\vdash)$ have not been applied in the proof. Indeed, a property of our *cut-free* calculus $\mathsf{GGL_C}$ is that once a (Boolean) connective is introduced in a proof it stays present in the proof. Because the conclusion of our proof does not contain Boolean connective, this entails that the Boolean rules have not been used.

Stage A: rules $(\circledast\vdash)$ and (dr_1). We start with a proof in $\mathsf{GGL_C}$ whose conclusion is of the form $\circledast(\varphi_1,\ldots,\varphi_k)\vdash\circledast'(\varphi'_1,\ldots,\varphi'_l)$ and we analyse its proof backwards and determine which rule(s) can be used as we proceed bottom–up. At the begining, it is not possible to apply rule $(\vdash\circledast)$ because the antecedent and the consequent of the consecution are both formulas. On the other hand, it is possible to apply rule (dr_2) or $(\mathsf{WI}\vdash)$ right at the beginning and in that case we go directly to stage B. Otherwise, it is also possible to apply the rules $(\circledast\vdash)$ and (dr_1) (possibly iteratively). We then obtain an expression of the form $S([\circledast_1],X_1,\ldots,X_m,\circledast_2(\psi_1,\ldots,\psi_n))$ or $S([\circledast_1],X_1,\ldots,X_m,[\circledast_2](Y_1,\ldots,Y_n))$ where $X_1,\ldots,X_m,Y_1,\ldots,Y_n$ belong to the language $\mathcal{L}^X$ built up from formulas φ, structural atoms and structural connectives $[\circledast]$. Hence, at the end of that stage, we have a consecution of the form $[\circledast_1](X_1,\ldots,X_n)\vdash\circledast_2(\psi_1,\ldots,\psi_n)$ (1) or $\circledast_1(\varphi_1,\ldots,\varphi_m)\vdash[\circledast_2](Y_1,\ldots,Y_n)$ (2) or $[\circledast_1](X_1,\ldots,X_m)\vdash[\circledast_2](Y_1,\ldots,Y_n)$ (3). Without loss of generality, let us deal with case (1) in what follows.

We can then go to stage B or to stage C.

Stage B: rules (dr_2) or $(\mathsf{WI}\vdash)$ and then structural rules. If rule (dr_2) is applied, we obtain

$$\frac{([\circledast_1](X_1,\ldots,X_m)\ ,\ *\circledast_2(\psi_1,\ldots,\psi_n))\vdash}{[\circledast_1](X_1,\ldots,X_m)\vdash\ \circledast_2(\psi_1,\ldots,\psi_n)}\ (\mathsf{dr}_2)$$

or

$$\frac{\vdash (\circledast_2(\psi_1,\ldots,\psi_n)\ ,\ *[\circledast_1](X_1,\ldots,X_m))}{[\circledast_1](X_1,\ldots,X_m) \vdash \circledast_2(\psi_1,\ldots,\psi_n)}\ (\mathsf{dr}_2).$$

If rule $(\mathsf{Cl}\vdash)$ is applied, we obtain

$$\frac{([\circledast_1](X_1,\ldots,X_m)\ ,\ [\circledast_1](X_1,\ldots,X_m)) \vdash \circledast_2(\psi_1,\ldots,\psi_n)}{[\circledast_1](X_1,\ldots,X_m) \vdash \circledast_2(\psi_1,\ldots,\psi_n)}\ (\mathsf{Wl}\vdash).$$

In both cases, we obtain a premise including the structural connective $,$. This means that we cannot apply rules (dr_1), $(\vdash\circledast)$ or $(\circledast\vdash)$ for the moment. We must use the other rules, the structural rules and (dr_2), in order to apply one of these rules. Indeed, for the proof to terminate, we have to apply these rules in order to reduce the complexity of the consecution. Since the structural rules and (dr_2) do not change the constituants of a consecution, the consecutions that we can obtain as a result of applying these rules in order to be able to apply rules (dr_1), $(\vdash\circledast)$ or $(\circledast\vdash)$ again are the following:

1. $\vdash \circledast_2(\psi_1,\ldots,\psi_n)$
2. $[\circledast_1](X_1,\ldots,X_m) \vdash$
3. $*\circledast_2(\psi_1,\ldots,\psi_n) \vdash [-\circledast_1](X_1,\ldots,X_n)$
4. $[\circledast_1](X_1,\ldots,X_m) \vdash \circledast_2(\psi_1,\ldots,\psi_n)$.

For each case, we replace the existing derivation by the following derivation:

1.

$$\frac{\vdash \circledast_2(\psi_1,\ldots,\psi_n)}{[\circledast_1](X_1,\ldots,X_m) \vdash \circledast_2(\psi_1,\ldots,\psi_n)}\ (\mathsf{K}\vdash)'$$

2.

$$\frac{[\circledast_1](X_1,\ldots,X_m) \vdash}{[\circledast_1](X_1,\ldots,X_m) \vdash \circledast_2(\psi_1,\ldots,\psi_n)}\ (\vdash\mathsf{K})'$$

3.

$$\frac{*\circledast_2(\psi_1,\ldots,\psi_n) \vdash [-\circledast_1](X_1,\ldots,X_m)}{[\circledast_1](X_1,\ldots,X_m) \vdash \circledast_2(\psi_1,\ldots,\psi_n)}\ (\mathsf{dr}_2')$$

4. We simply remove the existing derivation.

So, for all cases the Boolean display rule (dr_2) and the structural rules have been eliminated. In all cases, the proof (considered so far) can be transfomed into a proof where (dr_2) has been eliminated and replaced by $\left(\mathsf{dr}_2'\right)$, $(\mathsf{K}\vdash)'$ and $(\vdash\mathsf{K})'$.

In all cases, the last premise ends up to be a consecution of the form $S([\circledast_1], X_1, \ldots, X_m, \circledast_2(\psi_1, \ldots, \psi_n))$ or $\circledast_1(\varphi_1, \ldots, \varphi_m) \models \circledast_2(\psi_1, \ldots, \psi_n)$ (possibly with $\circledast_2(\psi_1, \ldots, \psi_n)$ empty). Then, we go to stage C.

Stage C: rules (dr_1) or $(\vdash\circledast)$. If rule (dr_1) is applied then we go back to stage A.

If rule $(\vdash\circledast)$ is applied,

$$\frac{U_1 \models V_1 \quad \ldots \quad U_n \models V_n}{S([\circledast_1], X_1, \ldots, X_n, \circledast_2(\psi_1, \ldots, \psi_n))}\ (\vdash\circledast)$$

then for all $j \in [\![1;n]\!]$, $U_j \models V_j$ are of the form $X_j \models \psi_j$ or $\varphi_j \models X_j$ where $X_j \in \mathcal{L}^X$. So, we apply inductively stages A, B and C to each $U_j \models V_j$.

Hence, applying these stages recursively, we are able to eliminate all structural rules and the Boolean display rule (dr_2) from the proof and replace them with the rules $\left(\mathsf{dr}_2'\right)$, $(\mathsf{K}\vdash)'$ and $(\vdash\mathsf{K})'$.

Stage D. At this stage we have transformed our initial proof in $\mathsf{GGL_C}$ into a proof in the calculus consisting in the rules $(\vdash\circledast)$, $(\circledast\vdash)$, (dr_1), $\left(\mathsf{dr}_2'\right)$, $(\mathsf{K}\vdash)'$ and $(\vdash\mathsf{K})'$. A requirement of rule $(\mathsf{K}\vdash)'$ $((\vdash\mathsf{K})')$ is that the antecedent (resp. consequent) of its premisse is empty. If we examine the other rules, we notice that an empty antecedent can only appear in rule $(\vdash\circledast)$ if one of its premise already contains an empty antecedent (see Expression (7)). As a matter of fact, because of our axioms (see Expressions (5) and (6)) and the other rules, this can never happen. Hence, rules $(\mathsf{K}\vdash)'$ and $(\vdash\mathsf{K})'$ are in fact never used in a proof. Therefore, the proof that we eventually obtain is actually a proof in $\mathsf{GGL}^0_\mathsf{C}$. □

Theorem 57 (Soundness and strong completeness)**.** *Let $\mathsf{C} \subseteq \mathbb{C}$ and let G be a group associated to C. The calculus $\mathsf{GL}^0_{\mathsf{C},G}$ ($\mathsf{GL}_{\mathsf{C},G}$) is sound and strongly complete for the (Boolean) basic gaggle logic $(\mathcal{S}^0_\mathsf{C}, \mathcal{M}_\mathsf{C}, \Vdash)$ (resp. $(\mathcal{S}_\mathsf{C}, \mathcal{M}_\mathsf{C}, \Vdash)$).*

Proof: We assume that we have a proof of a consecution $\circledast_1(\varphi_1, \ldots, \varphi_m) \models \circledast_2(\psi_1, \ldots, \psi_n) \in \mathcal{S}^0_\mathsf{C}$ in $\mathsf{GGL}^0_\mathsf{C}$ and we show that we can transform this proof into a proof of the same consecution in $\mathsf{GL}^0_{\mathsf{C},G}$. For that, we analyse the proof and perform a backward proof search. The first rule that we can apply (backwards) is $(\circledast\vdash)$ and we arrive at a consecution of the form $S([\circledast], \varphi_1, \ldots, \varphi_n, U)$. Then, we can directly apply $(\circledast\vdash)$ or a sequence of display rules in order to apply $(\circledast\vdash)$. In both cases, we arrive at a consecution of the form $S([\circledast'], X_1, \ldots, X_n, \circledast(\varphi_1, \ldots, \varphi_n))$ with $\circledast' \in \mathsf{C}$ (in order to apply $(\vdash\circledast)$). Since both $\circledast \in \mathsf{C}$ and $\circledast' \in \mathsf{C}$, the sequence of display rules is equivalent to a single application of rule (dr_3) and it suffices to replace this sequence by a single application of rule (dr_3) to obtain a proof in $\mathsf{GL}^0_{\mathsf{C},G}$. Then, we repeat this process inductively to the premises of the instance of the rule $(\vdash\circledast)$ applied. Hence, we obtain the result for $\mathsf{GL}^0_{\mathsf{C},G}$.

As for $\mathsf{GL}_{\mathsf{C},G}$, it suffices to observe that (dr_3) is derivable from (dr_1) and (dr_2) and that, vice versa, (dr_1) is derivable from (dr_3). $\square$

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The Reasonable Effectiveness of Model Theory in Mathematics

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Abstract

In this article we first provide some background on why the applicability of model theory across mathematics is reasonable and briefly describe some of the well-known results over the last seventy years (Sections 1 and 2). In the remainder we focus on three areas that have developed in the last five. Two are parallel developments of fundamental notions of stability theory and certain combinatorial notions in learning theory (Section 3) and in functional analysis (Section 4). Another is one of the many recent interactions between stability theory and counting problems in finite combinatorics (Section 5). Section 6 summarizes the argument.

1 Introduction

In his famous article, *The Unreasonable Effectiveness of Mathematics in the Natural Sciences* [78], Eugene Wigner asserts, 'The first point is that the enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and that there is no rational explanation for it.' In contrast, we will argue that applicability of model theory across mathematics is not mysterious but is easily understood in terms of the basic methodology and motivations of model theory[1]. In his *Introduction to Logic and the Methodology of the Deductive Sciences*, Tarski aimed 'to present to the educated layman . . . that powerful trend . . . modern logic . . . [which] seeks to create a common basis for the whole human knowledge' ([75], xi).

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[1]Unsurprisingly, I am not the first to appropriate Wigner's metaphor, although most writers maintain Wigner's *UN*reasonable. See [30] who refers to Corfield and Manders.

In his 1950 address to the International Conference of Mathematicians, Robinson [64] made this more goal more specific, '. . . we shall be concerned with the effective application of symbolic logic to mathematics proper, more particularly to abstract algebra. Thus, we may hope to find the answer to a genuine mathematical problem by applying a decision procedure to a certain formalized statement.'

After more than a half century of development, we argue that specific formalizations of areas of mathematics are fruitful for those areas. Moreover the technology of classification theory provides a uniform strategy to obtain results in many different contexts, extending well beyond Robinson's innovations in abstract algebra.

There are three key reasons for this effectiveness. The first is representing an area of mathematics as the study of a collection of similar structures for a fixed vocabulary. One attempts local (area dependent) rather than global foundations for mathematics. Second, rather than examining all subsets of those structures, restricting to those defined in a formal logic provides a principled way to isolate *tame* mathematics. Thirdly, the classification of theories introduced by Shelah [69], brings to the fore certain combinatorial features that play significant roles in widely distinct areas of mathematics.

At [4, page 2], I wrote,

> In short, the paradigm around 1950 concerned the study of *logics*; the principal results were completeness, compactness, interpolation and joint consistency theorems. Various semantic properties of theories were given syntactic characterizations but there was no notion of partitioning all theories by a family of properties. After the paradigm shift there is a systematic search for a *finite* set of syntactic conditions which divide first order theories into disjoint classes such that models of different theories in the same class have similar mathematical properties.

The finer analysis in the last ten years of the unstable section of the classification has converted the italicized 'finite' to infinite. This classification, which is roughly syntactic (certainly set theoretically absolute), drives what I call the *paradigm shift* [4, Introduction]. *Neo-stablity theory*[2] analyzes further patterns recognizable by formalized theories [71, 53, 73]. Model theory provides *reasonable effectiveness because this classification crosses traditional areas to provide* unification and generalization. As we'll see the patterns identified play important roles in a wide range of traditional mathematical settings.

[2]See the interactive map `http://www.forkinganddividing.com/#_02_54` and the report `https://pdfs.semanticscholar.org/b2c9/a98d0f1e26d336dc913358f45ec9f1a1f951.pdf`. The interactive feature allows one to see the classification hierarchy.

Model theory analyzes the structure of definable sets (solutions to formulas with free variables) in any model of a theory along two axes: the (quantifier)-complexity of the definition and the combinatorial complexity of the class of definable sets.

Restriction to definable sets is historically very natural. Euclid and Eudoxus developed the method of exhaustion to provide a framework for studying the relations among possibly incommensurable specific pairs of magnitudes such as the diagonal and side of a square. But each example relates to objects which are definable in the modern sense. It is Dedekind who posits that limits exists for *arbitrary* cuts. Speaking polemically, studying only the 'definable' objects in a structure means, 'studying the ones which actually arise'.

A natural way to 'tame a structure' is to look at definable subsets rather than all sets. This happens automatically in algebraic geometry where the study of solution sets of equations is essentially the study of all definable sets. Tarski and Robinson [74, 64] saw this result in full generality as quantifier elimination for a real closed or algebraically closed field, while Chevalley described the key inductive step: constructible sets are closed under projection. This *method of quantifier elimination* provides a general format unifying the Hilbert Nullensatz for a wide range of algebraic applications.

Combinatorial is not quite the right word for the second axis. The central idea is (non)-existence of certain configurations among the definable sets. One such configuration is simply an infinite decreasing sequence of definable sets. On the combinatorial side, replacing the (ascending (acc)) descending chain condition (dcc) (no such sequence exists) on subgroups (ideals) by the (ascending) descending chain condition on *definable* subgroups (ideals) provides a common framework across group theory, differential algebra, ring theory, etc. Thus, the Wedderburn theorem that certain rings satisfying the descending chain conditions on ideals are represented as matrix rings can be proved for rings, whose theories are stable, and so satisfy the dcc on principal (1-generated) ideals [6]. The general picture is further clarified by noting that similar variants on the chain condition (e.g., requiring infinite index at each step) for different areas are unified by noting the theory is stable, superstable or ω-stable.

What I refer to as 'traditional philosophy of mathematics' is dubbed 'philosophy of Mathematics' (Harris, page 30 of [31] or [4, page 5]) or 'Foundations of Mathematics' (Simpson in clarifying his view on the Foundations of Mathematics Listserve)). This distinction is transcended in Maddy's recent article, *What do we want a foundation to do?* [52]. She writes

> So my suggestion is that we replace the claim that set theory is a (or 'the') foundation for mathematics with a handful of more precise observa-

> tions: set theory provides *Risk Assessment* for mathematical theories, a *Generous Arena* where the branches of mathematics can be pursued in a unified setting with a *Shared Standard of Proof*, and a *Meta-mathematical Corral* so that formal techniques can be applied to all of mathematics at once.

I write from a similar perspective. I am not emphasizing the search for a reliable basis for all mathematics but investigating the organization of mathematics and how particular organizations[3] can productively impact mathematical practice. The clarification of such concepts as function, cardinality, and continuity in the late 19th century had immediate positive impact on mathematics. This effect is usually viewed from the lens of reliability. But Coffa places the relationship between 'reliability and clarity' in historical perspective:

> [We consider] the sense and purpose of foundationalist or reductionist projects such as the reduction of mathematics to arithmetic or arithmetic to logic. It is widely thought that the principle inspiring such reconstructive efforts were basically a search for certainty. This is a serious error. It is true, of course, that most of those engaging in these projects believed in the possibility of achieving something in the neighborhood of Cartesian certainty for principles of logic or arithmetic on which a priori knowledge was to be based. But it would be a gross misunderstanding to see in this belief the basic aim of the enterprise. A no less important purpose was the clarification of what was being said. . . .
>
> The search for rigor might be, and often was, a search for certainty, for an unshakable 'Grund'. But it was also a search for a clear account of the basic notions of a discipline. ([26], 26)

While Maddy argued that set theory met the criteria for a foundation listed above she said it failed a further criterion *essential guidance*: aiding the choice of solution and posing of problems across mathematics. We argue below that the flexibility of model theoretic axiomatizations and the exposure and clarification of common themes provides such essential guidance.

In the first part of this article we outline the paradigm of contemporary model theory and explain why this paradigm might be expected to be useful for proving results in traditional mathematics. In the remainder we sketch a number of such applications.

[3] I contrast model theory and category theory as different 'scaffolds' for mathematics in [5].

2 The Model Theoretic Approach

The distinctive feature of model theory is *formalization*: the description of various areas of mathematics in a formal language. The first two of the four theses of [4](where the argument is expanded in more detail) assert:

1. Contemporary model theory makes formalization of *specific mathematical areas* a powerful tool to investigate both mathematical problems and issues in the philosophy of mathematics (e.g. methodology, axiomatization, purity, categoricity and completeness).

2. Contemporary model theory enables systematic comparison of local formalizations for distinct mathematical areas in order to organize and do mathematics, and to analyze mathematical practice.

Tarski's term, *meta-mathematics* summarises the underlying motif of model theory. By meta-mathematics I mean both developing a general notion of a formal theory as an object of mathematical study and contributing to particular areas of mathematics by formalizing the area in an appropriate theory.

Definition 2.1. *A* full formalization *involves the following components.*

1. Vocabulary: specification of primitive notions.

2. Logic:

   (a) Specify a class[4] *of well formed formulas.*

   (b) Specify truth of a formula from this class in a structure.

   (c) Specify the notion of a formal deduction for these sentences

3. Axioms: specify the basic properties of the situation in question by sentences of the logic.

For much of model theory, compactness (consistency of a set of sentences X follows from consistency of finite subsets of X) is more important than explicit deductions and the completeness theorem. But there is an implicit reliance on completeness to transfer results on properties that are first order expressible consequence of axioms. For 'getting tight results', a recursive deduction system is important but

[4]In the instances treated here, this will be a set.

not sufficient. Even primitive recursive upper bounds are far too crude for mathematical applications. More sophisticated model theoretic techniques often obtain mathematically interesting upper bounds.

I have chosen the word 'vocabulary' rather than such rough synonyms as language, similarity type, signature or, even rougher, logic. Examining a particular mathematical topic, the investigator selects certain concepts as fundamental. The *vocabulary* is a set τ of relation symbols, function symbols, and constant symbols chosen to represent these basic concepts. A τ-structure with universe A assigns (e.g., to each n-ary relation symbol R a subset R^A of A^n). Thus, many situations in mathematics have led to the now nearly ubiquitous notion of a group. This notion can be formalized in such diverse vocabularies as a single binary function, a single ternary relation, or augmenting, say, the binary function with a unary function (inverse) and a constant symbol (identity). Clarifying when these different approaches are, or are not, equivalent is one of many important uses of the rigorous model theoretic definition of the notion of *interpretation*.

Crucially, fixing a vocabulary, even with suggestive names, has done little work. One must choose axioms that reflect the topic being studied. Calling a binary relation an order and then positing that it satisfies the axioms of an equivalence relation is madness. There has been no strict formal error, just an abuse of the mathematician's right to name concepts arbitrarily. However, a fruitful formalization will respect the previous terminology. Crucially, one must select an appropriate logic. Dedekind and Peano provided second order axioms which shed great light on the internal structure of the arithmetic of the natural numbers. While these axioms are particularly valued for determining a *categorical* (unique up to isomorphism[5]) structure, and give a uniform basis for various results in number theory proved by induction, they are not central in the great 20th century advances in number theory. Rather, these advances are based on considering the natural numbers as substructures of much more tame objects such as geometries over algebraically closed fields.

We focus here on *first order* logic ($L_{\omega,\omega}$) which allows finite Boolean combinations of formulas and quantification over finite strings of individuals. We will make occasional comparisons with *infinitary logic* ($L_{\kappa,\lambda}$) which allows Boolean combinations of $< \kappa$ formulas and quantification over $< \lambda$ individuals. But second order logic will get short shrift. On the one hand, first order set theory is a useful avatar of second order logic [76]; on the other there is almost no model theory of second order logic.

[5]Note that isomorphism is not well-defined unless one specifies the vocabulary. See Pierce's paradox in [4].

A seminal aspect of modern model theory is the focus on complete theories (usually in first order logic) and their models rather than on logics. Already in the 1950's (or even the 30's for real closed fields) such pioneers as Robinson and Tarski realized that showing that those subsets definable in a model of a theory T could be defined by formulas with low quantifier complexity was a powerful tool for studying the theory. This is an epistemological insight. If one can formalize an area of mathematics in a way that all definable sets are 'simple', then one has a much better understanding of the subject. Thus, while the formulas of first order Peano arithmetic have unbounded quantifier-complexity, every definable subset of the complex (or real field with order) is definable without quantifiers. The relation between this kind of simplicity and decidability is not obligatory. But many decision problems (e.g. the real field) were solved precisely by reducing to quantifier free formulas where a brute-force analysis was possible.

Morley [58] proved that for countable first order theories, categoricity in one uncountable cardinal κ (unique model with cardinality κ) is equivalent to categoricity in all uncountable cardinalities. He asked whether the number of models in cardinality $\aleph_\alpha$ ($\alpha \geqslant 1$) is an increasing function. Shelah generalized Morley's method and developed the classification to solve this problem. The classification provided tools for analyzing the structure of models whose impact across mathematics is discussed here.

The notion of *type* connects the syntactic and the semantic: the type of a finite sequence[6] of elements b over a set A in a model M is the collection $p = \mathrm{tp(b/A)}$ of formulas $\phi(x, a)$ (with $a \in A$) such that $\phi(b, a)$ is true ($M \models \phi(b, a)$). Thus, the collection $S(A)$ of types over A is the *Stone Space* of the Boolean algebra of formulas with parameters from A. A model is κ-saturated if every type over a set of cardinality less than κ is realized in M. In an $|A|^+$-saturated model M there is an automorphism of M fixing A pointwise and mapping b_1 to b_2 if both satisfy p. So each type determines a possible 'kind' of extension over A by a finite sequence in an elementary extension of M. A countable first order theory T is *κ-stable* if for $A \subseteq M \models T$, $|A| \leqslant \kappa$ implies $|S(A)| \leqslant \kappa$.

Shelah defined a stable theory and gave a long list of equivalent requirements for a theory to be stable (e.g. stable in some infinite cardinal κ). We give three of the criteria to emphasize the diversity of the notion.

Fact 2.2. *Each of the following is equivalent to 'T is stable'*

i) T is stable in every cardinal κ with $\kappa^{\aleph_0} = \kappa$.

[6] We use lower case Roman letters for finite sequences of variables or elements of a model.

ii) (fundamental theorem of stability theory) there is no formula $\phi(\mathbf{x}, \mathbf{y})$ that has the order property*: for every n*

$$T \models (\exists \mathbf{x}_1, \ldots \mathbf{x}_n \exists \mathbf{y}_1, \ldots \mathbf{y}_n) \bigwedge_{i<j} \phi(\mathbf{x}_i, \mathbf{y}_j) \wedge \bigwedge_{i \geqslant j} \neg\phi(\mathbf{x}_i, \mathbf{y}_j)$$

iii) there is a notion of independence on models of T which, locally, generalizes the notion of independence of a vector space.

This equivalence of i) a condition about models of arbitrarily large cardinality, ii) a syntactic condition that applies to theories across mathematics, and iii) an algebraic condition leading to a geometry shows the unifying powers of model theory.

Fact 2.2.i) demonstrates that there are fundamental mathematical properties which depend non-trivially on cardinality. In contrast most mathematical results are either very specific to structures of size less than the continuum, e.g., a complete separable ordered field is isomorphic to the real numbers, or completely independent of cardinality, e.g., any Desarguesian plane can be coordinatized by a division ring.

The syntactic Fact 2.2.ii) can clearly be checked on the countable models of T. Observe how it specializes: a) the ring of rationals and b) the real field each have the order property by defining $x < y$ a) if $y - x$ is the sum of four squares and b) if $y - x$ is a square. There are consequences for reliability. These notions are clearly described in second order arithmetic and do not depend on higher set theory. Fact 2.2.iii), manifests itself both in the general notions of forking and orthogonality of types which underlie global structure of models and more specifically in geometric stability as theory discussed after Definition 2.5.

Shelah extended Morley's global analysis of the Boolean algebra of all formulas, over a set A and its Stone Space $S(A)$ by localizing to instances $\phi(a, b)$ of a single formula $\phi(x, y)$, ϕ-types in $S_\phi(A)$. This idea led not only to the connection with the order property in Fact 2.2.ii) but to identifying those patterns which determine instability. The patterns encoded by the next two properties appear across mathematics. E.g., if the theory of a group G has NIP. Then G satisfies the descending chain condition on centralizers and if locally nilpotent is solvable [7]; Macintyre's proof [51] that ω-stable fields are algebraically closed uses Morley rank, the Cantor-Bendixson rank on the Stone space $S(M)$.

Definition 2.3. *Let M be a model of a first order theory T, and $\phi(x, y)$ a formula in the vocabulary of T.*

1. *ϕ has the* independence property *(IP) if for arbitrarily large finite sets I $\phi(x, a_i)$ for each $i \in I$ and each $X \subseteq I$, there is a b_X such that $M \models \phi(b_x, a_i)$ iff and only $i \in X$.*

2. *$\phi(x, y)$ has the* strict *order property (SOP) if there are $b_i \in M$, for $i < \omega$, such that*

$$M \models (\forall x)\phi(x, b_i) \to \phi(x, b_j) \text{ iff } i \leqslant j.$$

3. *The complete theory T has (IP) or (SOP) if some formula does in some model. T has NIP and NSOP if the property fails for each formula.*

We explore in Section 4 connections of functional analysis with Shelah's clarification of sources of instability.

Theorem 2.4. *T is stable if and only if every formula is both NIP (fails the independence property) and NSOP (fails the strict order property).*

Morley's rank on formulas (or types) and the variety of similar ranks introduced by Shelah provide a general tool that have been applied in many areas of mathematics. Berline [11] proved that Morley rank on algebraically closed fields coincides with the algebraic ranks defined by Krull (on ideals) and by Weil (on the associated algebraic varieties) and all definable sets by (Morley). Surprisingly, the underlying topologies providing the ranks are quite distinct. Morley works with a Stone topology which is totally disconnected and Hausdorff, while the Zariski topology is never Hausdorff.

The notion of *forking* provides a dependence notion in any stable theory that satisfies (almost) the Van der Waerden (vector space) axioms for a dependence relation (except cl(cl(A) may not be cl(A)). On a strongly minimal set algebraic closure satisfies all the axioms. In a *strongly minimal theory* such as vector spaces or algebraically closed fields the formula $x = x$ is strongly minimal. Strong minimality was introduced to study first order theories categorical in power κ.

Definition 2.5. 1. *An element a is said to be in the* algebraic closure *of a set B in a model M, $a \in \mathrm{acl(B)}$ if there is a formula $\phi(x, \mathbf{y})$ and a sequence $\mathbf{b} \in B$ such that $M \models \phi(a, \mathbf{b})$ and there are only finitely many solutions of $\phi(x, \mathbf{b})$ in M (written $(\exists^{\leqslant k} x)\phi(x, \mathbf{b})$.)*

2. *A definable set $D = \phi(M, a)$ of a model M is* strongly minimal *if every definable subset of D is finite or cofinite in any elementary extension of M. Equivalently, D has Morley degree and rank one. This implies that there is a unique non-algebraic type of elements in D.*

3. *A structure is said to have* trivial algebraic closure *if* $\mathrm{acl(A)} = \bigcup_{\mathrm{a} \in \mathrm{A}} \mathrm{acl(a)}$ *for every subset A.*

Zilber's *geometric stability theory* [61] highlighted specific properties of the algebraic closure relation (e.g. the lattice of algebraically closed sets is modular) as a further salient pattern. Zilber [81] conjectured the trichotomy: the geometry of any strongly minimal sets is trivial, vector space-like (modular), or field-like. This generalized a lemma in his proof that theories categorical in all infinite powers are not finitely axiomatizable. The conjecture failed in general [35], but its truth for specific cases such as differentially closed fields and o-minimal theories has immense consequences in traditional mathematics.

In [4, Chapters 4/5] (see [32, 57, 34, 67, 59]) we describe how the technology developed ostensibly for counting models (more profoundly, for proving structure theorems) and geometric stability theory underlie contributions to Diophantine geometry and differential equations (via Manin kernels [55, §5]). Completely abstract model theoretic conditions on a theory T (that hold in many different contexts) imply that (algebraic) groups (or fields) are interpreted in the theory. This technology, specific reference to DCF and the strictly stable theory of separably closed fields all contribute to E. Hrushovski's celebrated proof of the function field Mordell-Lang conjecture [19]. The realization that strongly minimal sets controlled differentially closed fields (DCF) led to solutions of problems in differential equations and transcendence theory stemming from Painlevè more than a century ago [60, 20, 66].

In the 1980's work of Steinhorn, Pillay, and Van Den Dries [63, 27] modified the epistemological approach around quantifier rank by considering the simplicity of definable sets. By Definition 2.5, each definable subset (finite or cofinite) of a strongly minimal set is definable with parameters using only =. Analogously, a theory whose models are linearly ordered is *o-minimal* if every definable subset of a model M is a finite union of points and intervals with endpoints in M (defined using only $\{<,=\}$). This definition captures the essential character of the collection of definable subsets of the real field. This essence is emphasized by the proof [79] that the real exponential field is also o-minimal and model complete. This work was followed by showing other expansions of reals (e.g., by the Γ function) remain o-minimal. Wilkie explains the sense in which o-minimality captures Grothendieck's notion of 'tame topology' in [80]; see also Marker [56]. The subject has been well-integrated with contemporary real algebraic geometry [14] and has had a significant impact in number theory. Half of the 2013 Karp prize[7] was awarded to Kobi Peterzil, Jonathan Pila, Sergei Starchenko, and Alex Wilkie for 'their efforts in turning the theory of o-minimality into a sharp tool for attacking conjectures in number theory, which culminated in the solution of important special cases of the André-Oort Conjecture by Pila.' Chambert-Loir's review of *o-Minimality and Diophantine Geometry* [25, 37]

[7] For award details see `http://vsl2014.at/2014/07/awards-at-the-logic-colloquium/`.

provides a poetic metaphor of unicorns and grasslands. Moreover, Matthias Aschenbrenner, Lou van den Dries, and Joris van der Hoeven were awarded the 2018 Karp prize for their work in model theory, especially on asymptotic differential algebra and the model theory of transseries [2]. The effectiveness of model theory results from a combination of a methodology applicable in many areas of mathematics and a deep understanding of the particular topic.

In the remainder of the paper we pass over the famous examples mentioned above and describe some recent interactions of model theory with other areas of mathematics.

3 Parallel Developments I: statistics and learning theory

A child who can recognize which of a collection of figures are squares has 'learned' the concept of square. Machine learning abstracts this notion. Fix a set X and write $\mathcal{P}(X)$ for its power set. A concept class $\mathcal{C}$ on X is a subset of $\mathcal{P}(X)$. We discuss several models for *learning* $\mathcal{C}$: being able to predict whether $A \subseteq X$ is in $\mathcal{C}$. The Sauer-Shelah Lemma describes an avatar of NIP that gives sufficient conditions when for $Y \subset X$, $|\mathcal{C}_Y| = |\{Y \cap S_i : S_i \in \mathcal{C}\}|$ grows polynomially in $|Y|$. Independently discovered by three investigators (Sauer (combinatorics of set systems), Shelah/Perles[8] (model theory/geometry), Vapnik-Chervonenkis (statistics)) around 1972, [44] connected these independent discoveries twenty years later. We adapt the set system terminology.

Definition 3.1. *Suppose $\mathcal{C} = \{S_1, S_2, \dots\}$ is a family of subsets of a set X and $Y \subseteq X$:*

1. *Y is* shattered *by $\mathcal{C}$ if $\mathcal{P}(Y) \subseteq \mathcal{C}_Y = \{Y \cap S_i : S_i \in \mathcal{C}\}$.*
2. *The Vapnik-Chervonenkis (VC) dimension of $\mathcal{C}$ is the largest cardinality n of a finite set Y shattered by $\mathcal{C}$. If such an n exists, $\mathcal{C} \subset \mathcal{P}(X)$ is a VC class.*

Lemma 3.2 (Sauer-Shelah)**.** *If $\mathcal{C}$ is a family of subsets of a set X and there is a $Y \subseteq X$ with $|Y| = n$ such that $|\mathcal{C}_Y| > \sum_{i=0}^{k-1} \binom{n}{i}$, then $\mathcal{C}$ shatters a set of size k. Equivalently, if the VC dimension of $\mathcal{C}$ is k and $|Y| = n$, then*

$$|\mathcal{C}_Y| \leqslant \textstyle\sum_{i=0}^{k} \binom{n}{i} = O(n^k).$$

[8] Shelah [68, 254] cites 'a little more complex result, of Perles and Shelah'.

To translate to model theory, take X to be the universe of M, a model of a first order theory T, and $\phi(x, y)$ a formula in the vocabulary of T. Let $\mathcal{C}^\phi = \{\phi(M, a)|a \in M\}$. The Sauer-Shelah Lemma asserts that if a formula ϕ *does not have the independence property* (NIP), the number of ϕ-types of a set of size n is a polynomial in n with order the VC dimension of ϕ. The following three properties of $\mathcal{C}^\phi$ are equivalent: i) has finite VC dimension, ii) has NIP and iii) is *PAC-learnable* as we now define.

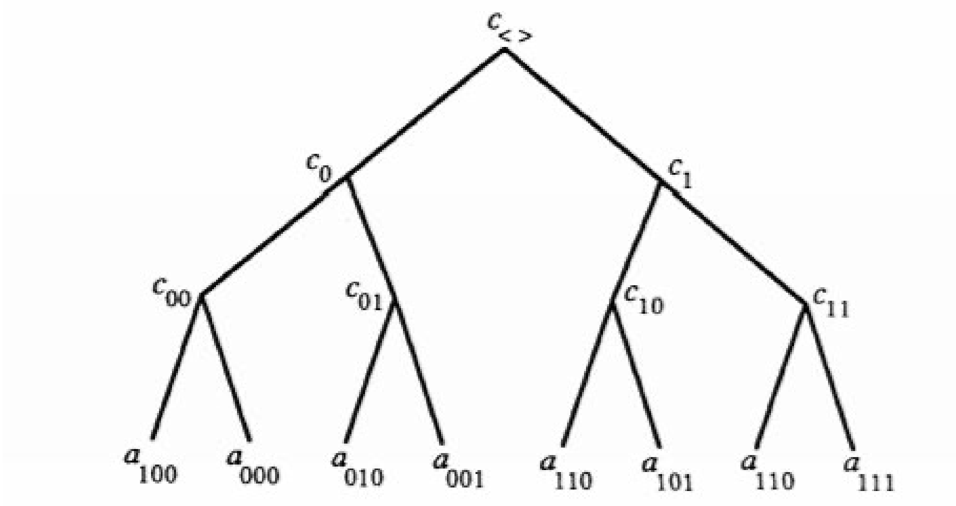
One model for measuring 'learnability by an algorithm', computes the probability that the algorithm will, for a large enough sample size, predict the target set arbitrarily well. If this probability can be made arbitrarily high predictions are *probably approximately correct* (Definition 3.3). In this model for some large n, a sample A of n elements of X is chosen randomly, and the learner is told which points belong to $C \in \mathcal{C}$. The goal is to use the sample to make a prediction $G(A)$ that estimates C with small error. More formally,

Definition 3.3 ([23]). *For a fixed $\epsilon > 0$ and measure μ, we say that the sample A estimates the set $C \in \mathcal{C}$ ϵ-well if $\mu(G(A) \triangle C) < \epsilon$. The class $\mathcal{C}$ is PAC-learnable (*probably approximately correct*) if for any δ there is a large enough n such that the measure of the samples of size n (computed using the product measure μ^n) that estimate the sample ϵ-well is greater than $1 - \delta$.*

Model theory provides a general setting and a wealth of new examples [23, Section 5] for the learning theory community. E.g., since the real field is o-minimal and so NIP, any definable family of subsets $\mathbb{R}$ is PAC-learnable. Until recently most of the inter-field transfer has been from learning theory to model theory (exception [49]). In particular, the learning theory notion of a *compression scheme* [50] was adapted to the stability theory context [36]. The abstract of [28] emphasizes this impact: 'Combining two results from machine learning theory we prove that a formula has NIP if and only if it satisfies uniform definability of types over finite sets. This settles a conjecture of Laskowski.'

However, PAC learning is only one of many models of machine learning. More recently a surprising new connection arose between three such models and *stable* theories. In the *online learning* setting, the learner is presented with a stream of elements and is asked to guess if they belong to the target set. A class is *online learnable* if there is some N such that the learner has a strategy to make at most N mistakes in learning any set in the class. In the *equivalence query (EQ) learning model*, a learner [21] attempts to identify a target set $A \in \mathcal{C}$ by means of a series of data requests called equivalence queries. The learner has full knowledge of $\mathcal{C}$, as well as a hypothesis class $\mathcal{H}$ with $\mathcal{C} \subseteq \mathcal{H} \subseteq \mathcal{P}(X)$. An equivalence query consists of the learner submitting a hypothesis $B \in \mathcal{H}$ to a teacher, who either returns yes if $A = B$, or a counterexample $x \in A \triangle B$. In the former case, the learner has learned A, and in the latter case, the learner uses the new information to update and submit a new hypothesis. [21] improve the upper bounds for the number of queries ($LC^{EQ}(C, H)$) required for EQ learning (and the related EQ+MQ: *equivalence and membership queries*) of a class $\mathcal{C}$ with hypotheses $\mathcal{H}$. In these cases the order the sample is chosen is central, while it is irrelevant for the *random* sample in PAC-learning. This distinction led to an ostensibly new rank on set systems: *Littlestone (or thicket) dimension* ($\mathrm{Ldim}(\mathcal{C})$).

The new insight in [21] is the discovery that if the set system is given by first order formulas Littlestone dimension is another version of Shelah 2-rank which takes account of the order information is presented. The maximum height k of a tree indexed by $\langle c_s : s \in 2^{<k} \rangle$, where elements $\langle a_t : t \in 2^k \rangle$ satisfy $\phi(a_i, c_s)$ exactly when $s \subset t$ is $\mathrm{rk}(\phi(\mathrm{x}, \mathrm{c}_{<>}))$. For VC, dimension c_s must equal $c_{s'}$ if $\lg(s) = \lg(s')$.



[1] proves that 'private PAC learning' (a variant on PAC-learning appropriate when the input data, such as medical records, need to be kept secret) implies finite Littlestone dimension and [18] shows the converse. The consistency dimension of $\mathcal{C}$, motivated by the model theoretic notion nfcp (the finite cover property fails), with respect to $\mathcal{H}$, denoted $C(\mathcal{C}, \mathcal{H})$ is new [21] to learning theory. Here is a sample result.

Theorem 3.4. *Suppose* $\mathrm{Ldim}(\mathcal{C}) = d < \infty$ *and* $1 < C(\mathcal{C}, \mathcal{H}) = c < \infty$. *Then* $LC^{EQ}(C, H) \leqslant c^d$.

Bhakasar [16] showed Y = Littlestone dimension solves for learning theory the following analogy.

$$\frac{Y}{stability} = \frac{VC\text{–}dimension}{NIP}.$$

Theorem 3.5 (Thicket Sauer-Shelah)**.** *[16] Let $\mathcal{C}$ be a set system of Littlestone dimension k. Then the maximum number of realized leaves,*

$$\rho_{\mathcal{C}}(n) \leqslant \sum_{i=0}^{k} \binom{n}{i}.$$

Chase and Freitag [22] introduce the notion of *banned sequences* to give a proof that specializes not only to each version of Sauer-Shelah considered here but further extends the Malliaris and Terry improvement [54] (using the stability classification to better organize the case analysis) on the bounds in a result of [24] on a case of the Erdős-Hajnal conjecture. The use of stability theory in online learning gives both better upper bounds and a unified framework.

4 Parallel Developments II: functional analysis

In this section we explore some striking analogies between functional analysis and stability theory that turn out to be not at all coincidental. After tracing some of the history we present some suggestions of Khanaki for new methods and problems in stability theory arising from analyzing these analogies.

In [12], Ben Yaacov argued that Grothendieck 'first' proved the fundamental theorem of stability theory (Fact 2.2). Like an earlier hybrid, the Gödel-Deligne completeness theorem[9], there is a kernel of truth here; there is a common core to the central argument. But in both cases the pairs of authors have different contexts. That is, as discussed in [12, 62], there is a topological (functional analytic) core to Shelah's proof that for a first order theory instability (i.e. failure of the order property) is equivalent to the non-definability of types[10]. Grothendieck had earlier isolated this argument as a theorem of *general* topology. Shelah rediscovered the argument in the much more *general* context of complete first order theories, by considering the Stone spaces $S_\phi(A)$ for $A \subset M \models T$.

I contrast the two uses of 'general' in the previous paragraph. Grothendieck found topological (function analytic) conditions for a certain result. The Stone space

[9] `https://ncatlab.org/nlab/show/Deligne+completeness+theorem`

[10] Ben Yaacov focuses on the equivalent we omitted from Fact 2.2: every complete ϕ-type $p \in S_\phi(B)$ is definable; there is a formula $\psi_\phi(y)$ over B such that $\phi(x,a) \in p$ if and only if $\psi_\phi(y)$. We connect a different equivalent with Grothendieck.

(compact, totally disconnected, Hausdorff) topology used by Shelah exemplifies this situation. But Shelah proved a general result about first order theories. Thus, he grounded the whole range of applications across mathematics mentioned in this article and so a context which may enable new applications of functional analytic concepts to model theory and then across mathematics.

This section reports the work of Khanaki [41, 42, 39] in transferring theorems of functional analysis to inspire new characterizations of some classes and new classes of first order theories. We isolate these topological phenomena, separating them from the linear space context [41] so as to focus on the core of the argument. By studying the action on a Stone space which is compact, we are able to study the space of functionals with the topology of pointwise convergence rather than various notions of weak topology.

We review some notions and results for the topology of pointwise convergence. If X is any set and A a subset of $\mathbb{R}^X$, then the topology of *pointwise convergence* on A is that inherited from the usual product topology of $\mathbb{R}^X$. A typical neighborhood of a function f is determined by a finite subset $\{x_1, \ldots, x_n\}$ of X and $\epsilon > 0$ as: $U_f(x_1, \ldots, x_n; \epsilon) = \{g \in \mathbb{R}^X : |f(x_i) - g(x_i)| < \epsilon \text{ for } i \leqslant n\}$. $C(X) \subseteq \mathbb{R}^X$ denotes the space of continuous functions from X into $\mathbb{R}$; it is naturally a linear space under pointwise addition and is equipped with sup norm. We describe the relevant function space following [42, 1,2] and [40, 2.1].

Notation 4.1. *Let T be a first order theory, M a model of T, and M^* an $|M|^+$-saturated elementary extension of M.*

1. *Fix $\phi(x, y)$ with $\lg(x) = n$ and A a set of n-tuples contained in M^*. $S_\phi(A)$ is the collection of types containing formulas $\phi(x, a)$ or $\neg\phi(x, a)$ for $a \in A$. $S_{\phi^{opp}}(A)$ reverses the roles of x and y; now formulas $\phi(a, y)$ are in the type.*

2. *Define a collection of functions $\phi(a, y)$ from $S_{\phi^{opp}}(A)$ into 2 by $\phi(a, q) = 1$ iff $\phi(a, y) \in q$. As ϕ is fixed* we can identify this set of functions with *A. Since each $f \in A$ maps into $\{0, 1\}$, A is uniformly bounded. Moreover, the totally disconnected Stone topology on $S_{\phi^{opp}}(A)$ ensures that each function $\phi(a, y)$ is continuous. So $A \subseteq C(S_{\phi^{opp}}(A))$.*

In general a space of functions from X to $\mathbb{R}$ has the interchangeable double limit property if for sequences of functions $f_n \in \mathbb{R}^X$ and points $x_m \in X$

$$\lim_n \lim_m f_n(x_m) = \lim_m \lim_n f_n(x_m)$$

when the limits on both sides exist. We translate this to our context:

Definition 4.2. *Let $A \subseteq M \models T$, $(A, S_{\phi^{\text{opp}}}(A))$ has the* interchangeable double limit property *if for any infinite sequences $\boldsymbol{a} = \langle a_n : n < \omega\rangle \in A$ and $\mathbf{b} = \langle b_n : n < \omega\rangle \in S_{\phi^{\text{opp}}}(A)$*

$$\lim_n \lim_m \phi(a_n, b_m) = \lim_m \lim_n \phi(a_n, b_m)$$

when the limits on both sides exist.

If $\boldsymbol{a}$ and $\mathbf{b}$ are infinite sequences we denote by $(\boldsymbol{a}\frown\mathbf{b})$ the sequence obtained by concatenating at each n, $\langle a_1 b_1, a_2 b_2, \ldots\rangle$.

Definition 4.3. *A sequence $A = \langle a_i : i \in I\rangle$ is ϕ-n-*order indiscernible *(over B) in a model M if for any n any pair of properly ordered n-tuples $a_1 \ldots a_n, a'_1 \ldots a'_n$ from A, $\mathrm{tp}_\phi(\mathrm{a}, \mathrm{B}) = \mathrm{tp}_\phi(\mathrm{a}', \mathrm{B})$. If this holds for arbitrary n, the sequence is ϕ-*order indiscernible *and if for all formulas ψ,* order indiscernible. *If the ordering of the a_i, a'_i does not affect the equality of types, we say* set-*indiscernible.*

Order-indiscernibility implies set-indiscernibility is one of the main equivalents to stability [69, I.2.3.1].

Observation 4.4. *T does not have the order property exactly if for each $A \subseteq M \models T$, $(A, S_{\phi^{\text{opp}}}(A))$ has the interchangeable double limit property.*

Proof. If there exist $\boldsymbol{a}, \mathbf{b}$ with the order property, $\phi(a_i, b_j)$ if and only if $i < j$, then $\lim_n \lim_m \phi(a_n, b_m) = 1$ since for fixed n and a tail of m $\phi(a_n, b_m)$ is true. But the value is 0 when the limit is taken in the opposite order.

Conversely, suppose T is stable, Fix $\boldsymbol{a}, \mathbf{b}$ so that both limits exist and fix $n > 2$. By the Ramsey theorem we can find a subsequence of $(\boldsymbol{a}\frown\mathbf{b})$ that is ϕ-n-indiscernible. Hence, the double limits on the subsequence are equal. And since the sequence has double limits it must be the limit of the subsequence. ■

Definition 4.5. *Let A be a subset of a topological space X, then the set A is* relatively compact *in X if its closure in X is compact.*

Fact 4.6 applies to A and $X = S_{\phi^{\text{opp}}}(A)$. See [12, 62].

Fact 4.6 (Grothendieck's criterion)**.** *Let X be a compact topological space. Then the following are equivalent for a norm-bounded subset $A \subseteq C(X)$:*

(i) A is relatively compact in $C(X)$.

(ii) A has the interchangeable double limit property.

Since the interchangeable double limit property is equivalent to: ϕ does not have the order property (Observation 4.4), we have:

Theorem 4.7 (stable). *ϕ does not have the order property if and only if for each model M of T and $A \subset M$, A is relatively compact in $C(S_{\phi^{\mathrm{opp}}}(A))$.*

Khanaki [39] refines Shelah's equivalence (Theorem 2.4) of stable with (NIP and NSOP) in several ways by characterizing various notions in functional analytic terms. For this, we introduce a property A_ϕ that yields a new characterization of failing the strict order property (NSOP).

Definition 4.8. *We say sequences $\boldsymbol{a} =< a_i : i < \omega >$ and $\mathbf{b} =< b_j : j < \omega >$ from a model M witness that ϕ satisfies A_ϕ in M if*

1. *the* independence property is uniformly blocked for $\phi(x, y)$ on $\boldsymbol{a}$. *That is, there exist $(N_{\phi,\mathbf{a}}, E_{\phi,\mathbf{a}})$ with $N < \omega$ and $E \subseteq \{0, \ldots N-1\}$ such that for any subset $(a_{i_1}, \ldots a_{i_j}, \ldots a_{i_N})$ of distinct elements of $\boldsymbol{a}$:*

$$\neg\exists y(\bigwedge_{j\in E} \phi(a_{i_j}, y) \wedge \bigwedge_{j\notin E} \neg\phi(a_{i_j}, y)).$$

2. *$\boldsymbol{a}, \mathbf{b}$ witness ϕ has the order property.*

Crucially, the uniformity gives that 'blocks' is preserved by elementary equivalence, so is a property of a theory. Khanaki shows by fairly standard model theoretic arguments:

Theorem 4.9.
1. *[39, Proposition 2.4] If A_ϕ holds witnessed by some $\boldsymbol{a}, \mathbf{b}$ then some Boolean combination of instances of ϕ has the strict order property (ϕ* engenders *the strict order property*[11] *(SOP)).*
2. *[39, Proposition 2.7] T has the NSOP if and if only there is no formula and sequence that witness A_ϕ is true.*

We deduce from Theorem 4.9 an 'intrinsic' characterization of those formulas ϕ which have the Independence Property but not the Strict Order Property. The characterization asserts that the type of a countable sequence $\boldsymbol{a}$ that indexes an independent family of sets is omitted and a second type of a countable sequence $\boldsymbol{a}\frown\mathbf{b}$ that witnesses the strict order property is realized in any $\aleph_1$-saturated model.

Theorem 4.10. *ϕ has NIP but engenders SOP if and only if*

for every $\boldsymbol{a}$ in an $\aleph_1$-saturated model, M^*, of T the independence property is uniformly blocked for $\phi(x, y)$ by some $(N_{\phi,\mathbf{a}}, E_{\phi,\mathbf{a}})$ with $N_{\phi,\mathbf{a}} < \omega$ on $\boldsymbol{a}$ and there exists $\boldsymbol{a}, \mathbf{b}$ that witness the order property for ϕ.

[11] This characterization was extracted from [39].

Note that, although $N_{\phi,\mathbf{a}}$ varies with $\boldsymbol{a}$, by compactness there must be a uniform bound N or there would be a sequence in M^* that is not bounded. This uniformity illustrates two instruments for the effectiveness of model theory: i) the compactness theorem allows one to 'concentrate' an unbounded phenomenon in a single instance and ii) the ability to choose models with special properties that focus a problem. In our case, we posit a saturated model to realize the concentrated phenomenon. In another situation, the prime model might show a certain configuration can be avoided.

In [4, Chapter 2.3] I distinguish between a virtuous property of a theory T and a dividing line. A property is *virtuous* if it has significant mathematical consequences for the theory or its models. A property is a *dividing line* if it and its negation are both virtuous. We now find some further virtuous properties suggested by the study of Baire functions in analysis.

Definition 4.11. 1. *A real valued function from a complete metric space is Baire-*1 *if it is a pointwise limit of a sequence of continuous functions.*

2. $f \in \mathbb{R}^X$ *is DBSC if it is a* difference of two bounded semi-continuous functions. *This is a proper subclass of the Baire-1 functions.*

It is standard (e.g. [73]) that any formula ϕ which does not have the independence property has an alternation number, the maximal number n_ϕ of elements such that there exists an indiscernible sequence $\boldsymbol{a}$ and a b such that $\phi(a_i, b) \leftrightarrow \neg\phi(a_{i+1}, b)$ for $i < n$. We use a *wider notion of alternation number* by not requiring $\boldsymbol{a}$ to be a sequence of indiscernibles. Khanaki shows in [39] a topological result which translates into model theory[12] as follows.

Fact 4.12. *If the independence property is uniformly blocked on a sequence $\boldsymbol{a}$ then ϕ has alternation number $n_{\phi,\boldsymbol{a}}$ on $\boldsymbol{a}$ and consequently $\phi(a_n, x)$ converges pointwise to a function $f \in \mathbb{R}^X$ that is a difference of two bounded semi-continuous functions (DBSC).*

Note the distinction in form between the two propositions in Theorem 4.13. The first is an unconditional statement that there is a subsequence whose limit is DBSC; the second is conditioned on the sequence being uniformly blocked.

[12] The 'consequently' in Lemma'4.12 is ii) implies iii) of the topological Lemma 2.6 in [39]. Additional assumptions, which here amount to the observation that the $\phi(a_n, x)$ are continuous and $S_{\phi^{opp}}(\boldsymbol{a})$ is a metric space, yield 'ii) implies iii)'. This last condition depends on the countability of $\boldsymbol{a}$. For large A, $S_{\phi^{opp}}(A)$ is not a metric space although it is compact.

Theorem 4.13. *1. (NIP) [39, Remark 2.11] ϕ has NIP if and only if for every sequence $\boldsymbol{a}$, there is a subsequence a_i such that $\phi(a_i, y)$ converges to an $f \in \mathbb{R}^X$ which is a difference of two bounded semi-continuous functions (DBSC).*

2. (NSOP) [39, Remark 2.8] A complete first order theory T is NSOP if and only if

for any formula ϕ and infinite sequence $\mathbf{a}$ if the independence property is uniformly blocked on $\mathbf{a}$ by some $(N_{\phi,\mathbf{a}}, E_{\phi,\mathbf{a}})$ then $\phi(a_i, x)$ converges to an f that is continuous.

Proof. 1) It is well known that NIP is equivalent to every sequence $\phi(a_n, x)$ has a subsequence with bounded alternation number and so the subsequence converges. The statement here just adds that the limit function is DSBC, which follows from Fact 4.12.

2) Suppose T has NSOP. Then, by Theorem 4.9.2) there is no formula ϕ and sequences $\boldsymbol{a}, \mathbf{b}$ that satisfy both conditions of A_ϕ. Suppose there is an $\boldsymbol{a}$ satisfying condition 1) of A_ϕ. Since condition 2) of A_ϕ fails, for any $\mathbf{b}$, the pair $\boldsymbol{a}, \mathbf{b}$ do not witness the order property. Pillay [62, Proposition 2.2] shows that if ϕ does not satisfy the order property in M, then for any sequence $\boldsymbol{a} \in M$, $\lim \phi(a_n, x)$ converges to a continuous function f.

Conversely, suppose T has SOP witnessed by the formula ϕ so there is a sequence $\boldsymbol{a}$ such that $\forall y \phi(a_i, y) \to \phi(a_j, y)$ if and only if $i < j$. Thus, if $j < i$, $\exists y(\phi(a_i, y) \wedge \neg\phi(a_j, y)$. In particular, there is a $\mathbf{b}$ so that $\boldsymbol{a}\mathbf{b}$ witness the order property for ϕ; so, condition 2) of A_ϕ holds. But then the independence property is blocked on $\boldsymbol{a}$ by $N = 2$ and $E = \{1\}$ and condition i) of A_ϕ is satisfied contrary to hypothesis. ∎

Definition 4.14. *Let A be a subset of a topological space X, then*

(RSC) The set A is relatively sequentially compact *(RSC) in X if each sequence of elements of A has a subsequence converging to an element of X.*

(SCP) The set A is sequentially complete in X if the limit of every convergent sequence from A is continuous.

A theory T has RSC (SCP) if for every $A \subseteq M \models T$ and every ϕ, $(A, S_{\phi^{opp}}(A))$ has RSC (SCP).

Note that SCP of a theory is a strengthening of the characterization of NSOP in Theorem 4.13 as SCP drops the hypothesis of the implication defining NSOP.

Using Definition 4.14, Khanaki states [39, Fact 3.1] the following version of the Eberlein-Šmulian theorem for the topology of pointwise convergence on $C(X)$. We are interested in the result when X and A are as described in Notation 4.1. As noted A is uniformly bounded. See [77] for a short proof.

Theorem 4.15 (Eberlein-Šmulian variant)**.** *A is relatively compact in $C(X)$ if and only if A is both relatively sequentially compact (RSC) and sequentially complete (SCP).*

Since we know stability is equivalent to the relative compactness of A in $C(X)$ the following theorem just states the model theoretic translation of Theorem 4.15.

Theorem 4.16. *[39, Remark 3.2] The following are equivalent:*

1. *ϕ is stable for T.*
2. *For every $A \subseteq M^*$ and every ϕ, the pair $(A, S_{\phi^{\mathrm{opp}}}(A))$ is both RSC and SCP.*

The novelty here is that SCP strictly implies NSOP and NIP is equivalent to RSC. This is a splitting of unstable into two classes (NRSC and NSCP) that overlap differently than IP and SOP do. The flagship IP theory, random graphs is SCP, but the theory of [39, Remark 3.5, Example 2.15] is NSOP and IP but does not have SCP.

Khanaki [41] introduced the notion of *NIP in a model* and with Pillay [42, 39] has demonstrated the interest of such 'stability properties' in a fixed model in both first order and continuous logic. Khanaki suggests in [39] that the Kechris-Louveau hierarchy of Baire-1 functions could be translated by the scheme outlined here to a hierarchy of theories defined analogously to RSC and SCP above. In particular, he suggests investigating the class of theories such that convergent sequences of functions $\phi(a_n, x)$ are DBSC. These suggestions are an interesting way in which functional analysis could aid in the neo-stability project.

Several questions arise. Are these properties virtuous? Are they dividing lines? Do they separate interesting theories? In particular, do they give applications in other areas of math? Shelah ([70] quoted at [4, 63]) assures us that one should explore the universe without worrying about this last question although expecting such consequences.

I have discussed here the use of functional analytic methods in refining the stability classification. Let me quickly mention some other applications of model theory to functional analysis. In particular there is a lot of work around C^* and Von Neumann algebras. Showing specific classes of function algebras are elementary in continuous logic is a key tool. Hart's web page `https://ms.mcmaster.ca/~bradd/#Research` contains links to many papers including the forthcoming Memoir of the American Mathematical Society, *Model theory of* C**-algebras* [29]. [13] provides the background in continuous logic. The study of metric abstract elementary classes provides another perspective and links to category theory [15, 33, 48].

5 Finite Combinatorics

I cannot attempt to survey all the interactions of model theory with combinatorics. Recently such topics include the Erdős-Hajnal conjecture, Szmerédi's theorem, approximate subgroups, and the Elekes-Szabó theorem. Examples come from various places in the stability hierarchy, especially the new notion of distal theories. Here I will concentrate on one particular investigation that involves very nicely behaved structures from a model theoretic standpoint.

Graph theorists count graphs that have a specified property. One standard sort of problem is to fix a class of finite graphs $\mathcal{H}$ that is *hereditary* (closed under substructure and isomorphism) and count. The model theorists eyes light up. One of the earliest theorems of model theory, the Łoś-Tarski theorem, asserts a class $\mathcal{H}$ is hereditary exactly if it is defined by a set of universal (only $\forall$ in prefix) sentences $T_{\mathcal{H}}$. And counting the number of models of each cardinality was the motivating problem for the stability classification. The *speed* of $\mathcal{H}$ is the function sending n to $|\mathcal{H}_n|$, where $|\mathcal{H}_n|$ is the number of members of $\mathcal{H}$ with universe n. Work in the 2000's by Alon, Balogh, Bollobás, Morris, Thomason, Weinreich (in various combinations) almost completely classified the possible speeds for an *hereditary class of graphs*:

Theorem 5.1. *Let $\mathcal{H}$ be an hereditary class of finite graphs.*

- *(1) (poly/exp) For some k, $|\mathcal{H}_n|$ is a sum of terms $p_i(n)i^n$ for $i < k$, where each $p_i(n)$ is a rational polynomial*
- *(2) (factorial) $|\mathcal{H}_n| = n^{(1-\frac{1}{k}-o(n))n}$ for some $k > 1$.*
- *(3) (penultimate) $|\mathcal{H}_n|$ is caught between a function growing slightly slower than n^n and one slightly below 2^{n^2}.*
- *(4) (exponential in n^2) $|\mathcal{H}_n|$ grows as $2^{Cn^2+o(n^2)}$.*

The penultimate, 'next to fastest' growth rate, class gives only a range. There is an $\mathcal{H}$ whose growth rate is close to the lower limit on one infinite set of natural numbers and close to the upper limit on another [9].

A graph is a structure with one symmetric binary relation. Can the kind of analysis carried out for graphs be extended to an arbitrary finite relational language[13]? Noting that $|\mathcal{H}_n|$ is counting the number of quantifier-free n-types of the theory $T_{\mathcal{H}}$ consisting of the universal sentences true in $\mathcal{H}$ links the problem with classical

[13] Spencer was surprised that the Shelah-Spencer 0-1 law for graphs with edge probability $n^{-\alpha}$[72] (α irrational) extended to arbitrary finite relational languages [8].

(1950's) model theory. Strikingly, the solution by Laskowski and Terry depends on the fine analysis of the stability hierarchy. Their work illustrates one of the themes underlying the effectiveness of model theory: approximating the finite by the infinite [3]: study the class $\mathcal{H}$ of finite models by studying infinite models of $T_{\mathcal{H}}$. We need a little history to see how more sophisticated model theory enters the picture.

In generalizing Morley's categoricity results to uncountable vocabularies, Shelah introduced the notion of a *weakly minimal* set: an infinite definable set $W(x)$ such that every complete type p over a model M with $W(x) \in p$ has a unique non-algebraic extension to any $N \succ M$. Strongly (weakly) minimal theories are the best behaved ω-stable (superstable) theories.

Laskowski [45] defines *a τ-formula $\phi(z)$ to be mutually algebraic* if there is an integer K so that $M \models \forall x \exists^{\leqslant K} y \phi(x; y)$ for every proper partition $z = x\frown y$. If every formula with parameters is equivalent to a Boolean combination of mutually algebraic formulas the *structure is mutually algebraic.* An incomplete theory T is mutually algebraic if and only if every completion is.

Simplifying (abusing) the original notation we say a quantifier-free n-type p over a finite set $A \subset M$ is *m-large* in M if there are m pairwise disjoint realizations of p. And T has *unbounded arrays* if for arbitrarily large m and N there is an $M \models T$ such that for some finite A there are at least N m-large types over A.

Theorem 5.2. *An incomplete theory T is mutually algebraic if and only if every atomic formula has uniformly bounded arrays in every model M of T [47].*

Each model of a complete theory T is mutually algebraic if and only if T is weakly minimal and algebraic closure is trivial on models of T [45].

Laskowski and Terry [46] use these model theoretic notions to obtain new results measuring speeds.

Theorem 5.3. *Let $\mathcal{H}$ be an hereditary class of finite structures in a language with finitely many relation symbols with maximal arity r.*

- *Classes 1) and 2) are as in graphs.*
- *But the higher speeds depend on the arity r.*
  - *(3) (penultimate) $|\mathcal{H}_n|$ is caught between a function growing slightly slower than n^n and one with growth approximately $2^{n^{r-\epsilon}}$.*
  - *(4) (exponential in n^r) $|\mathcal{H}_n|$ grows as $2^{Cn^r+o(n^r)}$.*

As in the graph case, there are examples showing that the range of solutions in the penultimate case actually occur. The argument divides into two main cases.

On the one hand the authors show theories with unbounded arrays (so not mutually algebraic by Theorem 5.2) fall into classes 3) and 4) and then analyze the distinction. On the other, they break the mutually algebraic theories into three classes using one further concept.

A countable structure M is *cellular* ([17]) if for some n it admits a partition into a finite set K and m families $\langle C_{i,j} : i < m, j < \omega\rangle$ of finite sets such that for each permutation σ of ω and $i < m$ there is $\sigma_i \in \mathrm{aut(M)}$ mapping each $C_{i,j}$ onto $C_{i,\sigma(j)}$, and fixing $K \cup \bigcup_{\ell \neq i, j<\omega} C_{\ell,j}$ pointwise. Braunfield and Laskowski prove a model M is cellular if and only if it is mutually algebraic and $\aleph_0$-categorical. Call $T_{\mathcal{H}}$ k-cellular if every model of the (incomplete) theory of $\mathcal{H}$ is k-cellular.

Laskowski and Terry (unpublished) have shown that if $T_{\mathcal{H}}$ is k-cellular where $k = \max\{|C_{i,1}| : i < m\}$ (k depends only on $T_{\mathcal{H}}$) then its growth rate is in class 1) if $k = 1$ and in class 2) if $k \geqslant 2$; if $T_{\mathcal{H}}$ is mutually algebraic but not k-cellular for any k then it is in class 3). Thus we have captured the growth rate of a class of finite structures by purely model theoretic properties of an associated theory. Here, the notion of cellularity was developed in response to a problem in graph theory but the characterization is entirely model theoretic.

This extension of a result for graphs to arbitrary relational languages uses not only a refinement ($\aleph_0$-categorical, mutually algebraic) of the stability classification that gives very precise control over definable sets but invokes the precise model theoretic notion of interpretation to control the mutually algebraic structures by ones which are 'totally bounded'. It uses deep model theoretic analysis to show many arguments that appeared to be 'graph-theoretic' are actually determined by properties of arbitrary finite relational structures.

6 The value of formalization

This article focuses on understanding why model theory has so many applications across mathematics. Our choice of topics was restricted by space and time, the desire to emphasize the widening range of applications, and the need to avoid areas where the technical mathematical prerequisites are huge. Two, more or less random examples of the last are [10, 20].

Applying formal definability to traditional mathematical topics is a central point. But, it is exploited by each branch of pure logic (not to mention logic/computer science). Definability plays not only a central role in exploring relations within *set theory* (V=L, determinacy, etc.) but via the notion of Borel isomorphism in classifying problems arising in many areas, e.g. [38, 65]. *Computability theory* has contributed to the general theory of randomness; the large literature was summarised in [Nie12]. In his retiring presidential address at the 2019 ASL meeting in Prague, Ulrich Kohlenbach, described proof-mining as 'local *proof theory*' [43]. In the general setting of abstract metric spaces, he describes results in fixed point and ergodic theory, convex optimization, geodesic geometry, Cauchy problems, game theory etc. General metatheorems are applied to the formal proof of theorem in specific areas that have been formalized in an appropriate way. This is analogous to applying results about ω-stable theories to differentially closed fields as well as compact complex manifolds.

The particular applicability of model theory stems from:

1. Axiomatization of specific theories yields better understanding.

2. The model theoretic classification is orthogonal to usual organizations (algebra, analysis, geometry ...); nevertheless, it provides essential guidance to transfer methods and results from one area to another.

3. Parallel developments in distinct areas illustrate the ubiquity of patterns isolated in model theory.

4. Compactness allows the concentration of unbounded phenomena into a single instance.

5. Specific kinds (e.g. Saturated or atomic) of models are more easily analyzed.

6. Ranks support proofs by induction of analogous properties in different areas.

7. Structure theory from general model theory induces structure theory that answers existing questions in various areas.

8. Abstract model theoretic conditions imply the existence of groups and fields. The algebraic connection is inevitable.

9. Collections of finite structures can be 'approximated' by well-behaved infinite structures.

Of course, the work involves both the model theoretic framework and the often even more complex technology of the particular area. Model theory is not an isolated subject but an integral part of the mathematical enterprise.

References

[1] Noga Alon, Roi Livni, Maryanthe Malliaris, and Shay Moran, *Private PAC learning implies finite Littlestone dimension*, 51st Symposium on the Theory of Computing (STOC), 2019, pp. 91–99.

[2] Mathias Aschenbrenner, Lou van den Dries, and Joris van der Hoeven, *Asymptotic differential algebra and model theory of transseries*, Annals of Math Studies, Princeton University Press, 2017.

[3] John T. Baldwin, *Finite and infinite model theory: An historical perspective*, Logic Journal of the IGPL **8** (2000).

[4] ———, *Model theory and the philosophy of mathematical practice: Formalization without foundationalism*, Cambridge University Press, 2018.

[5] ———, *Exploring the generous arena*, Maddy volume (J. Kennedy, ed.), Cambridge, 202x, invited: submitted.

[6] John T. Baldwin and B. Rose, $\aleph_0$*-categoricity and stability of rings*, Journal of Algebra **45** (1977), 1–17.

[7] John T. Baldwin and J. Saxl, *Logical stability in group theory*, Journal of the Australian Mathematical Society **21** (1976), 267–276.

[8] John T. Baldwin and S. Shelah, *Randomness and semigenericity*, Transactions of the American Mathematical Society **349** (1997), 1359–1376.

[9] József Balogh, Béla Bollobas, and David Weinreich, *The penultimate rate of growth for graph properties*, European Journal of Combinatorics **22** (2000a), 277–289.

[10] M. Bays, B. Hart, and A. Pillay, *Universal covers of commutative finite Morley rank groups*, Volume 19, Issue 3 May 2020 , pp. 767-799, *Journal of the Institute of Mathematics of Jussieu.* `https://www3.nd.edu/~apillay/papers/universalcovers-BHP.pdf`.

[11] Chantal Berline, *Déviation des types dans les corps algébriquement clos*, Seminaire Théories Stable: 1980-82 (Bruno Poizat, ed.), vol. 3, Poizat, 1982, pp. 3.01–3–10.

[12] I. Ben Yaacov, *Model theoretic stability and definability of types, after A. Grothendieck*, Bull. Symb. Log. **20** (2014), 491–496.

[13] I. Ben Yaacov, A. Berenstein, C. W. Henson, and A. Usvyatsov, *Model theory for metric structures*, Model theory with applications to algebra and analysis. Vol. 2 (Z. Chatzidakis and et al., eds.), London Math. Soc. Lecture Note Ser., vol. 350, Cambridge Univ. Press, Cambridge, 2008, pp. 315–427.

[14] Jacek Bochnak, Michel Coste, and Marie-Francoise Roy, *Real algebraic geometry*, A Series of Modern Surveys in Mathematics, Springer-Verlag, 1998.

[15] W. Boney, G. Grossberg, M. Lieberman, Jirí Rosický, and S. Vasey, μ*-abstract elementary classes and other generalizations*, Journal of Pure and Applied Algebra **220** (2016), 3048–66.

[16] Siddharth Bhaskar, *Thicket Density*, Math Archiv:1702.03956, 2018.

[17] Samuel Braunfield and Michael C. Laskowsk, *Mutual algebraicity and cellularity*, `https://arxiv.org/pdf/1911.06303.pdf`.

[18] Mark Bun, Roi Livni, and Shay Moran, *An equivalence between private classification and online prediction*, Math archiv: `https://arxiv.org/pdf/2003.00563.pdf`, 2020.

[19] E. Bouscaren (ed.), *Model theory and algebraic geometry : An introduction to E. Hrushovski's proof of the geometric mordell-lang conjecture*, Springer-Verlag, Heidelberg, 1999.

[20] Guy Casale, James Freitag, and Joel Nagloo, *Ax-Lindemann-Weierstrass with derivatives and the genus* 0 *Fuchsian groups*, To appear *Annals of Mathematics.* arXiv:1811.06583, 2008.

[21] H. Chase and J. Freitag, *Bounds in query learning*, arXiv:1904.10122, 2018.

[22] Hunter Chase and James Freitag, *Model theory and combinatorics of banned sequences*, Math arXiv:1904.10122, 2018.

[23] ———, *Model theory and machine learning*, Bulletin of Symbolic Logic **25** (2019), 319–332, Math arXiv:1801.06566.

[24] M. Chudnovsky, R. Kim, Sang-Il Oum, and P. Seymour, *Unavoidable induced subgraphs in large graphs with no homogeneous sets*, J. Combinatorial Theory, Ser. B, **118** (2016), Math ArXiv 1504.05322.

[25] Antoine Chambert-Loir, *Review of 'o-minimality and diophantine geometry'*, The Bulletin of Symbolic Logic **23** (2017), 115–117, DOI:10.1017/bsl.2017.3.

[26] Alberto Coffa, *The semantic tradition from Kant to Carnap: To the Vienna Station*, Cambridge University Press, 1991.

[27] L. van den Dries, *Remarks on Tarski's problem concerning* $(\mathbb{R}; +, \cdot, \exp)$, Logic Colloquium '82 (G. Lolli, G. Longo, and G. Marcja, eds.), Springer-Verlag, 1984, MR 86g:03052, pp. 97–121.

[28] Shlomo Eshel and Itay Kaplan, *On uniform definability of types over finite sets for NIP formulas*, 2019 manuscript arXiv:1904.10336v1, 2019.

[29] I. Farah, B. Hart, M. Lupini, L. Robert, A. Tikuisis, A. Vignati, and W. Winter, *Model theory of* C^**-algebras*, Memoirs of the American Mathematical Society, American Mathematical Society, Providence, 2020?

[30] Ian Hacking, *Why is there a philosophy of mathematics at all?*, Cambridge University Press, 2014.

[31] Michael Harris, *Mathematics without apologies: Portrait of a problematic vocation*, Princeton University Press, 2015.

[32] D. Haskell, E. Hrushovski, and H.D. MacPherson, *Stable domination and independence in algebraically closed valued fields*, Lecture Notes in Logic, Association for Symbolic Logic, 2007.

[33] Åsa Hirvonen and T. Hyttinen, *Metric abstract elementary classes with perturbations*, Fund. Math. **217** (2012), 123–170.

[34] Ehud Hrushovski and Anand Pillay, *Effective bounds for the number of transcendental points on subvarieties of semi-abelian varieties*, American Journal of Mathematics **122** (2000), 451–463.

[35] Ehud Hrushovski, *A new strongly minimal set*, Annals of Pure and Applied Logic **62** (1993), 147–166.

[36] Hunter R. Johnson and Michael C. Laskowski, *Compression schemes, stable definable families, and o-minimal structures*, Discrete & Computational Geometry **43** (2010), 914–926.

[37] G.O. Jones and A.J. Wilkie, *O-minimality and diophantine geometry*, London Mathe-

matical Society Lecture Note Series, vol. 421, Cambridge University Press, 2015.

[38] A. Kechris, *Global aspects of ergodic group actions*, Mathematical Surveys and Monographs, no. 160, American Mathematical Society, 2010.

[39] K. Khanaki, *Dividing lines in unstable theories and subclasses of Baire 1 functions*, preprint: 2019 arXiv:1904.09486v5, 2019.

[40] ______, *Remarks on the strict order property*, preprint: 2019 `https://arxiv.org/pdf/1902.05229v1.pdf`, 2019.

[41] ______, *Stability, NIP, and NSOP; model theoretic properties of formulas via topological properties of function spaces*, to appear: MLQ (on arxiv 2014) arXiv:1410.3339, 2019.

[42] K. Khanaki and A. Pillay, *Remarks on the NIP in a model*, Math. Log. Quart. **64** (2018), 429–434.

[43] U. Kohlenbach, *Proof-theoretic methods in nonlinear analysis*, Proceedings of the ICM, 2018, to appear.

[44] Michael C. Laskowski, *Vapnik-Chervonenkis classes of definable sets*, Journal of the London Mathematical Society **45** (1992), 377–384.

[45] ______, *Mutually algebraic structures and expansions by predicates*, Journal of Symbolic Logic **78** (2013), 185–194.

[46] Michael C. Laskowski and Caroline Terry, *Jumps in speeds of hereditary properties in finite relational languages*, preprint, 2019.

[47] ______, *Uniformly bounded arrays and mutually algebraic structuress*, preprint, 2019.

[48] M. Lieberman and Jirí Rosický, *Classification theory for accessible categories*, Journal of Symbolic Logic **81** (2016), 151–165.

[49] Roi Livni and Pierre Simon, *Honest compressions and their application to compression schemes*, JMLR Workshop and Conference Proceedings, vol. 30, Journal of Machine Learning Research, 2013, pp. 77–92.

[50] Nick Littlestone and Manfred Warmuth, *Relating data compression and learnability*, Tech. report, University of California, Santa Cruz, 1986.

[51] Angus J. Macintyre, *On ω_1-categorical theories of fields*, Fundamenata Mathematicae **71** (1971), 168–75.

[52] Penelope Maddy, *What do we want a foundation to do? Comparing set-theoretic, category-theoretic, and univalent approaches*, Reflections on Foundations: Univalent Foundations, Set Theory and General Thoughts (S. Centrone, D. Kant, and D. Sarikaya, eds.), Springer-Verlag, 2019.

[53] Maryanthe Malliaris and Saharon Shelah, *Keisler's order has infinitely many classes*, Israel Journal of Mathematics **224** (2018), 189–230.

[54] Maryanthe Malliaris and Caroline Terry, *On unavoidable induced subgraphs in large prime graphs*, J. Graph Theory **88** (2018), 255–270.

[55] David Marker, *Manin kernels*, Model Theory and Applications, Quaterna di matematica, 2000, pp. 1–21.

[56] ______, *Review of: Tame topology and o-minimal structures, by Lou van den Dries,*,

Bulletin (New Series) of American Mathematical Society **37** (2000), 351–357.

[57] ______, *The number of countable differentially closed fields*, Notre Dame Journal of Formal Logic **48** (2007), 99–113.

[58] M. Morley, *Categoricity in power*, Transactions of the American Mathematical Society **114** (1965), 514–538.

[59] Joel Nagloo, *Model theory and differential equations*, Notices of the AMS (2021).

[60] J. Nagloo and A. Pillay, *On algebraic relations between solutions of a generic Painlevè equation*, Journal für die reine und angewandete Mathematik (Crelle's journal) (2016), DOI: 10.1515/crelle-2014-0082.

[Nie12] Andres Nies, *Computability and randomness*, Oxford Logic Guides, Oxford, Oxford, 2012.

[61] A. Pillay, *Geometric stability theory*, Clarendon Press, Oxford, 1996.

[62] ______, *Generic stability and Grothendieck*, South American Journal of Logic **2** (2016), 437–442.

[63] A. Pillay and C. Steinhorn, *Definable sets in ordered structures I*, Transactions of the American Mathematical Society **295** (1986).

[64] A. Robinson, *On the application of symbolic logic to algebra*, Proceedings of the International Congress of Mathematicians, Cambridge, Massachusetts, U.S.A., August 30-September 6,1950, vol. 1, American Mathematical Society, Providence, Rhode Island, 1952, reprinted in Collected works Vol 1. Yale University press, 1971, pp. 686–694.

[65] Christian Rosendal, *Descriptive Classification Theory and Separable Banach Spaces*, Notices of the American Mathematical Society (2011), 1251–1262.

[66] Omar Sànchez and Anand Pillay, *Some definable Galois theory and examples*, Bulletin of Symbolic Logic **23** (2017), 145–159.

[67] T. Scanlon, *Counting special points: Logic, Diophantine geometry, and transcendence theory*, Bulletin of the American Mathematical Society (N.S.) **49** (2012), 51–71.

[68] S. Shelah, *A combinatorial problem; stability and order for models and theories in infinitary languages*, Pacific Journal of Math **41** (1972), 247–261.

[69] ______, *Classification Theory and the Number of Nonisomorphic Models*, North-Holland, 1978.

[70] ______, *Response to the award of the 2013 Steele prize for seminal research*, A.M.S. Prize Booklet, 2013 (2013), 50, `http://www.ams.org/profession/prizebooklet-2013.pdf`.

[71] ______, *Dependent theories and the generic pair conjecture*, Communications in Contemporary Math **17** (2015), 64 pp, Sh index 900: had circulated since early 2000's.

[72] Saharon Shelah and Joel Spencer, *Zero-one laws for sparse random graphs*, Journal of A.M.S. **1** (1988), 97–115.

[73] Pierre Simon, *A guide to NIP theories*, Lecture Notes in Logic, Cambridge, 2017.

[74] Alfred Tarski, *Sur les ensemble définissable de nombres réels I.*, Fundamenta Mathematica **17** (1931), 210–239.

[75] ———, *Introduction to Logic and to the Methodology of the Deductive Sciences*, Oxford: Galaxy Book, 1965, First edition: Oxford 1941.

[76] Jouko Väänänen, *Second order logic or set theory*, Bulletin of Symbolic Logic **18** (2012), 91–121.

[77] R. Whitley, *An elementary proof of the Eberlein-Šmulian theorem*, Mathematische Annalen (1967), 116–119.

[78] Eugene Wigner, *The unreasonable effectiveness of mathematics in the natural sciences*, Communications in Pure and Applied Mathematics **13** (1960).

[79] Alex Wilkie, *Model completeness results for expansions of the real field by restricted Pfaffian functions and exponentiation*, Journal of American Mathematical Society (1996), 1051–1094.

[80] ———, *O-minimal structures*, Séminaire Bourbaki **985** (2007), `http://eprints.ma.man.ac.uk/1745/01/covered/MIMS_ep2012_3.pdf`.

[81] B.I. Zilber, *The structure of models of uncountably categorical theories*, Proceedings of the International Congress of Mathematicians August 16-23, 1983, Warszawa, Polish Scientific Publishers, Warszawa, 1984, 359–68.

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Bilattice Basics

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1 Introduction

Stated abstractly, a bilattice is an algebraic structure with two orderings meeting various conditions, separately and jointly. Stated more intuitively, a bilattice is a space of truth values with two orderings, one intended to represent degrees of truth, the other intended to represent the degrees of information that lead us to assign these particular truth values. Bilattices were introduced by Matt Ginsberg, [33, 34, 35], with the idea that they would be useful in artificial intelligence. In fact many of the ideas behind bilattices have a substantial pre-history, with details to be found in [13, 31]. It has often happened in scientific history that, when the time is ripe, a range of ideas coalesces into a subject to be investigated for its own sake. As algebraic structures, bilattices can play a direct role in semantical investigations. But also they can be made to carry a logic, and so have an associated proof theory. Both the semantical and the proof theoretical roles will be discussed here.

2 The Most Important Example

Bilattices are as varied as lattices themselves, but one example stands out. It was introduced independently, is well-known, influential, and illustrates every nice feature any bilattice might have. We refer to the well-known Belnap-Dunn four-valued structure, [5], shown in Figure 1. In Belnap's paper two orderings were discussed, but independently. The diagram in Figure 1 is a double Hasse diagram, and shows Belnap's two orderings together, as is done for bilattices generally.

Following Belnap-Dunn, think of the four values as sets of classical truth values, consisting of values supplied to us by outside agents. Then $\mathbf{f}$ is that some said false and nobody said true, $\mathbf{t}$ is some said true and nobody said false, $\top$ is some said true and some said false, $\bot$ is nobody said anything. Moving upward involves an increase in information (not necessarily verified information, but information nonetheless).

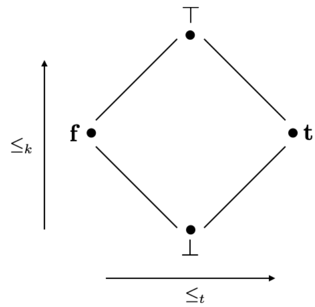


Figure 1: The Bilattice $\mathcal{FOUR}$

The increase goes from no information, to some, to too much. Moving from left to right involves an increase in truth or a decrease in falsity. Then $\mathbf{f} \leq_t \top$ because $\top$, like $\mathbf{f}$, includes false, but $\top$ also includes true, and so degree of truth has increased. Similarly $\mathbf{f} \leq_t \bot$ because degree of falsity has decreased. And so on.

In $\mathcal{FOUR}$, both orderings have the structure of a bounded lattice. Meets and joins exist. Following bilattice conventions, meet and join for the $\leq_t$ ordering are denoted $\wedge$ and $\vee$ and have properties meant to generalize logical conjunction and disjunction. Meet and join for the $\leq_k$ ordering are denoted $\otimes$ and $\oplus$. Commonly with bilattices, $\otimes$ is read as *consensus* because $x \otimes y$ is the least information common to both x and y. Likewise $\oplus$ is read as *gullability* or *accept all.* $x \oplus y$ simply lumps together what each of x and y tell us. Figure 2 has the tables for all these operations of $\mathcal{FOUR}$.

$\wedge$	$\mathbf{f}$	$\mathbf{t}$	$\bot$	$\top$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{f}$	$\mathbf{t}$	$\bot$	$\top$
$\bot$	$\mathbf{f}$	$\bot$	$\bot$	$\mathbf{f}$
$\top$	$\mathbf{f}$	$\top$	$\mathbf{f}$	$\top$

$\vee$	$\mathbf{f}$	$\mathbf{t}$	$\bot$	$\top$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{t}$	$\bot$	$\top$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$
$\bot$	$\bot$	$\mathbf{t}$	$\bot$	$\mathbf{t}$
$\top$	$\top$	$\mathbf{t}$	$\mathbf{t}$	$\top$

$\otimes$	$\mathbf{f}$	$\mathbf{t}$	$\bot$	$\top$
$\mathbf{f}$	$\mathbf{f}$	$\bot$	$\bot$	$\mathbf{f}$
$\mathbf{t}$	$\bot$	$\mathbf{t}$	$\bot$	$\mathbf{t}$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
$\top$	$\mathbf{f}$	$\mathbf{t}$	$\bot$	$\top$

$\oplus$	$\mathbf{f}$	$\mathbf{t}$	$\bot$	$\top$
$\mathbf{f}$	$\mathbf{f}$	$\top$	$\mathbf{f}$	$\top$
$\mathbf{t}$	$\top$	$\mathbf{t}$	$\mathbf{t}$	$\top$
$\bot$	$\mathbf{f}$	$\mathbf{t}$	$\bot$	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$

Figure 2: Binary Connectives of $\mathcal{FOUR}$

There is a plausible negation operation corresponding to left right symmetry. There is also a similar vertical operation, known as *conflation* in the general bilattice context. Figure 3 has the tables for these in $\mathcal{FOUR}$.

All this structure is quite tightly interconnected. For instance with four binary operations there are twelve possible distributive laws. Some, like $x \wedge (y \vee z) =$

$\neg$	
$\mathbf{f}$	$\mathbf{t}$
$\mathbf{t}$	$\mathbf{f}$
$\bot$	$\bot$
$\top$	$\top$

$-$	
$\mathbf{f}$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$
$\bot$	$\top$
$\top$	$\bot$

Figure 3: Unary Connectives of $\mathcal{FOUR}$

$(x \wedge y) \vee (x \wedge z)$, involve only one kind of operation, in this case those associated with the $\leq_t$ ordering. Some, like $x \wedge (y \oplus z) = (x \wedge y) \oplus (x \wedge z)$, involve operations associated with both orderings. In fact, all twelve distributive laws hold in $\mathcal{FOUR}$.

Going further, negation and conflation commute, $\neg - x = -\neg x$. There are De Morgan laws for both orderings.

$$\begin{aligned}
\neg(x \wedge y) &= \neg x \vee \neg y \\
\neg(x \vee y) &= \neg x \wedge \neg y \\
-(x \otimes y) &= -x \oplus -y \\
-(x \oplus y) &= -x \otimes -y
\end{aligned}$$

And there are what might be called pseudo De Morgan laws across orderings.

$$\begin{aligned}
\neg(x \otimes y) &= \neg x \otimes \neg y \\
\neg(x \oplus y) &= \neg x \oplus \neg y \\
-(x \wedge y) &= -x \wedge -y \\
-(x \vee y) &= -x \vee -y
\end{aligned}$$

Note that $\mathcal{FOUR}$ contains subsystems corresponding to familiar logics. $\{\mathbf{f}, \mathbf{t}\}$ behaves like classical truth using $\wedge$, $\vee$, and $\neg$. $\{\mathbf{f}, \bot, \mathbf{t}\}$ similarly behaves like Kleene's strong three valued logic. Actually, $\{\mathbf{f}, \top, \mathbf{t}\}$ has similar behavior, but informally $\bot$ is much like the value of undefined, $\mathfrak{u}$, that Kleene talked about in [38], while $\top$ is like the inconsistent truth value one finds in Priest's Logic of Paradox, LP. In a sense, $\mathcal{FOUR}$ contains multitudes.

All this tightly connected machinery is of more than formal interest. It plays a natural role in many of the applications of $\mathcal{FOUR}$ to issues of importance in both philosophical logic and computer science. But this is far enough to take this single example. It is time to move to the full bilattice family.

3 Bilattices

Bilattices are a family of structures having some, or all, of the signfiicant properties mentioned in the previous section as applicable to $\mathcal{FOUR}$. We give the general definitions below. We should note that the term *bilattice* is somewhat loose, and has varied in small ways over the years. For instance, negation is generally considered to be a basic part, but with different conditions than was the case originally. Bounds are built in below, but bilattices without bounds have been investigated. One must be careful to note what a particular author's practice is. Here we begin with structures having two orderings, with no connections between the orderings postulated.

Definition 3.1. The structure $\mathcal{B} = \langle \mathcal{B}, \leq_t, \leq_k \rangle$ is a *pre-bilattice* provided $\mathcal{B}$ is a non-empty set and $\leq_t$ and $\leq_k$ are partial orderings each giving $\mathcal{B}$ the structure of a bounded lattice. We will assume all pre-bilattices are non-trivial, meaning that the extreme elements, bottom and top in each ordering, are all distinct.

We refer to the $\leq_t$ order as the *truth ordering.* As was our practice with $\mathcal{FOUR}$, meet and join operations for the truth ordering are denoted by $\wedge$ and $\vee$, bottom is denoted by $\mathbf{f}$ and top by $\mathbf{t}$. Similarly the $\leq_k$ is the *information ordering.* For it, meet and join operations are denoted $\otimes$, *consensus* and $\oplus$, *gullability*, bottom is denoted by $\perp$ and represents total ignorance; top is denoted by $\top$ and represents total information, even allowing inconsistencies. Note that our non-triviality condition amounts to assuming that $\mathbf{t}$, $\mathbf{f}$, $\top$, and $\perp$ are all distinct.

Historically the $\leq_k$ ordering was referred to as a "knowledge ordering", hence the subscript k. Calling it an information ordering is better, but the k subscript has become somewhat traditional.

Pre-bilattices have two orderings, with no interconnecting conditions required. Things get interesting when connections between the orderings are imposed. Typically, a pre-bilattice with additional conditions is referred to just as a bilattice with those conditions.

Definition 3.2. A pre-bilattice $\langle \mathcal{B}, \leq_t, \leq_k \rangle$ is:

Interlaced if each of the operations $\wedge$, $\vee$, $\otimes$, and $\oplus$ is monotone with respect to both orderings;

Distributive if all 12 distributive laws connecting $\wedge$, $\vee$, $\otimes$, and $\oplus$ are valid.

Interlacing was not mentioned when discussing $\mathcal{FOUR}$ because it is a consequence of distributivity, which is a property that $\mathcal{FOUR}$ has. However, interlacing is strictly weaker than distributivity, as an example in the next section will show.

Then it is good to get some idea of just what the interlacing conditions are actually requiring. In any pre-bilattice, $a \leq_t b$ implies $a \wedge c \leq_t b \wedge c$ by the following lattice-theoretic argument. Suppose $a \leq_t b$. Then $a \wedge c \leq_t a$ because $a \wedge c$ is a lower bound for $\{a, c\}$ and hence is below a. Since $a \leq_t b$, then $a \wedge c \leq_t b$. Similarly $a \wedge c \leq_t c$. Then $a \wedge c$ is a lower bound for $\{b, c\}$ and hence $a \wedge c \leq_t b \wedge c$ because $b \wedge c$ is the *greatest* lower bound for $\{b, c\}$. In a similar way it can be shown that we always have $a \leq_t b$ implies $a \vee c \leq_t b \vee c$, and $a \leq_k b$ implies both $a \otimes c \leq_k b \otimes c$ and $a \oplus c \leq_k b \oplus c$. What interlacing adds is *cross conditions*, connecting operations pertinent to one of the orderings with the other order. Thus we have the following new items in an interlaced bilattice.

$$a \leq_t b \text{ implies } a \otimes c \leq_t b \otimes c$$
$$a \leq_t b \text{ implies } a \oplus c \leq_t b \oplus c$$
$$a \leq_k b \text{ implies } a \wedge c \leq_k b \wedge c$$
$$a \leq_k b \text{ implies } a \vee c \leq_k b \vee c$$

Distributivity is required to hold not only within each ordering, as in $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, but across the two orderings, as in $a \wedge (b \otimes c) = (a \wedge b) \otimes (b \wedge c)$. Distributivity implies interlacing. For instance, suppose $a \leq_k b$. Then $a \otimes b = a$, so $(a \otimes b) \wedge c = a \wedge c$, so assuming distributivity, $(a \wedge c) \otimes (b \wedge c) = a \wedge c$, and hence $a \wedge c \leq_k b \wedge c$. As we noted above, interlacing does not imply distributivity.

Finally, a negation is commonly assumed and, less frequently, an analogous second symmetry operation.

Definition 3.3. Let $\mathcal{B} = \langle \mathcal{B}, \leq_t, \leq_k \rangle$ be a pre-bilattice.

1. $\mathcal{B}$ has a *negation* if there is a mapping $\neg : \mathcal{B} \to \mathcal{B}$ such that

   (Neg-1) if $a \leq_t b$ then $\neg b \leq_t \neg a$,

   (Neg-2) if $a \leq_k b$ then $\neg a \leq_k \neg b$,

   (Neg-3) $\neg\neg a = a$.

2. $\mathcal{B}$ has a *conflation* if there is a mapping $- : \mathcal{B} \to \mathcal{B}$ such that

   (Con-1) if $a \leq_k b$ then $-b \leq_k -a$,

   (Con-2) if $a \leq_t b$ then $-a \leq_t -b$,

   (Con-3) $- - a = a$.

3. Negation and conflation commute if both are present, that is, $-\neg a = \neg - a$.

In the literature negation is commonly assumed. Conflation is less common, but it is not generally assumed unless negation is present, in which case commutation is also assumed. Our definitions are given as properties of the orderings, but the following shows that the De Morgan laws noted earlier for $\mathcal{FOUR}$ are consequences (indeed, equivalences).

Proposition 3.4 (De Morgan)**.** *Let $\mathcal{B} = \langle \mathcal{B}, \leq_t, \leq_k \rangle$ be a pre-bilattice.*

1. *If $\mathcal{B}$ has negation then $\neg(x \wedge y) = \neg x \vee \neg y$ and $\neg(x \vee y) = \neg x \wedge \neg y$, while $\neg(x \otimes y) = \neg x \otimes \neg y$ and $\neg(x \oplus y) = \neg x \oplus \neg y$.*
2. *If $\mathcal{B}$ has conflation then $-(x \otimes y) = -x \oplus -y$ and $-(x \oplus y) = -x \otimes -y$, while $-(x \wedge y) = -x \wedge -y$ and $-(x \vee y) = -x \vee -y$.*

Proof. We show one item, as representative of the rest. Suppose $\mathcal{B}$ has negation. Since $x \wedge y \leq_t x$, then by (Neg-1) $\neg x \leq_t \neg(x \wedge y)$. Similarly $\neg y \leq_t \neg(x \wedge y)$, and so $\neg x \vee \neg y \leq_t \neg(x \wedge y)$. Further, $\neg x \leq_t \neg x \vee \neg y$, so $\neg(\neg x \vee \neg y) \leq_t \neg\neg x = x$, by ($\mathsf{Neg}$-1) and ($\mathsf{Neg}$-3). Similarly $\neg(\neg x \vee \neg y) \leq_t y$, so $\neg(\neg x \vee \neg y) \leq_t x \wedge y$. Then $\neg(x \wedge y) \leq_t \neg\neg(\neg x \vee \neg y)$ by (Neg-1), so $\neg(x \wedge y) \leq_t \neg x \vee \neg y$ by (Neg-3). □

Conflation, when present, can be used to characterize interesting substructures of a bilattice, though we will see that these substructures may not exhaust the entire of a bilattice.

Definition 3.5. Let $\mathcal{B} = \langle \mathcal{B}, \leq_t, \leq_k \rangle$ be a pre-bilattice with a negation and a conflation that commute. A member $x \in \mathcal{B}$ is:

Consistent if $x \leq_k -x$,

Anticonsistent if $-x \leq_k x$,

Exact if $x = -x$.

Note that the exact members of $\mathcal{B}$ are also both consistent and anticonsistent. While the categories above are defined using the information ordering, it is with respect to the truth ordering that they are particularly nice.

Proposition 3.6. *Let $\mathcal{B} = \langle \mathcal{B}, \leq_t, \leq_k \rangle$ be an interlaced bilattice with a negation and a conflation that commute. Each of the consistent, anticonsistent, and exact subclasses of $\mathcal{B}$ are closed under $\wedge$, $\vee$, and $\neg$.*

Proof. We show a few parts of this; the rest are similar. Suppose we have a commuting negation and conflation, and suppose x and y are consistent, so that $x \leq_k -x$ and $y \leq_k -y$.

$x \wedge y \leq_k -x \wedge -y$ using interlacing, and $-x \wedge -y = -(x \wedge y)$ by Proposition 3.4, so $x \wedge y \leq_k -(x \wedge y)$; hence $x \wedge y$ is consistent. In a similar way $x \vee y$ is consistent. Finally, since $x \leq_k -x$ then $\neg x \leq_k \neg - x$ by (Neg-2), and so $\neg x \leq_k -\neg x$ since negation and conflation commute. Then $\neg x$ is consistent. □

It is easy to check that in the bilattice $\mathcal{FOUR}$ the exact members are $\{\mathbf{f}, \mathbf{t}\}$, the classical values; the consistent members are $\{\mathbf{f}, \bot, \mathbf{t}\}$, the ones we connected with Kleene's strong three valued logic; and the anticonsistent members are $\{\mathbf{f}, \top, \mathbf{t}\}$, the ones we connected with Priest's logic of paradox. In bilattices other than $\mathcal{FOUR}$ the three categories can be thought of as generalizations of classical logic, Kleene strong three valued logic, and logic of paradox. Many interesting properties carry over.

4 Examples

We have already seen $\mathcal{FOUR}$, a distributive (and hence interlaced) bilattice with commuting negation and conflation. When we come to infinitary operations, Section 9, it will be seen to be infinitarily distributive, and hence infinitarily interlaced, simply because it is finite. In short, $\mathcal{FOUR}$ is a model for everything we discuss. Now for some examples that are more complicated.

Figure 4 shows a bilattice taken from [33], where it was constructed to model default reasoning. In addition to the usual extreme values there are $d\mathbf{f}$ and $d\mathbf{t}$, for default falsehood and default truth. Also there is $d\top = d\mathbf{f} \oplus d\mathbf{t}$, for default both false and true. Ginsberg gives a thorough analysis of this. Here we only note that it provides us with an example of a bilattice that is not interlaced, and hence not distributive either. In it $\mathbf{f} \leq_t d\mathbf{f}$ but $\mathbf{f} \otimes d\top \not\leq_t d\mathbf{f} \otimes d\top$ because $\mathbf{f} \otimes d\top = d\top$ and $d\mathbf{f} \otimes d\top = d\mathbf{f}$ but $d\top \not\leq_t d\mathbf{f}$.

Figure 5 shows a nine-valued bilattice that is about as well-behaved as $\mathcal{FOUR}$. It is interlaced, indeed distributive, has negation and conflation, and these commute. Exact values are $\{\mathbf{f}, d\top, \mathbf{t}\}$, consistent values are $\{\mathbf{f}, d\top, \mathbf{t}, d\mathbf{f}, d\mathbf{t}, \bot\}$, and anticonsistent values are $\{\mathbf{f}, d\top, \mathbf{t}, o\mathbf{f}, o\mathbf{t}, \top\}$, which together exhaust the entire bilattice.

Finally something seriously more elaborate. Figure 6 shows a bilattice that, despite its forbidding appearance, actually has interesting theoretical applications. This bilattice is distributive, and has commuting negation and conflation. However, neither negation nor conflation corresponds to a simple symmetry of the diagram. While the node names will take on deeper meaning in Section 5, they do provide an

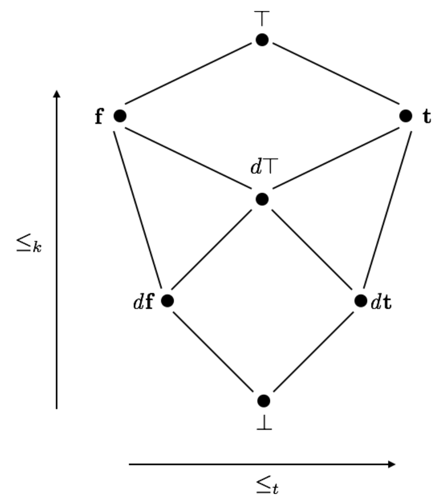


Figure 4: The Bilattice $\mathcal{DEFAULT}$

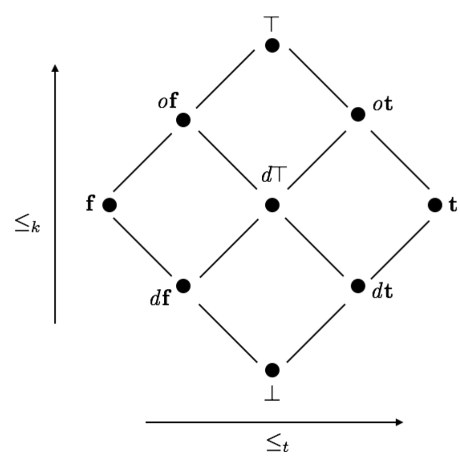


Figure 5: The Bilattice $\mathcal{NINE}$

easy description of negation: the negation of node $\langle x, y\rangle$ is node $\langle y, x\rangle$. Conflation has a similar but more complicated characterization, and we will come to it in detail in Proposition 5.2. For now, just assume conflation is given by the table in Figure 7.

As noted, it is the case that negation and conflation commute. For instance, $\neg - \langle \mathbf{t}, \bot\rangle = \neg\langle \bot, \mathbf{f}\rangle = \langle \mathbf{f}, \bot\rangle$ and $-\neg\langle \mathbf{t}, \bot\rangle = -\langle \bot, \mathbf{t}\rangle = \langle \mathbf{f}, \bot\rangle$, so $\neg - \langle \mathbf{t}, \bot\rangle = -\neg\langle \mathbf{t}, \bot\rangle$. Exact values are $\{\langle \mathbf{t}, \mathbf{f}\rangle, \langle \top, \top\rangle, \langle \bot, \bot\rangle, \langle \mathbf{f}, \mathbf{t}\rangle\}$. Consistent values are

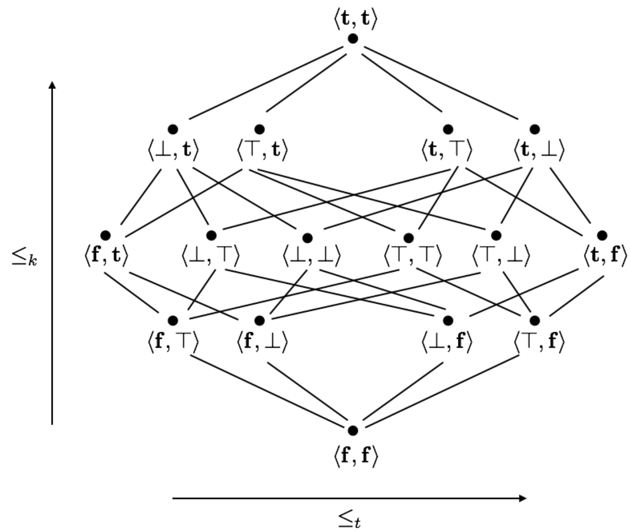


Figure 6: A Sixteen Valued Bilattice

x	$-x$
$\langle \mathbf{t}, \mathbf{t} \rangle$	$\langle \mathbf{f}, \mathbf{f} \rangle$
$\langle \bot, \mathbf{t} \rangle$	$\langle \mathbf{f}, \bot \rangle$
$\langle \top, \mathbf{t} \rangle$	$\langle \mathbf{f}, \top \rangle$
$\langle \mathbf{t}, \top \rangle$	$\langle \top, \mathbf{f} \rangle$
$\langle \mathbf{t}, \bot \rangle$	$\langle \bot, \mathbf{f} \rangle$
$\langle \mathbf{f}, \mathbf{t} \rangle$	$\langle \mathbf{f}, \mathbf{t} \rangle$
$\langle \bot, \top \rangle$	$\langle \top, \bot \rangle$
$\langle \bot, \bot \rangle$	$\langle \bot, \bot \rangle$
$\langle \top, \top \rangle$	$\langle \top, \top \rangle$
$\langle \top, \bot \rangle$	$\langle \bot, \top \rangle$
$\langle \mathbf{t}, \mathbf{f} \rangle$	$\langle \mathbf{t}, \mathbf{f} \rangle$
$\langle \mathbf{f}, \top \rangle$	$\langle \top, \mathbf{t} \rangle$
$\langle \mathbf{f}, \bot \rangle$	$\langle \bot, \mathbf{t} \rangle$
$\langle \bot, \mathbf{f} \rangle$	$\langle \mathbf{t}, \bot \rangle$
$\langle \top, \mathbf{f} \rangle$	$\langle \mathbf{t}, \top \rangle$
$\langle \mathbf{f}, \mathbf{f} \rangle$	$\langle \mathbf{t}, \mathbf{t} \rangle$

Figure 7: Sixteen Valued Conflation

$\{\langle \mathbf{t}, \mathbf{f} \rangle, \langle \top, \top \rangle, \langle \bot, \bot \rangle, \langle \mathbf{f}, \mathbf{t} \rangle, \langle \bot, \mathbf{f} \rangle, \langle \top, \mathbf{f} \rangle, \langle \mathbf{f}, \top \rangle, \langle \mathbf{f}, \bot \rangle, \langle \mathbf{f}, \mathbf{f} \rangle\}$. Anticonsistent values are $\{\langle \mathbf{t}, \mathbf{f} \rangle, \langle \top, \top \rangle, \langle \bot, \bot \rangle, \langle \mathbf{f}, \mathbf{t} \rangle, \langle \mathbf{t}, \bot \rangle, \langle \mathbf{t}, \top \rangle, \langle \top, \mathbf{t} \rangle, \langle \bot, \mathbf{t} \rangle, \langle \mathbf{t}, \mathbf{t} \rangle$. The values $\langle \bot, \top \rangle$

and $\langle \top, \bot \rangle$ do not fall into any of these categories. They are not exact because they are conflations of each other, and not of themselves. And they are neither consistent nor anticonsistent, since they are not comparable in the information ordering.

Function spaces provide a useful family of bilattices. We will see them in action when discussing applications connecting bilattices with Kripke-style theories of truth.

Definition 4.1. Let $\mathcal{B} = \langle \mathcal{B}, \leq_t, \leq_k \rangle$ be a bilattice, and let S be a set. The function space $\mathcal{B}^S$ is the bilattice whose domain is the set consisting of all functions from S to $\mathcal{B}$, and with pointwise orderings. That is, for $f, g \in \mathcal{B}^S$:

$$f \leq_t g \text{ if and only if } f(x) \leq_t g(x) \text{ for all } x \in S$$
$$f \leq_k g \text{ if and only if } f(x) \leq_k g(x) \text{ for all } x \in S$$

It is easy to check that $\mathcal{B}^S$ is a bilattice: both orderings are those of bounded lattices, and so on. Here are some further items whose verification is left to you.

Proposition 4.2. *Let $\mathcal{B} = \langle \mathcal{B}, \leq_t, \leq_k \rangle$ be a bilattice, let S be a set, and let $f, g \in \mathcal{B}^S$.*

1. *The least member of $\mathcal{B}^S$ in the pointwise $\leq_k$ ordering is the function that is identically $\bot$ on S, where $\bot$ is the least member of $\mathcal{B}$ in the $\leq_k$ ordering on $\mathcal{B}$. Similarly for the other three extreme elements.*
2. *$(f \wedge g)(x) = f(x) \wedge g(x)$, and similarly for $\vee$, $\otimes$, and $\oplus$.*
3. *If $\mathcal{B}$ is (infinitarily) interlaced, so is $\mathcal{B}^S$. Similarly for (infinitarily) distributive.*
4. *If $\mathcal{B}$ has negation, so does $\mathcal{B}^S$, and $(\neg f)(x) = \neg(f(x))$, and similarly for conflation.*
5. *If negation and conflation commute in $\mathcal{B}$ they also do in $\mathcal{B}^S$.*
6. *f is exact in $\mathcal{B}^S$ if and only if $f(x)$ is exact in $\mathcal{B}$ for every x, and similarly for consistent and anticonsistent.*

5 The Representation Theorem

There is a way of constructing bilattices, with conditions imposed, that is completely general in the sense that every bilattice is isomorphic to a bilattice constructed in this way. In [18, 17] it is called the *Ginsberg-Fitting product*, but we will just call it a *bilattice product*. It has a complicated history, see [31, 13].

Definition 5.1 (Bilattice Product)**.** Let $L_1 = \langle L_1, \leq_1 \rangle$ and $L_2 = \langle L_2, \leq_2 \rangle$ be bounded lattices. Their *bilattice product* is defined as follows.

$$L_1 \odot L_2 = \langle L_1 \times L_2, \leq_t, \leq_k \rangle$$
$$\langle a, b \rangle \leq_k \langle c, d \rangle \text{ iff } a \leq_1 c \text{ and } b \leq_2 d$$
$$\langle a, b \rangle \leq_t \langle c, d \rangle \text{ iff } a \leq_1 c \text{ and } d \leq_2 b$$

Note that in the definition of $\leq_t$ above, the order reverses for the second component. The term *twist structure* is rather common for such things.

The definition of bilattice product is strictly algebraic but there are intuitive, everyday examples involving groups of experts. Imagine we have one group, the *pros*, whose members can announce their opinions for something, or refrain from doing so, and another group, the *cons*, who similarly announce opinions against, or don't. The two groups could be distinct, overlap, or be identical. Think of $\mathcal{L}_1$ and $\mathcal{L}_2$ as consisting of all subsets of the first group and of the second group, respectively. That is, a member of $\mathcal{L}_1$ is a set of experts each of whom say 'yes' to some proposition, and a member of $\mathcal{L}_2$ is a set of experts who say 'no' to some proposition. For both $\mathcal{L}_1$ and $\mathcal{L}_2$ the ordering relations are simply subset. A member of the corresponding bilattice, then, is a generalized truth value that tells us who approves and who rejects, where approving and rejecting are independent actions. For such a bilattice we have an increase in information if additional experts declare their opinions, either for or against. We have an increase in degree of truth if additional experts declare in favor while some withdraw from declaring against. Of course not all bilattices are of this kind, but it is good to have it in mind while discussing the representation theorems that follow.

Proposition 5.2 (Bilattice Construction Properties)**.** *Let $L_1 = \langle L_1, \leq_1 \rangle$ and $L_2 = \langle L_2, \leq_2 \rangle$ be bounded lattices. In the following we write 0_1 and 0_2 for the smallest elements of L_1 and L_2 respectively, and 1_1 and 1_2 for the largest, $\sqcup_1$ and $\sqcup_2$ for the respective joins, and $\sqcap_1$ and $\sqcap_2$ for the meets.*

1. *$L_1 \odot L_2$ is a an interlaced bilattice.*
2. *The extreme elements of $L_1 \odot L_2$ are $\bot = \langle 0_1, 0_2 \rangle$, $\top = \langle 1_1, 1_2 \rangle$, $\mathbf{f} = \langle 0_1, 1_2 \rangle$, and $\mathbf{t} = \langle 1_1, 0_2 \rangle$.*
3. *The bilattice operations of $L_1 \odot L_2$ are the following.*

$$\langle a, b \rangle \wedge \langle c, d \rangle = \langle a \sqcap_1 c, b \sqcup_2 d \rangle$$
$$\langle a, b \rangle \vee \langle c, d \rangle = \langle a \sqcup_1 c, b \sqcap_2 d \rangle$$
$$\langle a, b \rangle \otimes \langle c, d \rangle = \langle a \sqcap_1 c, b \sqcap_2 d \rangle$$
$$\langle a, b \rangle \oplus \langle c, d \rangle = \langle a \sqcup_1 c, b \sqcup_2 d \rangle$$

4. If L_1 and L_2 are distributive lattices then $L_1 \odot L_2$ is a distributive bilattice.

5. If $L_1 = L_2 = L$ then $L \odot L$ is a bilattice with negation, where $\neg\langle a, b\rangle = \langle b, a\rangle$.

6. If $L_1 = L_2 = L$ is a non-distributive De Morgan algebra then $L \odot L$ is a bilattice with a conflation that commutes with negation, where $-\langle a, b\rangle = \langle \overline{b}, \overline{a}\rangle$, writing overbar for the De Morgan involution. (A non-distributive De Morgan algebra is like a De Morgan algebra, but distributivity is not required.)

Proof. This proof is really something of an exercise. We show the first part of item 3 as an illustration, and leave the rest to you. Assume $\langle a, b\rangle, \langle c, d\rangle \in L \odot L$.

$a \sqcap_1 c \leq_1 a$ and $b \leq_2 b \sqcup_2 d$ so $\langle a \sqcap_1 c, b \sqcup_2 d\rangle \leq_t \langle a, b\rangle$. Similarly $\langle a \sqcap_1 c, b \sqcup_2 d\rangle \leq_t \langle c, d\rangle$. So $\langle a \sqcap_1 c, b \sqcup_2 d\rangle$ is a lower bound for $\langle a, b\rangle$ and $\langle c, d\rangle$ in the $\leq_t$ ordering.

Suppose $\langle x, y\rangle \leq_t \langle a, b\rangle$ and $\langle x, y\rangle \leq_t \langle c, d\rangle$, so that $\langle x, y\rangle$ is a lower bound. Then $x \leq_1 a$ and $x \leq_1 c$, so $x \leq_1 a \sqcap c$. Similarly $b \sqcup d \leq_2 y$. Then $\langle x, y\rangle \leq_t \langle a \sqcap_1 c, b \sqcup_2 d\rangle$. So $\langle a \sqcap_1 c, b \sqcup_2 d\rangle$ is the *greatest* lower bound for $\langle a, b\rangle$ and $\langle c, d\rangle$.

It follows that $\langle a, b\rangle \wedge \langle c, d\rangle$ is $\langle a \sqcap_1 c, b \sqcup_2 d\rangle$. □

What is harder to show is that all these conditions reverse, and hence we have a completely general method for producing bilattices—we have a bilattice representation theorem. We discuss this in Section 7, after taking a fresh look at our bilattice examples.

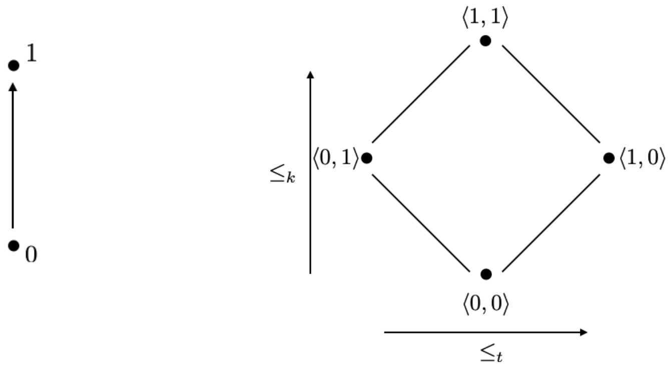
6 Examples Again

The bilattice product operation makes bilattice examples easy to come by, and indeed we have already seen some of the results. Suppose we start with the simplest, where we use the De Morgan lattice for classical logic, shown in Figure 8a.

The bilattice product of this with itself is, isomorphically, the bilattice $\mathcal{FOUR}$ from Figure 1, but in Figure 8b we have used as node labels the ones we get from the product construction. In a similar way the bilattice product of the standard three-valued lattice with itself is given in Figure 9. It is a version of what we saw in Figure 5.

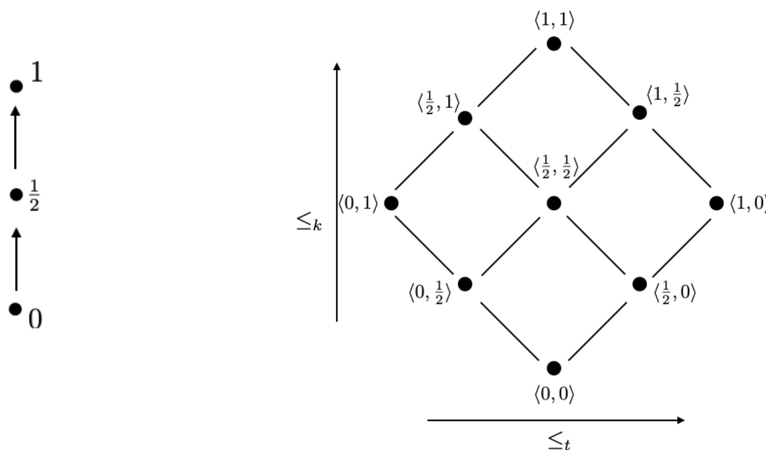
As more elaborate example, the 16 element bilattice shown in Figure 6 is the bilattice product of the bilattice $\mathcal{FOUR}$ with itself, considering $\mathcal{FOUR}$ as a lattice by using only the $\leq_t$ ordering. In the double Hasse diagram of Figure 6 we already made use of node labels corresponding to the product representation.

Recall that in Section 5 we used sets of experts as motivation. This can be made into a proper example. Let E be a finite set, informally called "experts". The



(a) The Classical Lattice (b) $\mathcal{FOUR}$ Again

Figure 8: The Prime Example



(a) Three-Valued Lattice (b) $\mathcal{NINE}$ Again

Figure 9: The Next Easiest Example

collection of all subsets of E is a lattice under the subset ordering, $\subseteq$. Then meet and join are simply intersection and union. It becomes a De Morgan lattice by using the complementation operation. If we form the bilattice product of this lattice with itself, we get exactly what was discussed in Section 5. A member of the product bilattice is a pair of sets of experts, and we can plausibly think of members of the first component as the set of those who say “yes” to something, and members of the second component as the set of those who say “no.”

Our final example has a somewhat different feel to it. Start with the lattice $[0, 1]$, the closed unit interval, with the usual ordering $\leq$, thus making meet and join be minimum and maximum. Use $x \mapsto 1 - x$ as DeMorgan involution. Form the bilattice product of this structure with itself. This is a bilattice in which each element represents a *degree of belief* and a *degree of doubt*, in a plausible sense.

7 The Representation Theorem Continued

Proposition 5.2 reverses, and thus the bilattice product is a very general piece of construction machinery. These results were proved over time, with it first being established for distributive bilattices in a succession of papers [33, 21, 22]. Finally the most general version, for interlaced bilattices, was shown in [3] and [46], essentially simultaneously. (It eventually was realized that the result already existed, using different terminology and in a purely algebraic setting. See [13] for the history of this.) Here we give a presentation based on the proof from [3]. Proposition 5.2 makes constructing bilattices easy. The Proposition below turns proofs about bilattices into fairly straightforward computations. (It is not necessary to go through the proof to make use of the result.)

Proposition 7.1 (Bilattice Representation)**.** *Let $\mathcal{B} = \langle \mathcal{B}, \leq_t, \leq_k \rangle$ be an interlaced bilattice.*

1. *There are bounded lattices $L_1 = \langle L_1, \leq_1 \rangle$ and $L_2 = \langle L_2, \leq_2 \rangle$, unique up to isomorphism, such that $\mathcal{B}$ is isomorphic to $L_1 \odot L_2$.*

2. *If $\mathcal{B}$ is a distributive bilattice, L_1 and L_2 will be distributive lattices.*

3. *If $\mathcal{B}$ has negation, we can take $L_1 = L_2$ and negation in $L_1 \odot L_2$ as $\neg\langle a, b\rangle = \langle b, a\rangle$. Negation is preserved by the isomorphism.*

4. *If $\mathcal{B}$ has negation and conflation, in addition to 3 we can take $L_1 (= L_2)$ to be a non-distributive De Morgan algebra and $-\langle a, b\rangle = \langle \overline{b}, \overline{a}\rangle$, writing overbar for the De Morgan involution. Conflation is preserved by the isomorphism.*

We begin with some observations that should help motivate the proof. Suppose we have a bilattice $L_1 \odot L_2$ that was constructed as a product of two lattices. Note that $\langle a, b\rangle \vee \bot = \langle a, b\rangle \vee \langle 0_1, 0_2\rangle = \langle a \sqcup 0_1, b \sqcap 0_2\rangle = \langle a, 0_2\rangle$. In a similar way,

$\langle a, b\rangle \wedge \perp = \langle 0_1, b\rangle$. This almost gets us the two components, a and b of the pair $\langle a, b\rangle$, and what we did get can be used as stand-ins for them. We need to reverse engineer what we just noted.

We begin with a fundamental result from [3], connecting machinery from the two bilattice orderings.

Proposition 7.2. *In an interlaced bilattice $\mathcal{B}$, for any $a \leq_t b$,*

$$a \leq_t x \leq_t b \text{ if and only if } a \otimes b \leq_k x \leq_k a \oplus b.$$

Likewise for any $a \leq_k b$,

$$a \leq_k x \leq_k b \text{ if and only if } a \wedge b \leq_t x \leq_t a \vee b.$$

Proof. Suppose $a \leq_t x \leq_t b$. By interlacing, $a \otimes (a \otimes b) \leq_t x \otimes (a \otimes b) \leq_t b \otimes (a \otimes b)$, or $a \otimes b \leq_t x \otimes (a \otimes b) \leq_t a \otimes b$. Then $x \otimes (a \otimes b) = a \otimes b$, and it follows that $a \otimes b \leq_k x$. By a similar argument using $\oplus$ instead of $\otimes$, $x \leq_k a \oplus b$.

Suppose $a \leq_t b$ and $a \otimes b \leq_k x \leq_k a \oplus b$. By interlacing, $a \wedge (a \otimes b) \leq_k a \wedge x \leq_k a \wedge (a \oplus b)$. Also since $a \leq_t b$ then again by interlacing $a \otimes a \leq_t a \otimes b$, so $a \leq_t a \otimes b$, and then $a \wedge (a \otimes b) = a$. Similarly $a \wedge (a \oplus b) = a$. Combining these, $a \leq_k a \wedge x \leq_k a$, so $a \wedge x = a$, and hence $a \leq_t x$. In a similar way, $x \leq_t b$.

The second item is similar. □

Here are some consequences, which will be of use of in proving Proposition 7.1.

Proposition 7.3. *In any interlaced bilattice,*

$$\begin{aligned} \mathbf{t} \otimes \mathbf{f} &= \perp \\ \mathbf{t} \oplus \mathbf{f} &= \top \\ \top \wedge \perp &= \mathbf{f} \\ \top \vee \perp &= \mathbf{t}, \end{aligned}$$

and for any x,

$$\begin{aligned} (x \vee \perp) \wedge \top &= x \wedge \top \\ (x \wedge \perp) \vee \top &= x \vee \top \\ (x \vee \top) \wedge \perp &= x \wedge \perp \\ (x \wedge \top) \vee \perp &= x \vee \perp \end{aligned}$$

and also

$$(x \wedge \perp) \oplus (x \vee \perp) = x.$$

Proof. We have $\mathbf{f} \le_t x \le_t \mathbf{t}$ for any x, so $\mathbf{f} \le_t \bot \le_t \mathbf{t}$. Then by Proposition 7.2, $\mathbf{f} \otimes \mathbf{t} \le_k \bot \le_k \mathbf{f} \oplus \mathbf{t}$. From the first of these two inequalities and the trivial $\bot \le_k \mathbf{f} \otimes \mathbf{t}$ we conclude $\mathbf{f} \otimes \mathbf{t} = \bot$. The next three items are similar.

Since $\bot \le_k x$, by interlacing $x \vee \bot \le_k x \vee x = x$. Then $x \vee \bot \le_k x \le_k \top$, and by the second part of Proposition 7.2 $(x \vee \bot) \wedge \top \le_t x \le_t (x \vee \bot) \vee \top$, so $(x \vee \bot) \wedge \top \le_t x$ and it follows by interlacing that $(x \vee \bot) \wedge \top \le_t x \wedge \top$. In the other direction, $x \le_t x \vee \top$, so $x \wedge \top \le_t (x \vee \bot) \wedge \top$. Then $x \wedge \top = (x \vee \bot) \wedge \top$. Again the next three items are similar.

By using interlacing, $(x \wedge \bot) \oplus (x \vee \bot) \le_k (x \wedge x) \oplus (x \vee x) = x \oplus x = x$. Also, $x \wedge \bot \le_t x \le_t x \vee \bot$ so by Proposition 7.2, $(x \wedge \bot) \otimes (x \vee \bot) \le_k x \le_k (x \wedge \bot) \oplus (x \vee \bot)$. Since we have $(x \wedge \bot) \oplus (x \vee \bot) \le_k x$ and $x \le_k (x \wedge \bot) \oplus (x \vee \bot)$ we have $x = (x \wedge \bot) \oplus (x \vee \bot)$. □

Remark 7.4. There are more results with similar proofs. For instance here is a list, given as Corollary 2.8 in [3]. In an interlaced bilattice, all of the following are equal to x: $(x \wedge \bot) \oplus (x \vee \bot)$, $(x \otimes \mathbf{f}) \oplus (x \otimes \mathbf{t})$, $(x \wedge \top) \otimes (x \vee \top)$, $(x \oplus \mathbf{f}) \otimes (x \oplus \mathbf{t})$, $(x \otimes \mathbf{f}) \vee (x \oplus \mathbf{f})$, $(x \wedge \bot) \vee (x \wedge \top)$, $(x \otimes \mathbf{t}) \wedge (x \oplus \mathbf{t})$, $(x \vee \bot) \wedge (x \vee \top)$.

Now, finally, we show the central item.

Proof. Of Proposition 7.1 Part 1. Throughout the following, $\mathcal{B} = \langle \mathcal{B}, \le_t, \le_k \rangle$ is an interlaced bilattice.

Existence: Two structures are specified, which will be shown to be bounded lattices. Recalling the remarks at the beginning of the section, let $L_1 = \langle \{x \vee \bot \mid x \in \mathcal{B}\}, \le_1 \rangle$ where $\le_1$ is $\le_t$ restricted to the set. Likewise let $L_2 = \langle \{x \wedge \bot \mid x \in \mathcal{B}\}, \le_2 \rangle$, where $\le_2$ is $\le_t$ *reversed* restricted to the set.

Lattice Structure: L_1 is a sublattice of $\mathcal{B}$ under the $\le_t$ ordering. Suppose $(x \vee \bot), (y \vee \bot) \in L_1$. We trivially have closure under $\vee$ since $(x \vee \bot) \vee (y \vee \bot) = (x \vee y) \vee \bot$, and this is in L_1. Closure under $\wedge$ is a bit more complicated. We have both $\bot \le_t x \vee \bot$ and $\bot \le_t y \vee \bot$, so $\bot \le_t (x \vee \bot) \wedge (y \vee \bot)$. Then $(x \vee \bot) \wedge (y \vee \bot) = ((x \vee \bot) \wedge (y \vee \bot)) \vee \bot$. The right hand side is in L_1 by definition, hence so is the left.

Lattices are Bounded: L_1 is a bounded lattice. Since the ordering agrees with $\le_t$, a largest member would be $\mathbf{t}$ if it were present, but it is because $\mathbf{t} = \top \vee \bot \in L_1$. The smallest member is $\bot$. It is in L_1 because it is $\bot \vee \bot$, and it is smallest because $\bot \le_t x \vee \bot$ for every $x \in \mathcal{B}$. In a similar way, L_2 is a bounded lattice; we omit the arguments.

Product and Mapping: We now have the existence of two bounded lattices L_1 and L_2, and so can form the bilattice product $L_1 \odot L_2$ (Definition 5.1). And we can define an isomorphism candidate $\theta : \mathcal{B} \to L_1 \times L_2$ by setting $\theta(x) = \langle x \vee \bot, x \wedge \bot \rangle$.

θ is an Injection: Suppose $x, y \in \mathcal{B}$ and $\theta(x) = \theta(y)$, that is, $\langle x \vee \bot, x \wedge \bot \rangle = \langle y \vee \bot, y \wedge \bot \rangle$. Then $x \vee \bot = y \vee \bot$ and $x \wedge \bot = y \wedge \bot$, so $x = (x \wedge \bot) \oplus (x \vee \bot) = (y \wedge \bot) \oplus (y \vee \bot) = y$, by Proposition 7.3. Thus we have that θ is 1-1.

θ is a Surjection: Suppose $\langle x, y \rangle \in L_1 \odot L_2$. Then $x \oplus y \in \mathcal{B}$, and it will be shown that $\theta(x \oplus y) = \langle x, y \rangle$, thus establishing surjectivity. Of course what must be shown is that $(x \oplus y) \vee \bot = x$ and $(x \oplus y) \wedge \bot = y$.

Since $x \in L_1$ then x is $a \vee \bot$ for some $a \in \mathcal{B}$, but $a \vee \bot = (a \vee \bot) \vee \bot$, so $x = x \vee \bot$. Similarly $y = y \wedge \bot$. Then $y = y \wedge \bot \leq_t \bot \leq_t x \vee \bot = x$, in short, $y \leq_t x$. Using interlacing, $x \oplus y \leq_t x \oplus x = x$, so $(x \oplus y) \vee \bot \leq_t x \vee \bot = x$. In the other direction, $\bot \leq_k x \leq_k x \oplus y$ so by Proposition 7.2, $x \leq_t (x \oplus y) \vee \bot$. We now have that $(x \oplus y) \vee \bot = x$. In a similar way, $(x \oplus y) \wedge \bot = y$, and thus θ is onto.

θ preserves both orderings:

Let $x, y \in \mathcal{B}$. We show that $x \leq_t y$ if and only if $\theta(x) \leq_t \theta(y)$. There is a similar result concerning $\leq_k$ whose proof we omit.

Suppose first that $x \leq_t y$. Then $x \vee \bot \leq_t y \vee \bot$ and $x \wedge \bot \leq_t y \wedge \bot$, so in L_1, $x \vee \bot \leq_1 y \vee \bot$ and in L_2, $y \wedge \bot \leq_2 x \wedge \bot$ (recall that for L_2 the $\leq_t$ ordering was reversed). Then $\theta(x) = \langle x \vee \bot, x \wedge \bot \rangle \leq_t \langle y \vee \bot, y \wedge \bot \rangle = \theta(y)$.

In the other direction, suppose that $\theta(x) \leq_t \theta(y)$. Then $\langle x \vee \bot, x \wedge \bot \rangle \leq_t \langle y \vee \bot, y \wedge \bot \rangle$, which means $x \vee \bot \leq_1 y \vee \bot$ and $y \wedge \bot \leq_2 x \wedge \bot$. Taking the order reversal into account, this means $x \vee \bot \leq_t y \vee \bot$ and $x \wedge \bot \leq_t y \wedge \bot$. But then using interlacing and Proposition 7.3, $x = (x \vee \bot) \oplus (x \wedge \bot) \leq_t (y \vee \bot) \oplus (y \wedge \bot) = y$.

Uniqueness Up To Isomorphism: We have shown that $\mathcal{B}$ is isomorphic to $L_1 \odot L_2$. Suppose it is also isomorphic to $M_1 \odot M_2$, where M_1 and M_2 are also bounded lattices. Of course then $L_1 \odot L_2$ and $M_1 \odot M_2$ would be isomorphic to each other. We observed, just after the statement of Proposition 7.1, that in product bilattices, $\langle a, b \rangle \vee \bot = \langle a, 0_2 \rangle$ and $\langle a, b \rangle \wedge \bot = \langle 0_1, b \rangle$. Since $L_1 \odot L_2$ and $M_1 \odot M_2$ are isomorphic, the subsets defined by $x \vee \bot$ in the two are isomorphic. In the first the set is $\{\langle a, 0_2 \rangle \mid a \in L_1\}$ (where 0_2 is the bottom of L_2), and this is clearly isomorphic to L_1. Similarly the subset of $M_1 \odot M_2$ defined by $x \vee \bot$ is isomorphic to M_1, so L_1 and M_1 are isomorphic. In a similar way L_2 and M_2 are isomorphic.

$\square$

We have just seen the central item. Verifying the rest of the representation properties is rather straightforward.

Proof. Of Proposition 7.1 Parts 2–4. Assume throughout that $\mathcal{B} = \langle \mathcal{B}, \leq_t, \leq_k \rangle$ is an interlaced bilattice, and is isomorphic to $L_1 \odot L_2$, where L_1 has as elements $\{x \vee \bot \mid x \in \mathcal{B}\}$ and ordering $\leq_1$ being $\leq_t$ restricted to the set, and L_2 is $\{x \wedge \bot \mid x \in \mathcal{B}\}$ with ordering $\leq_2$ being $\leq_t$ restricted to the set, and *reversed*.

2. *Distributivity* Suppose $\mathcal{B}$ is a distributive bilattice. We show $\langle L_1, \leq_1 \rangle$ and $\langle L_2, \leq_2 \rangle$ are distributive lattices. But this is trivial. L_1 is a sublattice of $\mathcal{B}$ under $\leq_t$, which is distributive by assumption, and similarly for L_2.

3. *Negation* Suppose $\mathcal{B}$ has a negation. It was claimed that we could take $L_1 = L_2$ but if we show that as defined, L_1 and L_2 are isomorphic lattices, the result as stated follows.

   If $x \in L_1$, $x = a \vee \bot$, so $\neg x = \neg(a \vee \bot) = \neg a \wedge \bot \in L_2$. Then the map $\eta : L_1 \to L_2$, given by $x\eta = \neg x$, is well-defined. We claim it is an isomorphism between L_1 and L_2.

   That the map is $1-1$ follows from the fact that $(x\eta)\eta = x$. If $x \in L_2$, $x = a \wedge \bot$ for some a, so $\neg x = \neg(a \wedge \bot) = \neg a \vee \bot \in L_1$. Now $(\neg x)\eta = \neg\neg x = x$, so it follows that η is onto. Finally, suppose $x, y \in L_1$. Then $x \leq_1 y \Leftrightarrow x \leq_t y \Leftrightarrow \neg y \leq_t \neg x \Leftrightarrow y\eta \leq_t x\eta \Leftrightarrow x\eta \leq_2 y\eta$. Thus η is an isomorphism between L_1 and L_2.

   Using η, $L_1 \odot L_2$ has a negation, $\neg\langle a, b \rangle = \langle b\eta, a\eta \rangle$. Now, using the bilattice isomorphism θ from the proof of part 1 of Proposition 7.1, for any $x \in \mathcal{B}$, $\neg(x\theta) = \neg\langle x \vee \bot, x \wedge \bot \rangle = \langle (x \wedge \bot)\eta, (x \vee \bot)\eta \rangle = \langle \neg x \vee \bot, \neg x \wedge \bot \rangle = (\neg x)\theta$. Thus θ preserves negation.

4. *Conflation* Suppose $\mathcal{B}$ has negation and conflation operations that commute with each other. Continuing from the previous part, instead of having $L_1 = L_2$ we assume L_1 and L_2 are isomorphic via the mapping η defined earlier. Note that if $x \in L_1$, then $x = a \vee \bot$ and so $x \vee \bot = a \vee \bot \vee \bot = a \vee \bot = x$. Now let $\mu : L_1 \to L_1$ be given by $x\mu = -\neg x \vee \bot$. We claim μ is a de Morgan complement.

By Proposition 7.3, for $x \in L_1$, $(x\mu)\mu = -\neg(-\neg x \vee \bot) \vee \bot = (x \wedge \top) \vee \bot = x$. Thus μ is an involution.

Next, suppose $x, y \in L_1$, and $x \leq_1 y$. By definition, $x \leq_t y$ so $\neg y \leq_t \neg x$, and then $-\neg y \leq_t -\neg x$. It follows that $-\neg y \vee \bot \leq_t -\neg x \vee \bot$, and thus $y\mu \leq_t x\mu$, so $y\mu \leq_1 x\mu$. Thus μ is order reversing.

Finally, $L_1 \odot L_2$ itself has a conflation, working out to $-\langle a, b\rangle = \langle (b\eta)\mu, (a\mu)\eta\rangle$. Then for any $x \in \mathcal{B}$, $-(x\theta) = (-x)\theta$ so θ preserves convlation. Here is the verifying calculation.

$$
\begin{aligned}
x\theta &= \langle x \vee \bot, x \wedge \bot\rangle \text{ so} \\
-(x\theta) &= \langle (x \wedge \bot)\eta\mu, (x \vee \bot)\mu\eta\rangle \\
&= \langle (\neg(x \wedge \bot))\mu, (-\neg(x \vee \bot) \vee \bot)\eta\rangle \\
&= \langle -\neg\neg(x \wedge \bot) \vee \bot, \neg(-\neg(x \vee \bot) \vee \bot)\rangle \\
&= \langle -(x \wedge \bot) \vee \bot, -(x \vee \bot) \wedge \bot\rangle \\
&= \langle (-x \wedge \top) \vee \bot, (-x \vee \top) \wedge \bot\rangle \\
&= \langle -x \vee \bot, -x \wedge \bot\rangle \text{ by Proposition 7.3} \\
&= (-x)\theta.
\end{aligned}
$$

□

8 The Logic of Bilattices

A bilattice is simply a particular kind of algebraic structure. In order to turn it into something characterizing a logic, one must look to the methodology of many valued logics in general. This was done in [2, 1], with significant consequences which we summarize here. At its most general, a *many valued logic* is defined semantically by giving a space of truth values, some operations on them designed to interpret the logical connectives, and a set of *designated* truth values, the good ones, so to speak. To evaluate logical formulas in such a structure we need a *valuation*, mapping atomic formulas to our space of truth values. A valuation is then extended to all formulas using the operations that interpret the logical connectives. A valuation *validates* a formula if it maps the formula to a member of the designated truth value set. If every valuation validates a formula, the formula is simply *valid* in the many-valued structure.

If we have not just a set of generalized truth values, but a (non-distributive) De Morgan algebra, we have readily available interpretations for the connectives $\wedge$, $\vee$, and $\neg$. In such a setting one commonly imposes special algebraic conditions on the set of designated truth values—specifically it should be a *prime filter*. This simply means that $a \wedge b$ should be designated exactly when both a and b are, and $a \vee b$ should be designated exactly when one of a or b are.

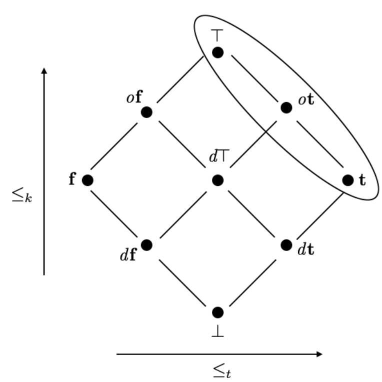
In [2, 1] this algebraic approach was extended to bilattices with negations, by a simple doubling. one uses the bilattice operations of the truth ordering, $\wedge$, $\vee$, and $\neg$ to interpret the corresponding logical connectives. And designated truth values should constitute a *prime bifilter*, essentially meaning that it should have the prime filter properties with respect to both the truth and the information orderings.

Definition 8.1. Let $\mathcal{B}$ be a blattice. A *prime bifilter* on $\mathcal{B}$ is a subset $\mathcal{F} \subseteq \mathcal{B}$ that is not empty, not the entire of $\mathcal{B}$, and that meets the following conditions. (The first two are for being a bifilter, the last two are for being prime.)

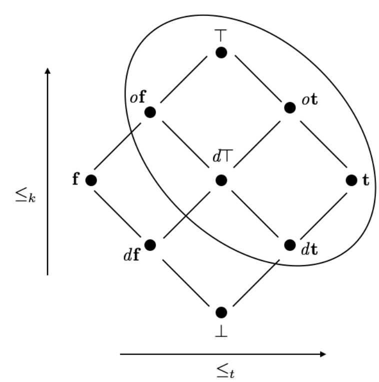
1. $(a \wedge b) \in \mathcal{F}$ if and only if $a \in \mathcal{F}$ and $b \in \mathcal{F}$
2. $(a \otimes b) \in \mathcal{F}$ if and only if $a \in \mathcal{F}$ and $b \in \mathcal{F}$
3. $(a \vee b) \in \mathcal{F}$ if and only if $a \in \mathcal{F}$ or $b \in \mathcal{F}$
4. $(a \oplus b) \in \mathcal{F}$ if and only if $a \in \mathcal{F}$ or $b \in \mathcal{F}$

A *logical bilattice* is a pair $\langle \mathcal{B}, \mathcal{F} \rangle$ where $\mathcal{F}$ is a prime bifilter on $\mathcal{B}$.

A bilattice can have more than one prime bifilter. For instance $\mathcal{NINE}$ in Figure 5 has two, as shown in Figure 10.



(a) Prime Bifilter $\mathcal{A}$



(b) Prime Bifilter $\mathcal{B}$

Figure 10: $\mathcal{NINE}$ and Two Prime Bifilters

It is helpful to see how one checks that the prime bifilters $\mathcal{A}$ and $\mathcal{B}$ are the only two that $\mathcal{NINE}$ has because some of the reasoning is of more general use.

First, for any prime bifilter $\mathcal{F}$ of any bilattice, if a is a member and $a \leq_k b$, then b is a member. This is because, if $a \leq_k b$ then $a = a \otimes b$, so if $a \in \mathcal{F}$ then $a \otimes b \in \mathcal{F}$ and hence $b \in \mathcal{F}$ by item 2 of Definition 8.1. It is similar if the ordering $\leq_t$ is involved. In short, prime bifilters are upward closed.

Since prime bifilters are upward closed, and are non-empty, $\top$ and $\mathbf{t}$ must always belong. And then $\{x \mid \top \leq_t x\}$ and $\{x \mid \mathbf{t} \leq_k x\}$ must be included. (In fact, in any interlaced bilattice these are the same set. This is a good exercise—try using the Representation Theorem.) And further, in any interlaced bilattice this set is always a subset of any prime bifilter, and is a prime bifilter by itself (again a good exercise). That is, it is always the smallest prime bifilter. For $\mathcal{NINE}$ this gives us prime bifilter $\mathcal{A}$.

Any other prime bifilter in $\mathcal{NINE}$ must extend $\mathcal{A}$. Let us try adding one more member and completing to a prime bifilter, if we can. If we add $\mathbf{f}$, upward closure of prime bifilters under $\leq_t$ would force us to add everything, but prime bifilters must not be the entire of the bilattice, so this is out. If we add either of $d\mathbf{f}$ or $\perp$, upward closure under $\leq_k$ would have us add $\mathbf{f}$, which we cannot do. So none of $\mathbf{f}$, $d\mathbf{f}$, or $\perp$ can be added to $\mathcal{A}$.

If we add any of $o\mathbf{f}$, $d\top$, or $d\mathbf{t}$ to $\mathcal{A}$, upward closure under one or another of $\leq_k$ and $\leq_t$ would force us to add all of them. So $\mathcal{B}$ is the only possible other prime bifilter for $\mathcal{NINE}$. Simple checking verifies that it is, in fact, a prime bifilter.

We examined a very specific case above. In [2, 1] one can find general results doing the job we just did as a special case. In particular they show the following. In an interlaced bilattice $\mathcal{B}$, a subset $\mathcal{F}$ is a prime bifilter if and only if it is a prime filter (in the usual algebraic sense) relative to the ordering $\leq_t$ and contains $\top$ if and only if it is a prime filter relative to the ordering $\leq_k$ and contains $\mathbf{t}$.

All the examples of logical bilattices so far have involved interlaced, or even distributive bilattices. Things are more general, though. In Figure 4 an interesting bilattice was presented that is not interlaced. It has exactly one prime bifilter, shown in Figure 11.

Figure 12 shows the simplest example of a bilattice with a prime bifilter. For reasons that will become clear shortly, this is more than just an interesting example—it is fundamental.

Each logical bilattice carries a logic with it. The logic based on the logical bilattice of Figure 12 has a significant history—it is called *First Degree Entailment*. At first it seems rather dreary, since there are no validities! That is, no formula maps to a designated value under every valuation. Here is the simple argument.

Let v be the valuation that maps every propositional letter to $\perp$. Extend v to

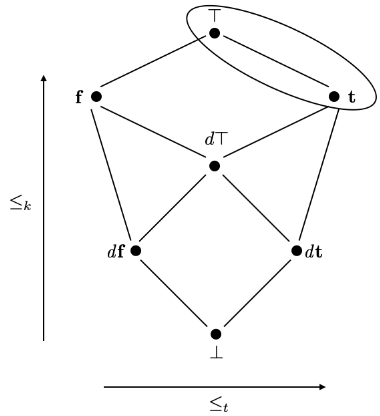


Figure 11: $\mathcal{DEFAULT}$ With Prime Bifilter

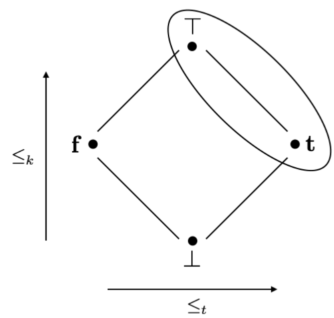


Figure 12: First Degree Entailment

all formulas using the operations $\wedge$, $\vee$, and $\neg$, as given in Figures 2 and 3. It is easy to see by induction on formula complexity that $v(X) = \bot$ for every formula X, so every formula maps to a non-designated value under v, and so is not valid.

But, validity is the wrong (that is, uninteresting) thing, and consequence is the right one. It is common to write consequence relations in sequent form, which we do here.

Definition 8.2 (Sequents)**.** A *sequent* is an ordered pair $\langle \Gamma, \Delta \rangle$ of (generally finite) sets of formulas. It is customary to write this sequent as $\Gamma \Rightarrow \Delta$, where Γ and Δ are written without enclosing curly braces. Thus, for example, $A, B, C \Rightarrow D, E$ is a sequent provided A, B, C, D, E are formulas. Commonly $\emptyset \Rightarrow \Delta$ is written as $\Rightarrow \Delta$, and similarly if Δ is $\emptyset$.

In a many-valued logic, a valuation v *validates* a sequent $\Gamma \Rightarrow \Delta$ provided, if v maps every member of Γ to a designated truth value, then v maps some member of Δ to a designated truth value. And a sequent is simply *valid* if every valuation validates it.

In particular, in a logical bilattice $\langle \mathcal{B}, \mathcal{F} \rangle$ a valuation v in $\mathcal{B}$ validates $\Gamma \Rightarrow \Delta$ provided, if v maps every member of Γ to the prime bifilter $\mathcal{F}$ then v maps some member of Δ to $\mathcal{F}$.

Now if $\langle \mathcal{B}, \mathcal{F} \rangle$ is a logical bilattice it validates a set of sequents, and we can refer to this as the logic of the logical bilattice. One can ask questions about these logics; what variation is possible. The remarkable answer is none! **They are all the same.** This is an important result from [2, 1]. In the rest of this section we present their proof that first degree entailment is the logic of every logical bilattice. Actually, it should not come as a deep surprise. There is a similar phenomenon with boolean algebras: all of them determine the same logic, namely classical logic. It does not mean the range of boolean algebras becomes uninteresting, and the same is true of the range of bilattices.

Definition 8.3. Let $\langle \mathcal{B}, \mathcal{F} \rangle$ be a logical bilattice, where $\mathcal{B}$ is a bilattice with negation, not necessarily interlaced or with conflation. Define a relation $\equiv_{\mathcal{F}}$ on $\mathcal{B}$ as follows. For $x, y \in \mathcal{B}$, $x \equiv_{\mathcal{F}} y$ if the following two items hold:

1. $x \in \mathcal{F}$ if and only if $y \in \mathcal{F}$

2. $\neg x \in \mathcal{F}$ if and only if $\neg y \in \mathcal{F}$.

The relation $\equiv_{\mathcal{F}}$ is obviously an equivalence relation, and so defines equivalence classes. There are always four of them. The following is a simple characterization, along with some interesting properties. In it, and subsequently, we write $\top_{\mathcal{B}}$ to mean the largest member of the bilattice $\mathcal{B}$ under the $\leq_k$ ordering, similarly for $\mathbf{t}_{\mathcal{B}}$, and so on.

Proposition 8.4. *Let $\langle \mathcal{B}, \mathcal{F} \rangle$ be a logical bilattice, where $\mathcal{B}$ is a bilattice with negation, not necessarily interlaced or with conflation. No two of $\top_{\mathcal{B}}$, $\bot_{\mathcal{B}}$, $\mathbf{t}_{\mathcal{B}}$, $\mathbf{f}_{\mathcal{B}}$ are equivalent using the relation $\equiv_{\mathcal{F}}$, but every $x \in \mathcal{B}$ is equivalent to one of them. Further, these members of $\mathcal{B}$ satisfy the truth tables of Figure 2 and for negation from Figure 3* up to equivalence. *(That is, the tables say that $\mathbf{f} \wedge \mathbf{t} = \bot$, while we have $\mathbf{f}_{\mathcal{B}} \wedge \mathbf{t}_{\mathcal{B}} \equiv_{\mathcal{F}} \bot_{\mathcal{B}}$.)*

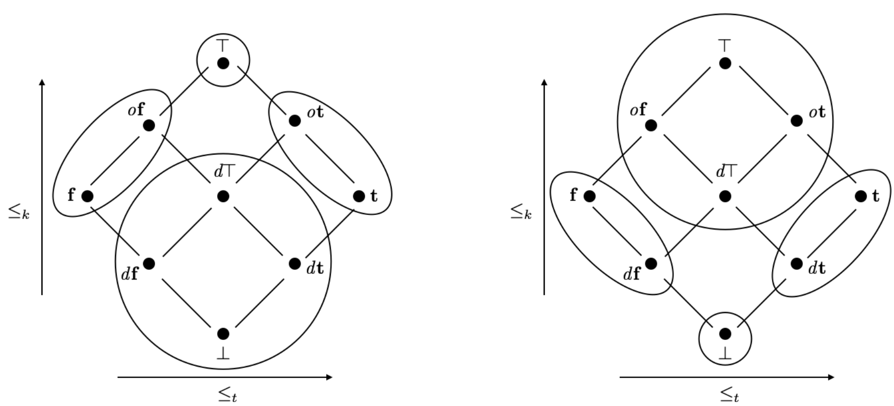
Proof. Much of the proof is essentially by case checking. First we show no two of $\top_{\mathcal{B}}$, $\bot_{\mathcal{B}}$, $\mathbf{t}_{\mathcal{B}}$, $\mathbf{f}_{\mathcal{B}}$ can be equivalent. By upward closure of prime bifilters, $\top_{\mathcal{B}}$ and

$\mathbf{t}_\mathcal{B}$ are in $\mathcal{F}$, but since a prime bifilter must be a proper subset of $\mathcal{B}$, neither $\bot_\mathcal{B}$ nor $\mathbf{f}_\mathcal{B}$ can be in $\mathcal{F}$. Then $\top_\mathcal{B}$ and $\mathbf{t}_\mathcal{B}$ cannot be equivalent to either of $\bot_\mathcal{B}$ or $\mathbf{f}_\mathcal{B}$. $\neg\top_\mathcal{B} = \top_\mathcal{B} \in \mathcal{F}$ but $\neg\mathbf{t}_\mathcal{B} = \mathbf{f}_\mathcal{B} \notin \mathcal{F}$, so $\top_\mathcal{B}$ and $\mathbf{t}_\mathcal{B}$ are not equivalent. Similarly $\neg\bot_\mathcal{B} = \bot_\mathcal{B} \notin \mathcal{F}$ but $\neg\mathbf{f}_\mathcal{B} = \mathbf{t}_\mathcal{B} \in \mathcal{F}$, so $\bot_\mathcal{B}$ and $\mathbf{f}_\mathcal{B}$ are not equivalent. This covers all the cases.

For any $x \in \mathcal{B}$ there are four possibilities. First, $x \in \mathcal{F}$ and $\neg x \in \mathcal{F}$. But then, using items just discussed, x and $\top_\mathcal{B}$ will be equivalent. Second, $x \in \mathcal{F}$ but $\neg x \notin \mathcal{F}$, in which case x and $\mathbf{t}_\mathcal{B}$ are equivalent. Third, $x \notin \mathcal{F}$, so $\neg x \in \mathcal{F}$, and x and $\mathbf{f}_\mathcal{B}$ are equivalent. And fourth, $x \notin \mathcal{F}$ and $\neg x \notin \mathcal{F}$, so x and $\bot_\mathcal{B}$ are equivalent.

Finally as a representative case we show that $\mathbf{t}_\mathcal{B} \otimes \mathbf{f}_\mathcal{B} \equiv_\mathcal{F} \bot_\mathcal{B}$, leaving the other cases to the reader. As we saw above, $\mathbf{f}_\mathcal{B} \notin \mathcal{F}$ and $\mathbf{t}_\mathcal{B} \in \mathcal{F}$. Then $\mathbf{t}_\mathcal{B} \otimes \mathbf{f}_\mathcal{B} \notin \mathcal{F}$ because $\mathbf{f}_\mathcal{B}$ is not present. Also $\neg(\mathbf{t}_\mathcal{B} \otimes \mathbf{f}_\mathcal{B}) = (\neg\mathbf{t}_\mathcal{B} \otimes \neg\mathbf{f}_\mathcal{B}) = (\mathbf{f}_\mathcal{B} \otimes \mathbf{t}_\mathcal{B}) \notin \mathcal{F}$, again because $\mathbf{f}_\mathcal{B}$ is not present. But also $\bot_\mathcal{B} \notin \mathcal{F}$, and $\neg\bot_\mathcal{B} = \bot_\mathcal{B} \notin \mathcal{F}$. Hence $\mathbf{t}_\mathcal{B} \otimes \mathbf{f}_\mathcal{B} \equiv_\mathcal{F} \bot_\mathcal{B}$. $\square$

By this proposition, every member of a logical bilattice falls into one of four equivalence classes. In Figure 13 we show the equivalence classes for the two logical bilattices from Figure 10, and in Figure 14 we show the equivalence classes for the default logical bilattice of Figure 11.



(a) $\langle \mathcal{NINE}, \mathcal{A} \rangle$ Equivalence Classes (b) $\langle \mathcal{NINE}, \mathcal{B} \rangle$ Equivalence Classes

Figure 13: $\mathcal{NINE}$ and Equivalence Classes

Suppose we write $\|x\|_\mathcal{F}$ for the equivalence class containing x. Then, very simply, the four equivalence classes of a logical bilattice $\langle \mathcal{B}, \mathcal{F} \rangle$ have the very suggestive representations: $\|\mathbf{f}_\mathcal{B}\|_\mathcal{F}$, $\|\mathbf{t}_\mathcal{B}\|_\mathcal{F}$, $\|\bot_\mathcal{B}\|_\mathcal{F}$, and $\|\top_\mathcal{B}\|_\mathcal{F}$.

Proposition 8.5. *Let $\langle \mathcal{B}, \mathcal{F} \rangle$ be a logical bilattice. Not only is $\equiv_\mathcal{F}$ an equivalence relation, it is a congruence as well. That is, for all $x, x', y, y' \in \mathcal{B}$, if $x \equiv_\mathcal{F} x'$ and*

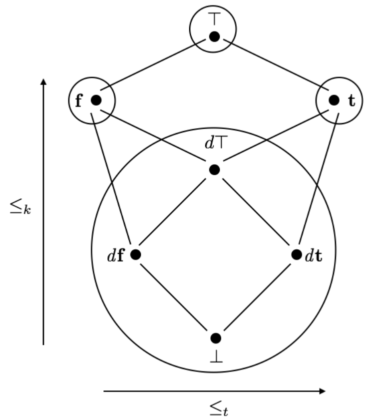


Figure 14: $\mathcal{DEFAULT}$ Equivalence Classes

$y \equiv_{\mathcal{F}} y'$ then:

$$
\begin{aligned}
x \wedge y &\equiv_{\mathcal{F}} x' \wedge y' \\
x \vee y &\equiv_{\mathcal{F}} x' \vee y' \\
x \otimes y &\equiv_{\mathcal{F}} x' \otimes y' \\
x \oplus y &\equiv_{\mathcal{F}} x' \oplus y' \\
\neg x &\equiv_{\mathcal{F}} \neg x'.
\end{aligned}
$$

Proof. We verify the $\wedge$ case. There are four possibilities, and we simply check each of them.

$x \wedge y \in \|\top_{\mathcal{B}}\|_{\mathcal{F}}$: Then $x \wedge y \in \mathcal{F}$ and $\neg(x \wedge y) \in \mathcal{F}$. Using the properties of a prime filter, from the first, $x \in \mathcal{F}$ and $y \in \mathcal{F}$, and from the second, since $\neg(x \wedge y) = \neg x \vee \neg y$, one of $\neg x \in \mathcal{F}$ or $\neg y \in \mathcal{F}$, say the first. By definition of $\equiv_{\mathcal{F}}$, $x' \in \mathcal{F}$, $y' \in \mathcal{F}$, and $\neg x' \in \mathcal{F}$. It follows that $x' \wedge y' \in \mathcal{F}$, and $\neg x' \vee \neg y' \in \mathcal{F}$, that is $\neg(x' \wedge y') \in \mathcal{F}$. Then $x \wedge y \equiv_{\mathcal{F}} x' \wedge y'$ in this case.

$x \wedge y \in \|\mathbf{t}_{\mathcal{B}}\|_{\mathcal{F}}$: Then $x \wedge y \in \mathcal{F}$ and $\neg(x \wedge y) \notin \mathcal{F}$. From the first, exactly as in the previous case, it follows that $x' \wedge y' \in \mathcal{F}$. From the second, $\neg x \vee \neg y \notin \mathcal{F}$. Using the prime bifilter properties, $\neg x \notin \mathcal{F}$ and $\neg y \notin \mathcal{F}$. Then by definition of $\equiv_{\mathcal{F}}$, $\neg x' \notin \mathcal{F}$ and $\neg y' \notin \mathcal{F}$, so $\neg x' \vee \neg y' \notin \mathcal{F}$, and so $\neg(x' \wedge y') \notin \mathcal{F}$. Then $x \wedge y \equiv_{\mathcal{F}} x' \wedge y'$ in this case too.

$x \wedge y \in \|\perp_{\mathcal{B}}\|_{\mathcal{F}}$: Similarly.

$x \wedge y \in \|\mathbf{f}_{\mathcal{B}}\|_{\mathcal{F}}$: Similarly.

The other cases are left to the reader. $\square$

We note that the result above does not extend to the conflation operation. However, it does get us the following central result.

Proposition 8.6. *Let $\langle \mathcal{B}, \mathcal{F} \rangle$ be a logical bilattice, and let* FDE *be the logical bilattice for First Degree Entailment, built on the bilattice $\mathcal{FOUR}$ and shown in Figure 12. Define a mapping $h : \mathcal{B} \to \mathcal{FOUR}$ as follows. h maps every member of $\|\mathbf{f}_{\mathcal{B}}\|_{\mathcal{F}}$ to $\mathbf{f}$, every member of $\|\mathbf{t}_{\mathcal{B}}\|_{\mathcal{F}}$ to $\mathbf{t}$, every member of $\|\bot_{\mathcal{B}}\|_{\mathcal{F}}$ to $\bot$, and every member of $\|\top_{\mathcal{B}}\|_{\mathcal{F}}$ to $\top$ (where $\mathbf{f}$, $\mathbf{t}$, $\bot$ and $\top$ are the extreme members of $\mathcal{FOUR}$). Then h is a homomorphism in the following sense. For every $x, y \in \mathcal{B}$:*

$$\begin{aligned} h(x \wedge y) &= h(x) \wedge h(y) \\ h(x \vee y) &= h(x) \vee h(y) \\ h(x \otimes y) &= h(x) \otimes h(y) \\ h(x \oplus y) &= h(x) \oplus h(y) \\ h(\neg x) &= \neg h(x). \end{aligned}$$

Further, the image of the prime bifilter $\mathcal{F}$ is the prime bifilter $\{\top, \mathbf{t}\}$ of FDE *while the complement of $\mathcal{F}$ in $\mathcal{B}$ maps to $\{\bot, \mathbf{f}\}$.*

Proof. We show the first item in the list of equalities; the rest are similar. Note that by the last part of Proposition 8.4, h is an isomorphism between the bilattice $\mathcal{FOUR}$ and $\{\top_{\mathcal{B}}, \bot_{\mathcal{B}}, \mathbf{t}_{\mathcal{B}}, \mathbf{f}_{\mathcal{B}}\}$. For instance, if we write $\bot$ for the least member of $\mathcal{FOUR}$ in the $\leq_k$ ordering and so on, we have $\mathbf{f}_{\mathcal{B}} \wedge \mathbf{t}_{\mathcal{B}} \equiv_{\mathcal{F}} \bot_{\mathcal{B}}$, so $h(\mathbf{f}_{\mathcal{B}} \wedge \mathbf{t}_{\mathcal{B}}) = h(\bot_{\mathcal{B}}) = \bot = \mathbf{f} \wedge \mathbf{t} = h(\mathbf{f}_{\mathcal{B}}) \wedge h(\mathbf{t}_{\mathcal{B}})$. Similarly for all the other cases.

Assume $x, y \in \mathcal{B}$. By Proposition 8.4, there are unique $a, b \in \{\top_{\mathcal{B}}, \bot_{\mathcal{B}}, \mathbf{t}_{\mathcal{B}}, \mathbf{f}_{\mathcal{B}}\}$ such that $x \equiv_{\mathcal{F}} a$ and $y \equiv_{\mathcal{F}} b$. By Proposition 8.5, $x \wedge y \equiv_{\mathcal{F}} a \wedge b$. From the definition of h, $h(x) = h(a)$, $h(y) = h(b)$, and $h(x \wedge y) = h(a \wedge b)$. Finally, $h(a) \wedge h(b) = h(a \wedge b)$ because h is an isomorphism on $\{\top_{\mathcal{B}}, \bot_{\mathcal{B}}, \mathbf{t}_{\mathcal{B}}, \mathbf{f}_{\mathcal{B}}\}$. It follows that $h(x \wedge y) = h(x) \wedge h(y)$.

Finally we consider the behavior of h on the prime bifilter $\mathcal{F}$. Suppose first that $x \in \mathcal{F}$. Then $h(x)$ is either $\top$ or $\mathbf{t}$ depending on whether $\neg x \in \mathcal{F}$ or not. Either way, $h(x) \in \mathcal{F}$, so h maps $\mathcal{F}$ into $\{\top, \mathbf{t}\}$. And since both $\top_{\mathcal{B}}, \mathbf{t}_{\mathcal{B}} \in \mathcal{F}$ and $h(\top_{\mathcal{B}}) = \top$ and $h(\mathbf{t}_{\mathcal{B}}) = \mathbf{t}$, h maps $\mathcal{F}$ onto $\{\top, \mathbf{t}\}$. Next, suppose $x \notin \mathcal{F}$. Then $h(x) = \bot$ or $h(x) = \mathbf{f}$ depending on whether $\neg x \in \mathcal{F}$ or not, so the complement of $\mathcal{F}$ maps to $\{\bot, \mathbf{f}\}$. $\square$

Corollary 8.7. *Let $\langle \mathcal{B}, \mathcal{F} \rangle$ and* FDE *be as in Proposition 8.6. For a valuation v in $\mathcal{B}$ define a valuation v' in $\mathcal{FOUR}$ by setting $v'(P) = h(v(P))$ for every propositional* *letter P. Extend v and v' to all formulas X built up using $\wedge$, $\vee$, and $\neg$. Then for every X, $v'(X) = h(v(X))$.*

Proof. An easy induction on the complexity of X. $\square$

Now, finally with all the work done, the connection between the logics of various bilattices, as shown in [2, 1], is easy going.

Proposition 8.8. *Let $\langle \mathcal{B}, \mathcal{F}\rangle$ be a logical bilattice. It validates the same formulas as the logical bilattice* FDE *(Figure 12).*

Proof. Throughout this proof, h is the homomorphism from $\mathcal{B}$ to $\mathcal{FOUR}$ defined in Proposition 8.6, and let $\Gamma \Rightarrow \Delta$ be any sequent.

Suppose $\Gamma \Rightarrow \Delta$ is not valid in $\langle \mathcal{B}, \mathcal{F}\rangle$, say using the valuation v. That is, $v(X) \in \mathcal{F}$ for every $X \in \Gamma$ and $v(Y) \notin \mathcal{F}$ for every $Y \in \Delta$. Let v' be the valuation in $\mathcal{FOUR}$ defined in Corollary 8.7. Then for each $X \in \Gamma$, since $v(X) \in \mathcal{F}$ we have $h(v(X)) \in \{\top, \mathbf{t}\}$ by Proposition 8.6, and since $v'(X) = h(v(X))$, we have $v'(X) \in \{\top, \mathbf{t}\}$ for all $X \in \Gamma$. Similarly we have $v'(Y) \notin \{\top, \mathbf{t}\}$ for all $Y \in \Delta$. It follows that $\Gamma \Rightarrow \Delta$ is not valid in FDE.

Suppose $\Gamma \Rightarrow \Delta$ is not valid in FDE. Then there is a valuation v' in $\mathcal{FOUR}$ such that for every $X \in \Gamma$, $v'(X) \in \{\top, \mathbf{t}\}$ and for every $Y \in \Delta$, $v'(Y) \notin \{\top, \mathbf{t}\}$. Define a valuation in $\mathcal{B}$ by setting $v(P) = \top_{\mathcal{B}}$ if $v'(P) = \top$, $v(P) = \mathbf{t}_{\mathcal{B}}$ if $v'(P) = \mathbf{t}$, and so on, for each propositional letter P. Then for each propositional letter P, $v'(P) = h(v(P))$ and hence for every formula Z, $v'(Z) = h(v(Z))$. If $X \in \Gamma$ we have $v'(X) \in \{\top, \mathbf{t}\}$, so $h(v(X)) \in \{\top, \mathbf{t}\}$. Then we must have $v(X) \in \mathcal{F}$ since h maps non-members of $\mathcal{F}$ to non-members of $\{\top, \mathbf{t}\}$. Similarly $v(Y) \notin \mathcal{F}$ for every $Y \in \Delta$. Then v does not validate $\Gamma \Rightarrow \Delta$ in $\langle \mathcal{B}, \mathcal{F}\rangle$. $\square$

9 Infinitary Operations

Throughout we have been using meets and joins with respect to the $\leq_t$ ordering of bilattices to interpret logical conjunction and disjunction. If quantification comes into it, something more is needed. In a logic model with an infinite domain quantification can be thought of as an infinite conjunction or disjunction. Up to this point meets and joins have combined two items and, by iteration, any finite number, but they can't handle infinite cases. For this, lattices must meet a special condition called completeness. We now extend earlier notation and terminology.

Definition 9.1. A lattice $\langle L, \leq\rangle$ is *complete* if the greatest lower and the least upper bound of *every* set exists (not just finite sets). For an arbitrary set, $\prod S$ is the greatest lower bound and $\bigsqcup S$ is the least upper bound of S. More specific notation will be introduced below for bilattices.

To be clear, the conditions that must be met are these. In a complete lattice, for every set S there must be members $\bigsqcap S$ and $\bigsqcup S$ such that

Lower Bound $\bigsqcap S \leq x$ for every $x \in S$;

Greatest (Lower Bound) If $b \leq x$ for every $x \in S$, then $b \leq \bigsqcap S$;

Upper Bound $x \leq \bigsqcup S$ for every $x \in S$;

Least (Upper Bound) If $x \leq b$ for every $x \in S$, then $\bigsqcup S \leq b$

A minor point: a complete lattice is automatically bounded, since the definition makes $\bigsqcap \emptyset$ the largest member, which thus must exist. Similarly $\bigsqcup \emptyset$ is the smallest member.

Definition 9.2. A pre-bilattice $\langle \mathcal{B}, \leq_t, \leq_k \rangle$ is *complete* if both lattices $\langle \mathcal{B}, \leq_t \rangle$ and $\mathcal{B}, \leq_k \rangle$ are complete as lattices. Thus in a complete pre-bilattice all meets and joins exist with respect to both orderings. In a pre-bilattice meet and join with respect to $\leq_t$ are symbolized by $\bigwedge$ and $\bigvee$, and with respect to the $\leq_k$ ordering by $\prod$ and $\sum$.

In Definition 3.3 conditions for negation and conflation were given. These carry over to complete pre-bilattices and we have the following extension of Proposition 3.4. In it we use the notation: for any set S, $\neg S = \{\neg x \mid x \in S\}$ and $-S = \{-x \mid x \in S\}$.

Proposition 9.3. *In any complete pre-bilattice with negation or conflation, for each set S,*

1. $\neg \bigwedge S = \bigvee(\neg S)$ *and* $\neg \bigvee S = \bigwedge(\neg S)$,
2. $\neg \prod S = \prod(\neg S)$ *and* $\neg \sum S = \sum(\neg S)$,
3. $-\prod S = \sum(-S)$ *and* $-\sum S = \prod(-S)$,
4. $-\bigwedge S = \bigwedge(-S)$ *and* $-\bigvee S = \bigvee(-S)$.

Proof. We show half of the first item and leave the rest to the reader. For each $x \in S$, $\bigwedge S \leq_t x$ because $\bigwedge S$ is a lower bound for S. Then $\neg x \leq_t \neg \bigwedge S$ for each $\neg x \in \neg S$, so $\neg \bigwedge S$ is an upper bound for $\neg S$. And then $\bigvee(\neg S) \leq_t \neg \bigwedge S$ because $\bigvee(\neg S)$ is the least upper bound for $\neg S$.

In the other direction, $\neg x \leq_t \bigvee(\neg S)$ for each $\neg x \in \neg S$ because $\bigvee(\neg S)$ is an upper bound for $\neg S$. Then $\neg \bigvee(\neg S) \leq_t \neg\neg x = x$, for all $x \in S$, so $\neg \bigvee(\neg S)$ is a lower bound for S. And then $\neg \bigwedge(\neg S) \leq_t \bigwedge S$ because $\bigwedge S$ is the greatest lower bound for S, so $\neg \bigwedge S \leq_t \neg\neg \bigvee(\neg S) = \bigvee(\neg S)$. □

Now Definition 3.2 is partially extended. We will have no need here for infinite distributive laws, but an infinite version of interlacing is fundamental. From now on connections between orderings will be common, and we only refer to bilattices, and not to pre-bilattices.

Definition 9.4. A bilattice $\langle \mathcal{B}, \leq_t, \leq_k \rangle$ is *infinitarily interlaced* if it is complete and all four infinitary meet and join operations are monotone with respect to both orderings, where we understand this as follows. If $\{a_i \mid i \in I\}$ and $\{b_i \mid i \in I\}$ are two indexed collections with a common indexing set I, then:

1. If $a_i \leq_t b_i$ for all $i \in I$ then
   (a) $\prod_{i \in I} a_i \leq_t \prod_{i \in I} b_i$
   (b) $\sum_{i \in I} a_i \leq_t \sum_{i \in I} b_i$
2. if $a_i \leq_k b_i$ for all $i \in I$ then
   (a) $\bigwedge_{i \in I} a_i \leq_k \bigwedge_{i \in I} b_i$
   (b) $\bigvee_{i \in I} a_i \leq_k \bigvee_{i \in I} b_i$

Obviously an infinitarily interlaced bilattice is interlaced, so earlier results apply. Making use of Proposition 9.3 part 4, Proposition 3.6 extends easily.

Proposition 9.5. *In an infinitarily interlaced bilattice with a negation and a conflation that commute, each of the consistent, anticonsistent, and exact subclasses of $\mathcal{B}$ are closed under $\bigwedge$ and $\bigvee$.*

Finally, the Representation Theorem from Sections 5 and 7 extends quite directly. We state results and omit proofs.

Proposition 9.6. *If L_1 and L_2 are bounded lattices that are complete, then $L_1 \odot L_2$ is a complete bilattice. Conversely, if $\mathcal{B}$ is a complete infinitarily interlaced bilattice, then there are complete bounded lattices, unique up to isomorphism, such that $\mathcal{B}$ is isomorphic to $L_1 \odot L_2$, where the isomorphism takes the infinitary operations into account.*

10 Lattice Fixed Point Theorems

When formulating a language containing its own truth predicate the liar sentence, that asserts its own non-truth, is a familiar stumbling block. It is common to say it should lack a truth value, or be undefined, or have some similar special status, because otherwise contradictions ensue. This is a negative approach and it would be better to be able to say which sentences do have standard truth values, rather than discover that some cannot. Semantical approaches making use of truth revision operators, and fixed points of them, have had considerable success. These originated in Kripke's very influential [39], and also in the simultaneous but less general [41]. Essentially these investigations took place in the consistent part of the bilattice $\mathcal{FOUR}$, though making use of the entire of $\mathcal{FOUR}$ was later seen to simplify things technically and to lead to interesting philosophical issues that we will not go into here (see [54]). We will look at some of this work in the next section, in the bilattice context. But here, as a preliminary, we examine two fixed point theorems for their own sakes, without consideration of philosophical applications. For this section only, we work entirely in lattices, and not in bilattices as such.

The Knaster-Tarski theorem is a remarkable example of a powerful result with a short self-contained proof. Actually it has two quite different proofs, one short and self-contained, the other longer, involving facts from set theory, and yet curiously providing more intuition. We give both proofs. The context is that of a complete lattice, Definition 9.1.

Definition 10.1. Let $\langle L, \leq \rangle$ be a lattice and let $f : L \to L$ be a mapping on the lattice. f is *monotone* if it is order preserving, that is, if $x \leq y$ then $f(x) \leq f(y)$. A *fixed point* of f is a value x such that $f(x) = x$.

Proposition 10.2 (Knaster-Tarski)**.** *Any monotone mapping on a complete lattice has a fixed point; in fact it has a smallest and a greatest fixed point.*

Actually, the full Knaster-Tarski theorem says that the set of fixed points will, itself, be a complete lattice, but this is more than we need here. We said two proofs would be given. We begin with the one from [53], which is elegant and short. Indeed, the only background needed is basic definitions of things like a lattice, and greatest lower and least upper bounds.

Proof. Version One. Let $\langle L, \leq \rangle$ be a complete lattice, and let $f : L \to L$ be monotone. We show the existence of a least fixed point. Let $C = \{x \in L \mid f(x) \leq x\}$. Since we have a complete lattice, the greatest lower bound of C exists, that is $\bigwedge C \in L$. We will show it is the least fixed point of f.

First we show C itself is closed under f. Suppose $x \in C$. Then $f(x) \le x$, so by monotonicity $f(f(x)) \le f(x)$, but this says that $f(x) \in C$.

Next we show $\bigwedge C \in C$, which we do by showing $f(\bigwedge C) \le \bigwedge C$. Suppose $x \in C$. Since $\bigwedge C$ is a lower bound for C, $\bigwedge C \le x$. By monotonicity, $f(\bigwedge C) \le f(x)$ and since $c \in C$, $f(x) \le x$, so $f(\bigwedge C) \le x$. Since x was an arbitrary member of C, this says that $f(\bigwedge C)$ is a lower bound for C. Since $\bigwedge C$ is the *greatest* lower bound, $f(\bigwedge C) \le \bigwedge C$, which is the condition for $\bigwedge C \in C$.

Since $\bigwedge C \in C$, and C is closed under f we have $f(\bigwedge C) \in C$. Since $\bigwedge C$ is a lower bound for C, $\bigwedge C \le f(\bigwedge C)$. Combining this with $f(\bigwedge C) \le \bigwedge C$ which we showed above, $f(\bigwedge C) = \bigwedge C$.

Finally we show $\bigwedge C$ is the *least* fixed point. Suppose $f(x) = x$. Then of course $f(x) \le x$ so $x \in C$, and then $\bigwedge C \le x$ since $\bigwedge C$ is a lower bound for C.

To show the existence of a greatest fixed point, let $D = \{x \in L \mid x \le f(x)\}$. Then carry out the dual of the argument above. Replace C with D, reverse all inequalities, use $\bigvee$ in place of $\bigwedge$, and talk about *greatest* fixed points instead of least fixed points. We leave the details to the reader, once again. □

The second proof provides richer intuition. It shows we can, in a certain sense, approximate to the smallest fixed point by starting at the bottom and iterating applications of the function f. Dually with the biggest fixed point. But some additional background material from set theory is required, and here we are somewhat sketchy about details, hoping the intuition is clear. One needs to make use of *ordinal numbers.* These start with the natural numbers, after which comes the smallest infinite ordinal, ω, followed by $\omega+1$, $\omega+2$ and so on. The ordinal ω is the first *limit* ordinal, in that it does not have an immediate predecessor. There are many limit ordinals beyond ω. Ordinals divide into three groups: 0 which, uniquely, is neither a limit ordinal nor a successor ordinal; successor ordinals, which are written as $\alpha+1$ where α is the immediate predecessor; and limit ordinals, which have predecessors but no immediate one. We will need an important fundamental fact about the ordinals: in set theory the collection of ordinals does not constitute a set, and so cannot be placed in a 1-1 correspondence with a set.

Proof. Version Two. Again let $\langle L, \le \rangle$ be a complete lattice (which we assume is a set), and let $f : L \to L$ be monotone. Once more we show the existence of a least fixed point, leaving to the reader the dual argument for the greatest one.

Define a function φ from ordinals to members of L, using transfinite recursion, as follows.

$$\begin{aligned}\varphi(0) &= 0_L \text{ the least member of } L\\ \varphi(\alpha+1) &= f(\varphi(\alpha)) \text{ for a successor ordinal } \alpha\\ \varphi(\lambda) &= \bigvee\{\varphi(\alpha) \mid \alpha < \lambda\} \text{ for a limit ordinal } \lambda\end{aligned}$$

The first thing we need is that this is a (weakly) increasing sequence. That is, if $\alpha < \beta$ for two ordinals then $\varphi(\alpha) \leq \varphi(\beta)$ in the ordering of L. Essentially this comes down to an induction for which we just sketch the three basic cases: 0, successor, and limit.

Trivially $\varphi(0) \leq \varphi(\alpha)$ for any α because 0_L is the smallest member of L, and hence is below anything.

Suppose $\varphi(\alpha) \leq \varphi(\alpha+1)$. Using the monotonicity of f, $f(\varphi(\alpha)) \leq f(\varphi(\alpha+1))$, and this tells us that $\varphi(\alpha+1) \leq \varphi(\alpha+2)$.

Finally, if λ is a limit ordinal and $\alpha < \lambda$ then $\varphi(\alpha) \leq \bigvee\{\varphi(\alpha) \mid \alpha < \lambda\} = \varphi(\lambda)$.

The sequence is increasing and so at some point we must repeat a value, because otherwise we would be pairing up the collection of ordinals with the members of a set, and this cannot happen. Let us say α_0 is the smallest ordinal such that $\varphi(\alpha_0) = \varphi(\beta)$ for some $\beta > \alpha_0$. Since the sequence is weakly increasing, it must be that all values from $\varphi(\alpha_0)$ to $\varphi(\beta)$ are the same. In particular, $\varphi(\alpha_0) = \varphi(\alpha_0 + 1)$. Then $\varphi(\alpha_0)$ is a fixed point because $f(\varphi(\alpha_0)) = \varphi(\alpha_0 + 1) = \varphi(\alpha_0)$.

Finally, $\varphi(\alpha_0)$ is the *least* fixed point, which can be shown as follows. Let F be any fixed point of f.

$\varphi(0) \leq F$ because $\varphi(0)$ is the least member of L.

Suppose $\varphi(\alpha) \leq F$. By monotonicity $f(\varphi(\alpha)) \leq f(F)$, and so $\varphi(\alpha+1) \leq F$ because F is a fixed point.

Suppose λ is a limit ordinal and $\varphi(\alpha) \leq F$ for every $\alpha < \lambda$. It follows that $\varphi(\lambda) = \bigvee\{\varphi(\alpha) \mid \alpha < \lambda\} \leq F$.

Then (using transfinite induction) $\varphi(\alpha) \leq F$ for every ordinal α, so in particular $\varphi(\alpha_0) \leq F$ and hence $\varphi(\alpha_0)$ must be the least fixed point. □

There is a variation of Proposition 10.2, not as well known, but one that will be of use to us later on. Instead of finding a fixed point for a function, one seeks two values between which it oscillates.

Definition 10.3. Let $\langle L, \leq\rangle$ be a lattice and let $f : L \to L$ be a mapping. f is *anti-monotone* if it is order reversing, that is, if $x \leq y$ then $f(y) \leq f(x)$. Two values $x, y \in L$ are *comparable* if $x \leq y$ or $y \leq x$. An *alternating fixpoint pair* for f is a pair of comparable values, x and y, such that $f(x) = y$ and $f(y) = x$. An alternating fixpoint pair is *extremal* if any other alternating fixpoint pair is between them.

Proposition 10.4. *Any anti-monotone mapping on a complete lattice has an extremal alternating fixpoint pair.*

Proof. Let $\langle L, \leq \rangle$ be a complete lattice, and let $f : L \to L$ be anti-monotone. Then the function f^2 defined by $f^2(x) = f(f(x))$ is clearly monotone and so has a least and a greatest fixed point, call them s and S. We note that if a is a fixed point of f^2 then $f(a)$ is also a fixed point of f^2, because $f^2(f(a)) = f(f^2(a)) = f(a)$. And further, if b is a fixed point of f^2, then $b = f(a)$ for some fixed point a of f^2. Simply take a to be $f(b)$. This is a fixed point by what we just showed, and $f(a) = f(f(b)) = f^2(b) = b$.

Now we show that s and S is an extremal alternating fixpoint pair for f.

Trivially s and S are comparable—because we have $s \leq S$.

We have that s is the smallest fixed point of f^2; we show $f(s)$ is the largest fixed point, S. Since f maps fixed points of f^2 to fixed points of f^2, $f(s)$ is a fixed point of f^2. Let b be any fixed point of f^2. Then $b = f(a)$ for some a that is a fixed point of f^2. Since s is the smallest fixed point, $s \leq a$. And since f reverses order, $f(a) \leq f(s)$, that is, $b \leq f(s)$. Since b was arbitrary, $f(s)$ is the largest fixed point of f^2, and hence $f(s) = S$. Then $f(f(s)) = f(S)$, that is, $s = f(S)$. Thus s and S are an alternating fixpoint pair for f.

Finally, suppose x and y are an alternating fixpoint pair for f. Then $f^2(x) = f(f(x)) = f(y) = x$ so x is a fixpoint of f^2. Similarly for y. Then x and y are between the least fixed point of f^2, s, and the greatest fixed point, S. Thus s and S are extremal. □

11 Fixed Point Theorems and Bilattices

In Section 10 we only looked at two lattice fixed point theorems. These are the only ones we will need here, but each takes on additional features of interest when seen in a bilattice setting, [20, 24, 26]. Applications will come in Section 12.

Proposition 11.1. *Suppose $\langle \mathcal{B}, \leq_t, \leq_k \rangle$ is a complete bilattice that is infinitarily interlaced, and $f : \mathcal{B} \to \mathcal{B}$ is a mapping that is monotone in* both *orderings. By the Knaster-Tarski theorem f has a least and greatest fixed point with respect to each ordering. Call these p_t, P_t, p_k, and P_k, where lower case indicates least and upper case indicates greatest fixed point, while the subscripts indicate the bilattice ordering* *involved. The following hold.*

$$\begin{aligned}P_t \otimes p_t &= p_k\\ P_t \oplus p_t &= P_k\\ P_k \wedge p_k &= p_t\\ P_k \vee p_k &= P_t\end{aligned}$$

Proof. Since p_t and P_t are least and greatest fixed points with respect to $\leq_t$, and p_k is a fixed point, $p_t \leq_t p_k \leq_t P_t$. By Proposition 7.2, $p_t \otimes P_t \leq_k p_k \leq_k p_t \oplus P_t$. The first of the two inequalities gives us $p_t \otimes P_t \leq_k p_k$. Also since p_k is the least fixed point under $\leq_k$, and p_t and P_t are fixed points, $p_k \leq_k p_t$ and $p_k \leq_k P_t$. But then, $p_k = p_k \otimes p_k \leq_k p_t \otimes P_t$, so we have the first equality. The other results are similar. □

The Proposition above not only has a resemblance to Proposition 7.3, it is a proper generalization of it. Let f be the identity map on interlaced bilattice $\mathcal{B}$. This is trivially monotone in both bilattice orderings, and everything is a fixed point, so the least and greatest under $\leq_t$ are $\mathbf{f}$ and $\mathbf{t}$, while least and greatest under $\leq_k$ are $\perp$ and $\top$, so Proposition 7.3 is a special case.

Proposition 11.2. *Suppose $\langle \mathcal{B}, \leq_t, \leq_k \rangle$ is a complete bilattice that is infinitarily interlaced, and $f : \mathcal{B} \to \mathcal{B}$ is a mapping that is monotone with respect to $\leq_k$ but anti-monotone with respect to $\leq_t$. Let p_k and P_k be the least and greatest fixed points with respect to $\leq_k$, and let p_t and P_t be the smaller and the larger of the extremal alternating fixpoint pair with respect to $\leq_t$. Then again the following hold.*

$$\begin{aligned}P_t \otimes p_t &= p_k\\ P_t \oplus p_t &= P_k\\ P_k \wedge p_k &= p_t\\ P_k \vee p_k &= P_t\end{aligned}$$

Proof. Very simply, the mapping f^2 is monotone with respect to both orderings, and now use Proposition 11.1. □

12 Languages with a Truth Predicate

Suppose we have a language that can 'talk about' its own syntax. Arithmetic is commonly used for this purpose, with Gödel numbering employed, and we will follow this tradition here. Details are not critical. From now on we have a first order language with propositional operators $\wedge$, $\vee$, and $\neg$ and quantifiers $\forall$ and $\exists$. Also it contains a constant symbol 0, a unary function symbol for successor, which we write as x^+, binary function symbols $+$ and $\times$ which we write in infix position, and a binary relation symbol $=$, also written in infix position. All this will be interpreted in the standard model for arithmetic, in the obvious way.

Now also suppose we add to this language a new unary predicate symbol T, with the intention of having it represent truth internally. It is here we make use of Gödel numbering and, for convenience, we make the simplifying but not necessary assumption that our Gödel numbering is onto: every number is the Gödel number of some sentence of the language of arithmetic extended with T. Since we have numerals in the language, 0, 0^+, $(0^+)^+$, ..., every Gödel number has a representation as a closed term of the language. (This happens in more than one way since, for example, $(0^+)^+$ and $(0^+ + 0^+)$ denote the same number in the standard model, but this causes no problems.)

For a sentence S in the arithmetic language extended with T, let s be any closed term of the language denoting the Gödel number of sentence S in the standard model. We abbreviate this by saying *s names S*. We want S and $\mathsf{T}(s)$ to be equivalent, that is, they should have the same truth value for every sentence S and every closed term s naming S. The well-known impediment to this is that we have enough machinery to construct a liar sentence, one that asserts its own non-truth, and our desires can't be met for such a sentence, if we use classical logic.

A familiar move at this point is to switch from two valued logic to one with more values, and announce that the liar does not have a standard truth value. In his influential paper [39] Kripke, in effect, worked with several three valued logics each having an 'undefined' third value. Of these Kleene's strong three-valued logic has been the one most commonly used subsequently by others. Graham Priest, in [44, 45] and elsewhere, argues for an 'overdefined' or 'contradictory' third value. Both approaches came together in [54] which used the logic of first degree entailment, $\mathcal{FOUR}$, having both underdefined and overdefined as values—values often called 'gaps' and 'gluts'. Bilattices generalize $\mathcal{FOUR}$, add no new complexities, and provide additional models of interest. Here we sketch how this is done, following the approach from [20].

Let $\langle \mathcal{B}, \leq_t, \leq_k \rangle$ be a complete bilattice that is infinitarily interlaced, with a negation and conflation that commute. This is intended to supply our generalized truth value space. Closed terms and sentences in the language described above will have their arithmetic features interpreted in $\mathcal{B}$ as if in the standard model for arithmetic. It is only the meaning of the T predicate that needs serious work, and for this we make use of *valuations*, which will be completely specified by their behavior with respect to T.

Definition 12.1. A $\mathcal{B}$ valuation is a mapping from closed atomic sentences of the form $\mathsf{T}(s)$ to $\mathcal{B}$ meeting the condition that if s_1 and s_2 are closed terms that designate the same number in the standard model for arithmetic, then $v(\mathsf{T}(s_1)) = v(\mathsf{T}(s_2))$.

A $\mathcal{B}$ valuation is extended to all sentences as follows, using the operations of $\mathcal{B}$ corresponding to the $\leq_t$ ordering.

1. If X is an atomic sentence not involving T, it will be a sentence of arithmetic. Set $v(X)$ to be $\mathbf{t}$ or $\mathbf{f}$ in $\mathcal{B}$ depending on whether X is true or false in the standard model for arithmetic.
2. $v(X \wedge Y) = v(X) \wedge v(Y)$
3. $v(X \vee Y) = v(X) \vee v(Y)$
4. $v(\neg X) = \neg v(X)$
5. $v((\forall x)F(x)) = \bigwedge\{v(F(t)) \mid t \text{ is a closed term}\}$
6. $v((\exists x)F(x)) = \bigvee\{v(F(t)) \mid t \text{ is a closed term}\}$

$\mathcal{B}$ is a complete, infinitarily interlaced bilattice. Valuations map the set of atomic sentences of the form $\mathsf{T}(s)$ to $\mathcal{B}$. By Proposition 4.2 and the conditions from Definition 9.4, the space of valuations is also a complete, infinitarily interlaced bilattice. The goal is to find valuations v such that for each sentence S, $v(S)$ and $v(\mathsf{T}(s))$ are the same whenever s is a closed term that names S. To this end we introduce a 'truth revision operator' that embodies an approximation condition. That is, if v is a valuation that meets our desired condition for some sentences, then applying the truth revision operator to v produces another valuation that does so for these sentences, and perhaps for more. Then we look for valuations that are unimprovable, that is, for fixed points of the truth revision operator in the space of valuations. And for this we can make use of the results in Section 11. It is the two orderings of a bilattice that provide exactly the appropriate machinery now—the truth ordering gives us what we need to evaluate formulas, as we just saw, and the information ordering gives us what we need to apply the Knaster-Tarski theorem, as we are about to see.

Definition 12.2. Let $f_{\mathcal{B}}$ be the mapping from $\mathcal{B}$ valuations to $\mathcal{B}$ valuations determined by the following. For each $\mathcal{B}$ valuation v, set $f_{\mathcal{B}}(v) = v'$ where v' is the $\mathcal{B}$ valuation such that $v'(\mathsf{T}(s)) = v(S)$, where the closed term s names the sentence S.

The key thing needed is monotonicity. That is, in the space of valuations,

$$v_1 \leq_k v_2 \Longrightarrow f_{\mathcal{B}}(v_1) \leq_k f_{\mathcal{B}}(v_2).$$

Then by the Knaster-Tarski result, Proposition 10.2, $f_{\mathcal{B}}$ has a smallest and a greatest fixed point. We leave it to you to verify monotonicity—the conditions for interlacing, infinitary interlacing, and negation give us exactly what is needed.

If we take for $\mathcal{B}$ the bilattice $\mathcal{FOUR}$, we are essentially working in the context of [54]. The space of valuations in $\mathcal{FOUR}$ has a negation and a conflation that commute, so Definition 3.5 applies. If we work in the space of valuations in $\mathcal{FOUR}$ but restrict our attention to those valuations that are consistent, we have essentially the setting for Kripke's work in [39] based on the strong Kleene three valued logic. Working with the anti-consistent part gives us the paraconsistent LP approach of Priest.

What bilattices add to the picture is a uniform treatment of a variety of generalizations of the fixed point approach to theories of truth. Here are two particularly interesting examples.

The first example is one with a continuum of truth values, as follows. We use the same language as above, an arithmetic language extended with a truth predicate. But for truth values we use the lattice whose domain is the closed unit interval $[0, 1]$ with the usual ordering $\leq$ on reals. Construct the bilattice product (Definition 5.1) of this lattice with itself. Truth values now are ordered pairs $\langle a, b\rangle$ in which a represents what mighty be called the 'degree of truth', and b the 'degree of falsehood', a kind of fuzzy bilattice. It is a complete, infinitarily interlaced bilattice with negation and conflation that commute, and hence all the machinery of the fixpoint construction applies.

The second example brings a modal operator into the picture. Assume we have a Kripke possible world frame for a modal logic. Say the set of possible worlds is $\mathcal{G}$, with $\mathcal{R}$ as the accessibility relation. (Good treatments of modal semantics can be found in [36, 30, 6] and many, many other places.) Our arithmetic language with a truth predicate is extended by the addition of two modal operators: if X is a formula so are $\Box X$ and $\Diamond X$. Let $\mathcal{B}$ be any complete, infinitarily interlaced bilattice with commuting negation and conflation. We allow closed formulas to take on truth values in $\mathcal{B}$ at possible worlds of the modal model.

We use a common domain for all worlds of the modal frame: the integers. Arithmetic operators are interpreted at each possible world as if they were in the standard model for arithmetic. As before, valuations are brought in to supply interpretations for the truth predicate, T. The difference from earlier is that valuations are now world-dependent; that is, they assign truth values to formulas *at possible worlds*. More formally, a valuation now is a mapping $v : \mathit{TruthSentences} \to (\mathcal{G} \to \mathcal{B})$, where *TruthSentences* is the set of all formulas of the form $\mathsf{T}(t)$, for a closed term t. We follow custom and for a sentence $\mathsf{T}(t)$ and a possible world w, we write $v(\mathsf{T}(t))(w)$ as $v(\mathsf{T}(t), w)$.

Since $\mathcal{B}$ is an infinitarily interlaced bilattice with commuting negation and conflation, then so is the set of functions mapping $\mathcal{G}$ to $\mathcal{B}$, and then the same again for the set of functions mapping *TruthSentences* to $\mathcal{G} \to \mathcal{B}$. That is, the set of valuations in the modal frame is, itself, an infinitarily interlaced bilattice with commuting negation and conflation.

In the modal setting, valuations are extended to all sentences using the connective and quantifier conditions given above, but relativized to possible worlds. For instance, item 2 becomes $v(X \wedge Y, w) = v(X, w) \wedge v(Y, w)$, where w is a possible world. But two more conditions must be added to take care of the modal operators.

7. $v(\Box X, w) = \bigwedge\{v(X, w') \mid w\mathcal{R}w'\}$
8. $v(\Diamond X, w) = \bigvee\{v(X, w') \mid w\mathcal{R}w'\}$

The earlier Definition 12.2 still applies, using the revised version of valuation, and it is still the case that the mapping $f_{\mathcal{B}}$ is monotone, and we still leave the proof of this to the reader. The Knaster-Tarski theorem thus applies, and we have appropriate truth assignments at possible worlds of a modal model, relative to any underlying bilattice of truth values. In particular, one could plausibly use $\mathcal{FOUR}$, or the 'fuzzy' one discussed above.

We conclude the discussion of bilattices and fixed point theories of truth with a brief look at applications of the results in Section 11. A fuller study can be found in [26, 27].

The mapping $f_{\mathcal{B}}$ from Definition 12.2, while monotone in the $\leq_k$ ordering, will not be monotone in the $\leq_t$ ordering because of the behavior of negation. But if we restrict our attention to sentences that do not involve negation we will have monotonicity with respect to both orderings. It follows that there will be least and greatest fixed points with respect to both orderings, and these will be interconnected as shown in Proposition 11.1. But of course, without negation there is much less of interest that we can say. In [26] a different approach was taken, based on stable model theory coming out of logic programming, The paper [24] is, in fact, very closely related to what we are about to present.

From now on assume all sentences are in negation normal form, that is, all occurrences of the negation symbol are at the atomic level. This is no real restriction since every sentence can have its negations 'pushed all the way in' using De Morgan's laws. It is not essential to do this, but it does make things easier to follow. Now think of occurrences of $\neg\mathsf{T}(x)$ as if they were occurrences of a new atom, a falsehood atom that can behave independently of $\mathsf{T}(x)$.

Definition 12.3 (Pseudo-Valuations)**.** Let v_1 and v_2 be valuations in the infinitarily interlaced bilattice with commuting negation and conflation, $\mathcal{B}$. A *pseudo-valuation*

$v_1 \triangle v_2$ is defined from them as follows.

$$(v_1 \triangle v_2)(\mathsf{T}(t)) = v_1(\mathsf{T}(t))$$
$$(v_1 \triangle v_2)(\neg\mathsf{T}(t)) = \neg v_2(\mathsf{T}(t))$$

Pseudo-valuations extend to all sentences in a straightforward way. Details are similar to those of Definition 12.1 and are omitted here. As we did with valuations, we use the same notation for pseudo-valuations and for their extensions to all sentences.

The truth revision operator $f_{\mathcal{B}}$ of Definition 12.2 is replaced with a two-input version, as follows.

Definition 12.4. Let v_1 and v_2 be valuations in the infinitarily interlaced bilattice with commuting negation and conflation $\mathcal{B}$. $g_{\mathcal{B}}(v_1, v_2) = v'$ where v' is the $\mathcal{B}$ valuation such that $g_{\mathcal{B}}(v_1, v_2)(\mathsf{T}(s)) = (v_1 \triangle v_2)(S)$, where the closed term s names the sentence S.

The basic facts about this new mapping follow. Proofs are omitted here but can be found in [32, 19].

1. $g_{\mathcal{B}}$ is monotone in both inputs, under $\leq_k$: $v_1 \leq_k v_2$ and $w_1 \leq_k w_2$ imply $g_{\mathcal{B}}(v_1, w_1) \leq_k g_{\mathcal{B}}(v_2, w_2)$.
2. $g_{\mathcal{B}}$ is monotone in its first input, under $\leq_t$: $v_1 \leq_t v_2$ implies $g_{\mathcal{B}}(v_1, w) \leq_t g_{\mathcal{B}}(v_2, w)$.
3. $g_{\mathcal{B}}$ is anti-monotone in its second input, under $\leq_t$: $w_1 \leq_t w_2$ implies $g_{\mathcal{B}}(v, w_1) \geq_t g_{\mathcal{B}}(v, w_2)$.

This two input mapping is really auxilliary to the following, which is central.

Definition 12.5. The *derived operator* of $g_{\mathcal{B}}$ is the single input function $g'_{\mathcal{B}}$ defined by: $g'_{\mathcal{B}}(v)$ is the smallest fixed point, in the $\leq_t$ ordering, of the function $(\lambda x)g_{\mathcal{B}}(x, v)$.

The smallest fixed points required in the definition above exist because of item 2 in the list of properties of $g_{\mathcal{B}}$ above. Since we use the *smallest* fixed point instead of the largest, an explicit bias towards falsehood has been introduced, and thus there is a close relation with the work in [55]. It can be shown that $g'_{\mathcal{B}}$ is monotone in the $\leq_k$ ordering, and so this is another candidate for a truth revision operator. In fact it can be shown that its fixed points are among those of the mapping $f_{\mathcal{B}}$, defined earlier. It can also be shown that $g'_{\mathcal{B}}$ is *anti-monotonic* in the $\leq_t$ ordering, and so Proposition 11.2 applies.

We have now seen some glimpses of the intricacies that can be discovered through the use of bilattices. A discussion of the meaning that might be attached to the work sketched above is more appropriate to a philosophical discussion. The mathematical possibilities are what concern us here, these have been presented, and we take things no further.

13 Whither From Here?

Unknown to the early bilattice community, some of the fundamental work already existed, see **History** below. We have not attempted to straighten this complex story out here.

We have seen bilattices as generalizations of the structure for first degree entailment, FDE, see **Logic** below. That important four valued structure contains natural substructures appropriate for Kleene's strong three valued logic and Priest's logic of paradox. We have seen that it is also possible to define the operations of Kleene's weak three valued logic, and of supervaluations. All this extends to interlaced bilattices in general. We have also seen how FDE also extends to the family of all logical bilattices. While that turned out to be not an extension at all, a negative result in a sense, it can also be considered a positive result in that it gives us a much richer set of models for first degree entailment. Of course generalizations of fixed point theories of truth are prime examples of bilattices applied to logic constructs.

Work has not stopped, of course, though this chapter does. Here are some references for further reading, which in turn will provide further references. The citations are chronological within each category. Some citations are included that appeared earlier in the chapter, but many occur here for the first time.

Artificial Intelligence: Applications in what would seem like a promising area are, in fact, rather sparse. One wonders to what extent this is a consequence of the current success of deep learning algorithms, with the general role of logical reasoning pushed aside: [33] [34] [35] [51].

Logic: This area has been notable for multiple applications of bilattices. Connections with, and generalizations of, the familiar logics of Kleene, Priest, and others, were to be expected. But as we saw, bilattices provided a natural setting for Kripke style fixed point theories of truth, allowing self reference. Of course the work of Arieli and Avron has provided coherence to the entire bilattice family by showing that $\mathcal{FOUR}$ plays a role in the interlaced bilattice family analogous to the one that the two element Boolean algebra plays in the family of Boolean algebras. Indeed, logic has been a major user of the bilattice structure: [20] [23] [25] [2] [26] [1] [27] [7] [8] [12] [47] [43] [4] [16] [52] [28] [29].

Algebra: One is tempted to say that, since lattices are a big area of research, bilattices should be twice as big. (That was meant to be a bit silly.) But in fact, bilattices have a substantial history of interest within the general lattice theoretic community. For instance the representation results, covered here starting in Section 5, have been shown in more than one way, and notably generalized as well. The following are fundamental, but not exhaustive: [3] [46] [42] [37] [14] [10] [9] [15] [11].

Logic Programming: The area of logic programming, after an initial period of general interest in the 1980's, became a niche subject. Still, it raised interesting questions, in particular concerning its 'negation as failure' construct. This led to a fixpoint treatment that had notable similarities with Kripke's work on the theory of truth. One offshoot of logic programming that is still active is that of *answer set programming*. Here so-called *stable* models are central, and these play a role both in logic programming and in continuations of Kripke's work on truth predicates where self reference is allowed: [21] [24] [40].

History: History is partly decided by who writes it. Bilattices actually have a complex history, and the following will give some idea of how complex: [31] [13].

Trilattices and Multilattices: A perhaps reasonable question about bilattices is, why stop at dimension two. Well, people have gone beyond. In trilattices, which have three axes, typically one of them represents degree of constructivity, but this is not all that is possible. This is currently very much in development: [49] [50] [48] [18].

References

[1] Ofer Arieli and Arnon Avron. The value of four values. *Artificial Intelligence*, 102:97–141, 1998.

[2] — Reasoning with logical bilattices. *Journal of Logic, Language, and Information*, 5(1):25–63, 1996.

[3] Arnon Avron. The structure of interlaced bilattices. *Mathematical Structures in Computer Science*, 6(3):287–299, 6 1996.

[4] Zeinab Bakhtiarinoodeh and Umberto Rivieccio. Epistemic updates on bilattices. In Wiebe van der Hoek, Wesley H. Holliday, and Wen-Fang Wang, editors, *Logic, Rationality, and Interaction*, pages 426–428, Berlin, Heidelberg, 2015. Springer Berlin Heidelberg.

[5] Nuel D. Belnap, Jr. A useful four-valued logic. In Jon Michael Dunn and George Epstein, editors, *Modern Uses of Multiple-Valued Logic*, pages 5–37. Springer Netherlands, 1977.

[6] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic*. Tracts in Theoretical Computer Science. Cambridge University Press, Cambridge, UK, 2001.

[7] Félix Bou and Umberto Rivieccio. The logic of distributive bilattices. *Logic Journal of IGPL*, 19(1):183–216, February 2011.

[8] —Bilattices with implications. *Studia Logica*, 101(4):651–675, 2013.

[9] Leonardo Manuel Cabrer, A. P. K. Craig, and Hilary Priestley. Product representation for default bilattices. *Journal of Pure and Applied Algebra*, 219(7):2962–2988, 2015.

[10] Leonardo Manuel Cabrer and Hilary Priestley. A general framework for product representations: bilattices and beyond. *Logic Journal of IGPL*, 23:816–841, 2015.

[11] — Natural dualities through product representations: bilattices and beyond. *Studia Logica*, 104:567–592, 2016.

[12] Genito Daniele and Gerla Giangiacomo. Connecting bilattice theory with multivalued logic. *Logic and Logical Philosophy*, 23(1):15–45, 2014.

[13] Brian Davey. The product representation theorem for interlaced pre-bilattices: Some historical remarks. *Algebra Universalis*, 70:403–409, 2013.

[14] Diana Davidova and Yuri Movsisyan. Weakly idempotent lattices and bilattices, non-idempotent Plonka functions. *Demonstratio Mathematica*, 48(4):509–535, 2015.

[15] —A set-theoretical representation for weakly idempotent lattices and interlaced weakly idempotent bilattices. *European Journal of Mathematics*, 2(3):853–873, 2016.

[16] Thomas Ferguson. Cut-down operations on bilattices. In *2015 IEEE International Symposium on Multiple-Valued Logic*, pages 24–29, 2015.

[17] — *The Proscriptive Principle and Logics of Analytic Implication*. Philosophy, Graduate Center, City University of New York, 2017.

[18] —Rivals to Belnap-Dunn logic in interlaced trilattices. *Studia Logica*, 105(6):1123–1148, December 2017.

[19] Kit Fine. The justification of negation as failure. In Jens Eric Fenstad, Ivan T. Frolov, and Risto Hilpinen, editors, *Logic, Methodology and Philosophy of Science VIII*, pages 263–301, Amsterdam, 1989. North-Holland.

[20] Melvin C. Fitting. Bilattices and the theory of truth. *Journal of Philosophical Logic*, 18:225–256, 1989.

[21] —Bilattices in logic programming. In George Epstein, editor, *The Twentieth International Symposium on Multiple-Valued Logic*, pages 238–246. IEEE, 1990.

[22] —Bilattices and the semantics of logic programming. *Journal of Logic Programming*, 11:91–116, 1991.

[23] —Kleene's logic, generalized. *Journal of Logic and Computation*, 1:797–810, 1992.

[24] —The family of stable models. *Journal of Logic Programming*, 17:197–225, 1993.

[25] —Kleene's three-valued logics and their children. *Fundamenta Informaticae*, 20:113–

131, 1994.

[26] —A theory of truth that prefers falsehood. *Journal of Philosophical Logic*, 26:477–500, 1997.

[27] —Bilattices are nice things. In Thomas Bolander, Vincent Hendricks, and Stig Andur Pederrsen, editors, *Self-Reference*, chapter 3, pages 53–77. Center for the Study of Language and Information, 2006.

[28] —The strict/tolerant idea and bilattices. In Ofer Arieli, Beata Konikowska, Alexander Rabinovich, and Anna Zamansky, editors, *For Arnon Avron*, Outstanding Contributions to Logic. Springer, 2020 (forthcoming).

[29] —A family of strict/tolerant logics. *Journal of Philosophical Logic*. Online, print version forthcoming, 2020.

[30] Melvin C. Fitting and Richard Mendelsohn. *First-Order Modal Logic*. Kluwer, 1998. Paperback, 1999. Errata at `http://melvinfitting.org/errata/errata.html`.

[31] George Gargov. Knowledge, uncertaity and ignorance in logic: Bilattices and beyond. *Journal of Applied Non-Classical Logics*, 9(2-3):195–283, 1999.

[32] Michael Gelfond and Vladimir Lifschitz. The stable model semantics for logic programming. In Robert Kowalski and Kenneth Bowen, editors, *Proc. of the Fifth Logic Programming Symposium*, pages 1070–1080, Cambridge, MA, 1988. MIT Press.

[33] Matthew L. Ginsberg. Multivalued logics: a uniform approach to reasoning in artificial intelligence. *Computational Intelligence*, 4:265–316, 1988.

[34] —Bilattices and modal operators. In *Proceedings of the Third Conference on Theoretical Aspects of Reasoning about Knowledge*, pages 273–287, San Francisco, CA, 1990. TARK, Morgan Kaufmann.

[35] —Bilattices and modal operators. *Journal of Logic and Computation*, 1(1):41–69, 1990.

[36] George Edward Hughes and Max J. Cresswell. *A New Introduction to Modal Logic*. Routledge, London, 1996.

[37] Achim Jung and Umberto Rivieccio. Priestley duality for bilattices. *Studia Logica*, 100:223–252, 2012.

[38] Stephen C. Kleene. *Introduction to Metamathematics*. D. Van Nostrand, Princeton, NJ, 1950.

[39] Saul Kripke. Outline of a theory of truth. *The Journal of Philosophy*, 72(19):690–716, 1975. Reprinted in *New Essays on Truth and the Liar Paradox*, R. L. Martin, ed., Oxford (1983).

[40] Yann Loyer and Umberto Straccia. Epistemic foundation of the well-founded semantics over bilattices. In Fiala J., Koubek V., and Kratochvíl J., editors, *Mathematical Foundations of Computer Science 2004. MFCS 2004*, volume 3153 of *Lecture Notes in Computer Science*, pages 513–524, Berlin, Heidelberg, 2004. International Symposium on Mathematical Foundations of Computer Science, Springer.

[41] Robert L. Martin and Peter W. Woodruff. On representing "true-in-L" in L. *Philosophia*, 5:217–221, 1975. Reprinted in *Recent Essays on Truth and the Liar paradox*, R. L. Martin editor, Oxford, 1984.

[42] Yuri Movsisyan. Interlaced, modular, distributive and boolean bilattices. *Armenian Journal of Mathematics*, 1(3):7–13, 2007.

[43] Sergei Odintsov and Heinrich Wansing. The logic of generalized truth values and the logic of bilattices. *Studia Logica*, 103(1):91–112, 2014.

[44] Graham Priest. The logic of paradox. *Journal of Philosophical Logic*, 8:219–241, 1979.

[45] —Semantic closure. *Studia Logica*, 43:117–129, 1984.

[46] Alexej P. Pynko. Regular bilattices. *Journal of Applied Non-Classical Logics*, 10:93–111, 2000.

[47] Umberto Rivieccio. Bilattice public announcement logic. In Rajeev Goré and Agi Kurucz, editors, *Advances in Modal Logic*, number 10, pages 459–477, 2014.

[48] Yaroslav Shramko, Jon Michael Dunn, and Tatsutoshi Takenaka. The trilattice of constructive truth values. *Journal of Logic and Computation*, 11(6):761–788, 2001.

[49] Yaroslav Shramko and Heinrich Wansing. Some useful 16-valued logics: how a computer network should think. *Journal of Philosophical Logic*, 34(121-153), 2005.

[50] — Hypercontradictions, generalized truth values, and logics of truth and falsehood. *Journal of Logic, Language, and Information*, 15(403-424), 2006.

[51] Kwang Mong Sim. Bilattices and reasoning in artificial intelligence: Concepts and foundations. *Artifical Intelligence Review*, 15(3):219–240, May 2001.

[52] Damian J. Szmuc. Track-down operations on bilattices. In Robert Wille and Martin Lukac, editors, *Proceedings of the 48th IEEE International Symposium on Multiple-Valued Logic*, pages 74–79, 2018.

[53] Alfred Tarski. A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics*, 5:285–309, 1955.

[54] Albert Visser. Four valued semantics and the liar. *Journal of Philosophical Logic*, 13:181–212, 1984.

[55] Stephen Yablo. Truth and reflection. *Journal of Philosophical Logic*, 14:297–349, 1985.

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Morphisms and Duality for Polarities and Lattices with Operators

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Abstract

Structures based on polarities provide relational semantics for propositional logics that are modelled algebraically by non-distributive lattices with additional operators. This article develops a first order notion of morphism between polarity-based structures that generalises the theory of bounded morphisms (or p-morphisms) for Boolean modal logics. It defines a category of such structures that is contravariantly dual to a given category of lattice-based algebras whose additional operations preserve either finite joins or finite meets. Two non-equivalent versions of the Goldblatt-Thomason theorem are derived in this setting.

1 Introduction and Overview

Duality in mathematics is not a theorem, but a "principle".

Michael Atiyah [1]

We develop here a new definition of 'bounded morphism' between certain structures that model propositional logics lacking the distributive law for conjunction and disjunction. Our theory adapts a well known semantic analysis of modal logic, which we now review.

There are two main types of semantical interpretation of propositional modal logics. In *algebraic* semantics, formulas of the modal language are interpreted as elements of a modal algebra $(\mathbb{B}, f)$, which comprises a Boolean algebra $\mathbb{B}$ with an additional operation f that interprets the modality $\Diamond$ and preserves finite joins. In *relational* semantics, formulas are interpreted as subsets of a Kripke frame (X, R), which comprises a binary relation R on a set X.

The relationship between these two approaches is explained by a *duality* that exists between algebras and frames. This is fundamentally category-theoretic in nature. The modal algebras are the objects of a category **MA** whose arrows are the standard algebraic homomorphisms. The Kripke frames are the objects of a category **KF** whose arrows are the *bounded morphisms*, $\alpha\colon (X,R) \to (X',R')$, i.e. functions $\alpha\colon X \to X'$ satisfying the 'back and forth' conditions

$$\text{(Forth):}\quad xRy \text{ implies } \alpha(x)R'\alpha(y). \tag{1.1}$$

$$\text{(Back):}\quad \alpha(x)R'z \text{ implies } \exists y(xRy \;\&\; \alpha(y)=z). \tag{1.2}$$

(Bounded morphisms are also known as p-morphisms. The adjective 'bounded' derives from the R-bounded existential quantification in (1.2).)

Each Kripke frame $\mathcal{F} = (X,R)$ has the dual modal algebra $\mathcal{F}^+ = (\mathscr{P}X, f_R)$ comprising the Boolean algebra of all subsets of X, with the additional operation f_R defined for all $A \subseteq X$ by

$$f_R A = \{x \in X : \exists y(xRy \;\&\; y \in A)\}.$$

Each modal algebra $\mathbb{A} = (\mathbb{B}, f)$ has the dual frame $\mathbb{A}_+ = (X_{\mathbb{B}}, R_f)$, where $X_{\mathbb{B}}$ is the set of ultrafilters of $\mathbb{B}$, and $xR_f y$ iff $\{f(a) : a \in y\} \subseteq x$. There is an injective homomorphism of $\mathbb{A}$ into $(\mathbb{A}_+)^+$ that acts by $a \mapsto \{x \in X_{\mathbb{B}} : a \in x\}$, extending the Stone representation of Boolean algebras to modal algebras. The mappings $\mathcal{F} \mapsto \mathcal{F}^+$ and $\mathbb{A} \mapsto \mathbb{A}_+$ form the basis of a pair of functors, from **KF** to **MA** and from **MA** to **KF** respectively, that are contravariant, i.e. direction-reversing. Each homomorphism $\theta\colon \mathbb{A} \to \mathbb{A}'$ induces a dual bounded morphism $\theta_+\colon \mathbb{A}'_+ \to \mathbb{A}_+$, while each bounded morphism $\alpha\colon \mathcal{F} \to \mathcal{F}'$ induces a dual homorphism $\alpha^+\colon (\mathcal{F}')^+ \to \mathcal{F}^+$. These induced maps act by forming preimages under θ and α respectively.

Composing the two functors produces objects of logical significance. The double dual algebra $(\mathbb{A}_+)^+$ is known as the *canonical extension* of $\mathbb{A}$, a construction first introduced by Jónsson and Tarski [38] for Boolean algebras with any number of *operators:* finitary operations that preserve joins in each coordinate. They proved that any completely join preserving n-ary operator f on $(\mathbb{A}_+)^+$ is determined by an $n+1$-ary relation S_f on the structure $\mathbb{A}_+$, and showed that many equationally definable properties of f correspond to first-order definable properties of S_f. This correpondence between algebras and relational structures provides tools for the semantic analysis of a range of logics with modalities. One key to this is that if $\mathbb{A}$ is the Lindenbaum-Tarski algebra for a modal logic, then $\mathbb{A}_+$ is isomorphic to the *canonical frame* for the logic whose points are maximally consistent sets of formulas [3, Theorem 5.42].

The study of canonical extensions has now evolved well beyond the Boolean situation. Gehrke and Jónsson extended it to distributive lattice expansions [20, 21, 22]. Then Gehrke and Harding [18] gave an abstract algebraic definition of the canonical extension of any algebra $\mathbb{A}$ that is based on a bounded lattice. They proved the uniqueness of this extension up to isomorphism and constructed it as an algebra of 'stable' subsets of a *polarity*, a structure (X, Y, R) comprising a binary relation R between two sets X and Y. Thus the polarity becomes the dual structure $\mathbb{A}_+$ of $\mathbb{A}$, and the canonical extension of $\mathbb{A}$ is the double dual $(\mathbb{A}_+)^+$, which is the stable set lattice of $\mathbb{A}_+$. Polarities are called *(formal) contexts* in Formal Concept Analysis [16]. The term 'polarity' is itself of geometric origin, as we explain in Remark 2 below.

Polarities with additional relational structure to represent additional algebraic operations have been used by Gehrke and co-workers to provide relational semantics for several logical systems, including the logic of residuated lattices [12, 17], the Lambek-Grishin calculus [6] and linear logic [10], with canonical extensions playing a central role. There have also been applications to logics with unary modalities [7, 8]. A fuller overview of the history of canonical extensions is given in the introduction to [30].

Our objective here is to develop a new kind of morphism from a polarity $P = (X, Y, R)$ to a polarity $P' = (X', Y,' R')$ that can accomodate expansion of the polarities by additional relational structure, and which provides a dual morphism θ_+ for any homomorphism θ of lattices with operators. Like (1.1) and (1.2), the definition of morphism we will use is *first order* relative to the structures involved, i.e. it quantifies only over elements of the structures and not over any higher order entities like subsets or sets of subsets.

The literature already contains several proposals for a notion of morphism between polarities. Erné [13, 14] investigated context morphisms as pairs of functions of the form $\alpha\colon X \to X'$ and $\beta\colon Y \to Y'$, and constructed functors between some categories of complete lattices with complete homomorphisms and categories of contexts with morphism-pairs having various properties. One of these, *concept continuity*, is equivalent to our notion of morphism for polarities without additional structure, as we explain in Remark 12. Hartung [36] studied mapping pairs α, β between contexts with topological structure, and used them to obtain duals for *surjective* homomorphisms θ. In [35] he obtained duals of arbitrary lattice homomorphisms by taking a morphism to be a pair of 'multivalued functions', binary relations forming subsets of $X \times X'$ and $Y \times Y'$. Hartonas and Dunn [34] defined morphisms as certain pairs of continuous functions between polarities with additional topological and partially ordered structure that characterises them as the duals of lattices (see Remark 12). There has also been work on polarity morphisms as pairs of subsets of $X \times Y'$ and $Y \times X'$ that provide duals of completely join preserving homomorphisms [12, 17, 9]. Jipsen [37] discusses a notion due to M. A. Moshier of a context morphism as a subset of $X \times Y'$, for which R itself is the identity morphism on P. Gehrke and van

Gool [25] studied polarity morphisms as pairs of functions satisfying back and forth properties similar to the modal frame conditions (1.1) and (1.2), showing that they give duals for lattice homomorphisms that preserve finite sets whose join distributes over meets, and ones whose meet distributes over joins.

Here we define a bounded morphism between polarity structures to be a pair α, β of functions that have back and forth properties that look different to conditions (1.1) and (1.2), and in fact are similar to what would result from those conditions if the relations R and R' were replaced by their *complements*. For instance we use the reflection (back) condition

$$\alpha(x)R'\beta(y) \text{ implies } xRy,$$

in place of the preservation (forth) condition (1.1). The motivation for this approach comes from earlier work of the author [27] in transforming polarity-style models of orthologic into Kripke models of modal logic by replacing the polarity relation by its complement. Thus, at least for 'ortho-polarities', the bounded morphisms we use are essentially equivalent to the modal bounded morphisms of their transforms. This is explained in more detail in Remark 9.

The new notion of bounded morphism allows us to carry out the kind of programme that was sketched above for the categories **MA** of modal algebras and **KF** of Kripke frames. We construct contravariant functors between a category Ω-**Lat** of homomorphisms between lattices with additional operators (and dual operators) and a category Ω-**Pol** of bounded morphisms between polarities with additional $n+1$-ary relations corresponding to additional n-ary lattice operations. Bounded morphisms also gives rise to a notion of P being an *inner substructure* of P', meaning that P is a substructure of P' for which the inclusions $X \hookrightarrow X'$ and $Y \hookrightarrow Y'$ form a bounded morphism. It is shown that the image of a bounded morphism is an inner substructure of its codomain (Corollary 19). Moreover, the dual of a surjective homomorphism is a bounded morphism whose domain is isomorphic to its image (Theorem 25).

On the other side of the duality to $(\mathbb{A}_+)^+$ is the double dual $(\mathcal{F}^+)_+$ of a frame $\mathcal{F}$, which we also call the *canonical extension* of $\mathcal{F}$. It plays a central role in a definability result from [32], generally known as the Goldblatt-Thomason theorem, which addresses the question of when a class of frames is definable by modal formulas. Here we consider the corresponding question for a class $\mathcal{S}$ of polarity-based structures and show in Theorem 38 that if $\mathcal{S}$ is closed under canonical extensions, then it is equal to the class $\{P : P^+ \in \mathscr{V}\}$ of all structures whose dual algebras belong to some equationally definable class of algebras $\mathscr{V}$ if, and only if, $\mathcal{S}$ reflects canonical extensions and is closed under images of bounded morphisms, inner substructures and direct sums. The direct sum construction, introduced for polarities by Wille

[42, 43], performs the same function here that disjoint unions perform for Kripke frames, namely it is dual to the formation of direct products of stable set lattices. We note that it is also a coproduct in the category **Ω-Pol** that we define.

The original definability theorem from [32] was concerned with modal definability of *first-order* definable classes of frames, and its proof used the fact that any frame $\mathcal{F}$ has an elementary extension $\mathcal{F}^*$ that can be mapped surjectively onto $(\mathcal{F}^+)_+$ by a bounded morphism. This $\mathcal{F}^*$ can be taken to be an ultrapower of $\mathcal{F}$, so the theorem's hypothesis can be taken to be that $\mathcal{S}$ is closed under ultrapowers. Along with closure under images of bounded morphisms this then yields the required closure under canonical extensions. Here we adapt the construction to polarities and find that there is a divergence from the modal case: the bounded morphism $P^* \to (P^+)_+$ may not have the surjectivity required for this argument. But it does have a weaker property that allows a modified proof that $\mathcal{S}$ is closed under canonical extensions. We call this property *maximal covering* (briefly: the points of the X-part of $(P^+)_+$ are the filters of P^+ and the image of a maximal covering morphism includes any filter that is maximally disjoint from some ideal). Thus we obtain a different definability characterisation (Theorem 39) for a class $\mathcal{S}$ that is closed under ultrapowers, in which closure under codomains of maximal covering morphisms replaces closure under images of bounded morphisms. An example is provided to show that this change is essential.

At the end of the article we briefly indicate how the theory can be extended to *quasi*-operators, functions that in each coordinate either preserve joins or change meets into joins.

2 Polarities and Stable Set Lattices

We assume that all lattices dealt with have universal bounds, and view them as algebras of the form $(\mathbb{L}, \wedge, \vee, 0, 1)$, with binary operations of meet $\wedge$ and join $\vee$, least element 0 and greatest element 1. The partial order of a lattice is denoted $\leqslant$, and the symbols $\bigvee$ and $\bigwedge$ are used for the join and meet of a set of elements, when these exist. If they exist for all subsets, the lattice is *complete*.

A *polarity* is a structure $P = (X, Y, R)$ having $R \subseteq X \times Y$. For $A \subseteq X$ and $B \subseteq Y$, write ARB if xRy holds for all $x \in A$ and $y \in B$. Abbreviate $AR\{y\}$ to ARy and $\{x\}RB$ to xRB. Define

$$\rho_R A = \{y \in Y : ARy\}, \quad \lambda_R B = \{x \in X : xRB\}.$$

The operations ρ_R and λ_R are inclusion reversing: $A \subseteq A'$ implies $\rho_R A' \subseteq \rho_R A$, and likewise for λ_R. They also have $A \subseteq \lambda_R \rho_R A$ and $B \subseteq \rho_R \lambda_R B$, so form a *Galois* *connection* between the posets $(\mathscr{P}(X), \subseteq)$ and $(\mathscr{P}(Y), \subseteq)$. They satisfy the 'De Morgan laws'

$$\rho_R \bigcup \mathcal{C} = \bigcap \{\rho_R A : A \in \mathcal{C}\}, \qquad \lambda_R \bigcup \mathcal{C} = \bigcap \{\lambda_R B : B \in \mathcal{C}\}, \tag{2.1}$$

but not the corresponding laws with $\bigcup$ and $\bigcap$ interchanged.

The composite operations $\lambda_R \rho_R$ on $\mathscr{P}(X)$ and $\rho_R \lambda_R$ on $\mathscr{P}(Y)$ are closure operations whose fixed points are called *stable* sets. Thus a subset A of X is *stable* if $A = \lambda_R \rho_R A$, and a subset B of Y is *stable* if $B = \rho_R \lambda_R A$. In general $\lambda_R \rho_R A$ is the smallest stable superset of A and $\rho_R \lambda_R B$ is the smallest stable superset of B, so to prove stability of A it is enough to prove $\lambda_R \rho_R A \subseteq A$, and similarly for B. The stable subsets of X are precisely the sets of the form $\lambda_R B$, and the stable subsets of Y are precisely the sets of the form $\rho_R A$. This uses that under composition, $\lambda_R \rho_R \lambda_R = \lambda_R$ and $\rho_R \lambda_R \rho_R = \rho_R$.

Let P^+ be the set of all stable subsets of X in P, partially ordered by set inclusion. P^+ forms a complete lattice in which $\bigwedge \mathcal{C} = \bigcap \mathcal{C}$, $\bigvee \mathcal{C} = \lambda_R \rho_R \bigcup \mathcal{C}$, $1 = X$ and $0 = \lambda_R \rho_R \emptyset = \lambda_R Y$. We call P^+ the *stable set lattice* of P. For any $A \in P^+$, we have

$$A = \bigvee\nolimits_{x \in A} \lambda_R \rho_R \{x\} = \bigcap\nolimits_{y \in \rho_R A} \lambda_R \{y\}. \tag{2.2}$$

A quasi-order $\preccurlyeq_1$ on X, with inverse $\succcurlyeq_1$, is defined by putting

$$x \preccurlyeq_1 x' \quad \text{iff} \quad \rho_R\{x\} \subseteq \rho_R\{x'\}. \tag{2.3}$$

Similarly, a quasi-order $\preccurlyeq_2$ on Y is given by

$$y \preccurlyeq_2 y' \quad \text{iff} \quad \lambda_R\{y\} \subseteq \lambda_R\{y'\}. \tag{2.4}$$

Then the following condition holds:

$$x' \succcurlyeq_1 xRy \preccurlyeq_2 y' \text{ implies } x'Ry'. \tag{2.5}$$

For $x \in X$ and $y \in Y$ put

$$[x)_1 = \{x' \in X : x \preccurlyeq_1 x'\}, \qquad [y)_2 = \{y' \in Y : y \preccurlyeq_2 y'\}.$$

A subset A of X is an *upset* under $\preccurlyeq_1$ if it is closed upwards under $\preccurlyeq_1$, i.e. $x \in A$ implies $[x)_1 \subseteq A$. Likewise a set $B \subseteq Y$ is a $\preccurlyeq_2$-*upset* if $y \in B$ implies $[y)_2 \subseteq B$.

Lemma 1. *Any stable subset of X is a $\preccurlyeq_1$-upset, and any stable subset of Y is a $\preccurlyeq_2$-upset. Hence $\rho_R A$ is a $\preccurlyeq_2$-upset for any $A \subseteq X$, and $\lambda_R B$ is a $\preccurlyeq_1$-upset for any $B \subseteq Y$.*

Proof. Let $A \in P^+$. If $x \in A$ and $x \preccurlyeq_1 x'$, then any $y \in \rho_R A$ has $x' \succcurlyeq_1 xRy \preccurlyeq_2 y$, hence $x'Ry$ by (2.5). So $x' \in \lambda_R \rho_R A = A$. This shows A is a $\preccurlyeq_1$-upset. The case of stable subsets of Y is similar. The second statement of the lemma follows as $\rho_R A$ and $\lambda_R B$ are always stable. □

A map $\alpha : (X, \preccurlyeq) \to (X', \preccurlyeq')$ between quasi-ordered sets is *isotone* if it preserves the orderings, i.e. $x \preccurlyeq z$ implies $\alpha(x) \preccurlyeq' \alpha(z)$. For such a map, if A is an $\preccurlyeq'$-upset of X', then $\alpha^{-1}A$ is an $\preccurlyeq$-upset of X.

An *antitone* α is one that reverses the orderings, i.e. $x \preccurlyeq z$ implies $\alpha(z) \preccurlyeq' \alpha(x)$. For example, ρ_R is antitone as a map $(\mathscr{P}(X), \subseteq) \to (\mathscr{P}(Y), \subseteq)$. Likewise for $\lambda_R \colon (\mathscr{P}(Y), \subseteq) \to (\mathscr{P}(X), \subseteq)$.

Remark 2 (Etymology of 'polarity'). In projective plane geometry, a polarity is an interchange of points and lines, with the line associated to a given point being the *polar* of the point, and the point associated to a given line being the *pole* of the line. A point x lies on a given line iff the pole of that line lies on the polar of x. The pole of the polar of a point is that point, and the polar of the pole of a line is that line. Two points are called *conjugate* if each lies on the polar of the other. The polar of point x can be identified with the set of points $\{y : xRy\}$ where R is the congugacy relation.

A polarity on a projective three-space interchanges points and planes as poles and polars, while associating lines with each other in pairs. Two associated lines are polars of each other. More generally, a polarity on a finite-dimensional projective space is an inclusion reversing permutation θ of the subspaces that is also an involution, i.e. $\theta(\theta A) = A$. Such a θ can be obtained from an inner product (symmetric bilinear function) $x \cdot y$ on the space by putting $\theta A = \rho_R A$, where xRy iff $x \cdot y = 0$.

The use of 'polarity' to refer to a binary relation derives from the work of Birkhoff [2, Section 32] who first defined the operations ρ_R and λ_R for an arbitrary $R \subseteq X \times Y$ and observed that they give a dual isomorphism between the lattices of stable subsets of X and Y. He suggested that $\rho_R A$ could in general be called the *polar* of A with respect to R, in view of the above geometric example. □

3 Operators and Relations

A finitary operation $f \colon \mathbb{L}^n \to \mathbb{L}$ on a lattice is an *operator* if it preserves binary joins in each coordinate. As such it preserves the ordering of $\mathbb{L}$ is each coordinate, which implies that it preserves the product ordering, i.e. it is isotone as an operation on $\mathbb{L}^n$.

A *normal operator* preserves the least element in each coordinate as well, hence preserves all finite joins in each coordinate, including the empty join 0. A *complete operator* preserves all existing non-empty joins in each coordinate, while a *complete normal operator* preserves the empty join as well. By iterating the join preservation in each coordinate successively, one can show that if f is a complete normal operator, then

$$f(\bigvee A_0, \dots, \bigvee A_{n-1}) = \bigvee\{f(a_0, \dots, a_{n-1}) : a_i \in A_i \text{ for all } i < n\}. \tag{3.1}$$

A *dual operator* (*normal* dual operator, *complete* dual operator, *complete normal* dual operator) is a finitary operation that preserves binary meets (finite meets, non-empty meets, all meets) in each coordinate. (Preservation of the empty meet means preservation of the greatest element 1.) A dual operator is isotone on $\mathbb{L}^n$. A complete normal dual operator satisfies the equation that results from (3.1) by replacing each $\bigvee$ by $\bigwedge$.

Fix a polarity $P = (X, Y, R)$. We are going to show that complete normal n-ary operators on the stable set lattice P^+ can be built from $n+1$-ary relations on P. For this we need to introduce some vectorial notation for handling tuples and relations (sets of tuples).

A tuple $(x_0, \dots, x_{n-1})$ will be denoted $\vec{x}$. Then $\vec{x}[z/i]$ denotes the tuple obtained from $\vec{x}$ by replacing x_i by z, while $(\vec{x}, y)$ denotes the $n+1$-tuple $(x_0, \dots, x_{n-1}, y)$. If $S \subseteq X^n \times Y$ is an $n+1$-ary relation, i.e. a set of $n+1$-tuples, we usually write $\vec{x}Sy$ when $(\vec{x}, y) \in S$. For $Z \subseteq X^n$ we write ZSy if $\vec{x}Sy$ holds for all $\vec{x} \in Z$.

If $\vec{A} = (A_0, \dots, A_{n-1})$ is a tuple of sets A_i, we write $\pi\vec{A}$ for the product set $A_0 \times \cdots \times A_{n-1}$. We sometimes write $\vec{x} \in_\pi \vec{A}$ when $\vec{x} \in \pi\vec{A}$, i.e. when $x_i \in A_i$ for all $i < n$. Similarly $\vec{A} \subseteq_\pi \vec{B}$ means that $A_i \subseteq B_i$ for all $i < n$. Various operations are lifted to tuples coordinate-wise, so that $\rho_R\vec{A} = (\rho_R A_0, \dots, \rho_R A_{n-1})$ while $\lambda_R\vec{A} = (\lambda_R A_0, \dots, \lambda_R A_{n-1})$, $\theta^{-1}\vec{A} = (\theta^{-1}A_0, \dots, \theta^{-1}A_{n-1})$, etc.

A *section* of a relation $S \subseteq X^n \times Y$ is any subset of X or Y obtained by fixing all but one of the coordinates and letting the unfixed coordinate vary arbitrarily. Thus each $\vec{x} \in X^n$ determines the section

$$S[\vec{x}, -] = \{y \in Y : \vec{x}Sy\}.$$

For $i < n$, sections that vary the i-th coordinate are defined, for $\vec{x} \in X^n$ and $y \in Y$, by letting

$$S[\vec{x}[-]_i, y] = \{x' \in X : \vec{x}[x'/i]Sy\}.$$

We illustrate this definition with a technical lemma that will be applied below. For each $x \in X$, let $|x| = \lambda_R\rho_R\{x\}$, the smallest member of P^+ to contain x. For an

n-tuple $\vec{x} = (x_0, \dots, x_{n-1})$, let $|\vec{x}| = (|x_0|, \dots, |x_{n-1}|)$. Note that since $x_i \in |x_i|$ for all $i < n$, we have $\vec{x} \in_\pi |\vec{x}|$.

Lemma 3. *Suppose that all sections of S of the form $S[\vec{x}[-]_i, y]$ are stable in X. Then for all $(\vec{x}, y) \in X^n \times Y$, if $\vec{x} S y$ then every $\vec{z} \in_\pi |\vec{x}|$ has $\vec{z} S y$, i.e. $(\pi|\vec{x}|)Sy$.*

Proof. Let $\vec{x} S y$ where $\vec{x} = (x_0, \dots, x_{n-1})$. We will show by induction on $i \leqslant n$ that

$$\text{if } z_j \in |x_j| \text{ for all } j < i, \text{ then } (z_0, \dots, z_{i-1}, x_i, \dots, x_{n-1})Sy. \tag{3.2}$$

Putting $i = n$ then gives the desired result that $(|x_0| \times \cdots \times |x_{n-1}|)Sy$.

If $i = 0$, then (3.2) holds by the assumption $\vec{x} S y$. Now suppose inductively that (3.2) holds for some $i < n$, and that $z_j \in |x_j|$ for all $j < i + 1$. Then this hypothesis gives $\vec{w} S y$, where $\vec{w} = (z_0, \dots, z_{i-1}, x_i, \dots, x_{n-1})$. Now $S[\vec{w}[-]_i, y]$ is a stable set containing x_i, because $\vec{w}[x_i/i] = \vec{w}$ and $\vec{w} S y$. Since $|x_i|$ is the smallest such stable set, we get $|x_i| \subseteq S[\vec{w}[-]_i, y]$. But $z_i \in |x_i|$, so this implies $\vec{w}[z_i/i]Sy$, i.e. $(z_0, \dots, z_i, x_{i+1}, \dots, x_{n-1})Sy$. Hence (3.2) holds with $i + 1$ in place of i, That completes the inductive proof that (3.2) holds for all $i \leqslant n$, as required. □

The symbols $[x)_1$ and $|x|$ have conceptually different meanings, but name the same set. For, $|x|$ is a $\preccurlyeq_1$-upset by Lemma 1, so includes $[x)_1$ as the smallest $\preccurlyeq_1$-upset to contain x. Conversely, $[x)_1$ is stable [7, Lemma 1], so includes $|x|$ as the smallest stable set to contain x. The $|x|$ notation is a little more convenient, particularly when lifted to tuples, and when the focus is on stability.

Now for $S \subseteq X^n \times Y$, define $f_S^\bullet \colon (P^+)^n \to \mathscr{P}Y$ by putting, for $\vec{A} \in (P^+)^n$,

$$\begin{aligned} f_S^\bullet \vec{A} &= \{y \in Y : (\pi\vec{A})Sy\}, \\ &= \bigcap \{S[\vec{x}, -] : \vec{x} \in \pi\vec{A}\}. \end{aligned} \tag{3.3}$$

Then define an n-ary operation f_S on P^+ by putting

$$f_S \vec{A} = \lambda_R f_S^\bullet \vec{A}.$$

This definition generalises the form of the binary fusion operation $\otimes$ defined in [17] from a relation $S \subseteq X^2 \times Y$ by

$$\begin{aligned} A_0 \otimes A_1 &= \bigcap \{\lambda_R\{y\} : (\forall x_0 \in A_0)(\forall x_1 \in A_1)\, S(x_0, x_1, y)\} \\ &= \lambda_R\{y \in Y : (A_0 \times A_1)Sy\}. \end{aligned}$$

Note that $f_S^\bullet$ is antitone in the i-th coordinate, i.e. if $A_i \subseteq B$ then $f_S^\bullet(\vec{A}) \supseteq f_S^\bullet(\vec{A}[B/i])$. Hence f_S is isotone in each coordinate. The condition for $x \in f_S \vec{A}$ is

$$\forall y \in Y[\forall \vec{z}(\vec{z} \in_\pi \vec{A} \to \vec{z} S y) \to xRy], \tag{3.4}$$

which can be spelt out as a first-order formula in the predicates $z_i \in A_i$, $\vec{z}Sy$ and xRy.

Theorem 4. *Let f be any n-ary complete normal operator on P^+ for a polarity P. Then f is equal to the operation f_{S_f} determined by some relation $S_f \subseteq X^n \times Y$.*

Proof. Recall that for $\vec{x} = (x_0, \dots, x_{n-1})$ we put $|\vec{x}| = (|x_0|, \dots, |x_{n-1}|)$ where $|x_i| = \lambda_R\rho_R\{x_i\} \in P^+$. Define $\vec{x}S_f y$ iff $y \in \rho_R f|\vec{x}|$. Then for $\vec{A} \in (P^+)^n$,

$$f^{\bullet}_{S_f}\vec{A} = \{y \in Y : \vec{x} \in_\pi \vec{A} \text{ implies } y \in \rho_R f|\vec{x}|\} \tag{3.5}$$

$$= \bigcap\{\rho_R f|\vec{x}| : \vec{x} \in_\pi \vec{A}\}. \tag{3.6}$$

But since f preserves joins in each coordinate, using the first equation from (2.2) we get

$$\begin{aligned} f\vec{A} &= f(\bigvee\nolimits_{x_0 \in A_0} |x_0|, \dots, \bigvee\nolimits_{x_{n-1} \in A_{n-1}} |x_{n-1}|) \\ &= \bigvee\{f(|\vec{x}|) : \vec{x} \in_\pi \vec{A}\} \qquad \text{by (3.1)}, \\ &= \lambda_R\rho_R(\bigcup\{f(|\vec{x}|) : \vec{x} \in_\pi \vec{A}\}), \quad \text{by definition of } \bigvee, \\ &= \lambda_R(\bigcap\{\rho_R f(|\vec{x}|) : \vec{x} \in_\pi \vec{A}\}) \quad \text{by (2.1)}, \\ &= \lambda_R f^{\bullet}_{S_f}\vec{A} \qquad \text{by (3.6)}\ , \\ &= f_{S_f}\vec{A}. \end{aligned}$$

□

Theorem 5. *If all sections of S are stable, then f_S is a complete normal operator, and S is equal to the relation S_{f_S} determined by f_S.*

Proof. Assume all sections of S are stable. To prove that f_S preserves joins in the i-th coordinate it is enough to prove that the inclusion

$$f_S(\vec{A}[\bigvee\nolimits_J B_j/i]) \subseteq \bigvee\nolimits_J f_S(\vec{A}[B_j/i]) \tag{3.7}$$

holds for any $\vec{A} \in (P^+)^n$ and any collection $\{B_j : j \in J\} \subseteq P^+$ with join $\bigvee_J B_j$. This is because the converse inclusion must hold, since f_S is isotone in the i-th coordinate, so $f_S(\vec{A}[B_j/i]) \subseteq f_S(\vec{A}[\bigvee_J B_j/i])$ for all $j \in J$. We will first show that

$$\bigcap\nolimits_{j \in J} f^{\bullet}_S(\vec{A}[B_j/i]) \subseteq f^{\bullet}_S(\vec{A}[\bigvee\nolimits_J B_j/i]). \tag{3.8}$$

To see this, let $y \in f^{\bullet}_S(\vec{A}[B_j/i])$ for all $j \in J$. Take any $\vec{x} \in_\pi \vec{A}[X/i]$. Then for each j, if $x' \in B_j$ then $\vec{x}[x'/i] \in \pi\vec{A}[B_j/i]$, so $\vec{x}[x'/i]Sy$ by the definition (3.5) of

$f^\bullet_S$. This shows that $B_j \subseteq S[\vec{x}[-]_i, y]$. But the latter section is a stable subset of X, hence belongs to P^+, so this implies that $\bigvee_J B_j \subseteq S[\vec{x}[-]_i, y]$. Hence $\vec{x}[z/i]Sy$ for all $z \in \bigvee_J B_j$. As that holds for all $\vec{x} \in_\pi \vec{A}[X/i]$, we get $(\pi\vec{A}[\bigvee_J B_j/i])Sy$, hence $y \in f^\bullet_S(\vec{A}[\bigvee_J B_j/i])$, proving (3.8).

Now as $f^\bullet_S(\vec{A}[\bigvee_J B_j/i])$ is stable, being an intersection of stable sections (3.3), we reason that

$$\begin{aligned}\bigcap\nolimits_{j\in J} f^\bullet_S(\vec{A}[B_j/i]) &= \bigcap\nolimits_{j\in J} \rho_R\lambda_R f^\bullet_S(\vec{A}[B_j/i])\\ &= \bigcap\nolimits_{j\in J} \rho_R f_S(\vec{A}[B_j/i])\\ &= \rho_R\big(\bigcup\nolimits_{j\in J} f_S(\vec{A}[B_j/i])\big),\end{aligned}$$

and therefore from (3.8) that

$$\rho_R\big(\bigcup\nolimits_{j\in J} f_S(\vec{A}[B_j/i])\big) \subseteq f^\bullet_S(\vec{A}[\bigvee\nolimits_J B_j/i]).$$

Hence $\lambda_R f^\bullet_S(\vec{A}[\bigvee_J B_j/i]) \subseteq \lambda_R\rho_R\big(\bigcup_{j\in J} f_S(\vec{A}[B_j/i])\big)$, which is (3.7).

To show that $S = S_{f_S}$, note first that $\vec{x}S_{f_S}y$ iff $y \in \rho_R f_S|\vec{x}|$, by definition of S_f, while $\rho_R f_S|\vec{x}| = \rho_R\lambda_R f^\bullet_S|\vec{x}| = f^\bullet_S|\vec{x}|$ since $f^\bullet_S|\vec{x}|$ is stable by (3.3) as all S-sections are stable. Thus $\vec{x}S_{f_S}y$ iff $y \in f^\bullet_S|\vec{x}|$ iff $(\pi|\vec{x}|)Sy$. But if $(\pi|\vec{x}|)Sy$, then $\vec{x}Sy$ since $\vec{x} \in \pi|\vec{x}|$. Conversely, if $\vec{x}Sy$ then $(\pi|\vec{x}|)Sy$ by Lemma 3. So altogether, $\vec{x}S_{f_S}y$ iff $\vec{x}Sy$. □

An alternative proof that f_S is a complete normal operator can be given by showing that the function $A' \mapsto f_S(\vec{A}[A'/i])$ has a right adjoint, namely the function $B' \mapsto g^i(\vec{A}[B/i])$, where

$$g^i\vec{A} = \bigcap\{S[\vec{x}[-]_i, y] : \vec{x} \in \vec{A}[X/i] \text{ and } y \in \rho_R A_i\}.$$

The adjointness means that $f_S(\vec{A}[A'/i]) \subseteq B$ iff $A' \subseteq g^i(\vec{A}[B/i])$ for any $A', B \in P^+$. It is a standard fact that any lattice operation with a right adjoin preserves all joins. The functions $B' \mapsto g^i(\vec{A}[B/i])$ for each $i < n$ generalise the two residual operations $A_0\backslash A_1$ and A_0/A_1 of the above binary fusion operation $A_0 \otimes A_1$, as given in [17]. These can be expressed as

$$\begin{aligned}A_0\backslash A_1 &= \bigcap\{S[x_0, -, y] : x_0 \in A_0 \ \&\ y \in \rho_R A_1\},\\ A_0/A_1 &= \bigcap\{S[-, x_1, y] : x_1 \in A_1 \ \&\ y \in \rho_R A_0\}.\end{aligned}$$

Next we consider the construction of dual operators on P^+. An m-ary dual operator can be obtained from a relation of the form $T \subseteq X \times Y^m$. We write $xT\vec{y}$

when $(x, \vec{y}) \in T$, and xTZ when $xT\vec{y}$ holds for all $\vec{y} \in Z$. Sections of T take the form

$$\begin{aligned} T[-, \vec{y}] &= \{x \in X : xT\vec{y}\} && \text{for } \vec{y} \in Y^m, \\ T[x, \vec{y}[-]_i] &= \{y' \in Y : xT\vec{y}[y'/i]\} && \text{for } x \in X,\ i < m,\ \vec{y} \in Y^m. \end{aligned}$$

Dual to Lemma 3 is a result about sequences of the form $|\vec{y}| = (|y_0|, \dots, |y_{m-1}|)$, where $|y_i| = \rho_R\lambda_R\{y_i\}$, the smallest stable subset of Y to contain y_i. Suppose all sections of T of the form $T[x, \vec{y}[-]_i]]$ are stable. Then if $xT\vec{y}$, we can show by induction on $i \leqslant m$ that if $z_j \in |y_j|$ for all $j < i$, then $xT(z_0, \dots, z_{i-1}, y_i, \dots, y_{m-1})$. Putting $i = m$ then gives

Lemma 6. *Suppose that all sections of T of the form $T[x, \vec{y}[-]_i]]$ are stable in Y. Then for all $(x, \vec{y}) \in X \times Y^m$, if $xT\vec{y}$ then every $\vec{z} \in_\pi |\vec{y}|$ has $xT\vec{z}$, i.e. $xT(\pi|\vec{y}|)$.* $\square$

We now define an m-ary function g_T on P^+. For $\vec{A} \in (P^+)^m$ put

$$\begin{aligned} g_T\vec{A} &= \{x \in X : xT(\pi\rho_R\vec{A})\}, \\ &= \{x \in X : \forall\vec{y}(\vec{y} \in \pi\rho_R\vec{A} \text{ implies } xT\vec{y})\} \\ &= \bigcap\{T[-, \vec{y}] : \vec{y} \in_\pi \rho_R\vec{A}\}. \end{aligned} \tag{3.9}$$

The condition for $x \in g_T\vec{A}$ can be spelt out as the first-order expression

$$\forall\vec{y}(\textstyle\bigwedge_{i<m} \forall z(z \in A_i \to zRy_i) \to xT\vec{y}). \tag{3.10}$$

We exemplify g_T with the case that $m = 1$. Then T is a binary relation from X to Y, inducing a unary operation on P^+ which we denote more suggestively by $\Box_T$. Thus

$$\Box_T A = \{x \in X : xT\rho_R A\}.$$

As a binary relation, T can also be viewed as a subset of $X^n \times Y$ with $n = 1$, so it induces a unary operation $\Diamond_T$ on P^+ having

$$\Diamond_T A = \lambda_R\{y \in Y : ATy\}.$$

When all sections of T are stable, $\Box_T$ is a dual operator and $\Diamond_T$ is an operator that is left adjoint to $\Box_T$ in the sense that for any $A, B \subseteq X$,

$$\Diamond_T A \subseteq B \quad \text{iff} \quad A \subseteq \Box_T B.$$

$\Box_T$ and $\Diamond_T$ can be viewed as modal operators. Defining a 'satisfaction' relation $x \models A$ to mean that $x \in A$, then (3.10) in the case $m = 1$ becomes the semantic condition

$$x \models \Box_T A \quad \text{iff} \quad \forall y(\forall z(z \models A \to zRy) \to xTy).$$

Likewise, from the definition of $\Diamond_T$ we obtain

$$x \models \Diamond_T A \quad \text{iff} \quad \forall y(\forall z(z \models A \to zTy) \to xRy).$$

See [7, 8] for discussion of pairs of 'modalities' like $\Box_T$ and $\Diamond_T$.

In a similar way, the above operator $A_0 \otimes A_1$ and its two residual operations $A_0 \backslash A_1$ and A_0/A_1 have been used in [12] and [17] to develop a relational semantic modelling for the implication-fusion fragments of a number of substructural logics, including linear logic, relevant logic, and intuitionistic logic. This methodology has been extended further to give a generalised Kripke semantics for the Lambek-Grishin calculus [6] and the full linear logic [10].

Theorem 7. *If all sections of a relation $T \subseteq X \times Y^m$ are stable, then g_T is a complete normal dual operator.*

Proof. If all sections of T are stable then $g_T\vec{A}$ is stable by (3.9), so g_T is an operation on P^+. g_T is isotone in each coordinate. Hence it satisfies the inclusion

$$g_T(\vec{A}[\textstyle\bigcap_J B_j/i]) \subseteq \bigcap_J g_T(\vec{A}[B_j/i]).$$

To show that g_T is a complete normal dual operator, we prove that the last inclusion is an equality for any $i < m$. Let $x \in \bigcap_{j\in J} g_T(\vec{A}[B_j/i])$, where $\{B_j : j \in J\} \subseteq P^+$. Then

$$\forall j \in J\ \forall \vec{y} \in_\pi \rho_R(\vec{A}[B_j/i]),\ xT\vec{y}. \tag{3.11}$$

Now take any $\vec{y} \in_\pi \rho_R(\vec{A}[\bigcap_J B_j/i])$. Then for any $j \in J$, if $y' \in \rho_R B_j$ then $\vec{y}[y'/i] \in_\pi \rho_R(\vec{A}[B_j/i])$, so $xT(\vec{y}[y'/i])$ by (3.11). This proves $\rho_R B_j \subseteq T[x, \vec{y}[-]_i]$. Therefore $\lambda_R T[x, \vec{y}[-]_i] \subseteq \lambda_R\rho_R B_j = B_j$. Hence $\lambda_R T[x, \vec{y}[-]_i] \subseteq \bigcap_J B_j$, giving

$$\rho_R \textstyle\bigcap_J B_j \subseteq \rho_R\lambda_R T[x, \vec{y}[-]_i] = T[x, \vec{y}[-]_i]$$

as $T[x, \vec{y}[-]_i]$ is stable. But $y_i \in \rho_R \bigcap_J B_j$, so then $y_i \in T[x, \vec{y}[-]_i]$, making $xT\vec{y}$. Altogether we have shown that

$$\vec{y} \in_\pi \rho_R(\vec{A}[\textstyle\bigcap_J B_j/i]) \text{ implies } xT\vec{y},$$

which means that $x \in g_T(\vec{A}[\bigcap_J B_j/i])$, completing the proof that g_T preserves all meets in the i-th coordinate. $\Box$

It can also be shown that if g is any complete normal dual operator on P^+, then g is equal to g_{T_g}, where $T_g \subseteq X \times Y^m$ is defined by

$$xT_g\vec{y} \quad \text{iff} \quad x \in g(\lambda_R\{y_0\}, \dots, \lambda_R\{y_{m-1}\}),$$

making $T_g[-, \vec{y}] = g(\lambda_R\{y_0\}, \dots, \lambda_R\{y_{m-1}\})$. Then using the second equation from (2.2), for any $\vec{A} \in (P^+)^m$ we get

$$\begin{aligned}
g\vec{A} &= g\Big(\bigcap\nolimits_{y_0 \in \rho_R A_0} \lambda_R\{y_0\}, \dots, \bigcap\nolimits_{y_{m-1} \in \rho_R A_{m-1}} \lambda_R\{y_{m-1}\}\Big) \\
&= \bigcap\{g(\lambda_R\{y_0\}, \dots, \lambda_R\{y_{m-1}\}) : \vec{y} \in_\pi \rho_R\vec{A}\} \quad \text{by the dual of (3.1),} \\
&= \bigcap\{T_g[-, \vec{y}] : \vec{y} \in_\pi \rho_R\vec{A}\} \\
&= g_{T_g}\vec{A} \qquad \text{by (3.9).}
\end{aligned}$$

Also, if all sections of $T \subseteq X \times Y^m$ are stable, then T is equal to the relation T_{g_T} determined by g_T. For, if $xT_{g_T}\vec{y}$ then by definition of T_{g_T}, $x \in g_T\vec{A}$ where $\vec{A} = (\lambda_R\{y_0\}, \dots, \lambda_R\{y_{m-1}\})$, hence by definition of g_T, we get $xT(\pi\rho_R\vec{A})$. But $\vec{y} \in \pi\rho_R\vec{A}$ since $y_i \in \rho_R\lambda_R\{y_i\}$ for $i < m$, so this implies $xT\vec{y}$. Conversely, if $xT\vec{y}$, then by Lemma 6, $xT(\pi|\vec{y}|)$. But here $\pi|\vec{y}| = \pi\rho_R\vec{A}$, so this gives $x \in g_T\vec{A}$ and hence $xT_{g_T}\vec{y}$. Altogether, $T = T_{g_T}$.

We are going to work with lattices having additional operators and dual operators, and we need a convenient notation for this. Let Ω be a set of function symbols with given finite arities. Define an Ω-*lattice* to be an algebra of the form

$$\mathbb{L} = (\mathbb{L}_0, \{\mathbf{f}^{\mathbb{L}} : \mathbf{f} \in \Omega\}),$$

where $\mathbb{L}_0$ is a lattice, and for n-ary $\mathbf{f} \in \Omega$, $\mathbf{f}^{\mathbb{L}}$ is an n-ary operation on $\mathbb{L}_0$. Furthermore, we will take Ω to be presented as the union $\Lambda \cup \Upsilon$ of disjoint subsets Λ and Υ of 'lower' and 'upper' symbols, respectively (the reason for these names will emerge later—see (6.7)). An Ω-lattice will be called a *normal lattice with operators*—an Ω-NLO or just NLO—if each lower symbol denotes a normal operator in $\mathbb{L}$ and each upper symbol denotes a normal dual operator.

We define an Ω-*polarity* to be a structure of the form

$$P = (X, Y, R, \{S_{\mathbf{f}} : \mathbf{f} \in \Lambda\}, \{T_{\mathbf{g}} : \mathbf{g} \in \Upsilon\}),$$

based on a polarity $P_0 = (X, Y, R)$, such that for any n-ary lower symbol $\mathbf{f} \in \Lambda$, $S_{\mathbf{f}} \subseteq X^n \times Y$ and all sections of $S_{\mathbf{f}}$ are stable; and for any m-ary upper symbol $\mathbf{g} \in \Upsilon$, $T_{\mathbf{g}} \subseteq X \times Y^m$ and all sections of $T_{\mathbf{g}}$ are stable. Then P gives rise to the Ω-NLO

$$P^+ = (P_0^+, \{f_{S_{\mathbf{f}}} : \mathbf{f} \in \Lambda\}, \{g_{T_{\mathbf{g}}} : \mathbf{g} \in \Upsilon\}), \tag{3.12}$$

where $f_{S_{\mathbf{f}}}$ is the complete normal operator determined by $S_{\mathbf{f}}$, as per Theorem 5, and $g_{T_{\mathbf{g}}}$ is the complete normal dual operator determined by $T_{\mathbf{g}}$, as per Theorem 7.

4 Bounded Morphisms

To simplify the exposition, we fix two arbitrary natural numbers n and m and assume from now that Λ consists of a single n-ary function symbol while Υ consists of a single m-ary one. Then an Ω-polarity has the typical form

$$P = (X, Y, R, S, T)$$

with $S \subseteq X^n \times Y$, $T \subseteq X \times Y^m$, and all sections of S and T being stable. We lift the relations $\preccurlyeq_1$ and $\preccurlyeq_2$ to tuples coordinate-wise, putting $\vec{x} \preccurlyeq_1 \vec{z}$ iff $x_i \preccurlyeq_1 z_i$ for all $i < n$; with $[\vec{x})_1 = \{\vec{z} \in X^n : \vec{x} \preccurlyeq_1 \vec{z}\}$ etc.

Let P and P' be Ω-polarities of the kind just described. For a function $\alpha : X \to X'$ we put

$$\alpha(\vec{x}) = (\alpha(x_0), \ldots, \alpha(x_{n-1})),$$
$$\alpha^{-1}[\vec{x'})_1 = \{\vec{x} \in X^n : \vec{x'} \preccurlyeq_1' \alpha(\vec{x})\} \text{ etc.}$$

Definition 8. A *bounded morphism from P to P'* is a pair α, β of *isotone* maps $\alpha\colon (X, \preccurlyeq_1) \to (X', \preccurlyeq_1')$ and $\beta : (Y, \preccurlyeq_2) \to (Y', \preccurlyeq_2')$ that satisfy the following back and forth conditions.

(1_R) $\alpha(x)R'\beta(y)$ implies xRy, all $x \in X, y \in Y$.

(2_R) $(\alpha^{-1}[x')_1)Ry$ implies $x'R'\beta(y)$, all $x' \in X', y \in Y$.

(3_R) $xR\beta^{-1}[y')_2$ implies $\alpha(x)R'y'$, all $x \in X, y' \in Y'$.

(1_S) $\alpha(\vec{x})S'\beta(y)$ implies $\vec{x}Sy$, all $\vec{x} \in X^n$, $y \in Y$.

(2_S) $(\alpha^{-1}[\vec{x'})_1)Sy$ implies $\vec{x'}S'\beta(y)$, all $\vec{x'} \in (X')^n$, $y \in Y$.

(1_T) $\alpha(x)T'\beta(\vec{y})$ implies $xT\vec{y}$, all $x \in X$, $\vec{y} \in Y^m$.

(2_T) $xT\beta^{-1}[\vec{y'})_2$ implies $\alpha(x)T'\vec{y'}$, all $x \in X$, $\vec{y'} \in (Y')^m$.

In condition (3_R), $\beta^{-1}[y')_2$ is the set $\{y \in Y : y' \preccurlyeq_2' \beta(y)\}$. Thus the condition can be expressed as

$$\text{if } \forall y(y' \preccurlyeq_2' \beta(y) \text{ implies } xRy), \text{ then } \alpha(x)R'y'.$$

Contrapositively this says

$$\text{if not } \alpha(x)R'y', \text{ then } \exists y(y' \preccurlyeq_2' \beta(y) \text{ and not } xRy). \tag{4.1}$$

Similar formulations hold for (2_R), (2_S) and (2_T). Note that the converse of (4.1) is equivalent to (1_R), which follows from this converse by putting $y' = \beta(y)$. To derive the converse, observe that if not xRy then (1_R) implies not $\alpha(x)R'\beta(y)$, so then if $y' \preccurlyeq_2' \beta(y)$ we get not $\alpha(x)R'y'$ by definition of $\preccurlyeq_2'$.

Remark 9 (Source of Definition 8). Suppose that $X = Y$ and R is symmetric. Then we have the kind of polarity used in [27] to provide a semantics for orthologic, in which the operation λ_R $(= \rho_R)$ is an orthocomplementation modelling a negation connective. An ortholattice representation was given in [26] in which the points of the representing space are filters of the lattice, and which can be seen as a precursor to the canonical structures of Section 7 below. A translation of orthologic into classical modal logic was obtained in [27] by transforming an 'orthoframe' (X, R) into the Kripke frame $(X, \widehat{R})$, where $\widehat{R} = X^2 \setminus R$ is the complementary relation to R. Taking $\alpha = \beta$, so that (2_R) and (3_R) become equivalent, then if the conditions (1_R)–(3_R) are re-expressed in terms of $\widehat{R}$ and $\widehat{R'}$, they become similar to the standard definition of a bounded morphism between the Kripke frames $(X, \hat{R})$ and $(X', \widehat{R'})$, with (1_R) being equivalent to (1.1) for $\widehat{R}$, and (3_R) in the form (4.1) amounting to (1.2) for $\widehat{R}$ (except for the relation $\preccurlyeq_2'$). The other conditions in Definition 8 give parallel back and forth properties for the relations S and T. This account may explain why it is often natural to use contrapositive reasoning in proofs of properties of bounded morphisms, as we shall see.

The use of quasi-orderings $\preccurlyeq_i$ is well established in theories of duality for non-Boolean lattices [39, 40] and relates to the fact that the points of dual structures are typically filters and/or ideals that may not be maximal. Such points are naturally quasi-ordered by the set inclusion relation $\subseteq$. The use of a quasi-order to formulate bounded morphism conditions like (4.1) is also well established [28, p.192], [4, p.698]. We could adopt a more axiomatic approach and let $\preccurlyeq_1$ and $\preccurlyeq_2$ be any additional quasi-orders that satisfy the condition (2.5), which is equivalent to requiring only that $x \preccurlyeq_1 x'$ implies $\rho_R\{x\} \subseteq \rho_R\{x'\}$ and $y \preccurlyeq_2 y'$ implies $\lambda_R\{y\} \subseteq \lambda_R\{y'\}$. But in a polarity, suitable quasi-orders can be defined as in (2.3) and (2.4), and shown to give the relation $\subseteq$ in a canonical structure: see Lemma 21.

The requirement that α and β be isotone is needed to ensure that the class of bounded morphisms is closed under functional composition and gives a category: see Lemma 14. □

We will show that a bounded morphism makes the following diagrams commute, where the operation $\mathscr{P}_{\preccurlyeq}$ gives the set of all $\preccurlyeq$-upsets.

$$\begin{tikzcd} \mathscr{P}_{\preccurlyeq_1} X \arrow[r, "\rho_R"] & \mathscr{P}_{\preccurlyeq_2} Y \\ \mathscr{P}_{\preccurlyeq'_1} X' \arrow[u, "\alpha^{-1}"] \arrow[r, "\rho_{R'}"] & \mathscr{P}_{\preccurlyeq'_2} Y' \arrow[u, "\beta^{-1}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathscr{P}_{\preccurlyeq_1} X & \mathscr{P}_{\preccurlyeq_2} Y \arrow[l, "\lambda_R"'] \\ \mathscr{P}_{\preccurlyeq'_1} X' \arrow[u, "\alpha^{-1}"] & \mathscr{P}_{\preccurlyeq'_2} Y' \arrow[l, "\lambda_{R'}"'] \arrow[u, "\beta^{-1}"'] \end{tikzcd}$$

Lemma 10. *Let $\alpha, \beta \colon P \to P'$ be a bounded morphism and $A \subseteq X'$ and $B \subseteq Y'$.*

(1) *$\beta^{-1}(\rho_{R'} A) \subseteq \rho_R(\alpha^{-1} A)$, with $\beta^{-1}(\rho_{R'} A) = \rho_R(\alpha^{-1} A)$ when A is a $\preccurlyeq'_1$-upset.*

(2) *$\alpha^{-1}(\lambda_{R'} B) \subseteq \lambda_R(\beta^{-1} B)$, with $\alpha^{-1}(\lambda_{R'} B) = \lambda_R(\beta^{-1} B)$ when B a $\preccurlyeq'_2$-upset.*

(3) *If $A \in (P')^+$ then $\alpha^{-1} A \in P^+$.*

Proof. (1) Let $y \in \beta^{-1}(\rho_{R'} A)$, so $AR'\beta(y)$. If $x \in \alpha^{-1} A$, then $\alpha(x) \in A$, so then $\alpha(x) R' \beta(y)$, hence xRy by (1_R). This shows $(\alpha^{-1} A) Ry$, making $y \in \rho_R(\alpha^{-1} A)$.

Suppose further that A is a $\preccurlyeq'_1$-upset. Let $y \in \rho_R(\alpha^{-1} A)$, so $(\alpha^{-1} A) Ry$. If $x' \in A$, then $[x')_1 \subseteq A$ as A is an upset, so $\alpha^{-1}[x')_1 \subseteq \alpha^{-1} A$, hence $\alpha^{-1}[x')_1 Ry$, and so $x' R' \beta(y)$ by (2_R). This shows $AR'\beta(y)$, so $\beta(y) \in \rho_{R'} A$ and $y \in \beta^{-1}(\rho_{R'} A)$.

(2) Like (1), but using (1_R) and (3_R).

(3) $A = \lambda_{R'}(\rho_{R'} A)$ as A is stable, so $\alpha^{-1} A = \alpha^{-1} \lambda_{R'}(\rho_{R'} A) = \lambda_R \beta^{-1}(\rho_{R'} A)$ by part (2) as $\rho_{R'} A$ is stable, therefore a $\preccurlyeq'_2$-upset. But any subset of X of the form $\lambda_R B$ is stable and so belongs to P^+. □

Corollary 11. (1) *A pair α, β satisfies (1_R) and (2_R) if, and only if,*

$$\beta^{-1}(\rho_{R'} A) = \rho_R(\alpha^{-1} A) \text{ for all stable } A \subseteq X'. \tag{4.2}$$

(2) *α, β satisfies (1_R) and (3_R) if, and only if,*

$$\alpha^{-1}(\lambda_{R'} B) = \lambda_R(\beta^{-1} B) \text{ for all stable } B \subseteq Y'. \tag{4.3}$$

Proof. (1): If (1_R) and (2_R) hold, then (4.2) follows by part (1) of the Lemma because stable sets are upsets. Conversely, assume (4.2). To prove (1_R): if $\alpha(x) R' \beta(y)$, then

$$y \in \beta^{-1} \rho_{R'}\{\alpha(x)\} = \beta^{-1} \rho_{R'} \lambda_{R'} \rho_{R'}\{\alpha(x)\} = \rho_R \alpha^{-1} \lambda_{R'} \rho_{R'}\{\alpha(x)\},$$

with the last equation holding by (4.2). But $x \in \alpha^{-1} \lambda_{R'} \rho_{R'}\{\alpha(x)\}$, so then xRy.

For (2_R), suppose not $x'R'\beta(y)$. Then $y \notin \beta^{-1}\rho_{R'}\{x'\} = \rho_R\alpha^{-1}\lambda_{R'}\rho_{R'}\{x'\}$ as above. Hence there exists $x \in X$ such that not xRy and $\alpha(x) \in \lambda_{R'}\rho_{R'}\{x'\}$. Then $\rho_{R'}\{x'\} = \rho_{R'}\lambda_{R'}\rho_{R'}\{x'\} \subseteq \rho_{R'}\{\alpha(x)\}$, so $x' \preccurlyeq_1' \alpha(x)$ and $x \in \alpha^{-1}[x')_1$. Since not xRy this gives not $(\alpha^{-1}[x')_1)Ry$ as required.

The proof of (2) is similar. □

Remark 12 (**On conditions** (4.2) **and** (4.3)**).** Formal Concept Analysis defines a *concept* of a context/polarity $P = (X, Y, R)$ to be a pair (A, B) of subsets, of X and Y respectively, with $A = \lambda_R B$ and $B = \rho_R A$. The set of concepts is partially ordered by putting $(A, B) \leqslant (C, D)$ iff $A \subseteq C$ (iff $D \subseteq B$), forming a complete lattice isomorphic to P^+. Erné [13] defined a pair $X \xrightarrow{\alpha} X', Y \xrightarrow{\beta} Y'$ to be *concept continuous* if $(\alpha^{-1}A, \beta^{-1}B)$ is a concept of P whenever (A, B) is a concept of P'. He showed in [13, Prop. 3.2] that α, β is concept continuous iff the follow conditions hold.

$$\text{not } x'R'\beta(y) \text{ iff there is an } x \text{ with not } xRy \text{ and } \rho_{R'}\{x'\} \subseteq \rho_{R'}\{\alpha(x)\}, \tag{4.4}$$

$$\text{not } \alpha(x)R'y' \text{ iff there is a } y \text{ with not } xRy \text{ and } \lambda_{R'}\{y'\} \subseteq \lambda_{R'}\{\beta(y)\}. \tag{4.5}$$

Now it is readily seen that α, β is concept continuous iff conditions (4.2) and (4.3) of our Corollary 11 hold, which is equivalent by that corollary to having (1_R)–(3_R). Condition (4.5) is equivalent to the combination of (4.1) and its converse, which we already noted is equivalent to having (1_R) and (3_R). Likewise, (4.4) is equivalent to having (1_R) and (2_R).

Hartonas [33], building on [34], studied morphisms between polarity-based structures in which $\preccurlyeq_1$ and $\preccurlyeq_2$ are complete partial orders, X and Y each carry a Stone space topology, the closed subsets of X are the sets $[x)_1$ while the open subsets are the sets $\lambda_R\{y\}$, and similarly for subsets of Y. A morphism is a pair $X \xrightarrow{\alpha} X', Y \xrightarrow{\beta} Y'$ of continuous functions that preserve all meets in X and Y respectively, and are such that α^{-1} and β^{-1} satisfy the equations in (4.2) and (4.3) for all clopen stable A and B. □

In the proof of the next result, and elsewhere, the notation $\alpha[Z]$ will be used for the *image* $\{\alpha(z) : z \in Z\}$ of a set Z under function α.

Theorem 13. *For any bounded morphism $\alpha, \beta \colon P \to P'$, the map $A \mapsto \alpha^{-1}A$ gives a homomorphism*

$$(\alpha, \beta)^+ \colon (P')^+ \to P^+$$

of Ω-lattices. If α is surjective, $(\alpha, \beta)^+$ is injective. If α is injective, $(\alpha, \beta)^+$ is surjective.

Proof. The map is well defined by Lemma 10(3). It preserves binary meets because inverse maps preserve intersections, hence $\alpha^{-1}(A \cap B) = \alpha^{-1}A \cap \alpha^{-1}B$. It preserves binary joins because

$\alpha^{-1}(A \vee B)$
$= \alpha^{-1}\lambda_{R'}\rho_{R'}(A \cup B)$ by definition of $A \vee B$,
$= \lambda_R\beta^{-1}\rho_{R'}(A \cup B)$ by Lemma 10(2),
$= \lambda_R\rho_R\alpha^{-1}(A \cup B)$ by Lemma 10(1),
$= \lambda_R\rho_R(\alpha^{-1}A \cup \alpha^{-1}B)$ by property of α^{-1},
$= \alpha^{-1}A \vee \alpha^{-1}B$.

It preserves greatest elements as $\alpha^{-1}X' = X$, and least elements as $\alpha^{-1}\lambda_{R'}Y' = \lambda_R\beta^{-1}Y' = \lambda_R Y$.

Next we show that $(\alpha, \beta)^+$ preserves the operations f_S and $f_{S'}$, first proving that for any $\overrightarrow{A} \in ((P')^+)^n$,

$$\beta^{-1}f^\bullet_{S'}\overrightarrow{A} = f^\bullet_S\alpha^{-1}\overrightarrow{A}. \tag{4.6}$$

Let $y \in \beta^{-1}f^\bullet_{S'}\overrightarrow{A}$, so $(\pi\overrightarrow{A})S'\beta(y)$ by (3.3). Then if $\overrightarrow{x} \in_\pi \alpha^{-1}\overrightarrow{A}$, then $\alpha(\overrightarrow{x}) \in \pi\overrightarrow{A}$, so $\alpha(\overrightarrow{x})S'\beta(y)$, hence $\overrightarrow{x}Sy$ by (1_S). This shows that $(\pi\alpha^{-1}\overrightarrow{A})Sy$, so $y \in f^\bullet_S(\alpha^{-1}\overrightarrow{A})$.

Conversely, let $y \in f^\bullet_S(\alpha^{-1}\overrightarrow{A})$, so $(\pi\alpha^{-1}\overrightarrow{A})Sy$. Take any $\overrightarrow{x'} \in \pi\overrightarrow{A}$. Then if $\overrightarrow{z} \in \alpha^{-1}[\overrightarrow{x'})_1$, we have $\overrightarrow{x'} \preccurlyeq_1 \alpha(\overrightarrow{z})$, so $\alpha(\overrightarrow{z}) \in \pi\overrightarrow{A}$ as each A_i is a $\preccurlyeq_1$-upset, hence $\overrightarrow{z} \in \pi\alpha^{-1}\overrightarrow{A}$, and therefore $\overrightarrow{z}Sy$. Thus $(\alpha^{-1}[\overrightarrow{x'})_1)Sy$. Hence $\overrightarrow{x'}S'\beta(y)$ by (2_S). Altogether this shows that $(\pi\overrightarrow{A})S'\beta(y)$. Therefore $\beta(y) \in f^\bullet_{S'}\overrightarrow{A}$, hence $y \in \beta^{-1}f^\bullet_{S'}\overrightarrow{A}$, which completes the proof of (4.6). Now we reason that

$$\begin{aligned} f_S(\alpha^{-1}\overrightarrow{A}) &= \lambda_R f^\bullet_S\alpha^{-1}\overrightarrow{A} \\ &= \lambda_R\beta^{-1}f^\bullet_{S'}\overrightarrow{A} && \text{by (4.6)} \\ &= \alpha^{-1}\lambda_{R'}f^\bullet_{S'}\overrightarrow{A} && \text{by Lemma 10(2)} \\ &= \alpha^{-1}f_{S'}\overrightarrow{A}, \end{aligned}$$

proving that $(\alpha, \beta)^+$ preserves the operations f_S and $f_{S'}$.

Now we prove that $(\alpha, \beta)^+$ preserves g_T and $g_{T'}$, i.e. for any $\overrightarrow{A} \in ((P')^+)^m$,

$$\alpha^{-1}g_{T'}\overrightarrow{A} = g_T\alpha^{-1}\overrightarrow{A}. \tag{4.7}$$

Let $x \in \alpha^{-1}g_{T'}\overrightarrow{A}$, so $\alpha(x) \in g_{T'}\overrightarrow{A}$, i.e. $\alpha(x)T'\pi\rho_{R'}\overrightarrow{A}$. Now if

$$\overrightarrow{y} \in \pi\rho_R\alpha^{-1}\overrightarrow{A} = \pi\beta^{-1}\rho_{R'}\overrightarrow{A}$$

(see Lemma 10(1)), then $\beta(\overrightarrow{y}) \in \pi\rho_{R'}\overrightarrow{A}$, hence $\alpha(x)T'\beta(\overrightarrow{y})$, and thus $xT\overrightarrow{y}$ by (1_T). This shows that $xT(\pi\rho_R\alpha^{-1}\overrightarrow{A})$, i.e. $x \in g_T\alpha^{-1}\overrightarrow{A}$.

Conversely, assume $x \in g_T\alpha^{-1}\vec{A}$. Let $\vec{y'} \in \pi\rho_{R'}\vec{A}$. Then if $\vec{y} \in \beta^{-1}[\vec{y'})_2$, then $\vec{y'} \preccurlyeq_2 \beta(\vec{y})$, so $\beta(\vec{y}) \in \pi\rho_{R'}\vec{A}$. Hence for all $i < m$, $y_i \in \beta^{-1}\rho_{R'}A_i = \rho_R\alpha^{-1}A_i$, and so $\vec{y} \in \pi\rho_R\alpha^{-1}\vec{A}$. That gives $xT\vec{y}$ because $x \in g_T\alpha^{-1}\vec{A}$. Altogether this show $xT\beta^{-1}[\vec{y'})_2$. Hence $\alpha(x)T'\vec{y}'$ by (2_T). So we have established that $\alpha(x)T'\pi\rho_{R'}\vec{A}$, which means that $\alpha(x) \in g_{T'}\vec{A}$. Hence $x \in \alpha^{-1}g_{T'}\vec{A}$, completing the proof of (4.7), and hence the proof that $(\alpha,\beta)^+$ is an Ω-lattice homomorphism.

The fact that $A \mapsto \alpha^{-1}A$ is injective when α is surjective is standard set theory: surjectivity of α implies that $\alpha[\alpha^{-1}Z] = Z$ in general. Hence if $\alpha^{-1}A = \alpha^{-1}B$, then $A = \alpha[\alpha^{-1}A] = \alpha[\alpha^{-1}B] = B$.

Finally, suppose that α is injective. For any $B \in P^+$, take $A = \lambda_{R'}\rho_{R'}\alpha[B] \in (P')^+$. Then by Lemma 10, $\alpha^{-1}A = \lambda_R\rho_R\alpha^{-1}(\alpha[B]) = \lambda_R\rho_R B$ as α is injective. But B is stable, so we get that $B = \alpha^{-1}A = (\alpha,\beta)^+A$, showing that $(\alpha,\beta)^+$ is surjective. $\square$

Let **Ω-Pol** be the category whose objects are the Ω-polarities and whose arrows are the bounded morphisms between such objects. The identity arrow id_P on each object P is the pair $\mathrm{id}_X, \mathrm{id}_Y$ of identity functions on X and Y. The composition of two arrows

$$P \xrightarrow{\alpha,\beta} P' \xrightarrow{\alpha',\beta'} P''$$

is given by the pair of functional compositions $\alpha' \circ \alpha \colon X \to X''$ and $\beta' \circ \beta \colon Y \to Y''$.

Lemma 14. *The pair $\alpha' \circ \alpha, \beta' \circ \beta$ is a bounded morphism from P to P''.*

Proof. It is straightforward that the composition of isotone functions is isotone. For the back and forth conditions we give the details for (1_S) and (2_S), since the others follow the same pattern.

For (1_S), observe that if $(\alpha' \circ \alpha)(\vec{x})S''(\beta' \circ \beta)(y)$, then $\alpha(\vec{x})S'\beta(y)$ by (1_S) for α', β', hence $\vec{x}Sy$ by (1_S) for α, β.

For (2_S), we argue contrapositively and take $\vec{x''} \in (X'')^n$ and $y \in Y$ such that not $\vec{x''}S''(\beta' \circ \beta)(y)$. Then as α', β' is a bounded morphism, there exists $\vec{x'} \in (X')^n$ such that $\vec{x''} \preccurlyeq_1 \alpha'(\vec{x'})$ and not $\vec{x'}S'\beta(y)$. Hence as α, β is a bounded morphism, there exists $\vec{x} \in X^n$ such that $\vec{x'} \preccurlyeq_1 \alpha(\vec{x})$ and not $\vec{x}Sy$. From $\vec{x'} \preccurlyeq_1 \alpha(\vec{x})$ we get $\alpha'(\vec{x'}) \preccurlyeq_1 \alpha'(\alpha(\vec{x}))$ *as α' is isotone*, hence its action on tuples is isotone. Since $\vec{x''} \preccurlyeq_1 \alpha'(\vec{x'})$ it follows that $\vec{x''} \preccurlyeq_1 \alpha'(\alpha(\vec{x}))$, so we have $\vec{x} \in (\alpha' \circ \alpha)^{-1}[\vec{x''})_1$ while not $\vec{x}Sy$, hence not $(\alpha' \circ \alpha)^{-1}[\vec{x''})_1Sy$, confirming (2_S) for the pair $\alpha' \circ \alpha, \beta' \circ \beta$.

The cases of (3_R) and (2_T) depend on β' being isotone. $\square$

Let **Ω-NLO** be the category whose objects are the normal Ω-lattices with operators and whose arrows are the algebraic homomorphisms between Ω-NLO's, with the composition of arrows being their functional composition and the identity arrows being the identity functions. Then with the help of Theorem 13 and Lemma 14 we see that the mappings $P \mapsto P^+$ and $\alpha, \beta \mapsto (\alpha, \beta)^+$ form a contravariant functor from **Ω-Pol** to **Ω-NLO**.

5 Isomorphism and Inner Substructures

Every category provides a definition of isomorphism between its objects. Thus the existence of **Ω-Pol** allows us to read off a description of isomorphisms between Ω-polarities. A bounded morphism $\mu \colon P \to P'$ is called an isomorphism if there exists a bounded morphism $\mu' \colon P' \to P$ such that $\mu' \circ \mu = \mathrm{id}_P$ and $\mu \circ \mu' = \mathrm{id}_{P'}$. Then μ' is the *inverse* of μ. It is itself an isomorphism, with inverse μ.

We say that a bounded morphism α, β *preserves polarity*, or *preserves* R, if xRy implies $\alpha(x)R'\beta(y)$, i.e. if the converse of (1_R) holds. Similarly α, β *preserves* S if $\vec{x}Sy$ implies $\alpha(\vec{x})S'\beta(y)$, and *preserves* T if $xT\vec{y}$ implies $\alpha(x)T'\beta(\vec{y})$.

The function α *reflects quasi-order* if $\alpha(x) \preccurlyeq_1' \alpha(z)$ implies $x \preccurlyeq_1 z$. Likewise β *reflects quasi-order* if $\beta(y) \preccurlyeq_2' \beta(w)$ implies $y \preccurlyeq_2 w$.

Lemma 15. *For any bounded morphism α, β the following statements are equivalent.*

(1) *α, β preserves polarity.*

(2) *α reflects quasi-order.*

(3) *β reflects quasi-order.*

Proof. (1) implies (2): Assume (1). Let $x, z \in X$ have $\alpha(x) \preccurlyeq_1' \alpha(z)$. Then for all $y \in Y$, if xRy, then $\alpha(x)R'\beta(y)$ by (1), and so $\alpha(z)R'\beta(y)$ by definition of $\alpha(x) \preccurlyeq_1' \alpha(z)$, hence zRy by (1_R). Thus $\rho_R\{x\} \subseteq \rho_R\{z\}$, i.e. $x \preccurlyeq_1 z$. Altogether this proves (2).

(2) implies (1): Assume (2). We prove (1) contrapositively. Suppose that not $\alpha(x)R'\beta(y)$. Then by (2_R), there exists $z \in X$ with $\alpha(x) \preccurlyeq_1' \alpha(z)$ and not zRy. Hence $x \preccurlyeq_1 z$ by (2), so by definition of $\preccurlyeq_1$ we get not xRy as required for (1).

That completes a proof that (1) is equivalent to (2). Similarly we can prove that (1) is equivalent to (3), using (3_R) in place of (2_R). □

Theorem 16. *A bounded morphism $\mu = (\alpha, \beta) \colon P \to P'$ is an isomorphism iff $\alpha \colon X \to X'$ and $\beta \colon Y \to Y'$ are bijective and preserve the relations R, S and T.*

Proof. Suppose there is a bounded morphism $\mu' = (\alpha', \beta')$ that is inverse to μ. Then $\alpha' \circ \alpha = \mathrm{id}_X$ and $\alpha \circ \alpha' = \mathrm{id}_{X'}$, so α has an inverse α' and therefore is bijective. Similarly β is bijective. Also, if $\vec{x} S y$, then $\alpha'(\alpha(\vec{x})) S \beta'(\beta(y))$, so $\alpha(\vec{x}) S' \beta(y)$ by (1_S) for the bounded morphism α', β'. This proves that S is preserved. The proof that R and T are preserved is similar, using (1_R) and (1_T).

Conversely, assume that α and β are bijective and preserve the relations. Then α and β have inverses $\alpha' \colon X' \to X$ and $\beta' \colon Y' \to Y$. We show that $\mu' = (\alpha', \beta')$ is a bounded morphism, which will then be an inverse for μ, as required for μ to be an isomorphism.

First, as R is preserved, α and β reflect quasi-orders by Lemma 15. If $x', z' \in X'$ have $x' \preccurlyeq_1' z'$, then there exist $x, z \in X$ with $x' = \alpha(x)$ and $z' = \alpha(z)$, so $x \preccurlyeq_1 z$ from reflection by α. But this means that $\alpha'(x') \preccurlyeq_1 \alpha'(z')$, and shows that α' is isotone. Similarly β' is isotone.

To prove the back condition (1_S) for α', β', let $\alpha'(\vec{x'}) S \beta'(y)$. Then as (α, β) preserves S we get $\alpha(\alpha'(\vec{x'})) S' \beta(\beta'(y))$, i.e. $\vec{x'} S' y$ as required. The cases of (1_R) and (1_T) for α', β' are similar, using the preservation of R and T.

For a forth condition for α', β', we prove (2_S). Let $\vec{x} \in X^n$ and $y' \in Y'$, and suppose that not $\vec{x} S \beta'(y')$. We have to show that not $(\alpha')^{-1}[\vec{x})_1 S' y'$, i.e. that there exists $\vec{w} \in (X')^n$ such that $\vec{x} \preccurlyeq_1 \alpha'(\vec{w})$ and not $\vec{w} S' y'$. Now by (1_S) for α, β, from not $\vec{x} S \beta'(y')$ we get that not $\alpha(\vec{x}) S' \beta(\beta'(y'))$, hence by (2_S) for α, β we get not $\alpha^{-1}[\alpha(\vec{x}))_1 S \beta'(y')$. So there exists $\vec{z} \in \alpha^{-1}[\alpha(\vec{x}))_1$, hence $\alpha(\vec{x}) \preccurlyeq_1' \alpha(\vec{z})$, such that not $\vec{z} S \beta'(y')$. Let $\vec{w} = \alpha(\vec{z})$. Then as α' is isotone, $\alpha'(\alpha(\vec{x})) \preccurlyeq_1 \alpha'(\alpha(\vec{z}))$, i.e. $\vec{x} \preccurlyeq_1 \alpha'(\vec{w})$. Also from not $\vec{z} S \beta'(y')$, by (1_S) for α, β we get not $\alpha(\vec{z}) S' \beta(\beta'(y'))$, i.e. not $\vec{w} S' y$ as required.

That proves (2_S) for α', β'. The cases of the other forth condition for α', β' are similar. $\square$

We define P to be an *inner substructure* of P' if the following holds.

(i) P is a substructure of P' in the usual sense that $X \subseteq X'$, $Y \subseteq Y'$, and the relations of P are the restrictions of those of P', i.e. $R = R' \cap (X \times Y)$, $S = S' \cap (X^n \times Y)$, and $T = T' \cap (X \times Y^m)$.

(ii) The pair of inclusion maps $X \hookrightarrow X$ and $Y \hookrightarrow Y'$ form a bounded morphism from P to P'.

Condition (i) here entails that (1_R), (1_S) and (1_T) hold when α and β are the inclusions, e.g. (1_R) asserts that $xR'y$ implies xRy when $x \in X$ and $y \in Y$. Given (i), condition (ii) is equivalent to requiring the following.

(2_R) $\{x \in X : x' \preccurlyeq_1' x\}Ry$ implies $x'R'y$, all $x' \in X', y \in Y$;

(3_R) $xR\{y \in Y : y' \preccurlyeq_2' y\}$ implies $xR'y'$, all $x \in X, y' \in Y'$;

(2_S) $\{\vec{x} \in X^n : \vec{x'} \preccurlyeq_1' \vec{x}\}Sy$ implies $\vec{x'}S'y$, all $\vec{x'} \in (X')^n,\ y \in Y$;

(2_T) $xT\{\vec{y} \in Y^m : \vec{y'} \preccurlyeq_2' \vec{y}\}$ implies $xT'\vec{y'}$, all $x \in X,\ \vec{y'} \in (Y')^m$.

Theorem 17. *It P is an inner substructure of P', then the map $A \mapsto A \cap X$ is an Ω-lattice homomorphism from $(P')^+$ onto P^+.*

Proof. If α is the inclusion $X \hookrightarrow X'$, then $\alpha^{-1}A = A \cap X$, so by Theorem 13, $A \mapsto A \cap X$ is a surjective homomorphism $(P')^+ \to P^+$ of Ω-lattices, since α is injective. □

The *image* of a bounded morphism $\alpha, \beta : P \to P'$ is defined to be the structure

$$\mathrm{Im}(\alpha, \beta) = (\alpha[X], \beta[Y], R'', S'', T''),$$

where the relations displayed are the restrictions of the corresponding relations of P', i.e. $R'' = R' \cap (\alpha[X] \times \beta[Y])$, $S'' = S' \cap (\alpha[X]^n \times \beta[Y])$, and $T'' = T' \cap (\alpha[X] \times \beta[Y]^m)$.

Lemma 18. (1) *The quasi-orders $\preccurlyeq_1''$ and $\preccurlyeq_2''$ defined from R'' are the restrictions of the relations $\preccurlyeq_1'$ and $\preccurlyeq_2'$ to $\alpha[X]$ and $\beta[Y]$, respectively.*

(2) *All sections of S'' and T'' are stable in* $\mathrm{Im}(\alpha, \beta)$.

Proof. For part (1), we show that $\alpha(x) \preccurlyeq_1'' \alpha(z)$ iff $\alpha(x) \preccurlyeq_1' \alpha(z)$ for all $x, z \in X$. Suppose first that $\alpha(x) \preccurlyeq_1' \alpha(z)$. By definition this means that $\alpha(x)R'y'$ implies $\alpha(z)R'y'$ for all $y' \in Y'$ (see (2.3)). In particular, for any $y \in Y$, if $\alpha(x)R''\alpha(y)$ then $\alpha(x)R'\alpha(y)$, hence $\alpha(z)R'\alpha(y)$ and so $\alpha(z)R''\alpha(y)$. This shows $\rho_{R''}\{\alpha(x)\} \subseteq \rho_{R''}\{\alpha(z)\}$, i.e. $\alpha(x) \preccurlyeq_1'' \alpha(z)$.

Conversely, let $\alpha(x) \preccurlyeq_1'' \alpha(z)$. For any $y' \in Y'$, if not $\alpha(z)R'y'$, then by (3_R) there exists $y \in Y$ such that $y' \preccurlyeq_2' \beta(y)$ and not zRy. Hence not $\alpha(z)R'\beta(y)$ by (1_R), and so not $\alpha(z)R''\beta(y)$. This implies not $\alpha(x)R''\beta(y)$ because $\alpha(x) \preccurlyeq_1'' \alpha(z)$. Hence not $\alpha(x)R'\beta(y)$, so then not $\alpha(x)R'y'$ as $y' \preccurlyeq_2' \beta(y)$. Altogether this proves $\rho_{R'}\{\alpha(x)\} \subseteq \rho_{R'}\{\alpha(z)\}$, i.e. $\alpha(x) \preccurlyeq_1' \alpha(z)$.

The proof that $\preccurlyeq_2''$ is the restriction of $\preccurlyeq_2'$ to $\beta[Y]$ is similar.

For part (2), consider a section of the form $S''[\alpha(\vec{x})[-]_i, \beta(y)]$. If an element $\alpha(z)$ of $\alpha[X]$ does not belong to this section, then it does not belong to $S'[\alpha(\vec{x})[-]_i, \beta(y)]$. But the latter section is stable in P', so there is some

$$y' \in \rho_{R'} S'[\alpha(\vec{x})[-]_i, \beta(y)] \tag{5.1}$$

such that not $\alpha(z)R'y'$. Then not $zR\beta^{-1}[y')_2$ by (3_R), so there is some $w \in Y$ such that $y' \preccurlyeq_2' \beta(w)$ and not zRw. Hence by (1_R), not $\alpha(z)R'\beta(w)$, and therefore not $\alpha(z)R''\beta(w)$. Now we show that

$$\beta(w) \in \rho_{R''}S''[\alpha(\vec{x})[-]_i, \beta(y)]. \tag{5.2}$$

For if $t \in S''[\alpha(\vec{x})[-]_i, \beta(y)]$, then $t \in S'[\alpha(\vec{x})[-]_i, \beta(y)]$, so by (5.1), $tR'y'$. But $y' \preccurlyeq_2' \beta(w)$, so then $tR'\beta(w)$ by (2.5). Hence $tR''\beta(w)$ as $t \in \alpha[X]$. This proves (5.2). Since not $\alpha(z)R''\beta(w)$, $\alpha(z) \notin \lambda_{R''}\rho_{R''}S''[\alpha(\vec{x})[-]_i, \beta(y)]$, completing the proof that $S''[\alpha(\vec{x})[-]_i, \beta(y)]$ is stable.

The argument for sections of the form $S''[\alpha(\vec{x}), -]$ is similar, using (2_R). The arguments for the sections of T'' follow the same patterns. □

Corollary 19. $\mathrm{Im}(\alpha, \beta)$ *is an inner substructure of* P' *.*

Proof. Part (2) of the Lemma confirms that $\mathrm{Im}(\alpha, \beta)$ is an Ω-polarity. By part (1), the inclusions $(\alpha[X], \preccurlyeq_1'') \hookrightarrow (X', \preccurlyeq_1')$ and $(\beta[Y], \preccurlyeq_2'') \hookrightarrow (Y', \preccurlyeq_2')$ are isotone. We show that they satisfy the back and forth properties of Definition 8, so they form a bounded morphism, making $\mathrm{Im}(\alpha, \beta)$ an inner substructure of P' by definition. Since $\mathrm{Im}(\alpha, \beta)$ is defined to be a substructure of P', the inclusions do satisfy the back conditions, as already noted.

For the forward conditions, we consider (2_R). This requires that for any $x' \in X'$ and $w \in \beta[Y]$, if not $x'R'w$ then there exists $z \in \alpha[X]$ with $x' \preccurlyeq_1' z$ and not $zR''w$. Now $w = \beta(y)$ for some $y \in Y$, and α and β satisfy (2_R), so if not $x'R'\beta(y)$ then there exists $x \in X$ with $x' \preccurlyeq_1' \alpha(x)$ and not xRy. Hence not $\alpha(x)R'\beta(y)$ by (1_R), and so not $\alpha(x)R''\beta(y)$. Thus putting $z = \alpha(x)$ fulfills our requirement for (2_R). The proofs that the inclusions satisfy the other forward conditions are similar. □

Thus we have the general fact that the image of a bounded morphism is an inner substructure of its codomain.

Theorem 20. *If* α *and* β *are injective and preserve the relations* R*,* S *and* T*, then they give an isomorphism between* P *and* $\mathrm{Im}(\alpha, \beta)$*.*

Proof. By Lemma 18(1), as α is isotone as a map from $(X, \preccurlyeq_1)$ to $(X', \preccurlyeq_1')$, it is isotone as a map from $(X, \preccurlyeq_1)$ to $(\alpha[X], \preccurlyeq_1'')$. Likewise β is isotone as a map from $(Y, \preccurlyeq_2)$ to $(\beta[Y], \preccurlyeq_2'')$, so α, β acts as a bounded morphism from P to $\mathrm{Im}(\alpha, \beta)$.

If α and β are injective, then they are bijective as maps to $\alpha[X]$ and $\beta[Y]$, so if they preserve the relations as well then Theorem 16 ensures that they give an isomorphism from P to $\mathrm{Im}(\alpha, \beta)$. □

6 Canonical Extensions

Lattice homomorphisms are assumed to preserve the bounds 0 and 1, as well as $\wedge$ and $\vee$. An injective homomorphism (*mono*morphism) may be denoted by $\rightarrowtail$, and a surjective one (*epi*morphism) by $\twoheadrightarrow$. A function $\theta \colon \mathbb{L} \to \mathbb{M}$ between lattices is called a *lattice embedding* if it is a lattice monomorphism. A lattice embedding is always *order invariant*, i.e. has $a \leqslant b$ iff $\theta a \leqslant \theta b$.

First we review the definition of a canonical extension of a lattice, as given in [18]. A *completion* of lattice $\mathbb{L}$ is a pair $(\theta, \mathbb{C})$ with $\mathbb{C}$ a complete lattice and $\theta \colon \mathbb{L} \rightarrowtail \mathbb{C}$ a lattice embedding. An element of $\mathbb{C}$ is *open* if it is a join of elements from the θ-image $\theta[\mathbb{L}]$ of $\mathbb{L}$ and *closed* if it is a meet of elements from $\theta[\mathbb{L}]$. Members of $\theta[\mathbb{L}]$ are both open and closed. The set of open elements of the completion is denoted $O(\mathbb{C})$, and the set of closed elements is $K(\mathbb{C})$.

A completion $(\theta, \mathbb{C})$ of $\mathbb{L}$ is *dense* if $K(\mathbb{C})$ is join-dense and $O(\mathbb{C})$ is meet-dense in $\mathbb{C}$, i.e. if every member of $\mathbb{C}$ is both a join of closed elements and a meet of open elements. It is *compact* if for any set Z of closed elements and any set W of open elements such that $\bigwedge Z \leqslant \bigvee W$, there are finite sets $Z' \subseteq Z$ and $W' \subseteq W$ with $\bigwedge Z' \leqslant \bigvee W'$. An equivalent formulation of this condition that we will use (in Theorem 23) is that for any subsets Z and W of $\mathbb{L}$ such that $\bigwedge \theta[Z] \leqslant \bigvee \theta[W]$ there are finite sets $Z' \subseteq Z$ and $W' \subseteq W$ with $\bigwedge Z' \leqslant \bigvee W'$.

A *canonical extension* of lattice $\mathbb{L}$ is a completion $(\theta_{\mathbb{L}}, \mathbb{L}^\sigma)$ of $\mathbb{L}$ which is dense and compact. Any two such completions are isomorphic by a unique isomorphism commuting with the embeddings of $\mathbb{L}$. This legitimises talk of "the" canonical extension, and the assignment of a name $\mathbb{L}^\sigma$ to it.

A function $f \colon \mathbb{L} \to \mathbb{M}$ between lattices can be lifted it to a function $\mathbb{L}^\sigma \to \mathbb{M}^\sigma$ between their canonical extensions in two ways, using the embeddings $\theta_{\mathbb{L}} \colon \mathbb{L} \rightarrowtail \mathbb{L}^\sigma$ and $\theta_{\mathbb{M}} \colon \mathbb{M} \rightarrowtail \mathbb{M}^\sigma$ to form the *lower* canonical extension $f^\triangledown$ and *upper* canonical extension $f^\vartriangle$ of f (in [18] these are denoted f^σ and f^π respectively). Let $\mathbb{I}$ be the set of all intervals of the form $\{x : p \leqslant x \leqslant q\}$ in $\mathbb{L}^\sigma$ with $p \in K(\mathbb{L}^\sigma)$ and $q \in O(\mathbb{L}^\sigma)$. Then for $x \in \mathbb{L}^\sigma$,

$$f^\triangledown x = \bigvee \{ \bigwedge \{\theta_{\mathbb{M}}(fa) : a \in \mathbb{L} \text{ and } \theta_{\mathbb{L}}(a) \in E\} : x \in E \in \mathbb{I}\}. \tag{6.1}$$

$$f^\vartriangle x = \bigwedge \{ \bigvee \{\theta_{\mathbb{M}}(fa) : a \in \mathbb{L} \text{ and } \theta_{\mathbb{L}}(a) \in E\} : x \in E \in \mathbb{I}\}. \tag{6.2}$$

The functions $f^\triangledown$ and $f^\vartriangle$ have $f^\triangledown x \leqslant f^\vartriangle x$. They both extend f in the sense that the diagram

$$\begin{tikzcd} \mathbb{L} \arrow[r, "f"] \arrow[d, tail, "\theta_{\mathbb{L}}"'] & \mathbb{M} \arrow[d, tail, "\theta_{\mathbb{M}}"] \\ \mathbb{L}^\sigma \arrow[r, "h"] & \mathbb{M}^\sigma \end{tikzcd}$$

commutes when $h = f^{\triangledown}$ or $h = f^{\vartriangle}$. For *isotone* f, these extensions can also be specified as follows [18, Lemma 4.3].

$$f^{\triangledown}p = \bigwedge\{\theta_{\mathbb{M}}(fa) : a \in \mathbb{L} \text{ and } p \leqslant \theta_{\mathbb{L}}(a)\}, \quad \text{for all } p \in K(\mathbb{L}^{\sigma}). \tag{6.3}$$

$$f^{\triangledown}x = \bigvee\{f^{\triangledown}p : p \in K(\mathbb{L}^{\sigma}) \text{ and } p \leqslant x\}, \quad \text{for all } x \in \mathbb{L}^{\sigma}. \tag{6.4}$$

$$f^{\vartriangle}q = \bigvee\{\theta_{\mathbb{M}}(fa) : a \in \mathbb{L} \text{ and } q \geqslant \theta_{\mathbb{L}}(a)\} \quad \text{for all } q \in O(\mathbb{L}^{\sigma}). \tag{6.5}$$

$$f^{\vartriangle}x = \bigwedge\{f^{\vartriangle}q : q \in O(\mathbb{L}^{\sigma}) \text{ and } q \geqslant x\}, \quad \text{for all } x \in \mathbb{L}^{\sigma}. \tag{6.6}$$

If $f\colon \mathbb{L}^n \to \mathbb{L}$ is an n-ary operation on $\mathbb{L}$, then $f^{\triangledown}$ and $f^{\vartriangle}$ are maps from $(\mathbb{L}^n)^{\sigma}$ to $\mathbb{L}^{\sigma}$. But $(\mathbb{L}^n)^{\sigma}$ can be identified with $(\mathbb{L}^{\sigma})^n$, since the natural embedding $\mathbb{L}^n \rightarrowtail (\mathbb{L}^{\sigma})^n$ is dense and compact, so this allows $f^{\triangledown}$ and $f^{\vartriangle}$ to be regarded as an n-ary operations on $\mathbb{L}^{\sigma}$. Moreover it is readily seen that $K((\mathbb{L}^{\sigma})^n) = (K(\mathbb{L}^{\sigma}))^n$, i.e. a closed element of $(\mathbb{L}^{\sigma})^n$ is an n-tuple of closed elements of $\mathbb{L}^{\sigma}$, and likewise $O((\mathbb{L}^{\sigma})^n) = (O(\mathbb{L}^{\sigma}))^n$. This will be important below, where we apply the lower canonical extension to operators on $\mathbb{L}$, and the upper extension to dual operators.

For any Ω-lattice $\mathbb{L}$, based on a lattice $\mathbb{L}_0$, we define a canonical extension $\mathbb{L}^{\sigma}$ for $\mathbb{L}$ by taking the canonical extension of $\mathbb{L}_0$, applying the lower extension to operations denoted by members of Λ, and the upper extension to operations denoted by members of Υ, to form

$$\mathbb{L}^{\sigma} = (\mathbb{L}_0^{\sigma}, \{(\mathbf{f}^{\mathbb{L}})^{\triangledown} : \mathbf{f} \in \Lambda\} \cup \{(\mathbf{g}^{\mathbb{L}})^{\vartriangle} : \mathbf{g} \in \Upsilon\}). \tag{6.7}$$

It is shown in [18, Section 4] (see also [41, 2.2.14]) that if $\mathbb{L}$ is an NLO, then so is $\mathbb{L}^{\sigma}$, with each $(\mathbf{f}^{\mathbb{L}})^{\triangledown}$ being a complete normal operator, and each $(\mathbf{g}^{\mathbb{L}})^{\vartriangle}$ being a complete normal dual operator.

7 Canonical Structures

The existence of $\mathbb{L}^{\sigma}$ as a canonical extension was established in [18] by taking it to be the stable set lattice of a certain polarity between filters and ideals of $\mathbb{L}$, with the additional operations of $\mathbb{L}$ being extended to $\mathbb{L}^{\sigma}$ by the abstract lattice-theoretic definitions (6.1) and (6.2). We will now see that if $\mathbb{L}$ is an Ω-NLO, then the polarity can be expanded to an Ω-polarity, which we call the *canonical structure* of $\mathbb{L}$, and whose stable set Ω-lattice, as in (3.12), is a canonical extension of $\mathbb{L}$.

Recall that we are assuming that $\Lambda = \{\mathbf{f}\}$ and $\Upsilon = \{\mathbf{g}\}$. In what follows we write the n-ary operator $\mathbf{f}^{\mathbb{L}}$ just as f and the m-ary dual operator $\mathbf{g}^{\mathbb{L}}$ just as g. Let $\mathscr{F}_{\mathbb{L}}$ be the set of non-empty filters of $\mathbb{L}$ and $\mathscr{I}_{\mathbb{L}}$ be the set of non-empty ideals of

$\mathbb{L}$. For $F \in \mathscr{F}_{\mathbb{L}}$ and $D \in \mathscr{I}_{\mathbb{L}}$, write $F \between D$ to mean that F and D *overlap*, i.e. $F \cap D \neq \emptyset$. Define the *canonical structure of* $\mathbb{L}$ to be the structure

$$\mathbb{L}_+ = (\mathscr{F}_{\mathbb{L}}, \mathscr{I}_{\mathbb{L}}, \between, S_{\mathbb{L}}, T_{\mathbb{L}},),$$

where, for $\vec{F} \in \mathscr{F}_{\mathbb{L}}{}^n$ and $D \in \mathscr{I}_{\mathbb{L}}$,

$$\vec{F} S_{\mathbb{L}} D \quad \text{iff} \quad \text{there exists } \vec{a} \in_\pi \vec{F} \text{ with } f(\vec{a}) \in D;$$

while, for $F \in \mathscr{F}_{\mathbb{L}}$ and $\vec{D} \in \mathscr{I}_{\mathbb{L}}{}^m$,

$$F T_{\mathbb{L}} \vec{D} \quad \text{iff} \quad \text{there exists } \vec{a} \in_\pi \vec{D} \text{ with } g(\vec{a}) \in F.$$

Lemma 21. *In $\mathbb{L}_+$, if $F, F' \in \mathscr{F}_{\mathbb{L}}$ then $F \preccurlyeq_1 F'$ iff $F \subseteq F'$; and if $D, D' \in \mathscr{I}_{\mathbb{L}}$ then $D \preccurlyeq_2 D'$ iff $D \subseteq D'$.*

Proof. If $F \subseteq F'$, then $F \between D$ implies $F' \between D$, so $\rho_\between\{F\} \subseteq \rho_\between\{F'\}$, i.e. $F \preccurlyeq_1 F'$ by (2.3). Conversely, suppose $\rho_\between\{F\} \subseteq \rho_\between\{F'\}$. If $a \in F$, let D be the ideal $\{b \in \mathbb{L} : b \leqslant a\}$ generated by a. Then $a \in F \cap D$, so $D \in \rho_\between\{F\}$, hence there exists $b \in F' \cap D$, so $b \leqslant a$ and thus $a \in F'$. This shows $F \subseteq F'$. The case of $\preccurlyeq_2$ is the order-dual of this argument. □

Assume from now that $\mathbb{L}$ is an Ω-NLO.

Lemma 22. *All sections of the relations $S_{\mathbb{L}}$ and $T_{\mathbb{L}}$ are stable in $\mathbb{L}_+$, making $\mathbb{L}_+$ an Ω-polarity.*

Proof. First consider a section of the form $S_{\mathbb{L}}[\vec{F}, -]$ with $\vec{F} \in \mathscr{F}_{\mathbb{L}}{}^n$. Let G be the filter of $\mathbb{L}$ generated by $\{f(\vec{a}) : \vec{a} \in_\pi \vec{F}\}$. For any $D \in S_{\mathbb{L}}[\vec{F}, -]$ there exists $\vec{a} \in_\pi \vec{F}$ such that $f(\vec{a}) \in D$. But then $f(\vec{a}) \in G$, so $G \between D$. This proves $G \in \lambda_\between S_{\mathbb{L}}[\vec{F}, -]$. Now take any $D \in \rho_\between \lambda_\between S_{\mathbb{L}}[\vec{F}, -]$. Then $G \between D$ so there exists a $d \in G$ with $d \in D$. By definition of G there is a finite subset Z of $\pi\vec{F}$ such that

$$\bigwedge\{f(\vec{a}) : \vec{a} \in Z\} \leqslant d. \tag{7.1}$$

For all $i < n$ put $b_i = \bigwedge\{a_i : \vec{a} \in Z\} \in F_i$, and let $\vec{b} = (b_0, \ldots, b_{n-1}) \in_\pi \vec{F}$. Then for all $\vec{a} \in Z$ we have $\vec{b} \leqslant \vec{a}$, so $f(\vec{b}) \leqslant f(\vec{a})$ as operators are isotone. Hence by (7.1) $f(\vec{b}) \leqslant d \in D$, so $f(\vec{b}) \in D$. As $\vec{b} \in_\pi \vec{F}$, this gives $\vec{F} S_{\mathbb{L}} D$, so $D \in S_{\mathbb{L}}[\vec{F}, -]$. We have now shown that $\rho_\between \lambda_\between S_{\mathbb{L}}[\vec{F}, -] \subseteq S_{\mathbb{L}}[\vec{F}, -]$, which is enough to conclude that $S_{\mathbb{L}}[\vec{F}, -]$ is stable.

Next take a section of the form $S_{\mathbb{L}}[\overrightarrow{F}[-]_i, D]$ with $D \in \mathscr{I}_{\mathbb{L}}$. Let E be the ideal generated by the set

$$E_0 = \{b \in \mathbb{L} : \exists \overrightarrow{a} \in_\pi \overrightarrow{F}[\mathbb{L}/i](f(\overrightarrow{a}[b/i]) \in D)\}.$$

For any $G \in S_{\mathbb{L}}[\overrightarrow{F}[-]_i, D]$ we have $\overrightarrow{F}[G/i]S_{\mathbb{L}}D$ so there exist $\overrightarrow{a} \in \overrightarrow{F}[\mathbb{L}/i]$ and $b \in G$ such that $\overrightarrow{a}[b/i] \in \overrightarrow{F}[G/i]$ and $f(\overrightarrow{a}[b/i]) \in D$. Then $b \in E_0$, so $G \between E$. This proves that $E \in \rho_\between S_{\mathbb{L}}[\overrightarrow{F}[-]_i, D]$.

Now let $G \in \lambda_\between \rho_\between S_{\mathbb{L}}[\overrightarrow{F}[-]_i, D]$. Then there exists $d \in G \cap E$. By definition of E there is a finite set $Z \subseteq E_0$ with $d \leqslant \bigvee Z$. Hence $\bigvee Z \in G$ as d belongs to the filter G. For each $b \in Z$ there exists $\overrightarrow{a_b} \in_\pi \overrightarrow{F}[\mathbb{L}/i]$ such that $f(\overrightarrow{a_b}[b/i]) \in D$. Now for all $j < n$, put $c_j = \bigwedge\{(a_b)_j : b \in Z\}$. Then $c_j \in F_j$ provided $j \neq i$. Let $\overrightarrow{c} = (c_0, \dots, c_{n-1})$. Then for all $b \in Z$, $\overrightarrow{c}[b/i] \leqslant \overrightarrow{a_b}[b/i]$, so $f(\overrightarrow{c}[b/i]) \leqslant f(\overrightarrow{a_b}[b/i]) \in D$, hence $f(\overrightarrow{c}[b/i]) \in D$. *Since f is a normal operator*, we conclude that

$$f(\overrightarrow{c}[\textstyle\bigvee Z/i]) = \bigvee\{f(\overrightarrow{c}[b/i]) : b \in Z\} \in D.$$

But $\bigvee Z \in G$, so $\overrightarrow{c}[\bigvee Z/i] \in \overrightarrow{F}[G/i]$, implying that $\overrightarrow{F}[G/i]S_{\mathbb{L}}D$ and thus $G \in S_{\mathbb{L}}[\overrightarrow{F}[-]_i, D]$. This proves that $\lambda_\between \rho_\between S_{\mathbb{L}}[\overrightarrow{F}[-]_i, D] \subseteq S_{\mathbb{L}}[\overrightarrow{F}[-]_i, D]$, so $S_{\mathbb{L}}[\overrightarrow{F}[-]_i, D]$ is stable.

The arguments for the stability sections of $T_{\mathbb{L}}$ are essentially the duals of those for $S_{\mathbb{L}}$, but we go through the details, first for a section of the form $T_{\mathbb{L}}[-, \overrightarrow{D}]$ with $\overrightarrow{D} \in \mathscr{I}^m$.

Let E be the ideal generated by $\{g(\overrightarrow{a}) : \overrightarrow{a} \in_\pi \overrightarrow{D}\}$. For any $G \in T_{\mathbb{L}}[-, \overrightarrow{D}]$ there exists $\overrightarrow{a} \in_\pi \overrightarrow{D}$ such that $g(\overrightarrow{a}) \in G$. But then $g(\overrightarrow{a}) \in E$, so $G \between E$. This proves $E \in \rho_\between T_{\mathbb{L}}[-, \overrightarrow{D}]$. Now take any $G \in \lambda_\between \rho_\between T_{\mathbb{L}}[-, \overrightarrow{D}]$. Then $G \between E$ so there exists a $b \in G$ with $b \in E$. By definition of E there is a finite subset Z of $\pi\overrightarrow{D}$ such that

$$b \leqslant \bigvee\{g(\overrightarrow{a}) : \overrightarrow{a} \in Z\}. \tag{7.2}$$

For all $i < m$ put $d_i = \bigvee\{a_i : \overrightarrow{a} \in Z\} \in D_i$, and let $\overrightarrow{d} = (d_0, \dots, d_{n-1}) \in_\pi \overrightarrow{D}$. Then for all $\overrightarrow{a} \in Z$ we have $\overrightarrow{a} \leqslant \overrightarrow{d}$, so $g(\overrightarrow{a}) \leqslant g(\overrightarrow{d})$ as dual operators are isotone. Hence by (7.2) $b \leqslant g_{\mathbb{L}}(\overrightarrow{d})$. Then $g_{\mathbb{L}}(\overrightarrow{d}) \in F$ as $b \in F$. As $\overrightarrow{d} \in_\pi \overrightarrow{D}$, this gives $FT_{\mathbb{L}}\overrightarrow{D}$, so $F \in T_{\mathbb{L}}[-, \overrightarrow{D}]$. We have now shown that $\lambda_\between \rho_\between T_{\mathbb{L}}[-, \overrightarrow{D}] \subseteq T_{\mathbb{L}}[-, \overrightarrow{D}]$, hence $T_{\mathbb{L}}[-, \overrightarrow{D}]$ is stable.

Finally we consider a section of the form $T_{\mathbb{L}}[F, \overrightarrow{D}[-]_i]$ with $F \in \mathscr{F}_{\mathbb{L}}$. Let G be the filter generated by the set

$$G_0 = \{d \in \mathbb{L} : \exists \overrightarrow{a} \in_\pi \overrightarrow{D}[\mathbb{L}/i](g(\overrightarrow{a}[d/i]) \in F)\}.$$

For any $E \in T_{\mathbb{L}}[F, \vec{D}[-]_i]$ we have $FT_{\mathbb{L}}\vec{D}[E/i]$ so there exist $\vec{a} \in \vec{D}[\mathbb{L}/i]$ and $d \in E$ such that $\vec{a}[d/i] \in \vec{D}[E/i]$ and $g(\vec{a}[d/i]) \in F$. Then $d \in G_0$, so $G \between E$. This proves that $G \in \lambda_{\between} T_{\mathbb{L}}[F, \vec{D}[-]_i]$. Now let $E \in \rho_{\between}\lambda_{\between} T_{\mathbb{L}}[F, \vec{D}[-]_i]$. Then there exists $b \in G \cap E$. By definition of G there is a finite set $Z \subseteq G_0$ with $\bigwedge Z \leqslant b$. Hence $\bigwedge Z \in E$ as b belongs to the ideal E. For each $d \in Z$ there exists $\vec{a_d} \in_\pi \vec{D}[\mathbb{L}/i]$ such that $g(\vec{a_d}[d/i]) \in F$. Now for all $j < m$, put $c_j = \bigvee\{(a_d)_j : d \in Z\}$. Then $c_j \in D_j$ provided $j \neq i$. Let $\vec{c} = (c_0, \dots, c_{m-1})$. Then for all $d \in Z$, $\vec{a_d}[d/i] \leqslant \vec{c}[d/i]$, so $g(\vec{a_d}[d/i]) \leqslant g(\vec{c}[d/i])$, hence $g(\vec{c}[d/i]) \in F$. *Since g is a normal dual operator*, we conclude that

$$g(\vec{c}[\textstyle\bigwedge Z/i]) = \textstyle\bigwedge\{g(\vec{c}[d/i]) : d \in Z\} \in F.$$

But $\bigwedge Z \in E$, so $\vec{c}[\bigwedge Z/i] \in \vec{D}[E/i]$, implying that $FT_{\mathbb{L}}\vec{D}[E/i]$ and thus $E \in T_{\mathbb{L}}[F, \vec{D}[-]_i]$. This proves that $\rho_{\between}\lambda_{\between} T_{\mathbb{L}}[F, \vec{D}[-]_i] \subseteq T_{\mathbb{L}}[F, \vec{D}[-]_i]$, so $T_{\mathbb{L}}[F, \vec{D}[-]_i]$ is stable. □

Theorem 23. *$(\mathbb{L}_+)^+$ is a canonical extension of $\mathbb{L}$ as a lattice.*

Proof. This means that the bounded lattice underlying $(\mathbb{L}_+)^+$ is a canonical extension of the lattice $\mathbb{L}_0$ underlying $\mathbb{L}$. Urquhart [40] defined a dual space of $\mathbb{L}_0$ whose points are certain maximally disjoint pairs of filters and ideals of $\mathbb{L}_0$, and gave a representation of $\mathbb{L}_0$ as a lattice of stable subsets of a Galois connection over this dual space. Hartung [36] built on this to give an embedding of $\mathbb{L}_0$ into the stable set lattice of the overlap polarity between the set of filters of $\mathbb{L}_0$ that are maximally disjoint from some ideal and the set of ideals of $\mathbb{L}_0$ that are maximally disjoint from some filter. Hartonas and Dunn [34] observed that this embedding still obtains if the overlap polarity is taken between all filters and all ideals, as we have done in defining $\mathbb{L}_+$. Gehrke and Harding [18] introduced the general notion of canonical extension, as already explained, and showed that $(\mathbb{L}_+)^+$ is one. They also noted in [18, Remark 2.10] that the embedding of $\mathbb{L}_0$ into the complete lattice of all stable subsets of its Urquhart dual is a canonical extension. In [11] there is a proof of this as part of an exploration of the relationships between several ways of constructing canonical extensions. Here we will go over a proof that $(\mathbb{L}_+)^+$ is a canonical extension, since we make further use of its ideas.

For $a \in \mathbb{L}$, define $\mathscr{F}_a = \{F \in \mathscr{F}_{\mathbb{L}} : a \in F\}$ and $\mathscr{I}_a = \{I \in \mathscr{I}_{\mathbb{L}} : a \in I\}$. Then $\mathscr{F}_a = \lambda_{\between}\mathscr{I}_a$ and $\mathscr{I}_a = \rho_{\between}\mathscr{F}_a$. For the first equation, if $F \in \mathscr{F}_a$, then any $D \in \mathscr{I}_a$ has $a \in F \cap D$ so $F \between D$, hence $F \in \lambda_{\between}\mathscr{I}_a$. For the converse, let $(a] = \{b \in \mathbb{L} : b \leqslant a\} \in \mathscr{I}_a$ be the ideal generated by a. Then any $F \in \lambda_{\between}\mathscr{I}_a$ has $F \between (a]$, hence $a \in F$ and so $F \in \mathscr{F}_a$. The second equation is similar.

Thus $\mathscr{F}_a = \lambda_{\between}\mathscr{I}_a = \lambda_{\between}\rho_{\between}\mathscr{F}_a$, showing $\mathscr{F}_a$ is stable. The map $\theta(a) = \mathscr{F}_a$ gives a lattice embedding of $\mathbb{L}$ into the stable set lattice of the polarity $(\mathscr{F}_{\mathbb{L}}, \mathscr{I}_{\mathbb{L}}, \between)$, hence into $(\mathbb{L}_+)^+$. To show this, observe that $\mathscr{F}_{a\wedge b} = \mathscr{F}_a \cap \mathscr{F}_b$, so θ preserves binary meets. Since $\mathscr{F}_1 = \mathscr{F}_{\mathbb{L}}$ and $\mathscr{F}_0 = \lambda_{\between}\mathscr{I}_0 = \lambda_{\between}\mathscr{I}_{\mathbb{L}}$, it preserves the universal bounds. Also $\mathscr{I}_{a\vee b} = \mathscr{I}_a \cap \mathscr{I}_b$, so $\mathscr{F}_{a\vee b} = \lambda_{\between}(\mathscr{I}_a \cap \mathscr{I}_b) = \lambda_{\between}(\rho_{\between}\mathscr{F}_a \cap \rho_{\between}\mathscr{F}_b) = \lambda_{\between}\rho_{\between}(\mathscr{F}_a \cup \mathscr{F}_b) = \mathscr{F}_a \vee \mathscr{F}_b$, so θ preserves binary joins. Moreover, if $a \nleqslant b$, then the filter $[a) = \{b' \in \mathbb{L} : a \leqslant b'\}$ belongs to $\theta(a) \setminus \theta(b)$, so θ is an order-embedding.

Next we show that $\theta\colon \mathbb{L} \to (\mathbb{L}_+)^+$ is a compact and dense embedding. For compactness it suffices to take any subsets Z, W of $\mathbb{L}$ such that $\bigcap \theta[Z] \subseteq \bigvee \theta[W]$ in $(\mathbb{L}_+)^+$, and show that there are finite sets $Z' \subseteq Z$ and $W' \subseteq W$ with $\bigwedge Z' \leqslant \bigvee W'$ [18, 2.4]. Given such Z and W, let F be the filter of $\mathbb{L}$ generated by Z. Then $F \in \bigcap \theta[Z] \subseteq \bigvee \theta[W]$, so

$$F \in \bigvee \theta[W] = \lambda_{\between}\rho_{\between} \bigcup \theta[W] = \lambda_{\between} \bigcap_{b\in W} \rho_{\between}\mathscr{F}_b = \lambda_{\between} \bigcap_{b\in W} \mathscr{I}_b.$$

Now if D is the ideal generated by W, then $D \in \bigcap_{b\in W} \mathscr{I}_b$, so then $F \between D$. This means there is some $a \in F \cap D$ and so by the nature of generated filters and ideals there are finite sets $Z' \subseteq Z$ and $W' \subseteq W$ with $\bigwedge Z' \leqslant a \leqslant \bigvee W'$, hence $\bigwedge Z' \leqslant \bigvee W'$ as required.

Density of θ requires that each member of $(\mathbb{L}_+)^+$ is both a join of meets and a meet of joins of members of $\theta[\mathbb{L}]$. We use the fact (2.2) that in any polarity, a member of P^+ is both a join of elements of the form $\lambda\rho\{x\}$ and a meet of elements of the form $\lambda\{y\}$.

If $F \in \mathscr{F}_{\mathbb{L}}$ and $D \in \mathscr{I}_{\mathbb{L}}$, then $F \between D$ iff $\exists a \in F(D \in \mathscr{I}_a)$. So $\rho_{\between}\{F\} = \bigcup_{a\in F} \mathscr{I}_a$. Hence $\lambda_{\between}\rho_{\between}\{F\} = \lambda_{\between} \bigcup_{a\in F} \mathscr{I}_a = \bigcap_{a\in F} \lambda_{\between}\mathscr{I}_a = \bigcap_{a\in F} \mathscr{F}_a = \bigcap_{a\in F} \theta(a)$. Combining this with (2.2) gives that if $A \in (\mathbb{L}_+)^+$, then

$$A = \bigvee_{F\in A} \lambda_{\between}\rho_{\between}\{F\} = \bigvee_{F\in A} \bigcap_{a\in F} \theta(a). \tag{7.3}$$

Also, $F \between D$ iff $\exists a \in D(F \in \mathscr{F}_a)$, so $\lambda_{\between}\{D\} = \bigcup_{a\in D} \theta(a)$. Since $\lambda_{\between}\{D\}$ is stable, this union is a join. Together with (2.2) we then get that if $A \in (\mathbb{L}_+)^+$, then

$$A = \bigcap_{D\in\rho_{\between}A} \bigvee_{a\in D} \theta(a). \tag{7.4}$$

(7.3) and (7.4) show that θ is dense as required. □

Theorem 24. $(\mathbb{L}_+)^+$ *is a canonical extension of* $\mathbb{L}$ *as an* Ω*-lattice.*

Proof. We need to supplement Theorem 23 by showing that its embedding θ preserves f and g, and that the operations $f_{S_{\mathbb{L}}}$ and $g_{T_{\mathbb{L}}}$ on $(\mathbb{L}_+)^+$ are the canonical extensions of f and g as defined in (6.3)–(6.6). We will denote $(\mathbb{L}_+)^+$ more briefly as $\mathbb{L}^\sigma$, as justified by Theorem 23.

Preservation of f requires that for any $\vec{a} \in \mathbb{L}^n$,

$$\theta(f(\vec{a})) = f_{S_{\mathbb{L}}}(\theta(\vec{a})), \tag{7.5}$$

i.e. $\mathscr{F}_{f(\vec{a})} = \lambda_{\between} f^{\bullet}_{S_{\mathbb{L}}}(\theta(\vec{a}))$, where $f^{\bullet}_{S_{\mathbb{L}}}(\theta(\vec{a})) = \{D \in \mathscr{I}_{\mathbb{L}} : (\pi\theta(\vec{a}))S_{\mathbb{L}}D\}$. It is enough to show that

$$f^{\bullet}_{S_{\mathbb{L}}}(\theta(\vec{a})) = \mathscr{I}_{f(\vec{a})}, \tag{7.6}$$

since that implies that $\lambda_{\between} f^{\bullet}_{S_{\mathbb{L}}}(\theta(\vec{a})) = \lambda_{\between}\mathscr{I}_{f(\vec{a})} = \mathscr{F}_{f(\vec{a})}$, as desired. Note that

$$\pi\theta(\vec{a}) = \mathscr{F}_{a_0} \times \cdots \times \mathscr{F}_{a_{n-1}} = \{\vec{F} \in \mathscr{F}_{\mathbb{L}}{}^n : \vec{a} \in_\pi \vec{F}\},$$

so (7.6) amounts to the claim that for any $D \in \mathscr{I}_{\mathbb{L}}$,

$$f(\vec{a}) \in D \quad \text{iff} \quad \forall \vec{F} \in \mathscr{F}_{\mathbb{L}}{}^n(\vec{a} \in_\pi \vec{F} \text{ implies } \vec{F} S_{\mathbb{L}} \vec{D}). \tag{7.7}$$

If $f(\vec{a}) \in D$, then if $\vec{a} \in_\pi \vec{F}$ it is immediate that $\vec{F} S_{\mathbb{L}} D$ by definition of $S_{\mathbb{L}}$. Conversely, for each $i < n$, let $F_i = [a_i)$, the filter of $\mathbb{L}$ generated by a_i, so that $F_i \in \mathscr{F}_{a_i}$, and let $\vec{F} = (F_0, \ldots, F_{n-1})$. Then $\vec{a} \in_\pi \vec{F}$, so if the right side of (7.7) holds then $\vec{F} S_{\mathbb{L}} D$, hence there exists some $\vec{b} \in_\pi \vec{F}$ such that $f(\vec{b}) \in D$. Then $\vec{a} \leqslant \vec{b}$. But any operator is isotone, so this implies $f(\vec{a}) \leqslant f(\vec{b}) \in D$, hence $f(\vec{a}) \in D$. That completes the proof of (7.7), and therefore of (7.5).

Next we dualise this argument to show that for any $\vec{a} \in \mathbb{L}^m$,

$$\theta(g(\vec{a})) = g_{T_{\mathbb{L}}}(\theta(\vec{a})), \tag{7.8}$$

i.e. $\mathscr{F}_{g(\vec{a})} = \{F \in \mathscr{F}_{\mathbb{L}} : F T_{\mathbb{L}} \pi\rho_{\between}\theta(\vec{a})\}$. Note that

$$\pi\rho_{\between}\theta(\vec{a}) = \rho_{\between}\mathscr{F}_{a_0} \times \cdots \times \rho_{\between}\mathscr{F}_{a_{m-1}} = \{\vec{D} \in \mathscr{I}_{\mathbb{L}}{}^m : \vec{a} \in_\pi \vec{D}\},$$

so what we want for (7.8) is that for any $F \in \mathscr{F}_{\mathbb{L}}$,

$$g(\vec{a}) \in F \quad \text{iff} \quad \forall \vec{D} \in \mathscr{I}_{\mathbb{L}}{}^m(\vec{a} \in_\pi \vec{D} \text{ implies } F T_{\mathbb{L}} \vec{D}). \tag{7.9}$$

Now if $g(\vec{a}) \in F$, then if $\vec{a} \in_\pi \vec{D}$ it is immediate that $F T_{\mathbb{L}} \vec{D}$ by definition of $T_{\mathbb{L}}$. For the converse, for each $i < m$ let $D_i = (a_i]$, the ideal of $\mathbb{L}$ generated by a_i, and put $\vec{D} = (D_0, \ldots, D_{m-1})$. Then $\vec{a} \in_\pi \vec{D}$, so if the right side of (7.9) holds then $f T_{\mathbb{L}} \vec{D}$, hence there exists $\vec{b} \in_\pi \vec{D}$ with $g(\vec{b}) \in F$. Then $g(\vec{b}) \leqslant g(\vec{a})$ as dual operators are isotone, hence $g(\vec{a}) \in F$ as F is a filter. This proves (7.9) and hence (7.8).

Now we want to show that the lower canonical extension of f on $\mathbb{L}^\sigma = (\mathbb{L}_+)^+$ is just $f_{S_{\mathbb{L}}}$, i.e. $f^\triangledown(\vec{Z}) = f_{S_{\mathbb{L}}}(\vec{Z})$ for all $\vec{Z} \in (\mathbb{L}^\sigma)^n$. First we show this for the case that $\vec{Z}$ is any *closed* element of $(\mathbb{L}^\sigma)^n$, which means that for all $i < n$, Z_i is a closed element of $\mathbb{L}^\sigma$, hence there is some subset $A_i \subseteq \mathbb{L}$ such that

$$Z_i = \bigcap \theta[A_i] = \bigcap\{\mathscr{F}_a : a \in A_i\} = \{F \in \mathscr{F}_\mathbb{L} : A_i \subseteq F\}. \tag{7.10}$$

Since $\vec{Z} \in K((\mathbb{L}^\sigma)^n)$, (6.3) gives

$$f^\nabla(\vec{Z}) = \bigcap\{\theta(f(\vec{a})) : \vec{a} \in \mathbb{L}^n \text{ and } \vec{Z} \subseteq_\pi \theta(\vec{a})\}. \tag{7.11}$$

Now $f_{S_\mathbb{L}}$ is isotone, being an operator, so if $\vec{Z} \subseteq_\pi \theta(\vec{a})$, then

$$f_{S_\mathbb{L}}(\vec{Z}) \subseteq f_{S_\mathbb{L}}\theta(\vec{a}) = \theta(f(\vec{a}))$$

by (7.5). Hence $f_{S_\mathbb{L}}(\vec{Z}) \subseteq \bigcap\{\theta(f(\vec{a})) : \vec{Z} \subseteq_\pi \theta(\vec{a})\} = f^\nabla(\vec{Z})$ by (7.11).

For the converse inclusion, suppose $G \in f^\nabla(\vec{Z})$. For all $i < n$, let F_i be the filter generated by A_i. Then $F_i \in Z_i$ by (7.10), so the tuple $\vec{F} = (F_0, \dots, F_{n-1})$ belongs to $\pi\vec{Z}$. Thus for any $D \in f^\bullet_{S_\mathbb{L}}(\vec{Z})$ we have $\vec{F} S_\mathbb{L} D$ and so there is some $\vec{a} \in_\pi \vec{F}$ such that $f(\vec{a}) \in D$. But $Z_i = \{F \in \mathscr{F}_\mathbb{L} : F_i \subseteq F\}$, by (7.10) and the definition of F_i, so as $a_i \in F_i$, we get $a_i \in \bigcap Z_i$, so $Z_i \subseteq \mathscr{F}_{a_i} = \theta(a_i)$. Thus $\vec{Z} \subseteq_\pi \theta(\vec{a})$, so by (7.11) $f^\nabla(\vec{Z}) \subseteq \theta(f(\vec{a}))$. As $G \in f^\nabla(\vec{Z})$, this gives $f(\vec{a}) \in G$. But $f(\vec{a}) \in D$, so $G \between D$. Altogether this proves that $G \in \lambda_\between f^\bullet_{S_\mathbb{L}}(\vec{Z}) = f_{S_\mathbb{L}}(\vec{Z})$.

That completes the proof that f^∇ and $f_{S_\mathbb{L}}$ agree on all closed members of $(\mathbb{L}^\sigma)^n$. To show that they agree on an arbitrary $\vec{Z} \in (\mathbb{L}^\sigma)^n$ we use the fact that each Z_i is a join of closed members of $\mathbb{L}^\sigma$, so $Z_i = \bigvee \mathscr{Z}_i$ for some $\mathscr{Z}_i \subseteq K(\mathbb{L}^\sigma)$. Now f^∇ is a complete normal operator as it is the lower extension of a normal operator f [18, Sec. 4], and $f_{S_\mathbb{L}}$ is a complete normal operator by Theorem 5, so we reason that

$$\begin{aligned}
f^\nabla(\vec{Z}) &= f^\nabla(\textstyle\bigvee \mathscr{Z}_0, \dots, \bigvee \mathscr{Z}_{n-1}) \\
&= \textstyle\bigvee\{f^\nabla(\vec{Z}') : Z'_i \in \mathscr{Z}_i \text{ for all } i < n\} \qquad \text{by (3.1) for } f^\nabla, \\
&= \textstyle\bigvee\{f_{S_\mathbb{L}}(\vec{Z}') : Z'_i \in \mathscr{Z}_i \text{ for all } i < n\} \qquad \text{as } f^\nabla = f_{S_\mathbb{L}} \text{ on } K((\mathbb{L}^\sigma)^n), \\
&= f_{S_\mathbb{L}}(\textstyle\bigvee \mathscr{Z}_0, \dots, \bigvee \mathscr{Z}_{n-1}) \qquad \text{by (3.1) for } f_{S_\mathbb{L}}, \\
&= f_{S_\mathbb{L}}(\vec{Z}).
\end{aligned}$$

Finally, we show that $g_{T_\mathbb{L}}$ is the upper canonical extension $g^\triangle$. First we prove that $g^\triangle(\vec{Z}) = g_{T_\mathbb{L}}(\vec{Z})$ whenever $\vec{Z}$ is any *open* element of $(\mathbb{L}^\sigma)^n$, which means that for all $i < m$, Z_i is an open element of $\mathbb{L}^\sigma$, so $Z_i = \bigvee \theta[B_i]$ for some $B_i \subseteq \mathbb{L}$. Hence by the definition of $\bigvee$ in $\mathbb{L}^\sigma$,

$$Z_i = \lambda_\between \rho_\between \textstyle\bigcup_{a \in B_i} \mathscr{F}_a = \lambda_\between \bigcap_{a \in B_i} \mathscr{I}_a = \lambda_\between \{D \in \mathscr{I}_\mathbb{L} : B_i \subseteq D\}. \tag{7.12}$$

Since $\vec{Z} \in O((\mathbb{L}^\sigma)^n)$, (6.5) gives

$$g^\triangle(\vec{Z}) = \bigvee\{\theta(g(\vec{a})) : \vec{a} \in \mathbb{L}^n \text{ and } \theta(\vec{a}) \subseteq_\pi \vec{Z}\}. \tag{7.13}$$

But if $\theta(\vec{a}) \subseteq_\pi \vec{Z}$, then $\theta(g(\vec{a})) = g_{T_\mathbb{L}}(\theta(\vec{a})) \subseteq g_{T_\mathbb{L}}(\vec{Z})$ as the dual operator $g_{T_\mathbb{L}}$ is isotone. Hence we get $g^\triangle(\vec{Z}) \subseteq g_{T_\mathbb{L}}(\vec{Z})$ by (7.13).

For the converse inclusion, suppose $F \in g_{T_\mathbb{L}}(\vec{Z})$. For $i < m$, let D_i be the ideal of $\mathbb{L}$ generated by B_i. Then by (7.12), $\rho_\between Z_i = \rho_\between \lambda_\between\{D \in \mathscr{I}_\mathbb{L} : B_i \subseteq D\}$, and so as $B_i \subseteq D_i$ we get $D_i \in \rho_\between Z_i$. Thus if $\vec{D} = (D_0, \dots, D_{m-1})$, then $\vec{D} \in \pi\rho_\between\vec{Z}$, so $FT_\mathbb{L}\vec{D}$ as $F \in g_{T_\mathbb{L}}(\vec{Z})$. Hence there is some $\vec{a} \in \vec{D}$ such that $g(\vec{a}) \in F$. For each $i < m$, we have $a_i \in D_i$ and so any filter containing a_i intersects every ideal including D_i, i.e. $\theta(a_i) \subseteq \lambda_\between\{D \in \mathscr{I}_\mathbb{L} : D_i \subseteq D\} = Z_i$. Thus $\theta(\vec{a}) \subseteq_\pi \vec{Z}$, implying $\theta(g(\vec{a})) \subseteq g^\triangle(\vec{Z})$ by (7.13). But $F \in \theta(g(\vec{a}))$, so then $F \in g^\triangle(\vec{Z})$.

That completes the proof that $g^\triangle$ and $g_{T_\mathbb{L}}$ agree on all open members of $(\mathbb{L}^\sigma)^n$. Since every member of $(\mathbb{L}^\sigma)^n$ is a meet of open members, and $g^\triangle$ and $g_{T_\mathbb{L}}$ are both complete normal dual operators preserving all meets in each coordinate, we can then show that $g^\triangle$ and $g_{T_\mathbb{L}}$ are identical by using the order dual of (3.1). □

Theorem 24 justifies the equation $\mathbb{L}^\sigma = (\mathbb{L}_+)^+$. In the case that $\mathbb{L}$ is the stable set lattice P^+ of an Ω-polarity, we will call the canonical structure $(P^+)_+$ the *canonical extension of* P. Its stable lattice $((P^+)_+)^+$ is the canonical extension $(P^+)^\sigma$ of the Ω-lattice P^+.

8 Dual Categories

At the end of Section 4 it was shown that there is a contravariant functor from **Ω-Pol** to **Ω-NLO**. We now construct such a functor in the reverse direction.

Let $\theta\colon (\mathbb{L}, f_\mathbb{L}, g_\mathbb{L}) \to (\mathbb{M}, f_\mathbb{M}, g_\mathbb{M})$ be an Ω-homomorphism between two Ω-NLO's. If E is a filter or ideal of $\mathbb{M}$, then $\theta^{-1}E$ is a filter or ideal of $\mathbb{L}$, respectively, so we can define $\alpha_\theta : \mathscr{F}_\mathbb{M} \to \mathscr{F}_\mathbb{L}$ and $\beta_\theta : \mathscr{I}_\mathbb{M} \to \mathscr{I}_\mathbb{L}$ by putting $\alpha_\theta F = \theta^{-1}F$ and $\beta_\theta D = \theta^{-1}D$.

Theorem 25. *The pair $\theta_+ = (\alpha_\theta, \beta_\theta)$ is a bounded morphism from $\mathbb{M}_+$ to $\mathbb{L}_+$. If θ is injective, α_θ and β_θ are surjective. If θ is surjective, then α_θ and β_θ are injective and θ_+ is an isomorphism from $\mathbb{M}_+$ to the inner substructure* $\operatorname{Im}\theta_+$ *of $\mathbb{L}_+$.*

Proof. Since θ^{-1} preserves set inclusion, α_θ and β_θ are isotone by Lemma 21. We show that they fulfil the conditions of Definition 8, with $R = \between$.

(1_R): Let $F \in \mathscr{F}_\mathbb{M}$ and $D \in \mathscr{I}_\mathbb{M}$ have $\alpha_\theta F \between \beta_\theta D$. Then there is some $a \in \theta^{-1}F \cap \theta^{-1}D$. Then $\theta a \in F \cap D$, showing that $F \between D$.

(2_R): Let $F \in \mathscr{F}_{\mathbb{L}}$ and $D \in \mathscr{I}_{\mathbb{M}}$ have $\alpha_\theta^{-1}[F)_1 \between D$, where $[F)_1 = \{G \in \mathscr{F}_{\mathbb{L}} : F \subseteq G\}$. We have to show that $F \between \beta_\theta(D)$.

Now as θ preserves finite meets, the subset $\theta[F]$ of $\mathbb{M}$ is closed under finite meets. Hence the filter it generates is its upward closure in $\mathbb{M}$, i.e. the set

$$G = \{b \in \mathbb{M} : \exists a \in F(\theta(a) \leqslant b)\}. \tag{8.1}$$

Since $\theta[F] \subseteq G$ we have $F \subseteq \theta^{-1}(G) = \alpha_\theta(G)$, and so $G \in \alpha_\theta^{-1}[F)_1$. But $\alpha_\theta^{-1}[F)_1 \between D$, so there exists $b \in G \cap D$, and so for some $a \in F$, $\theta(a) \leqslant b \in D$. As D is an ideal, this gives $\theta(a) \in D$. Thus $a \in F \cap \theta^{-1}(D) = F \cap \beta_\theta(D)$, giving the desired result $F \between \beta_\theta(D)$.

(3_R): This is just the order-dual of the argument for (2_R). Let $F \in \mathscr{F}_{\mathbb{M}}$ and $D \in \mathscr{I}_{\mathbb{L}}$ have $F \between \beta_\theta^{-1}[D)_2$, where $[D)_2 = \{E \in \mathscr{I}_{\mathbb{L}} : D \subseteq E\}$. We have to show that $\alpha_\theta(F) \between D$.

The subset $\theta[D]$ of $\mathbb{M}$ is closed under finite joins, so the ideal it generates is

$$E = \{b \in \mathbb{M} : \exists a \in D(b \leqslant \theta(a))\}.$$

Since $\theta[D] \subseteq E$ we have $D \subseteq \theta^{-1}(E) = \beta_\theta(E)$, and so $E \in \beta_\theta^{-1}[D)_2$. But $F \between \beta_\theta^{-1}[D)_2$, so there exists $b \in F \cap E$, and so for some $a \in D$, $b \leqslant \theta(a)$. As F is a filter containing b, this implies $\theta(a) \in F$, hence $a \in \theta^{-1}(F) \cap D = \alpha_\theta(F) \cap D$, showing $\alpha_\theta(F) \between D$ as desired.

(1_S): Let $\vec{F} \in \mathscr{F}_{\mathbb{M}}{}^n$ and $D \in \mathscr{I}_{\mathbb{M}}$ have $\alpha_\theta(\vec{F}) S_{f_{\mathbb{L}}} \beta_\theta D$. Then there is some $\vec{a} \in_\pi \alpha_\theta(\vec{F}) = \theta^{-1}(\vec{F})$ with $f_{\mathbb{L}}(\vec{a}) \in \beta_\theta D$. Then $\theta(\vec{a}) \in_\pi \vec{F}$, and $f_{\mathbb{M}}(\theta(\vec{a})) = \theta(f_{\mathbb{L}}(\vec{a})) \in D$, showing that $\vec{F} S_{f_{\mathbb{M}}} D$.

(2_S): Let $\alpha_\theta^{-1}[\vec{F})_1 S_{f_{\mathbb{M}}} D$, where $\vec{F} \in \mathscr{F}_{\mathbb{L}}{}^n$ and $D \in \mathscr{I}_{\mathbb{M}}$. For all $i < n$, let G_i be the filter of $\mathbb{M}$ generated by $\theta[F_i]$. As in the case of (2_R), we have

$$G_i = \{b \in \mathbb{M} : \exists a \in F_i\, (\theta(a) \leqslant b)\}. \tag{8.2}$$

Let $\vec{G} = (G_0, \dots, G_{n-1})$. Since in general $F_i \subseteq \theta^{-1} G_i = \alpha_\theta(G_i) \in \mathscr{F}_{\mathbb{L}}$, we get $\vec{F} \subseteq_\pi \alpha_\theta(\vec{G})$, hence $\vec{G} \in \alpha_\theta^{-1}[\vec{F})_1$. But $\alpha_\theta^{-1}[\vec{F})_1 S_{f_{\mathbb{M}}} D$, so there must exist $\vec{b} \in_\pi \vec{G}$ such that $f_{\mathbb{M}}(\vec{b}) \in D$. For all $i < n$, as $b_i \in G_i$, by (8.2) there exists $a_i \in F_i$ with $\theta(a_i) \leqslant b_i$. Putting $\vec{a} = (a_0, \dots, a_{n-1})$, we have $\theta(\vec{a}) \leqslant \vec{b}$, hence $f_{\mathbb{M}}(\theta(\vec{a})) \leqslant f_{\mathbb{M}}(\vec{b}) \in D$, and so $f_{\mathbb{M}}(\theta(\vec{a})) \in D$, i.e. $\theta(f_{\mathbb{L}}(\vec{a})) \in D$. Thus $f_{\mathbb{L}}(\vec{a}) \in \theta^{-1}(D) = \beta_\theta(D)$. But $\vec{a} \in_\pi \vec{F}$, so this proves $\vec{F} S_{f_{\mathbb{L}}} \beta_\theta(D)$ as required.

(1_T): Let $\alpha_\theta(F) T_{g_{\mathbb{L}}} \beta_\theta(\vec{D})$, where $F \in \mathscr{F}_{\mathbb{M}}$ and $\vec{D} \in \mathscr{I}_{\mathbb{M}}{}^m$, so that there is some $\vec{a} \in_\pi \beta_\theta(\vec{D})$ with $g_{\mathbb{L}}(\vec{a}) \in \alpha_\theta(F)$. Then $\theta(\vec{a}) \in_\pi \vec{D}$, and $g_{\mathbb{M}}(\theta(\vec{a})) = \theta(g_{\mathbb{L}}(\vec{a})) \in F$, showing that $F T_{g_{\mathbb{M}}} \vec{D}$.

(2_T): Let $FT_{g_{\mathbb{M}}}\beta_\theta^{-1}[\vec{D})_2$, where $F \in \mathscr{F}_{\mathbb{M}}$ and $\vec{D} \in \mathscr{I}_{\mathbb{L}}{}^m$. Let E_i be the ideal of $\mathbb{M}$ generated by $\theta[D_i]$, so that

$$E_i = \{b \in \mathbb{M} : \exists a \in D_i(b \leqslant \theta(a))\}.$$

Let $\vec{E} = (E_0, \dots, E_{m-1})$. Then $D_i \subseteq \theta^{-1}(E_i) = \beta_\theta(E_i)$ for all $i < m$, so $\vec{D} \subseteq_\pi \beta_\theta(\vec{E})$, hence $\vec{E} \in \beta_\theta^{-1}[\vec{D})_2$, and therefore $FT_{g_{\mathbb{M}}}(\vec{E})$. So there must exist $\vec{b} \in_\pi \vec{E}$ such that $g_{\mathbb{M}}(\vec{b}) \in F$. For all $i < m$, as $b_i \in E_i$ there exists $a_i \in D_i$ with $b_i \leqslant \theta(a_i)$. Putting $\vec{a} = (a_0, \dots, a_{m-1})$, we have $\vec{b} \leqslant \theta(\vec{a})$, hence $g_{\mathbb{M}}(\vec{b}) \leqslant g_{\mathbb{M}}(\theta(\vec{a}))$. As F is a filter containing $g_{\mathbb{M}}(\vec{b})$, this implies $g_{\mathbb{M}}(\theta(\vec{a})) \in F$, i.e. $\theta(g_{\mathbb{L}}(\vec{a})) \in F$. Thus $g_{\mathbb{L}}(\vec{a})) \in \alpha_\theta(F)$. Since $\vec{a} \in_\pi \vec{D}$, this proves $\alpha_\theta(F)T_{g_{\mathbb{L}}}\vec{D}$ as required.

That completes the proof that the pair $\alpha_\theta, \beta_\theta$ is a bounded morphism. Now suppose that θ is injective. To show α_θ is surjective, take any $F \in \mathscr{F}_{\mathbb{L}}$. Let G be the filter of $\mathbb{M}$ generated by $\theta[F]$, as given in (8.1). Then $F \subseteq \theta^{-1}G$. To prove the converse inclusion, let $b \in \theta^{-1}G$. Then by (8.1), there exists $a \in F$ such that $\theta(a) \leqslant \theta(b)$. But θ, being an injective homomorphism, is order invariant, so this implies $a \leqslant b$, hence $b \in G$. Thus $G = \theta^{-1}F = \alpha_\theta F$, showing that $\alpha_\theta : \mathscr{F}_{\mathbb{M}} \to \mathscr{F}_{\mathbb{L}}$ is surjective. A dual argument shows that $\beta_\theta : \mathscr{I}_{\mathbb{M}} \to \mathscr{I}_{\mathbb{L}}$ is surjective: if $E \in \mathscr{I}_{\mathbb{L}}$, then $E = \beta_\theta D$ where D is the ideal of $\mathbb{M}$ generated by $\theta[E]$.

Finally, suppose that θ is surjective. If $\alpha_\theta(F) = \alpha_\theta(G)$, then $F = \theta[\theta^{-1}F] = \theta[\theta^{-1}G] = G$, so α_θ is injective. Similarly β_θ is injective. Then by Theorem 20, to show that θ_+ makes $\mathbb{M}_+$ isomorphic to $\operatorname{Im}\theta_+$, it suffices to show that it preserves the relations.

Preservation of R: Let $F \in \mathscr{F}_{\mathbb{M}}$ and $D \in \mathscr{I}_{\mathbb{M}}$ have $F \between D$. Then there is some $b \in F \cap D$. But $b = \theta(a)$ for some a as θ is surjective. Then $a \in \theta^{-1}F \cap \theta^{-1}G$, so $\alpha_\theta F \between \beta_\theta D$.

Preservation of S: Let $\vec{F} \in \mathscr{F}_{\mathbb{M}}{}^n$ and $D \in \mathscr{I}_{\mathbb{M}}$ have $\vec{F}S_{f_{\mathbb{M}}}D$. Then there is some $\vec{b} \in_\pi \vec{F}$ with $f_{\mathbb{M}}(\vec{b}) \in D$. But $\vec{b} = \theta(\vec{a})$ for some $\vec{a}$. Then $\vec{a} \in \theta^{-1}(\vec{F}) = \alpha_\theta(\vec{F})$, and $\theta(f_{\mathbb{L}}(\vec{a})) = f_{\mathbb{M}}(\vec{b})$, so $f_{\mathbb{L}}(\vec{a}) \in \theta^{-1}(D) = \beta_\theta D$. Hence $\alpha_\theta(\vec{F})S_{f_{\mathbb{L}}}\beta_\theta D$.

The preservation of T is similar. □

Using this result we can infer that the mappings $\mathbb{L} \mapsto \mathbb{L}_+$ and $\theta \mapsto \theta_+$ give a contravariant functor from **Ω-NLO** to **Ω-Pol**.

9 Direct Sums

Let $\{P_j : j \in J\}$ be an indexed set of Ω-polarities, with $P_j = (X_j, Y_j, R_j, S_j, T_j)$. We define a structure

$$\textstyle\sum_J P_j = (\sum_J X_j, \sum_J Y_j, R_J, S_J, T_J)$$

whose stable set lattice $(\sum_J P_j)^+$ is isomorphic to the direct product $\prod_J P_j^+$ of the stable set lattices of the P_j's. This $\sum_J P_j$ is called the *direct sum* of the P_j's. The polarity part of its definition is due to Wille [42, 43] and is given also in [16, p.184].

For each $j \in J$, let $\dot{X}_j = X_j \times \{j\}$ and $\dot{Y}_j = Y_j \times \{j\}$. Then $\sum_J X_j = \bigcup_J \dot{X}_j$ is the disjoint union of the X_j's, and $\sum_J Y_j = \bigcup_J \dot{Y}_j$ is the disjoint union of the Y_i's. However, unlike the case of Kripke modal frames, R is not the disjoint union of the R_j's. Rather, we put

$$\begin{aligned}
\dot{R}_j &= \{((x,j),(y,j)) : xR_jy\}, \\
\dot{S}_j &= \{((x_0,j),\dots,(x_{n-1},j),(y,j)) : \vec{x}S_jy\}, \\
\dot{T}_j &= \{((x,j),(y_0,j),\dots,(y_{m-1},j)) : xT_j\vec{y}\},
\end{aligned}$$

and then define

$$\begin{aligned}
R_J &= \textstyle\bigcup_J \dot{R}_j \ \cup \ \bigcup\{\dot{X}_j \times \dot{Y}_k : j \neq k\}, \\
S_J &= \textstyle\bigcup_J \dot{S}_j \ \cup \ \bigcup\{\dot{X}_{j_0} \times \cdots \times \dot{X}_{j_{n-1}} \times \dot{Y}_k : (\exists i < n)\, j_i \neq k\}, \\
T_J &= \textstyle\bigcup_J \dot{T}_j \ \cup \ \bigcup\{\dot{X}_k \times \dot{Y}_{j_0} \times \cdots \times \dot{Y}_{j_{m-1}} : (\exists i < m)\, j_i \neq k\}.
\end{aligned}$$

Spelling this out, we have that $(x,j)R_J(y,k)$ iff either $j \neq k$ or else $j = k$ and xR_ky. Likewise, $((x_0,j_0),\dots,(x_{n-1},j_{n-1}))S_J(y,k)$ iff either $j_i \neq k$ for some $i < n$, or else $(x_0,\dots,x_{n-1})S_ky$. The description of T_J is similar.

Lemma 26. *$\sum_J P_j$ is an Ω-polarity.*

Proof. This requires that all sections of S_J and T_J are stable. We prove stability for any section of the form $S_J[\overrightarrow{xJ}[-]_i,(y,k)]$, where

$$\overrightarrow{xJ} = ((x_0,j_0),\dots,(x_{n-1},j_{n-1})) \in (\textstyle\sum_J X_j)^n,$$

$(y,k) \in \sum_J Y_j$, and $i < n$. Let $\vec{x} = (x_0,\dots,x_{n-1})$.

If an element (u,l) of $\sum_J X_j$ is not in $S_J[\overrightarrow{xJ}[-]_i,(y,k)]$, then it is not the case that $\overrightarrow{xJ}[(u,l)/i]S_J(y,k)$, so by definition of S_J we have that

$$\{j_0,\dots,j_{i-1},l,j_{i+1},\dots,j_{n-1}\} = \{k\}$$

and not $\vec{x}[u/i]S_ky$. So u is not in the section $S_k[\vec{x}[-]_i,y]$, which is stable in P_j. Hence there is some $z \in Y_k$ with $z \in \rho_{R_k}S_k[\vec{x}[-]_i,y]$ but not uR_kz. Now we show that

$$(z,k) \in \rho_{R_J}S_J[\overrightarrow{xJ}[-]_i,(y,k)]. \tag{9.1}$$

Take any $(w, j) \in \sum_J X_j$ with $(w, j) \in S_J[\overrightarrow{xj}[-]_i, (y, k)]$. If $j \neq k$, then we have $(w, j)R_J(z, k)$ by definition of R_J. If $j = k$, then $w \in X_k$ and as $\overrightarrow{xj}[(w, j)/i]S(y, k)$ we have $\vec{x}[w/i]S_k y$. Since $z \in \rho_{R_k} S_k[\vec{x}[-]_i, y]$, this implies $wR_k z$, and so again $(w, j)R_J(z, k)$. That proves (9.1). But not $uR_k z$ and $l = k$, so not $(u, l)R_J(z, k)$. By (9.1) then, $(u, l) \notin \lambda_{R_J}\rho_{R_J} S_J[\overrightarrow{xj}[-]_i, (y, k)]$, completing the proof that the section $S_J[\overrightarrow{xj}[-]_i, (y, k)]$ is stable.

The cases of other sections of S_J, and those of T_J, are similar to this. □

Now for each $k \in J$, define functions $\alpha_k \colon X_k \to \sum_J X_j$ and $\beta_k \colon Y_k \to \sum_J Y_j$ by putting $\alpha_k(x) = (x, k)$ and $\beta_k(y) = (y, k)$.

Lemma 27. *The pair α_k, β_k is a bounded morphism $P_k \to \sum_J P_j$ whose image is an inner substructure of $\sum_J P_j$ isomorphic to P_k.*

Proof. First we show that α_k is isotone. Let $\preccurlyeq_1^k$ be the quasi-order of X_k determined by R_k, and $x \preccurlyeq_1^k x'$, i.e. $\rho_{R_k}\{x\} \subseteq \rho_{R_k}\{x'\}$. We have to show that $\rho_{R_J}\{\alpha_k(x)\} \subseteq \rho_{R_J}\{\alpha_k(x')\}$. But if $(x, k)R_J(y, j)$ then either $k \neq j$ and hence $(x', k)R_J(y, j)$, or else $k = j$ and $xR_k y$, hence $x'R_k y$ as $x \preccurlyeq_1^k x'$, which again gives $(x', k)R_J(y, j)$. The proof that β_k is isotone is similar.

Now given any $\vec{x} \in X_k^n$ and $y \in Y_k$, the definitions of α_k, β_k and S_J make it immediate that $\alpha_k(\vec{x})S_J\beta(y)$ iff $\vec{x}S_k y$. So α_k, β_k satisfies (1_S) and preserves S.

To show that (2_S) is satisfied, suppose that not $\overrightarrow{xj}S_J\beta_k(y)$, where $y \in Y_k$ and $\overrightarrow{xj} = ((x_0, j_0), \ldots, (x_{n-1}, j_{n-1}))$. Then $j_i = k$ for all $i < n$, and not $\vec{x}S_k y$, where $\vec{x} = (x_0, \ldots, x_{n-1})$. Then $\alpha_k(\vec{x}) = \overrightarrow{xj}$, so $\vec{x} \in \alpha_k^{-1}[\overrightarrow{xj})_1$, hence not $\alpha_k^{-1}[\overrightarrow{xj})_1 S_k y$ as required by (2_S).

The proofs that α_k, β_k satisfies the other back and forth conditions and also preserves R and T are similar to the above. Thus α_k, β_k is a bounded morphism. Since α_k and β_k are both injective, the rest of this theorem follows by Theorem 20. □

Theorem 28. $(\sum_J P_j)^+$ *is isomorphic to* $\prod_J P_j^+$.

Proof. For each $k \in J$, the bounded morphism α_k, β_k induces a Ω-homomorphism $\theta_k : (\sum_J P_j)^+ \to P_k^+$ by Theorem 13. The direct product of these θ_k's is the Ω-homomorpism $\theta : (\sum_J P_j)^+ \longrightarrow \prod_J P_j^+$ defined by $\theta(A)(k) = \theta_k A = \alpha_k^{-1} A$. θ is injective, for suppose $\theta(A) = \theta(B)$ and take any $(x, k) \in \sum_J X_j$. Then $\theta(A)(k) = \theta(B)(k)$, i.e. $\alpha_k^{-1} A = \alpha_k^{-1} B$, so $(x, k) \in A$ iff $(x, k) \in B$. Hence $A = B$.

Thus if θ is also surjective, it provides the desired isomorphism. To prove this surjectivity, let $\langle B_j : j \in J \rangle$ be any member of $\prod_I P_i^+$. Put $B = \bigcup_J \alpha_j[B_j] \subseteq \sum_J X_j$. If $B \in (\sum_I P_i)^+$, then $\theta(B)$ is defined with $\theta(B)(j) = \alpha_j^{-1} B = B_j$ for all j, hence $\theta(B) = \langle B_j : j \in J \rangle$.

Thus it remains to prove that B is stable. Take any $(x,k) \in \sum_J X_j$ with $(x,k) \notin B$. We want $(x,k) \notin \lambda_{R_J}\rho_{R_J}B$. We have $(x,k) \notin \alpha_k[B_k]$, so $x \notin B_k$. But $B_k \in P_k^+$, so there exists a $y \in \rho_{R_k}B_k$ with not xR_ky. Hence not $(x,k)R_J(y,k)$.

Now we show that $(y,k) \in \rho_{R_J}B$. Any member of B has the form $(z,j) \in \alpha_j[B_j]$ for some j with $z \in B_j$. If $j \neq k$, then $(z,j)R_J(y,k)$ by definition of R_J. But if $j = k$, then $z \in B_k$, hence zR_ky as $y \in \rho_{R_k}B_k$, giving $(z,j) = (z,k)R_J(y,k)$ again. This proves that $(y,k) \in \rho_{R_J}B$. Since not $(x,k)R_J(y,k)$, we have $(x,k) \notin \lambda_{R_J}\rho_{R_J}B$ as required to prove that B is stable and complete the proof that θ is surjective. $\square$

It is notable that the direct sum $\sum_J P_j$ and the bounded morphisms $\{\alpha_j, \beta_j : j \in J\}$ form a *coproduct* of $\{P_j : j \in J\}$ in the category **Ω-Pol**. This means that for any Ω-polarity P and bounded morphisms $\{\alpha'_j, \beta'_j \colon P_j \to P : j \in J\}$, there is exactly one bounded morphism $\alpha, \beta \colon \sum_J P_j \to P$ that factors each α'_j, β'_j through α_j, β_j, i.e. $\alpha'_j, \beta'_j = (\alpha, \beta) \circ (\alpha_j, \beta_j)$. The only maps that could do this are given by $\alpha(x,j) = \alpha'_j(x)$ and $\beta(y,j) = \beta'_j(y)$. It is left as an exercise for the reader to confirm that α, β as thus defined is indeed a bounded morphism.

10 Saturated Extensions of Ω-Polarities

Recall that we take the *canonical extension* of an Ω-polarity P to be the structure $(P^+)_+$. Regarding P as a model for a first-order language, we will now show that a sufficiently saturated elementary extension of P can be mapped to the canonical extension $(P^+)_+$ by a bounded morphism.

Let $\mathscr{L} = \{\overline{X}, \overline{Y}, \overline{R}, \overline{S}, \overline{T}\}$ be a signature consisting of relation symbols corresponding to the different components of an Ω-polarity. Fix an $\mathscr{L}$-structure $P = (X, Y, R, S, T)$ that is an Ω-polarity, and let $\mathscr{L}_P = \{\overline{A} : A \in P^+\}$, where each $\overline{A}$ is a unary relation symbol. Then P expands to an $\mathscr{L}_P$-structure, which we continue to call P, by interpreting each symbol $\overline{A}$ as the set A.

Now let

$$P^* = (X^*, Y^*, R^*, S^*, T^*, \{A^* : A \in P^+\})$$

be an $\mathscr{L}_P$-structure that is an ω-saturated elementary extension of P. Then P and P^* satisfy the same first-order $\mathscr{L}_P$-sentences. For each $A \in P^+$, A^* is the subset of X^* interpreting the symbol $\overline{A}$. To explain ω-saturation, consider the process of taking a set Z of elements of P^*, expanding $\mathscr{L}_P$ to $\mathscr{L}_P^Z$ by adding a set $\{\overline{z} : z \in Z\}$ of individual constants, and expanding P^* to an $\mathscr{L}_P^Z$-structure (P^*, Z) by interpreting each constant $\overline{z}$ as z. Then ω-saturation of P^* means that for any finite set Z of elements of P^*, and any set Γ of $\mathscr{L}_P^Z$-formulas, if each finite subset of Γ is satisfiable in (P^*, Z), then Γ is satisfiable in (P^*, Z).

Lemma 29. *P^* is an Ω-polarity.*

Proof. The sentence $\forall v_0 \forall v_1 (v_0 \overline{R} v_1 \to \overline{X}(v_0) \wedge \overline{Y}(v_1))$ is true in P, hence is true in P^*, implying that $R^* \subseteq X^* \times Y^*$. Similarly we can show that $S^* \subseteq (X^*)^n \times Y^*$ and $T^* \subseteq X^* \times (Y^*)^m$.

As is common, we write $\varphi(\vec{v})$ to indicate that the list $\vec{v}$ of variables includes all the free variables of formula φ. Then $\varphi(\vec{w})$ denotes the formula obtained by freely replacing each free occurrence of each v_i in φ by w_i.

Now for a formula $\varphi(\vec{v}, w)$ let $\rho\varphi(\vec{v}, w)$ be the formula

$$\forall u(\varphi(\vec{v}, u) \to u\overline{R}w),$$

and let $\lambda\varphi(\vec{v}, w)$ be

$$\forall u(\varphi(\vec{v}, u) \to w\overline{R}u),$$

where u is some fresh variable. The sentence $\forall w(\lambda\rho\overline{A}(w) \to \overline{A}(w))$ is true in P when $A \in P^+$, since A is stable. Hence the sentence is true in P^*, giving that $\lambda_{R^*}\rho_{R^*}A^* \subseteq A^*$, showing that $A^* \in (P^*)^+$.

Now let $\varphi(\vec{v}, w)$ be the atomic formula $\overline{S}(\vec{v}, w)$. The sentence

$$\forall \vec{v} \forall w(\lambda\rho\overline{S}(\vec{v}, w) \to \overline{S}(\vec{v}, w))$$

is true in P and hence in P^*, giving that all sections of the form $S^*[\vec{x}, -]$ are stable in P^*. Similar arguments establish the stability of all other sections of S^* and all sections of T^*. Thus P^* is an Ω-polarity. $\square$

We will construct a bounded morphism $\alpha, \beta \colon P^* \to (P^+)_+$ from P^* to

$$(P^+)_+ = (\mathscr{F}_{P^+}, \mathscr{I}_{P^+}, \between, S_{P^+}, T_{P^+}),$$

the canonical structure of P^+. The dual $((P^+)_+)^+ \to (P^*)^+$ of this bounded morphism, given by Theorem 13, proves to be a lattice embedding of the canonical extension $(P^+)^\sigma$ of P^+ into the stable set lattice of P^*. The construction of this bounded morphism α, β follows the methodology used in [28, Section 3.6] for the corresponding result for standard relational structures.

For $x \in X^*$, define

$$\alpha(x) = \{A \in P^+ : x \in A^*\}.$$

Then $\alpha(x)$ is non-empty, since it contains X. For $A, B \in P^+$, the sentence

$$\forall v(\overline{A \cap B}(v) \leftrightarrow \overline{A}(v) \wedge \overline{B}(v))$$

is true in P, hence in P^*, making $(A \cap B)^* = A^* \cap B^*$. So $A \cap B \in \alpha(x)$ iff $A \in \alpha(x)$ and $B \in \alpha(x)$. Thus $\alpha(x)$ is a filter of P^+, and so α maps X^* into $\mathscr{F}_{P^+}$.

For $y \in Y^*$, define

$$\beta(y) = \{A \in P^+ : y \in (\rho_R A)^*\}.$$

In P we have $\rho_R \lambda_R Y = Y$ and $\rho_R(A \vee B) = \rho_R A \cap \rho_R B$, the latter because

$$\rho_R(A \vee B) = \rho_R \lambda_R \rho_R(A \cup B) = \rho_R(A \cup B) = \rho_R A \cap \rho_R B.$$

So the sentences $\forall v(\overline{Y}(v) \to \overline{\rho_R \lambda_R Y}(v))$ and

$$\forall v(\,\overline{\rho_R(A \vee B)}(v) \leftrightarrow \overline{\rho_R A}(v) \wedge \overline{\rho_R B}(v))$$

are true in P, hence in P^*. Thus any $y \in Y^*$ has $\lambda_R Y \in \beta(y)$, and $A \vee B \in \beta(y)$ iff $A \in \beta(y)$ and $B \in \beta(y)$, i.e. $\beta(y)$ is an ideal of P^+. This shows that β maps Y^* into $\mathscr{I}_{P^+}$.

For each $A \in P^+$, we have

$$(\rho_R A)^* = \rho_{R^*} A^*. \tag{10.1}$$

This follows because the sentence $\forall w(\overline{\rho_R A}(w) \leftrightarrow \forall v(\overline{A}(v) \to v\overline{R}w)$ is true in P, hence in P^*.

Theorem 30. *The pair α, β is a bounded morphism from P^* to $(P^+)_+$.*

Proof. To show that the map α is isotone, first define $v \preccurlyeq_1 w$ to be an abbreviation for the formula $\forall u(v\overline{R}u \to w\overline{R}u)$. This formula defines the relation $\preccurlyeq_1$ on X determined by R as in (2.3), and the corresponding relation $\preccurlyeq_1^*$ on X^* determined by R^*. But if $A \in P^+$ then A is a $\preccurlyeq_1$-upset, so the sentence $\forall v \forall w(\overline{A}(v) \wedge v \preccurlyeq_1 w \to \overline{A}(w))$ is true in P, hence in P^*, showing that A^* is a $\preccurlyeq_1^*$-upset of X^*. Thus if $x, x' \in X^*$ have $x \preccurlyeq_1^* x'$, then $A \in \alpha(x)$ implies $A \in \alpha(x')$, showing that $\alpha(x) \subseteq \alpha(x')$, hence by Lemma 21, $\alpha(x) \preccurlyeq_1 \alpha(x')$ as required.

A similar argument shows that as $\rho_R A$ is stable, hence a $\preccurlyeq_2$-upset of Y, $(\rho_R A)^*$ is a $\preccurlyeq_2^*$-upset of Y^*. Thus if $y \preccurlyeq_2^* y'$, then $A \in \beta(y)$ implies $A \in \beta(y')$, so $\beta(y) \subseteq \beta(y')$. Hence β is isotone.

Our main task is to show that α, β satisfy the back and forth conditions of Definition 8. For (1_R), suppose $\alpha(x) \between \beta(y)$. Then there is some $A \in \alpha(x) \cap \beta(y)$, so $x \in A^*$ and $y \in (\rho_R A)^*$. Hence $y \in \rho_{R^*} A^*$ by (10.1), so xR^*y as required for (1_R).

For (2_R), take $F \in \mathscr{F}_{P^+}$ and $y \in Y^*$. We have to show that $\alpha^{-1}[F)_1 R^* y$ implies $F \between \beta(y)$, where $[F)_1 = \{F' \in \mathscr{F}_{P^+} : F \subseteq F'\}$. We prove the contrapositive of this implication.

Suppose that $F \between \beta(y)$ fails, i.e. $F \cap \beta(y) = \emptyset$. Consider the set of formulas

$$\Gamma = \{\neg(v\overline{R}\overline{y})\} \cup \{\overline{A}(v) : A \in F\}$$

in the single variable v, where $\overline{y}$ is a constant denoting y. We show Γ is finitely satisfiable in (P^*, y). Given any finite $Z \subseteq F$, let $A = \bigcap Z \in F$. Then by assumption $A \notin \beta(y)$, so $y \notin (\rho_R A)^*$. Hence by (10.1) there exists an $x \in A^*$ such that not xR^*y. Then for all $B \in Z$, as $A \subseteq B$ we get $A^* \subseteq B^*$ by the truth of the sentence $\forall w(\overline{A}(w) \to \overline{B}(w))$, so $x \in B^*$. Thus x satisfies $\{\neg(v\overline{R}\overline{y})\} \cup \{\overline{B}(v) : B \in Z\}$.

This proves that Γ is finitely satisfiable in (P^*, y). By saturation it follows that Γ itself is satisfiable in P^* by some $x \in \bigcap\{A^* : A \in F\}$ with not xR^*y. Then $F \subseteq \alpha(x)$, so $x \in \alpha^{-1}[F)_1$, and hence not $\alpha^{-1}[F)_1 R^* y$, giving (2_R).

The proof of (3_R) is similar: if $x \in X^*$, $D \in \mathscr{I}_{P+}$ and not $\alpha(x) \between D$, let

$$\Delta = \{\neg(\overline{x}\overline{R}v)\} \cup \{\overline{\rho_R A}(v) : A \in D\}.$$

For any finite $Z \subseteq D$, let $A = \bigvee Z \in D$. Then as $\alpha(x) \cap D = \emptyset$, $A \notin \alpha(x)$ and hence $x \notin A^*$. Since A^* is stable in P^*, there is some $y \in Y^*$ with $y \in \rho_{R^*}(A^*)$ and not xR^*y. Then for all $B \in Z$, as $B \subseteq A$ we get $B^* \subseteq A^*$, hence $y \in \rho_{R^*}(B^*)$. Thus y satisfies $\{\neg(\overline{x}\overline{R}v)\} \cup \{\overline{\rho_R B}(v) : B \in Z\}$ in (P^*, x).

This proves that Δ is finitely satisfiable in (P^*, x). By saturation it follows that Δ is satisfiable in (P^*, x) by some $y \in \bigcap\{(\rho_R A)^* : A \in D\}$ with not xR^*y. Then $D \subseteq \beta(y)$, so $y \in \beta^{-1}[D)_2$, and hence not $xR^*\beta^{-1}[D)_2$, giving (3_R).

For (1_S), suppose $\alpha(\vec{x})S_{P+}\beta(y)$. We want $\vec{x}S^*y$. We have some $\vec{A} \in_\pi \alpha(\vec{x})$ with $f_S(\vec{A}) \in \beta(y)$. Then $A_i \in \alpha(x_i)$, i.e. $x_i \in A_i^*$, for all $i < n$, while $y \in (\rho_\between f_S(\vec{A}))^* = (\rho_\between \lambda_\between f_S^\bullet(\vec{A}))^* = (f_S^\bullet(\vec{A}))^*$ as $f_S^\bullet(\vec{A})$ is stable. Now the sentence

$$\forall\vec{v}\forall w[\overline{f_S^\bullet(\vec{A})}(w) \wedge \textstyle\bigwedge_{i<n} \overline{A}(v_i) \to \overline{S}(\vec{v}, w)]$$

is true in P by definition of $f_S^\bullet(\vec{A})$, hence is true in P^*. This implies $\vec{x}S^*y$, as required for (1_S).

For (2_S), we must show that $\alpha^{-1}[\vec{F})_1 S^* y$ implies $\vec{F}S_{P+}\beta(y)$, where $\vec{F} \in (\mathscr{F}_{P+})^n$ and $y \in Y^*$. Suppose that $\vec{F}S_{P+}\beta(y)$ fails. Then for all $\vec{A} \in_\pi \vec{F}$, $f_S(\vec{A}) \notin \beta(y)$ and so $y \notin (\rho_\between f_S(\vec{A}))^* = (f_S^\bullet(\vec{A}))^*$. Now let

$$\Gamma' = \{\neg\overline{S}(\vec{v}, \overline{y})\} \cup \{\overline{A}(v_0) : A \in F_0\} \cup \cdots \cup \{\overline{A}(v_{n-1}) : A \in F_{n-1}\}.$$

We show Γ is finitely satisfiable in (P^*, y). As each filter F_i is closed under finite intersections, it is enough to show that if $A_i \in F_i$ for all $i < n$, then the set

$$\Gamma'_0 = \{\neg\overline{S}(\vec{v}, \overline{y})\} \cup \{\overline{A}_0(v_0), \ldots, \overline{A}_{n-1}(v_{n-1})\}$$

of formulas in the free variables $v_0, \dots, v_{n-1}$ is satisfiable. For such A_i we have $\overrightarrow{A} = (A_0, \dots, A_{n-1}) \in_\pi \overrightarrow{F}$, so $y \notin (f^\bullet_S(\overrightarrow{A}))^*$ by the above. But the sentence

$$\forall w [\, \overline{Y}(w) \wedge \neg \overline{f^\bullet_S(\overrightarrow{A})}(w) \to \exists \overrightarrow{v} (\, \textstyle\bigwedge_{i<n} \overline{A}(v_i) \wedge \neg \overline{S}(\overrightarrow{v}, w))]$$

is true in P, hence in P^*, so we infer that there exist $x_i \in A_i^*$ for all $i < n$ such that not $S^*(\overrightarrow{x}, y)$. Thus $\overrightarrow{x}$ satisfies Γ'_0 in (P^*, y).

By saturation it follows that Γ' is satisfied by some n-tuple $\overrightarrow{x}$. Then for all $i < n$ we have $x_i \in \bigcap\{A^* : A \in F_i\}$, so $F_i \subseteq \alpha(x_i)$. Thus $\overrightarrow{x} \in \alpha^{-1}[\overrightarrow{F})_1$. But $\overrightarrow{x} S^* y$ fails, therefore so does $\alpha^{-1}[\overrightarrow{F})_1 S^* y$. This completes the proof of (2_S).

The cases of (1_T) and (2_T) are similar to the above. For (1_T), suppose that $\alpha(x) T_{P+} \beta(\overrightarrow{y})$. We want $x T^* \overrightarrow{y}$. We have some $\overrightarrow{A} \in_\pi \beta(\overrightarrow{y})$ with $g_T(\overrightarrow{A}) \in \alpha(x)$. Then $A_i \in \beta(y_i)$, i.e. $y_i \in (\rho_R A_i)^*$, for all $i < m$, while $x \in (g_T(\overrightarrow{A}))^*$. Now the sentence

$$\forall v \forall \overrightarrow{w} [\, \overline{g_T(\overrightarrow{A})}(v) \wedge \textstyle\bigwedge_{i<m} \overline{\rho_R A_i}(w_i) \to \overline{T}(v, \overrightarrow{w})]$$

is true in P by definition of $g_T(\overrightarrow{A})$, hence is true in P^*. This implies $x T^* \overrightarrow{y}$, as required for (1_T).

For (2_T), suppose that not $\alpha(x) T_{P+} \overrightarrow{D}$, where $x \in X^*$ and $\overrightarrow{D} \in (\mathscr{I}_{P+})^m$. We show that not $x T^* \beta^{-1}[\overrightarrow{D})_2$. We have for all $\overrightarrow{A} \in_\pi \overrightarrow{D}$ that $g_T(\overrightarrow{A}) \notin \alpha(x)$ and so $x \notin (g_T(\overrightarrow{A}))^*$. Now let

$$\Delta' = \{\neg \overline{T}(\overline{x}, \overrightarrow{w})\} \cup \{\overline{\rho_R A}(w_0) : A \in D_0\} \cup \cdots \cup \{\overline{\rho_R A}(w_{m-1}) : A \in D_{m-1}\}.$$

Take any finite sets $E_0, \dots, E_{m-1}$ such that $E_i \subseteq D_i$ for all $i < m$ We show that the finite set

$$\Delta'_0 = \{\neg \overline{T}(\overline{x}, \overrightarrow{w})\} \cup \{\overline{\rho_R B}(w_0) : B \in E_0\} \cup \cdots \cup \{\overline{\rho_R B}(w_{m-1}) : B \in E_{m-1}\}$$

of formulas in the free variables $w_0, \dots, w_{m-1}$ is satisfiable in (P^*, x). For each $i < m$, let $A_i = \bigvee E_i \in D_i$. Then $\overrightarrow{A} = (A_0, \dots, A_{m-1-1}) \in_\pi \overrightarrow{D}$, and so $x \notin (g_T(\overrightarrow{A}))^*$ by the above. But the sentence

$$\forall v [\, \overline{X}(v) \wedge \neg \overline{g_T(\overrightarrow{A})}(v) \to \exists \overrightarrow{w} (\, \textstyle\bigwedge_{i<m} \overline{\rho_R A_i}(w_i) \wedge \neg \overline{T}(v, \overrightarrow{w}))]$$

is true in P, hence in P^*, so we infer that there exist $y_i \in (\rho_R A_i)^*$ for all $i < m$ such that not $T^*(x, \overrightarrow{y})$. Then for each i, any $B \in E_i$ has $B \subseteq A_i$, so $\rho_R A_i \subseteq \rho_R B$, hence $(\rho_R A_i)^* \subseteq (\rho_R B)^*$ and thus $y_i \in (\rho_R B)^*$. Thus $\overrightarrow{y}$ satisfies Δ'_0 in (P^*, x).

This proves that Δ' is finitely satisfiable in P^*. By saturation it follows that Δ' is satisfied in P^* by some m-tuple $\overrightarrow{y}$. Then for all $i < m$ we have $y_i \in \bigcap\{(\rho_R A)^* : A \in D_i\}$, so $D_i \subseteq \beta(y_i)$. Thus $\overrightarrow{y} \in \beta^{-1}[\overrightarrow{D})_2$. But $x T^* \overrightarrow{y}$ fails, therefore so does $x T^* \beta^{-1}[\overrightarrow{D})_2$. This completes the proof of (2_T). □

Example 31. The bounded morphism of Theorem 30 does not in general have surjective first component, in spite of the saturation of P^*. To see this, consider a P having the property $\lambda_R Y = \emptyset$, i.e. for all $x \in X$ there exists $y \in Y$ such that not xRy. This is a first-order condition, so it is preserved by elementary extensions. It is also preserved by images of bounded morphisms with surjective first component. For, if $\alpha, \beta : P \to P'$ is any bounded morphism with surjective α, and $\lambda_{R'} Y' \neq \emptyset$, then by the surjectivity there must exist an element of $\alpha^{-1}\lambda_{R'}Y'$, which equal to $\lambda_R(\beta^{-1}Y')$, so $\lambda_R Y \neq \emptyset$.

Now there are many P satisfying $\lambda_R Y = \emptyset$ (e.g. any with polarity of the form $(X, X, \neq)$). For such P, if there was *any* bounded morphism $P^* \to (P^+)_+$ with surjective α, then $(P^+)_+$ would satisfy the preserved condition. But that is not so, as no canonical structure $\mathbb{L}_+$ has $\lambda_{\between}\mathscr{I}_{\mathbb{L}} = \emptyset$. This is because $\mathscr{F}_{\mathbb{L}}$ contains the filter $\mathbb{L}$ which intersects every ideal of $\mathbb{L}$, so $\mathbb{L} \in \lambda_{\between}\mathscr{I}_{\mathbb{L}}$.

Thus if P has $\lambda_R Y = \emptyset$, then there is no bounded morphism $P^* \to (P^+)_+$ with surjective α, where P^* is any elementary extension of P.

11 Maximal Covering Morphisms

In the case of Kripke frames, for Boolean modal logics or distributive substructural logics, the corresponding version of Theorem 30 produces a bounded morphism that *is* surjective. The proof depends on the points of $(P^+)_+$ being *prime* filters [28, 3.6]. But here, in dealing with possibly non-distributive lattices, we admit arbitrary filters as points in canonical structures.

In the lattice representations developed by Urquhart [40] and Hartung [36], the points of representing spaces are filters or ideals, or filter-ideal pairs, that have certain mutual maximality properties. As these papers point out, this does not lead to a good duality construction for lattice homomorphisms, because the preimages of maximal filters under lattice homomorphisms need not be maximal. Here we have seen in Theorem 25 that admitting arbitrary filters leads to a notion of dual morphism for lattice homomorphisms that has good properties.

Now surjective bounded morphisms are logically important because they preserve validity of formulas in a semantics based on the structures involved. Typically, the validity of a formula in P will be equivalent to the satisfaction of some corresponding equation by the algebra P^+. So the preservation of formula validity in passing from P to P' will be secured if equation satisfaction is preserved in passing from P^+ to $(P')^+$. This will hold if $(P')^+$ is isomorphic to a subalgebra of P^+. That will in turn hold if the dual $(\alpha, \beta)^+$ of some bounded morphism $\alpha, \beta : P \to P'$ is injective. For this it suffices, by Theorem 13, that α be surjective. But when P' is a canonical structure, we can ensure this injectivity of $(\alpha,\beta)^+$ by a condition that is weaker than the surjectivity of α.

To define this condition, first define a filter F of a lattice $\mathbb{L}$ to be *D-maximal*, where D is an ideal of $\mathbb{L}$, if F is maximal in the set of all filters disjoint from D. Call F *i-maximal* if it is D-maximal for some ideal D. Now define a *maximal covering morphism* to be a bounded morphism $\alpha,\beta : P \to \mathbb{L}_+$ into some canonical structure such that the image $\alpha[X]$ of the map $\alpha : X \to \mathscr{F}_{\mathbb{L}}$ includes all i-maximal filters of $\mathbb{L}$. This condition holds immediately if α is surjective.

Theorem 32. *If $\alpha,\beta : P \to \mathbb{L}_+$ is a maximal covering morphism, then the Ω-lattice homomorphism $(\alpha,\beta)^+ \colon (\mathbb{L}_+)^+ \to P^+$ is injective.*

Proof. $(\alpha,\beta)^+$ is the map $A \mapsto \alpha^{-1}A$. Suppose that A and B are stable subsets of $\mathscr{F}_{\mathbb{L}}$ in $\mathbb{L}_+$, with $A \neq B$. Then, say, $A \nsubseteq B$, and there is some $F \in A$ with $F \notin B$. By stability of B there is some $D \in \rho_{\between} B$ such that not $F \between D$, hence $F \cap D = \emptyset$. By Zorn's Lemma, F can be extended to an $F' \in \mathscr{F}_{\mathbb{L}}$ that is D-maximal. By Lemmas 1 and 21, A is a $\subseteq$-upset, so we get $F' \in A$. Since $D \in \rho_{\between} B$ and $F' \cap D = \emptyset$, we have $F' \notin \lambda_{\between}\rho_{\between} B = B$. Since F' is i-maximal and α,β is maximal covering , there exists an $x \in X$ such that $\alpha(x) = F'$. So $x \in \alpha^{-1}A \setminus \alpha^{-1}B$. Hence $\alpha^{-1}A \neq \alpha^{-1}B$. Thus $(\alpha,\beta)^+$ is injective. □

Theorem 33. *The bounded morphism $\alpha,\beta \colon P^* \to (P^+)_+$ of Theorem 30 is maximal covering. Hence there is an Ω-lattice monomorphism $(P^+)^\sigma \rightarrowtail (P^*)^+$.*

Proof. Let $F \in \mathscr{F}_{P^+}$ be i-maximal. Then there is some ideal $D \in \mathscr{I}_{P^+}$ such that F is D-maximal. Now let

$$\Gamma = \{\overline{A}(v) : A \in F\} \cup \{\neg\overline{B}(v) : B \in D\}.$$

We show that Γ is finitely satisfiable in P^*. Given finite sets $G \subseteq F$ and $E \subseteq D$, put $A = \bigcap G \in F$ and $B = \bigvee E \in D$. If $A \subseteq B$, then $B \in F$ (and $A \in D$), contradicting $F \cap D = \emptyset$. So there must be some $x \in X$ with $x \in A \setminus B$. Hence $x \in A' \setminus B'$ for all $A' \in G$ and $B' \in E$. So x satisfies $\{\overline{A'}(v) : A' \in G\} \cup \{\neg\overline{B'}(v) : B' \in E\}$ in P.

This shows that each finite subset of Γ is satisfiable in P, hence is satisfiable in its elementary extension P^*. By saturation it follows that Γ is satisfiable in P^* by some $x \in \bigcap\{A^* : A \in F\}$, with $x \notin B^*$ for all $B \in D$. Thus the filter $\alpha(x)$ includes F and is disjoint from D. The maximality of F in this respect implies that $\alpha(x) = F$. Thus the image of α includes all i-maximal filters of P^+, as required for the morphism to be maximal covering.

It follows from Theorem 32 that the Ω-lattice homomorphism

$$(\alpha,\beta)^+ \colon ((P^+)_+)^+ \to (P^*)^+$$

given by Theorem 13 is injective. But $((P^+)_+)^+ = (P^+)^\sigma$. $\square$

We can strengthen this construction as follows.

Corollary 34. *Let $\alpha_1, \beta_1 \colon P \to P_1$ be a bounded morphism with $\alpha_1 \colon X \to X_1$ being surjective. Then there is a maximal covering morphism from P^* to $(P_1^+)_+$.*

Proof. We have the situation

$$P^* \xrightarrow{\alpha,\beta} (P^+)_+ \xrightarrow{\alpha_2,\beta_2} (P_1^+)_+ ,$$

where α, β is the maximal covering morphism of Theorem 33, and α_2, β_2 is the double dual $((\alpha_1, \beta_1)^+)_+$ of α_1, β_1. In particular, $\alpha_2(G) = \{A \in P_1^+ : \alpha_1^{-1}A \in G\}$ for any filter G of P^+. The composite pair $\alpha_2 \circ \alpha, \beta_2 \circ \beta$ is a bounded morphism $P^* \to (P_1^+)_+$, so it suffices to show that it is maximal covering. We adapt and extend the argument of Theorem 33. Let $F \in \mathscr{F}_{P_1^+}$ be D-maximal, where $D \in \mathscr{I}_{P_1^+}$. Put

$$\Gamma = \{\overline{\alpha_1^{-1}A}(v) : A \in F\} \cup \{\neg\overline{\alpha_1^{-1}B}(v) : B \in D\}.$$

For any finite sets $G \subseteq F$ and $E \subseteq D$, as in the proof of Theorem 33 there is some $z \in X_1$ with $z \in A \setminus B$ for all $A \in G$ and $B \in E$. As α_1 is surjective, there exists $x \in X$ with $\alpha_1(x) = z$, so $x \in \alpha_1^{-1}A \setminus \alpha_1^{-1}B$ for all $A \in G$ and $B \in E$. So x satisfies $\{\overline{\alpha_1^{-1}A}(v) : A \in G\} \cup \{\neg\overline{\alpha_1^{-1}B}(v) : B \in E\}$ in P, hence in P^*.

By saturation it follows that X^* has a point x that satisfies Γ, so belongs to $(\alpha_1^{-1}A)^*$ for all $A \in F$ but not to $(\alpha_1^{-1}B)^*$ for any $B \in D$. Thus $\alpha(x)$ includes $\{\alpha_1^{-1}A : A \in F\}$ and is disjoint from $\{\alpha_1^{-1}B : B \in D\}$. But $\alpha_2(\alpha(x)) = \{A \in P_1^+ : \alpha_1^{-1}A \in \alpha(x)\}$, which includes F and is disjoint from D. Hence $\alpha_2(\alpha(x)) = F$ by D-maximality of F.

This proves that the image of $\alpha_2 \circ \alpha$ includes all i-maximal filters of P_1^+, as required. $\square$

There is a further significant corollary to Theorem 33, which follows from the fact that saturated models can be obtained as ultrapowers. By the theory of [5, Section 6.1], for any P there is an ultrafilter U such that P^U is ω-saturated, where P^U is the ultrapower of P modulo U. We take P^U as P^* in Theorem 33 and Corollary 34:

Corollary 35. *For any Ω-polarity P and bounded morphism $\alpha_1, \beta_1 \colon P \to P_1$ with α_1 surjective, there is an ultrafilter U such that there is a maximal covering morphism $P^U \to (P_1^+)_+$ and an Ω-lattice monomorphism $(P_1^+)^\sigma \rightarrowtail (P^U)^+$.* $\square$

In [31], this corollary with $P_1 = P$ was obtained for a polarity P for which P^+ has additional Ω-indexed complete normal operators and dual operators that are all first-order definable over P in the sense indicated in (3.4) and (3.10) for the operations f_S and g_T. The method used in [31] was focused on the algebraic side of the duality between algebras and structures. It showed that $(P^U)^+$ is a MacNeille completion of the ultrapower algebra $(P^+)^U$ and then appealled to a result from [19] stating that there is an embedding of $(P^+)^\sigma$ into the MacNeille completion of $(P^+)^U$ for a suitable U that has $(P^+)^U$ sufficiently saturated.

Corollary 35 gives one of the properties that define the notion of a *canonicity framework*, as introduced in [30]. Such a framework describes a set of relationships between a class Σ of structures and a *variety*, i.e. equationally definable class, $\mathscr{C}$ of algebras equipped with operations $(-)^\sigma : \mathscr{C} \to \mathscr{C}$ and $(-)^+ : \Sigma \to \mathscr{C}$, that are sufficient to ensure that the following holds.

(‡) if $\mathcal{S}$ is any subclass of Σ that is closed under ultraproducts, then the variety of algebras generated by $\mathcal{S}^+ = \{P^+ : P \in \mathcal{S}\}$ is closed under the operation $(-)^\sigma$.

This provides an axiomatic formulation of a result about the generation of varieties closed under canonical extensions that was first proven in [28] for Boolean algebras with operators, and which was itself an algebraic generalisation of a theorem of Fine [15] stating that a first-order definable class of Kripke frames characterises a modal logic that is valid in its canonical frames.

A canonicity framework can be formed by taking Σ to be the class of Ω-polarities, $\mathscr{C}$ to be the variety of Ω-**NLO**'s, $\mathbb{L}^\sigma$ to be the canonical extension (6.7), and P^+ the stable set lattice (3.12). Hence the conclusion of (‡) holds if $\mathcal{S}$ is any class of Ω-polarities that is closed under ultraproducts.

12 Goldblatt-Thomason Theorem

This theorem [32, Theorem 8] was originally formulated as an answer to the question: which first-order definable properties of a binary relation can be expressed by modal axioms? It gave structural closure conditions on a first-order definable class of Kripke frames that are necessary and sufficient for that class to be the class of all frames that validate the theorems of some propositional modal logic. In this section we will derive two results of this kind in the present setting of polarity-based structures. The article [9] contains another result of this type, based on a different notion of polarity morphism (as a pair of binary relations) and different notions of morphic image and inner substructure.

Now a Kripke frame validates a particular modal formula iff the dual algebra of the frame satisfies some modal algebraic equation [3, Prop. 5.24]. Hence there is a correspondence between modal logics and varieties of modal algebras [3, Theorem 5.27]. Accordingly, the kind of result we seek is one that characterises when a class of Ω-polarities is the class of all such structures whose dual algebras P^+ belong to some variety. If $\mathscr{V}$ is a variety of Ω-lattices, let

$$\mathcal{S}_{\mathscr{V}} = \{P : P^+ \in \mathscr{V}\}$$

be the class of all Ω-polarities whose stable set lattice belongs to $\mathscr{V}$. We will give structural closure conditions on a class $\mathcal{S}$ of structures that characterise when it is of the form $\mathcal{S}_{\mathscr{V}}$.

We say that $\mathcal{S}$ is *closed under direct sums* if, whenever $\{P_j : j \in J\} \subseteq \mathcal{S}$, then $\sum_J P_j \in \mathcal{S}$. $\mathcal{S}$ is *closed under inner substructures* if, whenever $P' \in \mathcal{S}$ and P is an inner substructure of P', then $P \in \mathcal{S}$. $\mathcal{S}$ is *closed under images of surjective morphisms* when, for any bounded morphism $\alpha, \beta \colon P \to P'$ with α surjective, if $P \in \mathcal{S}$, then $P' \in \mathcal{S}$. $\mathcal{S}$ is *closed under codomains of maximal covering morphisms* if, whenever there is an maximal covering morphism from P to $\mathbb{L}_+$ and $P \in \mathcal{S}$, then $\mathbb{L}_+ \in \mathcal{S}$. $\mathcal{S}$ *reflects canonical extensions* if $P \in \mathcal{S}$ whenever $(P^+)_+ \in \mathcal{S}$ (equivalently, if the complement of $\mathcal{S}$ is closed under canonical extensions).

Lemma 36. *Suppose that $\mathcal{S}$ reflects canonical extensions.*

(1) *If $\mathcal{S}$ is closed under canonical extensions and codomains of maximal covering morphisms, then it is closed under images of surjective morphisms.*

(2) *If $\mathcal{S}$ is closed under ultrapowers and codomains of maximal covering morphisms, then it is closed under images of surjective morphisms.*

Proof. Let $P \in \mathcal{S}$, and suppose there is a bounded morphism $\alpha_1, \beta_1 \colon P \to P_1$ with α surjective.

(1): By Theorems 13 and 25, $(\alpha_1^+)_+, (\beta_1^+)_+ \colon (P^+)_+ \to (P_1^+)_+$ is a bounded morphism, and is maximal covering as $(\alpha_1^+)_+$ is surjective. Thus if $\mathcal{S}$ is closed under canonical extensions and codomains of maximal covering morphisms, then $P \in \mathcal{S}$ implies $(P_1^+)_+ \in \mathcal{S}$, hence $P_1 \in \mathcal{S}$ as $\mathcal{S}$ reflects canonical extensions.

(2) Taking P^* to be an ω-saturated ultrapower of P, by Corollary 34 there is a maximal covering morphism from P^* to $(P_1^+)_+$. Thus if $\mathcal{S}$ is closed under ultrapowers and codomains of maximal covering morphisms, then $P \in \mathcal{S}$ implies $(P_1^+)_+ \in \mathcal{S}$, hence $P_1 \in \mathcal{S}$. $\square$

Theorem 37. *For any variety $\mathscr{V}$, the class $\mathcal{S}_{\mathscr{V}}$ reflects canonical extensions and is closed under direct sums, inner substructures, codomains of maximal covering morphisms, and images of surjective morphisms.*

Proof. We use the fact that $\mathscr{V}$ is closed under direct products, subalgebras, and homomorphic images (including isomorphic images).

Reflection of canonical extensions: suppose $(P^+)_+ \in \mathcal{S}_{\mathscr{V}}$, so $(P^+)^\sigma$, which is $((P^+)_+)^+$, belongs to $\mathscr{V}$. But P^+ is isomorphic to a subalgebra of $(P^+)^\sigma$, so then $P^+ \in \mathscr{V}$, hence $P \in \mathcal{S}_{\mathscr{V}}$.

Closure under direct sums: if $\{P_j : j \in J\} \subseteq \mathcal{S}_{\mathscr{V}}$, then $\{P_j^+ : i \in J\} \subseteq \mathscr{V}$, so by closure of $\mathscr{V}$ under products and isomorphism and Theorem 28 we get $(\sum_J P_j)^+ \in \mathscr{V}$, so $\sum_J P_j \in \mathcal{S}_{\mathscr{V}}$.

Closure under inner substructures: suppose P is an inner substructure of $P' \in \mathcal{S}_{\mathscr{V}}$. By Theorem 17 there is a surjective homomorphism $(P')^+ \to P^+$. Since $(P')^+ \in \mathscr{V}$ this implies $P^+ \in \mathscr{V}$, hence $P \in \mathcal{S}_{\mathscr{V}}$.

Closure under images of surjective morphisms: this is dual to the previous case, using the result of Theorem 13 that if a morphism $\alpha, \beta \colon P \to P'$ has α surjective, then it induces an injective homomorphism $(P')^+ \to P^+$. Hence $P^+ \in \mathscr{V}$ implies $(P')^+ \in \mathscr{V}$.

Closure under codomains of maximal covering morphisms: suppose there is a maximal covering morphism from P to $\mathbb{L}_+$ with $P \in \mathcal{S}_{\mathscr{V}}$. Then by Theorem 32 there is an injective homomorphism making $(\mathbb{L}_+)^+$ isomorphic to a subalgebra of $P^+ \in \mathscr{V}$. Hence $(\mathbb{L}_+)^+ \in \mathscr{V}$ and so $\mathbb{L}_+ \in \mathcal{S}_{\mathscr{V}}$. □

Our first definability result is this:

Theorem 38. *Let $\mathcal{S}$ be closed under canonical extensions. Then the following are equivalent.*

(1) *$\mathcal{S}$ is equal to $\mathcal{S}_{\mathscr{V}}$ for some variety $\mathscr{V}$.*

(2) *$\mathcal{S}$ reflects canonical extensions and is closed under direct sums, inner substructures and codomains of maximal covering morphisms.*

(3) *$\mathcal{S}$ reflects canonical extensions and is closed under direct sums, inner substructures and images of surjective morphisms.*

Proof. (1) implies (2): By Theorem 37.

(2) implies (3): Assume (2). Then in particular $\mathcal{S}$ reflects canonical extensions and is closed under canonical extensions and codomains of maximal covering morphisms. These imply that $\mathcal{S}$ is closed under images of surjective morphisms by Lemma 36(1). Hence (3) holds.

(3) implies (1): Assume (3). Then we show that $\mathcal{S} = \mathcal{S}_{\mathscr{V}}$ where $\mathscr{V}$ is the variety generated by $\mathcal{S}^+ = \{P^+ : P \in \mathcal{S}\}$, i.e. the smallest variety that includes $\mathcal{S}^+$. It is immediate that $\mathcal{S} \subseteq \mathcal{S}_{\mathscr{V}}$. Conversely, suppose $P \in \mathcal{S}_{\mathscr{V}}$. Then $P^+ \in \mathscr{V}$, so from

the well known analysis of the generation of varieties, P^+ is a homomorphic image of some algebra $\mathbb{L}$ which is isomorphic to a subalgebra of a direct product $\prod_J P_j^+$ with $\{P_j : j \in J\} \subseteq \mathcal{S}$. But $\prod_J P_j^+ \cong (\sum_J P_j)^+$ (Theorem 28), and thus there are homomorphisms θ and χ having the configuration

$$P^+ \xtwoheadleftarrow{\theta} \mathbb{L} \xrightarrowtail{\chi} (\textstyle\sum_J P_j)^+ \ ,$$

with θ surjective and χ injective. By Theorem 25, there exist bounded morphisms

$$(P^+)_+ \xrightarrowtail{\alpha_\theta, \beta_\theta} \mathbb{L}_+ \xtwoheadleftarrow{\alpha_\chi, \beta_\chi} ((\textstyle\sum_J P_j)^+)_+ \ ,$$

with α_θ and β_θ injective and α_χ and β_χ surjective. But $((\sum_J P_j)^+)_+ \in \mathcal{S}$, by closure under direct sums and canonical extensions. Hence $\mathbb{L}_+ \in \mathcal{S}$ by closure under images of surjective morphisms. But Theorem 25 also gives that $\alpha_\theta, \beta_\theta$ makes $(P^+)_+$ isomorphic to an inner substructure of $\mathbb{L}_+$. By closure of $\mathcal{S}$ under inner substructures and images of isomorphisms (as a special case of images of surjective morphisms), this implies that $(P^+)_+ \in \mathcal{S}$. Finally then $\mathcal{S}$ contains P as it reflects canonical extensions. Thus $\mathcal{S} = \mathcal{S}_{\mathscr{V}}$ as required for (1). □

Now the equivalence of (1) and (3) of this theorem for Kripke frames in [32, Theorem 8] has the hypothesis that $\mathcal{S}$ is closed under first-order equivalence. Together with closure under images of bounded morphisms, this implies that $\mathcal{S}$ is closed under canonical extensions, and the proof of the main theorem proceeds from there. Thus, although closure under first-order equivalence is already weaker than being first-order definable, the theorem for Kripke frames can be stated with the still weaker hypothesis of closure under canonical extensions, as has been done here for Ω-polarities. In [32] the closure under elementary equivalence was used to show that a Kripke frame $\mathcal{F}$ has a saturated elementary extension $\mathcal{F}^*$ that is mapped by a surjective bounded morphism to $(\mathcal{F}^+)_+$. This $\mathcal{F}^*$ can be taken to be an ultrapower of $\mathcal{F}$, so an alternative hypothesis is that $\mathcal{S}$ is closed under ultrapowers. In the present situation with polarities we do not get a surjection to $(P^+)_+$, but rather a maximal covering morphism as in Theorem 33. But we can apply Lemma 36 to Theorem 38 to give the following definability characterisation.

Theorem 39. *Let $\mathcal{S}$ be a class of Ω-polarities that is closed under ultrapowers. Then the following are equivalent.*

(1) *$\mathcal{S}$ is equal to $\mathcal{S}_{\mathscr{V}}$ for some variety $\mathscr{V}$.*

(2) *$\mathcal{S}$ reflects canonical extensions and is closed under direct sums, inner substructures and codomains of maximal covering morphisms.*

Proof. Let $\mathcal{S}$ be closed under ultrapowers. (1) implies (2) again by Theorem 37. Conversely, assume (2). Then the closure of $\mathcal{S}$ under ultrapowers and codomains of maximal covering morphisms ensures that $\mathcal{S}$ is closed under images of surjective morphisms by Lemma 36(2), so (3) of Theorem 38 holds. But it also ensures that $\mathcal{S}$ is closed under canonical extensions, by Corollary 35, which implies (with $P_1 = P$) that there is a maximal covering morphism from an ultrapower of P to $(P^+)_+$. Hence (1) holds by Theorem 38. □

There is a good reason why part (3) of Theorem 38 is not part of this result. Although the equivalence of parts (1) and (3) holds for ultrapower-closed classes of modal Krikpe frames, it fails to hold in general for ultrapower-closed classes of polarities. For some such classes, (3) is strictly weaker than (1) and (2). Thus the replacement of images of bounded morphisms by codomains of maximal covering morphisms is essential here.

An example of this failure is the class $\mathcal{S}_0$ of all polarities that satisfy $\lambda_R Y = \emptyset$. It was observed in Example 31 that this is a first-order definable condition. Hence $\mathcal{S}_0$ is closed under ultrapowers. It was also noted that $\mathcal{S}_0$ is closed under images of surjective morphisms. It can be readily checked that it is closed under direct sums and inner substructures as well. Moreover it reflects canonical extensions, vacuously, because it contains no canonical structures $\mathbb{L}_+$, hence none of the form $(P^+)_+$, as Example 31 explained. Thus $\mathcal{S}_0$ fulfills part (3). However, since it is non-empty, it is not closed under canonical extensions. Hence by Corollary 35 it is not closed under codomains of maximal covering morphisms, so it fails to satisfy part (2), and thus fails (1) as well.

13 Further Studies

We conclude by pointing out two possible directions for further study of morphisms of polarities. One concerns the topological representation of lattices, imposing topologies on the sets X and Y in order to define a category of topological Ω-polarities and continuous bounded morphisms that is *dually equivalent* to Ω-**Lat**. This would involve functorial mappings $\mathbb{A} \mapsto \mathbb{A}_+$ and $P \mapsto P^+$ such that $\mathbb{A}$ is naturally isomorphic to $(\mathbb{A}_+)^+$ and P is naturally isomorphic to $(P^+)_+$. Guidance on how to go about topologising can be found in such papers as [39, 26, 40, 28, 36, 35, 34, 25].

The other development is to generalise from operators to *quasi*operators, operations that in each coordinate either preserve joins or map meets to joins [23, 24, 8, 9]. For instance, any 'negation' operation $\neg$ satisfying the De Morgan law $\neg(a \wedge b) = \neg a \vee \neg b$ is a unary quasioperator. A *dual quasioperator* is an operation that in each coordinate either preserves meets or maps joins to meets. The negation operation of a Heyting algebra is a dual quasioperator that is not in general a quasioperator.

Operations of these types can be characterised by using the *order dual* $\mathbb{L}^\partial$ of a lattice $\mathbb{L}$. The partial order of $\mathbb{L}^\partial$ is the inverse of that of $\mathbb{L}$, so the join and meet in $\mathbb{L}^\partial$ of a set of elements are the meet and join, respectively, of the same set in $\mathbb{L}$. An *n-ary monotonicity type* is an n-tuple $\varepsilon \in \{1, \partial\}^n$ whose terms will be denoted $\varepsilon(i)$ for $i < n$. Putting $\mathbb{L}^1 = \mathbb{L}$, we can then define $\mathbb{L}^\varepsilon$ to be the direct product lattice $\prod_{i<n} \mathbb{L}^{\varepsilon(i)}$. A function with domain $\mathbb{L}^n$ can also be viewed as a function on $\mathbb{L}^\varepsilon$, and $f : \mathbb{L}^n \to \mathbb{L}$ is an *ε-operator* if $f : \mathbb{L}^\varepsilon \to \mathbb{L}$ is an operator. An n-ary f is a *quasioperator* if it is an ε-operator for some $\varepsilon \in \{1, \partial\}^n$.

Given a polarity $P = (X, Y, R)$, let X^ε be the product set $\prod_{i<n} X_i$, where X_i is X if $\varepsilon(i) = 1$ and is Y if $\varepsilon(i) = \partial$. Then X^ε is quasi-ordered by the product relation $\leqslant_\varepsilon$, where $\vec{z} \leqslant_\varepsilon \vec{z'}$ iff $z_i \leqslant^{\varepsilon(i)} z'_i$ for all $i < n$, and $\leqslant^{\varepsilon(i)}$ is $\leqslant_1$ when $\varepsilon(i) = 1$ and is $\leqslant_2$ when $\varepsilon(i) = \partial$. This yields upsets of the form $[\vec{z})_\varepsilon = \{\vec{w} \in X^\varepsilon : \vec{z} \leqslant_\varepsilon \vec{w}\}$.

An ε-operator f_S on P^+ can be defined from a relation $S \subseteq X^\varepsilon \times Y$. For $\vec{A} \in (P^+)^n$, let $\vec{A}^\varepsilon = \left(A_0^{\varepsilon(0)}, \dots, A_{n-1}^{\varepsilon(n-1)}\right)$ where $A_i^{\varepsilon(i)}$ is A_i if $\varepsilon(i) = 1$ and is $\rho_R A_i$ if $\varepsilon(i) = \partial$. Then $\pi\vec{A}^\varepsilon \subseteq X^\varepsilon$, and we put

$$f_S\vec{A} = \lambda_R\{y \in Y : (\pi\vec{A}^\varepsilon)Sy\}.$$

Let P' be a second polarity with a relation $S' \subseteq (X')^\varepsilon \times Y'$, and let $\alpha : X \to X'$ and $\beta : Y \to Y'$ be isotone maps that satisfy (1_R)–(3_R). Define $\alpha_\varepsilon : X^\varepsilon \to (X')^\varepsilon$ by putting $\alpha_\varepsilon(\vec{z}) = \vec{w}$, where w_i is $\alpha(z_i)$ if $\varepsilon(i) = 1$ and is $\beta(z_i)$ if $\varepsilon(i) = \partial$. The back and forth conditions to make α, β a bounded morphism are then these:

(1_S) $\alpha_\varepsilon(\vec{z})S'\beta(y)$ implies $\vec{z}Sy$, all $\vec{z} \in X^\varepsilon,\ y \in Y$.

(2_S) $(\alpha^{-1}[\vec{w})_\varepsilon)Sy$ implies $\vec{w}S'\beta(y)$, all $\vec{w} \in (X')^\varepsilon,\ y \in Y$.

The constructions and notation have become more intricate, but there appears to be no impediment to carrying through the same analysis for quasioperators that we completed for operators, and to adapting it to dual quasioperators, which can be constructed on a stable set lattice from relations of the form $T \subseteq X \times Y^\varepsilon$. Details are left to the interested reader.

References

[1] Michael Atiyah. *Duality in Mathematics and Physics.* Lecture notes from the Institut de Matematica de la Universitat de Barcelona. `https://fme.upc.edu/ca/arxius/butlleti-digital/riemann/071218_conferencia_atiyah-d_article.pdf`, 2007.

[2] Garrett Birkhoff. *Lattice Theory*. American Mathematical Society, New York, first edition, 1940.

[3] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic*. Cambridge University Press, 2001.

[4] Sergio Celani and Ramon Jansana. Priestley duality, a Sahlqvist theorem and a Goldblatt-Thomason theorem for positive modal logic. *Logic Journal of the IGPL*, 7(6):683–715, 1999.

[5] Chen Chung Chang and Howard Jerome Keisler. *Model Theory*. North-Holland, Amsterdam, 1973.

[6] Anna Chernilovskaya, Mai Gehrke, and Lorijn van Rooijen. Generalized Kripke semantics for the Lambek–Grishin calculus. *Logic Journal of the IGPL*, 20(6):1110–1132, 2012.

[7] Willem Conradie, Sabine Frittella, Alessandra Palmigiano, Michele Piazzai, Apostolos Tzimoulis, and Nachoem M. Wijnberg. Categories: How I learned to stop worrying and love two sorts. In J. Väänänen et al., editor, *WoLLIC 2016*, volume 9803 of *Lecture Notes in Computer Science*, pages 145–164. Springer-Verlag, 2016.

[8] Willem Conradie and Alessandra Palmigiano. Algorithmic correspondence and canonicity for non-distributive logics. `arXiv:1603.08515`, 2016.

[9] Willem Conradie, Alessandra Palmigiano, and Apostolos Tzimoulis. Goldblatt-Thomason for LE-logics. `arXiv:1809.08225`, 2018.

[10] Dion Coumans, Mai Gehrke, and Lorijn van Rooijen. Relational semantics for full linear logic. *Journal of Applied Logic*, 12(1):50–66, 2014.

[11] Andrew Craig and Miroslav Haviar. Reconciliation of Approaches to the Construction of Canonical Extensions of Bounded Lattices. *Mathematica Slovaca*, 64(6):1335–1356, 2014.

[12] Jon Michael Dunn, Mai Gehrke, and Alessandra Palmigiano. Canonical extensions and relational completeness of some substructural logics. *The Journal of Symbolic Logic*, 70(3):713–740, 2005.

[13] Marcel Erné. Categories of contexts. `http://www.iazd.uni-hannover.de/~erne/preprints/CatConts.pdf`.

[14] — Tensor products of contexts and complete lattices. *Algebra Universalis*, 31:36–65, 1994.

[15] Kit Fine. Some connections between elementary and modal logic. In Stig Kanger, editor, *Proceedings of the Third Scandinavian Logic Symposium*, pages 15–31. North-Holland, 1975.

[16] Bernhard Ganter and Rudolf Wille. *Formal Concept Analysis*. Springer-Verlag, 1999.

[17] Mai Gehrke. Generalized Kripke frames. *Studia Logica*, 84:241–275, 2006.

[18] Mai Gehrke and John Harding. Bounded lattice expansions. *Journal of Algebra*, 239:345–371, 2001.

[19] Mai Gehrke, John Harding, and Yde Venema. MacNeille completions and canonical extensions. *Transactions of the American Mathematical Society*, 358:573–590, 2006.

[20] Mai Gehrke and Bjarni Jónsson. Bounded distributive lattices with operators. *Mathematica Japonica*, 40(2):207–215, 1994.

[21] — Monotone bounded distributive lattice expansions. *Mathematica Japonica*, 52(2):197–213, 2000.

[22] — Bounded distributive lattice expansions. *Mathematica Scandinavica*, 94:13–45, 2004.

[23] Mai Gehrke and Hilary A. Priestley. Canonical extensions of double quasioperator algebras: An algebraic perspective on duality for certain algebras with binary operations. *Journal of Pure and Applied Algebra*, 209(1):269–290, 2007.

[24] — Duality for double quasioperator algebras via their canonical extensions. *Studia Logica*, 86(1):31–68, 2007.

[25] Mai Gehrke and Samuel J. van Gool. Distributive envelopes and topological duality for lattices via canonical extensions. *Order*, 31:435–461, 2014.

[26] Robert I. Goldblatt. The Stone space of an ortholattice. *Bulletin of the London Mathematical Society*, 7:45–48, 1975.

[27] — Semantic analysis of orthologic. *Journal of Philosophical Logic*, 3:19–35, 1974. Reprinted in [29].

[28] — Varieties of complex algebras. *Annals of Pure and Applied Logic*, 44:173–242, 1989.

[29] — *Mathematics of Modality*. CSLI Lecture Notes No. 43. CSLI Publications, Stanford University, 1993.

[30] — *Canonical extensions and ultraproducts of polarities*. Algebra Universalis, 79(4), Article 80, 2018. `https://doi.org/10.1007/s00012-018-0562-4`.

[31] — *Definable operators on stable set lattices*. Studia Logica, 108(6):1263–1280, published online 5 February 2020, `https://doi.org/10.1007/s11225-020-09896-0`.

[32] Robert I. Goldblatt and S. K. Thomason. Axiomatic classes in propositional modal logic. In J. N. Crossley, editor, *Algebra and Logic*, volume 450 of *Lecture Notes in Mathematics*, pages 163–173. Springer-Verlag, 1975.

[33] Chrysafis Hartonas. Stone duality for lattice expansions. *Logic Journal of the IGPL*, 26(5):475–504, 2018.

[34] Chrysifas Hartonas and Jon Michael Dunn. Stone duality for lattices. *Algebra Universalis*, 37:391–401, 1997.

[35] Gerd Hartung. An extended duality for lattices. In K. Denecke and H.-J. Vogel, editors, *General Algebra and Applications*, pages 126–142. Heldermann-Verlag, Berlin, 1993.

[36] — A topological representation of lattices. *Algebra Universalis*, 29:273–299, 1992.

[37] Peter Jipsen. Categories of algebraic contexts equivalent to idempotent semirings and domain semirings. In W. Kahl and T.G. Griffin, editors, *Relational and Algebraic Methods in Computer Science (RAMiCS 2012)*, volume 7560 of *Lecture Notes in Computer Science*, pages 195–206. Springer-Verlag, 2012.

[38] Bjarni Jónsson and Alfred Tarski. Boolean algebras with operators, part I. *American Journal of Mathematics*, 73:891–939, 1951.

[39] Hilary A. Priestley. Representations of distributive lattices by means of ordered Stone

spaces. *Bull. London Math. Soc.*, 2:186–190, 1970.

[40] Alasdair Urquhart. A topological representation theory for lattices. *Algebra Universalis*, 8:45–58, 1978.

[41] Jacob Vosmaer. *Logic, Algebra and Topology: Investigations into canonical extensions, duality theory and point-free topology.* PhD thesis, Institute for Logic, Language and Computation, Universiteit van Amsterdam, 2010. ILLC Dissertation Series DS-2010-10.

[42] Rudolph Wille. Restructuring lattice theory: an approach based on hierarchies of concepts. In Ivan Rival, editor, *Ordered Sets*, pages 445–470. Reidel, 1982.

[43] — Subdirect product construction of concept lattices. *Discrete Mathematics*, 63:305–313, 1987.

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Twins in Logic – Identical and Otherwise

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Abstract. Connectives are twins in a logic, according to a metaphor of Łukasiewicz, when they behave 'in the same way' according to that logic. There are, however, looser and stricter ways of understanding that phrase, informally contrasted in §1 and then precisely defined and illustrated in §3, after a glance at related work by Michael Byrd and Evgeni Zolin in §2. §4 returns to the motivating case of the Ł-modal system, with its 'twin possibility operators.' One rather detailed discussion arising from §3 is deferred to an Appendix, so as not to interrupt the flow.

Keywords: Connectives; Łukasiewicz.

1 Introduction

Yes, the reader is right to be reminded by the title – its first three words, at least – of Łukasiewicz and the intriguing 'twin possibility' operators of his Ł-modal logic, to which we shall come in due course. First, though, as background for the present exploration of what exactly those three words might be taken to mean, we look at an example from the literature in which Łukasiewicz [42] is recalled not quite accurately. The details of the example – such as the proof (or even the correctness) of Proposition 1.1 below – are not important for later sections. The example serves only to make intelligible the remark embodying the inaccuracy just alluded to. (And in any case, the Łukasiewicz example serves chiefly as a lively prompt to investigate the 'twins' issue, which we illustrate with other examples in the following two sections before returning to Łukasiewicz to tidy up the discussion of that example.)

Humberstone [28] discusses the modal logic S5, formulated in some language with a functionally complete stock of Boolean connectives, of which one to be taken as primitive is the material conditional, to be written as $\supset$ just for this section (later

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we use $\to$ in this capacity), and the modal primitive $\Box$. The best known distinctively modal relative of the material conditional is the strict conditional $A \strictif B$, defined as $\Box(A \supset B)$, but [28] takes an interest, instead, in two other implicational connectives with a distinctly modal flavour, written here as $\Rightarrow$ and $\to$, defined thus, with $\equiv$ and $\vee$ being the usual material biconditional and (inclusive) disjunction connectives:

$$A \Rightarrow B = \Box(A \equiv B) \vee B \qquad\qquad A \to B = \Box A \supset B.$$

For the moment, restrict attention to the implicational fragment of classical (non-modal) propositional logic – or CL for short – whose formulas we call pure $\supset$-formulas. We denote by $C[\Rightarrow]$ and $C[\to]$ the results, respectively, of replacing all occurrences of $\supset$ in a such a formula with $\Rightarrow$ and with $\to$, and for uniformity we write C itself as $C[\supset]$. By $\vdash_{\mathsf{CL}}$ is meant the consequence relation most commonly associated with CL, sometimes called tautological (or truth-functional) consequence, though for the moment we consider only the consequences of the empty set, as usual writing "$\vdash_{\mathsf{CL}} A$" for "$\varnothing \vdash_{\mathsf{CL}} A$". Similarly in Proposition 1.1 $\vdash_{\mathsf{S5}}$ indicates provability in $\mathsf{S5}$; a simplified version of Proposition 1.5 from [28] reads as follows:

Proposition 1.1. *For any pure $\supset$-formula $A[\supset]$, the following are equivalent: (i) $\vdash_{\mathsf{CL}} A[\supset]$; (ii) $\vdash_{\mathsf{S5}} A[\Rightarrow]$; (iii) $\vdash_{\mathsf{S5}} A[\to]$.*

What has been simplified away here is restored in a footnote for those interested.[1]

An earlier publication (namely [25]) concentrated on the equivalence of *(ii)* and *(iii)* (or their more general versions in note 1) to throw light on the relation between the two implicational connectives in Matthew Spinks' BCSK logic,[2] in which they are far from interchangeable, the equivalence of *(ii)* and *(iii)* (in which each occurs without the other) notwithstanding. In view of this, a comment from [25] (p. 6, with some grammatical garbling corrected here) is recalled in [28] (p. 439), likening this situation to

[1]Proposition 1.5 of [28] is actually the following considerably stronger claim concerning the classical consequence relation $\vdash_{\mathsf{CL}}$ and the global consequence relation $\vdash^{glo}_{\mathsf{S5}}$ associated with $\mathsf{S5}$ (which we can think of syntactically as saying that the formula on the right can be obtained from those on the left together with the theorems of $\mathsf{S5}$ and the rules of Modus Ponens and Necessitation): The following are equivalent, for all pure $\supset$-formulas $C_1, \ldots, C_n, A$: *(i)* $C_1[\supset], \ldots, C_n[\supset] \vdash_{\mathsf{CL}} A[\supset]$; *(ii)* $C_1[\Rightarrow], \ldots, C_n[\Rightarrow] \vdash^{glo}_{\mathsf{S5}} A[\Rightarrow]$; *(iii)* $C_1[\to], \ldots, C_n[\to] \vdash^{glo}_{\mathsf{S5}} A[\to]$. Proposition 1.1 in the text is the $n = 0$ case of this more general claim. We do not need the "*glo*" superscript for that, because the global and local consequences of $\varnothing$ for $\mathsf{S5}$ (or any normal modal logic) coincide. (A corresponding syntactic formulation of the local version would drop the reference to necessitation figuring in the gloss given for the global relation.)

[2]See for instance p. 6 of Veroff and Spinks [70] where the two connectives appear as notated here.

> that which prompted Łukasiewicz in [42] to speak of two connectives as being like a pair of identical twins on the grounds that any formula in which only one of them made an appearance (perhaps alongside further connectives) did not have its (un)provability affected when it was replaced (throughout) by the other, even though the provability status of a formula containing both was prone to be affected by interchanging them. The intended analogy is with those identical human twins said to be indistinguishable except when appearing together.

The reference to identical twins in [25] even found its way into the title of [28].

There is only one problem with all this: Łukasiewicz makes no mention of *identical twins* – at least not in [42], the only reference cited in [25] and [28]. (We return to this point at the end of Section 4.) He speaks there only of *twins*. Now, is this inaccuracy in paraphrase just an excusable flourish in getting the gist of Łukasiewicz's point across? Perhaps so; but perhaps not: the choice between thinking of twins in general and thinking of identical twins in particular is naturally connected with two different ways of taking the idea that the connectives in question can only be told apart when both are present. On the human side of this simile, with identical twins, especially those making a point of exploiting this effect, one may not only fail to realise that there are two of them if they are seen separately, but even on encountering them together, be able to tell them apart only in the weak sense of seeing that they are different people. On the other hand, non-identical twins can still be sufficiently similar in appearance that when met singly, they may be taken to be the same person, while if one encounters them together, one can see not only *that* they are different – as in the identical twins case – but *how* they are different: one, for example, is now seen to be slightly taller than the other.[3] Might a similar contrast be operative in the logical case? And if so, on which side of the contrast would Łukasiewicz's own example (of the twin possibility operators in [42]) fall? We will look at that example more closely in Section 4, sampling some more recent literature bearing on the issue and honing the concepts needed, in Sections 2 and 3, respectively.

[3]That is: one can tell them apart in the sense of being aware of some differentiating respect which is intrinsic to the pair. In the identical twins case these last words are intended to exclude such a response as this: yes, I can tell them apart – Tweedledum is the one on the left and Tweedledee is the one on the right.

2 Concepts and Results from Byrd and Zolin

The authors whose names feature in the present section title will supply us with useful materials for approaching the subject of twins in logic.[4] Each of them distinguishes an internal from an external perspective. Thus Zolin [72], p. 861, writes as follows, understanding by a *modality* any formula $A(p)$ in the language of monomodal logic in which only the only sentence letter to appear is p.[5]

> According to the first, or *internal*, approach, modalities are identified if they are equivalent in L, i.e., if the equivalence of formulas they are induced by is a theorem of L. (...) The second, or *external*, approach prescribes not to distinguish between modalities having an identical "behaviour" over L.

Zolin uses ∇, Δ as variables over arbitrary modalities,[6] and explains the distinctive "$L(\nabla)$" notation he makes use of in the passage to be quoted as soon as we get this announcement out of the way:

[4]Both of them put in an appearance on p. 483 of Humberstone [30], which the present discussion complements.

[5]All propositional languages considered here have the same set of sentence letters, $p_1, p_2, \ldots, p_n \ldots$, the first three of which we write for convenience as p, q, r. Zolin takes $\Box$ as primitive, with $\Diamond$ defined from it (and $\neg$) in the usual way. What is often meant by a modality given such a choice of primitives, namely a (possibly empty) string of occurrence of $\Box$ and $\neg$, Zolin calls a *linear* modality.

[6]Except when discussing the specific noncontingency modality $\Delta p = \Box p \vee \Box\neg p$. (Historically, the literature on noncontingency-based logics uses Δ and ∇ for noncontingency and contingency, respectively; Zolin himself has, incidentally, been a prominent contributor this literature, the striking paper [73] being only one example.) For present purposes the notational difficulties become more acute still – and Convention 2.1 will reduce them somewhat – since Łukasiewicz [42] uses Δ as a possibility operator, rather than $\Diamond$, and then when he wants to introduce his 'twin possibility' operator, he writes the latter as ∇. Similarly Łukasiewicz writes Γ in place of $\Box$. It is for this reason that when we want to use (specifically) Greek capital letters for sets of formulas we reach for Θ in the following section. Notationally, matters would be even more complicated had the suggestion of Schock [56], p. 12, note 1, caught on to any extent: that one use upward and downward pointing triangles in place of $\Box$ and $\Diamond$ (for continuity with "$\wedge$" and "$\vee$" and their occasional enlarged use as universal and existential quantifiers). In saying this, differences in respect of stroke modulation – e.g., between a capital delta and an upward pointing triangle of similar size – are ignored, since this is often a typesetter's decision: for instance the former, from Łukasiewicz [42] appear as the latter in [43] (and Γ turns into a rotated version of what would now be taken as a negation symbol). By Chapter 7 of Łukasiewicz [44] Γ and Δ have become L and M, with the latter (more or less inverted) W in place of ∇. (Simons [59] has the idea of writing L – we shall use this later in the form "L" – as the inverted form of the earlier Γ for the dual of Łukasiewicz's ∇.) L and M in this usage gained considerable currency especially through the books of Hughes and Cresswell from 1968 onward. But a reader coming across Prior's discussion of Łukasiewicz in [52] is at some risk of misinterpreting the the symbol "M", especially on p. 204, which Prior uses as a contingency operator rather than a possibility operator.

Convention 2.1. *From now, in view of the potential for confusion mentioned note 6 from the triple use of "Δ" and "∇" mentioned there, whenever Zolin's use of these symbols for arbitrary modalities is being echoed, even in direct quotation, they will be written with a dot beneath them: $\underset{\cdot}{\Delta}$ and $\underset{\cdot}{\nabla}$. (This still leaves the double use of the undotted versions, for noncontingency and contingency on the one had, and for Łukasiewicz's two possibility operators on the other, but no use is made of them here in the first of these two roles.)*

We can now proceed with the passage from Zolin ([72], p. 862) explaining the (as we now call it) "$L(\underset{\cdot}{\nabla})$" notation he makes use of in characterizing several further concepts:

> [W]e define a logic $L(\underset{\cdot}{\nabla})$ of a modality $\underset{\cdot}{\nabla}$ over a logic L as the set of all formulas whose $\underset{\cdot}{\nabla}$-translations are theorems of L.

The $\underset{\cdot}{\nabla}$-translation of a formula A, $\tau_{\underset{\cdot}{\nabla}}(A)$, is defined inductively on the complexity of A by setting it to be the identity map except for the case in which A is $\Box B$ for some formula B, in which case $\tau_{\underset{\cdot}{\nabla}}(A)$ is $\underset{\cdot}{\nabla}(\tau_{\underset{\cdot}{\nabla}}(B))$. Thus, $L(\underset{\cdot}{\nabla})$ is the logic that says about $\Box$ what L says about $\underset{\cdot}{\nabla}$, and L is itself $L(\Box)$. Zolin's paper includes many examples of this relationship, which in the terminology of Humberstone [33] is put by saying that $\tau_{\underset{\cdot}{\nabla}}$ is a $\Box$-definitional translation faithfully embedding the logic $L(\underset{\cdot}{\nabla})$ into L.[7] Cases mentioned in Zolin [72] include (i) and (ii) of the following; all of these concern normal modal logics:

Examples 2.2. (i) $\mathsf{K}(\boxdot) = \mathsf{KT}$, *where* $\boxdot$ *is the modality induced by the formula* $\Box p \wedge p$.
(ii) $\mathsf{K4}(\boxdot) = \mathsf{KD4}(\boxdot) = \mathsf{S4}$.
(iii) $\mathsf{S4.2}(\Diamond\Box) = \mathsf{KD45}$. ◀

Examples 2.2(i) and (ii) are sufficiently well known and easily established as to be describable as folklore. A generalization subsuming them, given (as Lemma 5.9) in Zolin [72] says that whenever for normal L, $L \subseteq L(\boxdot)$, then $L(\boxdot)$ is the normal extension of L by $\Box p \to p$. The result for (iii) is due independently to E. E. Dawson and W. Lenzen, for further details on which, see Example 4.4.27 of [33] and §4.2 in French [16]. More discussion concerning such embeddings (including proofs of their fidelity), whether or not expressed using Zolin's $L(\cdot)$ notation, can be found in French [14], [16], Humberstone [29], [33] (passim), and – of special significance for $\tau_{\boxdot}$ – Jeřábek [36] (settling a question from French and Humberstone [15]).

[7] The $\tau_{\underset{\cdot}{\nabla}}$-style notation is common in the works cited below; Zolin actually uses "$\mathtt{tr}$" rather than "τ".

Finally, giving the equivalence relations of focal of interest for the internal and external perspectives, respectively, Zolin defines (p. 863) modalities $\underset{\cdot}{\nabla}$ and $\underset{\cdot}{\Delta}$ to be *equivalent in* L if $\vdash_L \underset{\cdot}{\nabla}(p) \leftrightarrow \underset{\cdot}{\Delta}(p)$, and to be *analogous over* L when $L(\underset{\cdot}{\nabla}) = L(\underset{\cdot}{\Delta})$. With respect to any congruential modal logic (i.e., an L for which $\vdash_L A \leftrightarrow B$ implies $\vdash_L \Box A \leftrightarrow \Box B$) any equivalent modalities are analogous, though not in general conversely. An interesting case in which the converse does hold is that of $\mathsf{S5}$. In 1946, Carnap [7] showed that this logic provides sixteen non-equivalent modalities. Zolin shows that no two of them are even analogous (over $\mathsf{S5}$), in the course of proving Theorem 4.21 of [72].[8] What bears more obviously on the Łukasiewicz 'twins' theme, though, is the other outcome: modalities which are analogous over L but not equivalent in L. Zolin provides numerous examples of this phenomenon, citing from the literature a proof that $\Box$ and $\Box\Box$ are analogous over – though evidently not equivalent in – K, and giving, himself, a proof that the same is so for KTB. Chellas [9], Exercise 7.8 on p. 211, had implicitly made a similar observation concerning the modalities $\Box$ and $\Diamond$ in E, the smallest congruential modal logic. In fact the points about $\Box$ and $\Box\Box$ hold, as Zolin notes, for any $\Box^m$ and $\Box^n$ with $m, n \geq 1$ and $m \neq n$ – setting the $m = n$ aside since we want *non-equivalent* (and so certainly distinct) though analogous modalities – and he also mentions at Remark 3.2 in [72], generalizing Chellas's example, that there are only three non-analogous linear modalities over E. We turn our attention presently to Byrd [5], though readers with no concern for the background of that discussion should skim the paragraphs after Remark 2.3, absorbing only the bold italic notation, down to Proposition 2.4. (Although, coincidentally, all the letters K, T and B will appear in this font, they have no connection with the K, T, and B appearing in the label "KTB" just seen.)

[8] The theorem itself reads: "$\varepsilon(\mathsf{S5}) = \alpha(\mathsf{S5}) = 16$." Here $\varepsilon(L)$ (resp. $\alpha(L)$) denotes the number of non-equivalent (resp. non-analogous) modalities in L. Zolin remarks in the course of the proof of Theorem 4.21 pertaining to non-equivalent modalities (which incidentally does not mention Carnap): "It is interesting to observe that each of these modalities is equivalent in $\mathsf{S5}$ to a boolean combination of $\bigcirc$ and $\odot$." Here $\bigcirc$ is the null modality and (quoting Zolin's own shorthand description) $\odot$ is $\boxplus \wedge \Diamond$, where $\boxplus$ is $\bigcirc \to \Box$. A quick way of seeing this is to notice that the Hasse diagram of the 16 $\mathsf{S5}$ modalities, partially ordered by provable implication – as at the base of p. 605 of Humberstone [27] – depicts the 16-element Boolean algebra, so taking the equivalence class of a sentence letter, this Boolean algebra is freely generated by that element and any other element at the same level in the diagram other than that element's complement. In [27] the sentence letter chosen was q and Zolin's $\odot q$ appears there as $X(q)$. (Put for convenience in alethic terms, this amounts to: it is either necessarily true or contingently false that q.) The same modality (labelled 'Q') was noted to be capable of playing this role in Canty and Scharle [6], where it was also erroneously claimed to be the only such candidate, overlooking the remaining three options in the same 'row'. (Remaining, that is, after setting aside the equivalence class of the chosen sentence letter, as well as its complement, and $\odot q$, from the six elements there.) The mistake was pointed out in Massey [46].

Remark 2.3. A unilateral version of Zolin's (symmetric) *analogousness* relation appears as the *subconnective* relation in Humberstone [22], p. 37*f*., where the 'logically loaded' notion of a connective is in play: a connective as an equivalence class of pairs $\langle\underset{\cdot}{\nabla}, L\rangle$, w.r.t. the equivalence relation "is analogous to" – to use Zolin's notation and terminology, so that (the equivalence class of) $\langle\Box, \mathsf{S4}\rangle$ can be thought of as "$\Box$-as-it-behaves-in-$\mathsf{S4}$". In the case where a single logic L is involved rather than a cross-logical comparison, then what in Zolin's notation would be written as $L(\underset{\cdot}{\nabla}) \subseteq L(\underset{\cdot}{\Delta})$ would be read as: "$\underset{\cdot}{\nabla}$ is a subconnective of $\underset{\cdot}{\Delta}$ in L", though to avoid various complications, as when this comes up in the following section, primitive (rather than defined) n-ary connectives κ and κ'; a definition in the 'one logic' case can by-pass the abstraction to equivalence classes: see Definition 3.6(*iii*) (for purposes of which, logics will be taken as consequence relations rather than sets of formulas). When $L = L'$ – or looking ahead to the consequence relation case, $\vdash = \vdash'$ – the possibility arises of a confusion between κ's being deductively stronger than (or: at least as strong as) κ' in L, in the sense that for all $A_1, \ldots, A_n$, we have $\vdash_L \kappa(A_1, \ldots, A_n) \rightarrow \kappa'(A_1, \ldots, A_n)$ (or, in the consequence relational version, which does not presume the availability of a suitably behaving $\rightarrow$ connective: $\kappa(A_1, \ldots, A_n) \vdash \kappa'(A_1, \ldots, A_n)$, all A) and κ's being a subconnective of κ in (or 'over') L (i.e., $\langle\kappa, L\rangle$'s being a subconnective of $\langle\kappa', L\rangle$). This particular external/internal confusion – the *subconnective fallacy*, we might call it – was illustrated in [22] of binary κ, κ' with an example treated in a conference presentation by Phil Staines, who cited evidence of writers' thinking along the following lines. Since a connective representing a natural language conditional construction – for example, the indicative conditional in English – would plausibly be taken as deductively stronger than the material conditional, any argument valid for the former would be valid for the latter (replacing the one conditional by the other). This amounts to passing without further ado – and thus fallaciously – from an internal deductive strength comparison to a claim that subconnective relation holds.[9] ◀

Turning now to Byrd, we are concerned with the discussion in [5] of a claim in Hintikka [19] to the effect that knowledge and true belief have different logics. To assess this contention, Byrd has to contend with another of Hintikka's claims: that the verbs *believe* and *know* are each amenable to either a weaker or a stronger interpretation. The detailed references to Hintikka's discussion are all to be found in Byrd [5], where they are treated using Hintikka's rather clunky 'model set' analysis

[9]Staines [65] is a published record of much of the material from the conference presentation cited in [22], though not, it appears, of the examples of those committing the 'subconnective fallacy' in respect of conditionals. For further discussion, see subsection 3.24 in Humberstone [30]; a bilateralized version of the subconnective fallacy would amount to confusing Zolin-style analogousness with equivalence in a logic – cf. the criticism of E. E. Dawson before Remark 4.5.1 on p. 279 of [33].

with all of its various conditions (A.PK*, A.PKK*, A.CBB*, etc.); explaining the contrast in what are now more familiar terms, we start with the representation, for a given cogniser a, of knowledge and belief attributions to a using respectively $\boldsymbol{K}_a$ and $\boldsymbol{B}_a$, taken as normal $\Box$ operators. The strong versions are those for which the 4 schema $\Box A \to \Box\Box A$ is appropriate, and the weak versions are those for which this is not so. Byrd shows that if we stick to the strong versions of both belief and knowledge, or else to the weak versions of both, then we find that knowledge and true belief in Hintikka's treatment, "when the two notions are considered in isolation (...) have the same internal logic."[10] The phrase "in isolation" is reminiscent of Łukasiewicz's earlier discussion of twins encountered one at a time, while the "internal logic" matches Zolin's later discussion of matters in those terms.

Byrd's official formulation of the result he is interested in is couched in terms of another Hintikka-specific notion, that of *defensibility* (and indefensibility), but this can be put for present purposes in the more familiar and less problematic terminology of consistency (and inconsistency), a set of formulas being understood as consistent in a particular logic if the logic does not prove any conjunction of formulas from the set.[11] For a formula A, Byrd writes $\boldsymbol{TB}_aA$ to abbreviate $\boldsymbol{B}_aA \wedge A$, but in order to recall the '$\boxdot$' notation of the discussion above, as well as to avoid the impression (with $\boldsymbol{T}$ followed by $\boldsymbol{B}$) of a composite notation, let us instead write $\dot{\boldsymbol{B}}_aA$ for this. From this point on, we drop the subscript a, since the knower/believer is taken as fixed for present purposes.[12]

The reference to inconsistency as defined above can be understood relative to the uniformly weaker ("4-less") logic or to the uniformly stronger one, and where Byrd's formulation speaks of epistemic operators because both $\boldsymbol{K}$ and its dual – the epistemic $\Diamond$ operator written, as $\boldsymbol{P}$ – are taken as primitive, of these we take only

[10]Byrd [5], p. 183. Byrd's discussion, following Hintikka's lead in this respect, abounds with talk of the strong and weak epistemic operators, and the strong and weak doxastic operators, where strength is a matter of being subject to the 4 axiom for the operator in question. Humberstone [30] accordingly complains (p. 483) that such talk is a standing invitation to – if not already an instance of – the subconnective fallacy, as Remark 2.3 puts it; if one operator has a logic that is stronger than (satisfies a proper superset of logical principles) that of another, this of course does *not* imply that when considered together the former is (deductively) stronger than the latter, or indeed vice versa.

[11]A classic critical discussion of indefensibility as originally explained by Hintikka is provided by Johnson Wu [37], though the end of the second last paragraph of Geach [17] is also interesting.

[12]Byrd follows Hintikka in writing the epistemic and doxastic operators in italic rather than in bold italic, which is done here to prevent them looking like the italic capitals $A, B, \ldots$ used here as schematic letters. For this reason Humberstone [33] switches to $\varphi, \psi, \ldots$ for schematic letters when a clash of this kind threatens. Hintikka (with Byrd following suit) uses $p, q, \ldots$ in this capacity – highly confusing in view of their use by almost everyone else as specifically sentence letters (propositional variables) rather than (standing in for) formulas of arbitrary complexity.

$\boldsymbol{K}$ as primitive, so as to have a standard bimodal logic under consideration. Lightly paraphrasing [5], p. 183*f.*, we have:

(†) Let Σ be a set of formulas in which the only non-Boolean connective to appear is $\boldsymbol{K}$. Then Σ is inconsistent if and only if the set Σ' is inconsistent, where Σ' differs from Σ only by having $\dot{\boldsymbol{B}}$ where Σ has $\boldsymbol{K}$.

The stronger logic favoured by Hintikka in [19] is $\mathsf{S4}$ for $\boldsymbol{K}$ and either $\mathsf{K4}$ or $\mathsf{KD4}$ for $\boldsymbol{B}$, along with the bridging axiom $\boldsymbol{K}p \to \boldsymbol{B}p$, while the weaker logic combines KT for $\boldsymbol{K}$ with either K or KD for $\boldsymbol{B}$, and the same bridging axiom as before. What is all this "either/or"? Hintikka's discussion is less than ideally clear as to what is intended. On p. 26 of [19] he writes that the formula (to put it in the present notation) $\boldsymbol{B}p \wedge \boldsymbol{B}\neg\boldsymbol{B}p$ "is easily seen to be inconsistent by means of (A.CBB*), together with [some conditions relating only to the Boolean connectives]", which amounts to saying that $\boldsymbol{B}p$ provably implies $\neg\boldsymbol{B}\neg\boldsymbol{B}p$ (or $\boldsymbol{C}\boldsymbol{B}p$, where $\boldsymbol{C}$ is the doxastic $\Diamond$ operator, taken, as with $\boldsymbol{P}$ as a defined symbol rather than a separate primitive), which is certainly not the case in $\mathsf{K4}$ as the logic of $\boldsymbol{B}$, though it is when we pass to to $\mathsf{KD4}$, as Hintikka does only later in the book, on p. 48, with the introduction of the condition (C.b*) – which conspicuously fails to put in any appearance in the official list of labelled conditions at the back (pp.169–173) of the book; compare, in this connection, note 3 (and the text to which it is appended) in Johnson Wu [37]. Nor does that condition figure in Byrd's own proof of the above result, though Lemmon explicitly takes D for $\boldsymbol{B}$, (i.e., $\boldsymbol{B}p \to \boldsymbol{C}p$) to be part of Hintikka's doxastic logic in his review [39] (p. 382, line 4).

Let us reformulate (†) above in the style of Proposition 1.1, for one specific case, the strongest of the logics mentioned above (with $\mathsf{4}$ for both operators and D replacing T for $\boldsymbol{B}$):

Proposition 2.4. *Let* S *be the smallest normal bimodal logic in* $\boldsymbol{K}$ *and* $\boldsymbol{B}$ *containing the formulas* $\boldsymbol{K}p \to p$, $\boldsymbol{K}p \to \boldsymbol{K}\boldsymbol{K}p$, $\boldsymbol{B}p \to \boldsymbol{C}p$, $\boldsymbol{B}p \to \boldsymbol{B}\boldsymbol{B}p$ *and* $\boldsymbol{K}p \to \boldsymbol{B}p$. *Then where* $A[\boldsymbol{K}]$ *is any formula in which* $\boldsymbol{K}$ *is the only non-Boolean connective to appear:*

$$\vdash_{\mathsf{S}} A[\boldsymbol{K}] \text{ if and only if } \vdash_{\mathsf{S}} A[\dot{\boldsymbol{B}}].$$

Although we are here considering a bimodal logic and the discussion of Zolin above pertained to monomodal logic, the simplest proof of Proposition 2.4 appeals to the embedding results reported in Example 2.2(*ii*), more specifically that concerning $\mathsf{KD4}$, since the latter is the logic of the doxastic fragment. (We will not directly deal with Byrd's further 'weak' version of Proposition 2.4, with the $\mathsf{4}$ principles $\boldsymbol{K}p \to \boldsymbol{K}\boldsymbol{K}p$ and $\boldsymbol{B}p \to \boldsymbol{B}\boldsymbol{B}p$ omitted.)

Let us reformulate the relevant part of Example 2.2(ii) in the notation used for Proposition 1.1, which involves treating $\Box$ and $\boxdot$ on a par, since they correspond to the $\boldsymbol{K}$ and $\dot{\boldsymbol{B}}$ of the current discussion:

$$\vdash_{\mathsf{S4}} A[\Box] \text{ if and only if } \vdash_{\mathsf{KD4}} A[\boxdot],$$

for any formula $A[\Box]$. We do not need to add here that the only non-Boolean vocabulary (if any) used in its construction is $\Box$, since that is the only such vocabulary in the language of $\mathsf{S4}$. To get from this to Proposition 2.4, which replaces both subscripts with "S" as specified there, we need to check that the epistemic and doxastic fragments of S are given by the epistemic and doxastic subsystems of that axiomatic specification. In other words, we should be sure that (1) adding the $\boldsymbol{B}$-involving axioms listed in Prop. 2.4 – $\boldsymbol{B}p \to \boldsymbol{C}p$, $\boldsymbol{B}p \to \boldsymbol{BB}p$, $\boldsymbol{K}p \to \boldsymbol{B}p$ – does not produce any new $\boldsymbol{B}$-free formulas over and above those yielded by only the $\boldsymbol{B}$-free axioms $\boldsymbol{K}p \to p$, $\boldsymbol{K}p \to \boldsymbol{KK}p$, and (2) adding the $\boldsymbol{K}$-involving axioms does not yield any new $\boldsymbol{K}$-free theorems other than those provable using the $\boldsymbol{K}$-free axioms.[13] But this is straightforward, since for (1), we can simply interpret $\boldsymbol{B}$ as $\boldsymbol{K}$ itself and all the $\boldsymbol{B}$-involving axioms are already provable from $\boldsymbol{K}p \to p$, $\boldsymbol{K}p \to \boldsymbol{KK}p$, and thus could not have the envisaged non-conservative effects, while for (2) we similarly read $\boldsymbol{K}$ as $\dot{\boldsymbol{B}}$.

It was, as already mentioned, with a view to treating $\Box$ and $\boxdot$ 'on a par' above that we wrote the claim inset above as "$\vdash_{\mathsf{S4}} A[\Box]$ iff $\vdash_{\mathsf{KD4}} A[\boxdot]$," rather than, in something closer to the style used by Zolin, as "$\vdash_{\mathsf{S4}} A$ iff $\vdash_{\mathsf{KD4}} A[\boxdot]$". Similarly, what Zolin writes as (2.1)

$$L(\underset{\cdot}{\nabla}) = L'(\underset{\cdot}{\Delta}) \tag{2.1}$$

as the special case of $L = L'$ in the definition of the relation of being analogous over L, would in the style of Proposition 1.1 appear as (2.2), understood as prefaced with "for all formulas A":

$$\vdash_L A[\underset{\cdot}{\nabla}] \text{ if and only if } \vdash_{L'} A[\underset{\cdot}{\Delta}] \tag{2.2}$$

The "$A[\underset{\cdot}{\nabla}]$","$A[\underset{\cdot}{\Delta}]$", here represent the result of replacing every occurrence of $\Box$ with an application of $\underset{\cdot}{\nabla}$, $\underset{\cdot}{\Delta}$, respectively, and so $\Box$ is in effect here playing what we might call an *anchoring* role in the sense that it marks a position in which it awaits replacement at the hands of $\tau_{\underset{\cdot}{\nabla}}$ by $\underset{\cdot}{\nabla}$, and at the hands of $\tau_{\underset{\cdot}{\Delta}}$ by $\underset{\cdot}{\Delta}$, giving the formulas schematically indicated on the left and right sides of (2.2).[14] The fact that the anchor $\Box$ is itself a primitive connective of the object language in modal logic –

[13]If we did not have the bridging axiom $\boldsymbol{K}p \to \boldsymbol{B}p$ to contend with, we could just simply appeal to Thomason [66]. Note also that for brevity, the formulation here mentions only axioms but rules – the necessitation rules for the two modal primitives – should also be understood as included.

[14]In Proposition 2.4, $\boldsymbol{K}$ plays both the anchoring role and the $\underset{\cdot}{\nabla}$ role.

and hence to be able to appear in the replacing formulas themselves – is in a way incidental, and even, for current purposes, potentially confusing, and the anchoring role might be better played by a special purpose 'dummy connective' not in the object language of the logic under discussion, though we make only incidental reference to such devices in the main body of the present discussion. Such a connective merely marks a spot for the insertion of the modalities or (primitive or compositionally derived) connectives of the object language – we may call the formula-like expression in which dummy connectives occur *preformulas* for the object language – and does not appear in what replaces it.[15] The symmetrical treatment thus afforded to $\underset{\cdot}{\nabla}$ and $\underset{\cdot}{\Delta}$, whether or not the symbol playing the $\Box$ role in specifying what these formulas are is itself part of the object language, seems in the end not to live up to its initial promise, as we shall see in the final paragraph of the following section.

On the final page of Byrd [5] one reads the following, in which the reference to specific results from earlier in the paper is not relevant to the point to be made about the passage:

> More generally, Lemma 6 and 7 together show that if a set whose sole non-truth-functional operators are true belief operators is indefensible, then so is the set obtained from it by replacing 'TB' throughout with 'K'.

Byrd is saying that if a formula in which the only non-Boolean connective to occur is $\dot{\boldsymbol{B}}$, then so is the formula obtained from it by replacing '$\dot{\boldsymbol{B}}$' throughout with '$\boldsymbol{K}$'. But $\dot{\boldsymbol{B}}$ can't be the only non-Boolean connective to occur in a formula since any subformula $\dot{\boldsymbol{B}}A$ is just the formula $\boldsymbol{B}A \wedge A$, so $\boldsymbol{B}$ itself occurs in any formula in which $\dot{\boldsymbol{B}}$ occurs. Perhaps Byrd meant that only non-Boolean connective occurring was $\boldsymbol{B}$, and all of the latter's occurrences were in subformulas of the form $\boldsymbol{B}A \wedge A$ (i.e. $\dot{\boldsymbol{B}}A$).

The minor criticism of Byrd in the preceding paragraph arises from his taking $\dot{\boldsymbol{B}}$ (or $\boldsymbol{TB}$, as he writes it) to be defined by $\dot{\boldsymbol{B}}A =_{\mathsf{df}} \boldsymbol{B}A \wedge A$, along with one of two possible views of definition, and in particular, the metalinguistic view rather

[15] In general one would want to allow the preformulas to be constructed with the aid of dummy connectives $*_n^m$ – the n^{th} m-place dummy connective, for all $n \geq 1, m \geq 0$ (in addition to the vocabulary of object language under discussion). Occasionally below, we write just "$*$" for $*_1^1$. Dummy connectives in essentially the present sense can be found on p. 361 of Makinson [45] (and no doubt elsewhere), where they are used for a convenient formulation of conditions on the form of rules; compare also the 'nominal symbols' of Schütte [57], p. 11, used for a convenient description of positive and negative occurrences of a formula within a formula, though not themselves part of the language from which those formulas are drawn. Though the phrase is not used and the setting is slightly different, the symbol $\Box$ itself plays such a dummy (1-ary) connective role in the discussion in Williamson [71] (esp. pp. 101–106), for purposes of explaining what principles are satisfied by this or that *bona fide* sentence operator.

than the object-linguistic view of definition. On the former view the defined symbol is introduced into the metalanguage to abbreviate reference to the formulas of the object language, and the "$=_{\mathsf{df}}$" (or ":=") is just a special case of "=" as identity: one and the same thing – in the present case a linguistic expression – is spoken of in two ways. On the object-linguistic view of definition, by contrast, a definition adds a new symbol to the object language which is intended to be interreplaceable with the material in the *definiens* according to the logic (or the non-logical theory) under discussion.[16] These are really two different approaches to definition, whose relative convenience depends on the purposes at hand, rather than two 'views' in sense of conflicting opinions. For current purposes the object-linguistic approach to definition does not seem particularly convenient, since it means we have three separate non-Boolean connectives to keep track of in the language under discussion: $\boldsymbol{K}$, $\boldsymbol{B}$, and now $\dot{\boldsymbol{B}}$ as well. Accordingly our default understanding of the defined connectives will be the metalinguistic one: they are not new symbols of the object language but functions from that language to itself derived by composing applications of the primitive connectives. However, it is often more desirable to proceed to (what in the present case constitutes) the bimodal setting by throwing away (what appears in the present case as) $\boldsymbol{B}$ by first promoting $\dot{\boldsymbol{B}}$ to the status of a primitive – even though definable – connective, and then passing to the $\boldsymbol{B}$-free fragment of the resulting logic, so as to compare directly with $\boldsymbol{K}$ without interference from $\boldsymbol{B}$: see Example 3.3(i) below on the kind of interference at issue here.

3 Twins

We turn to the distinction gestured at in the final paragraph of Section 1 between twins in general and identical twins in particular, using some of the discussion of Section 2 to illuminate the topic here. The $A[\Rightarrow]/A[\to]$ style of notation, as it appeared in Proposition 1.1 is not ideal if we want, as we shall, to consider formulas corresponding to those which in the present case are constructed using both $\Rightarrow$ and $\to$ at once. Our version of Byrd's result, Proposition 2.4 had us asserting the equi-provability in a certain logic of $A[\boldsymbol{K}]$ and $A[\dot{\boldsymbol{B}}]$, but the condition on the first of these was that $\boldsymbol{K}$ was the only non-Boolean connective to appear in $A[\dot{\boldsymbol{B}}]$, so again we ended up comparing certain monomodal formulas drawn from an originally bimodal language. (In that case we took $A[\dot{\boldsymbol{B}}]$ as defined but exploited the fact that $\boldsymbol{K}$ was primitive and so could perform the 'anchoring' role played by $\Box$ in Zolin's discussion, and envisaged for $*$, in notes 15 above and 18 below.) To address the

[16]For more on this, see the index entries in Humberstone [30] under 'defined connectives: object-linguistic *vs.* metalinguistic view'.

topic of twins along the lines suggested in Section 1 we need a notation which will apply whether only one, or instead both, of the connectives being compared appear in a given formula. This will provide a convenient way of drawing the distinction foreshadowed there between connectives behaving as twins and connectives behaving as identical twins (in a given logic).

Let us use κ, λ, sometimes decorated (κ' etc.) to stand for primitive connectives, with $ar(\kappa)$ denoting the arity of κ. Where $ar(\kappa) = ar(\kappa')$ it is convenient to have a notation for the result of interchanging occurrences of κ in a formula A with occurrences of κ', the official notation for which will be $A^{\kappa \bowtie \kappa'}$ though when the identity of κ and κ' is clear from the context, we will simply write $A^{\bowtie}$ ("A swap"). There are conceptual difficulties in trying to work with derived connectives (or Zolin's 'modalities') in settings like this, some taken up in the final paragraph of this section. For the moment we indicate how the issue raises its head in the present instance after giving an inductive definition of this $\bowtie$ notation. Here let us fix n as the arity of κ, κ', and suppress reference to those two connectives in the $\bowtie$ superscript:

- $(p_i)^{\bowtie} = p_i$
- $\big(\kappa(B_1, \ldots, B_n)\big)^{\bowtie} = \kappa'(B_1^{\bowtie}, \ldots, B_n^{\bowtie})$
- $\big(\kappa'(B_1, \ldots, B_n)\big)^{\bowtie} = \kappa(B_1^{\bowtie}, \ldots, B_n^{\bowtie})$
- $\big(\lambda(B_1, \ldots, B_{ar(\lambda)})\big)^{\bowtie} = \lambda(B_1^{\bowtie}, \ldots, B_{ar(\lambda)}^{\bowtie})$, for all primitive λ other than κ, κ'.

Remark 3.1. The need to restrict attention to primitive connectives in a treatment like this, rather than trying to exchange derived connectives such as Zolin's modalities, is clear from the inductive clauses here. Suppose we have $\boxdot$, for instance, not as a primitive connective, but instead used, as in Example 2.2(i), for the modality that maps A to $\Box A \wedge A$ – often indicated in Zolin [72] by writing the value of this function as $\boxdot(A)$ rather than $\boxdot A$. A serious problem then arises with the idea of interchanging $\Box$ and $\boxdot$ in a formula using the above explanation of $(\cdot)^{\bowtie}$: the fate of (non-primitive) $\boxdot$ is already settled by the inductive steps concerning $\Box$ and $\wedge$, so any attempt to settle it again is either redundant or inconsistent, depending on whether it agrees with or differs from the composite story already told for the primitive connectives. (There would be a similar problem in trying to add in an inductive clause for the compositionally derived connectives in defining the $\tau_{\triangledown}$ translations in play in the preceding section.) ◀

By induction (if necessary), we see that taking $\bowtie$ as $\kappa \bowtie \kappa'$ as above, for any formula A of a language in which these are primitive connectives of the same arity,

$(A^{\bowtie})^{\bowtie}$ is the same formula as A, and if A is constructed with the aid of only one of κ, κ' (and any further connectives), $A^{\bowtie}$ is the result of replacing (all occurrences of) κ with κ' or vice versa. Thus in the notation of Proposition 1.1, if A is $A[\kappa]$, then $A^{\bowtie}$ is $A[\kappa']$ – though for Prop. 1.1 there were stipulated to be no connectives present other than κ = "$\Rightarrow$", κ' = "$\rightarrow$" (or κ = "$\supset$"); no such restriction is in force here. Again allowing for the possible appearance of primitive connectives other than a given pair κ, κ' (with $ar(\kappa) = ar(\kappa')$), let us call a formula *mixed* if it is constructed with the aid of both κ, κ' and *unmixed* if it is constructed with the aid of at most one out of κ, κ'.

Definitions 3.2. *Suppose that L is a logic on a language with (possibly inter alia) primitive n-ary connectives κ, κ' (in terms of which the 'unmixed' terminology and the "$\bowtie$" in what follows is to be understood). Then, meaning by 'formula', 'formula of the language of L':*

(i) κ, κ' are twins in L if for all unmixed formulas A, we have $\vdash_L A$ iff $\vdash_L A^{\bowtie}$;

(ii) κ, κ' are identical twins in L if for all formulas A, we have $\vdash_L A$ iff $\vdash_L A^{\bowtie}$.

Note that the 'iff's in (i) and (ii) can be replaced by 'if's (or by 'only if's) without loss. For example, in the case of (i) the "all unmixed formulas A" covers the case of $A = A[\kappa]$, with $A^{\bowtie}$ being $A[\kappa']$, as well as the case of $A[\kappa']$, with $(\cdot)^{\bowtie}$ returning us to $A[\kappa]$. The present notion of twinhood is a close relation of Zolin's notion of analogousness, though the the role played $\Box$ in [72] would have to be played by a dummy connective of suitable arity (see note 15 above and note 18 below).

The idea, from Section 1, of twins being indistinguishable on the basis of separate encounters with them is embodied in Definition 3.2(i) by considering only formulas in which one of the candidate twins appears at a time: the logic in question returns the same provability verdict on all such formulas when either candidate is replaced with the other. But the stronger kind of indistinguishability raised in the introduction – there being no difference to be registered even when both twins are present – is embodied in Definition 3.2(ii), with its admission of mixed formulas to those in which interchanging them still leads to identical verdicts.

Here we take no interest in an 'inter-logical' version of the twin concept because the extra effort involved would have no pay-off for cases like that for which Łukasiewicz originally introduced this idea, though of course Definition 3.2(i) could be loosened up to allow for κ-in-L to be a 'twin' of κ'-in-L', with two one-directional versions of the $\bowtie$ notation, and the remaining vocabulary of L, L' were suitably constrained. (This amounts to something along the lines of (2.2), though the present κ, κ', unlike $\underset{\cdot}{\nabla}, \underset{\cdot}{\Delta}$ there, are supposed to primitive connectives.) Similarly, to apply this vocabulary to describe Byrd's observations in [5], L would be taken, not as the

S of Proposition 2.4, but as the $\{\boldsymbol{K}, \dot{\boldsymbol{B}}\}$-fragment of that S, now taking $\dot{\boldsymbol{B}}$ as a primitive connective in its own right – and getting rid of $\boldsymbol{B}$ itself for reasons explained in the first of the following illustrations of the 'twin' terminology:

Examples 3.3. (*i*) For S from Prop. 2.4 as just modified, we have: $\boldsymbol{K}$ and $\dot{\boldsymbol{B}}$ are twins, though not identical twins, in S. The first claim is just a restatement of Prop. 2.4, in the current terminology. But the change of terminology forces the new understanding of what S is: its language must have as the only non-Boolean connectives, $\boldsymbol{K}$ and $\dot{\boldsymbol{B}}$, throwing $\boldsymbol{B}$ out of the language and taking $\dot{\boldsymbol{B}}$ to be taken as primitive. We need the latter as primitive in any case, as observed in Remark 3.1, but we also need to exclude $\boldsymbol{B}$ from the language (of the new S) since if it were still present, taking $\bowtie$ as $\boldsymbol{K} \bowtie \dot{\boldsymbol{B}}$, we should have for $A = (\boldsymbol{B}p \wedge p) \to \dot{\boldsymbol{B}}p$

$$\vdash_{\mathsf{S}} A, \text{ whereas } \nvdash_{\mathsf{S}} A^{\bowtie},$$

since $A^{\bowtie}$ is $(\boldsymbol{B}p \wedge p) \to \boldsymbol{K}p$. (If we call connectives other than the κ, κ' being swapped by $\bowtie$, *extraneous* connectives – relative to the twinhood question under consideration – then what we have here is what might be called *interference* with the twin status of $\boldsymbol{K}$ and $\dot{\boldsymbol{B}}$ by the extraneous connective $\boldsymbol{B}$.) The second claim, that though twins in (revised) S, $\boldsymbol{K}$ and $\dot{\boldsymbol{B}}$ are not identical twins, is just a reflection of the fact that

$$\vdash_{\mathsf{S}} \boldsymbol{K}p \to \dot{\boldsymbol{B}}p \text{ whereas } \nvdash_{\mathsf{S}} \dot{\boldsymbol{B}}p \to \boldsymbol{K}p \text{ (i.e. } \nvdash_{\mathsf{S}} (\boldsymbol{K}p \to \dot{\boldsymbol{B}}p)^{\bowtie}).$$

(Byrd adds, p. 186: "There are other more interesting contrasts between knowledge and true belief in their strong senses [i.e., with the $\mathsf{4}$ principles in force], if mixed sets are allowed." He cites – putting it in the present terminology and notation – the provability of $\boldsymbol{B}p \to \dot{\boldsymbol{B}}\boldsymbol{B}p$ alongside the unprovability of $\boldsymbol{B}p \to \boldsymbol{K}\boldsymbol{B}p$ as an example. But allowing such 'mixed cases' involves going back to the language with $\boldsymbol{B}$ present in addition to $\boldsymbol{K}$ and $\dot{\boldsymbol{B}}$, which would lead straight back to the case of the A just considered, preventing $\boldsymbol{K}$ and $\dot{\boldsymbol{B}}$ from being even non-identical twins.)

(*ii*) For this example, we assume familiarity with propositional tense logic and use A. N. Prior's notation G, H for the future and past tense $\Box$-operators, to be interpreted by accessibility relations which are each other's converses. It is G and H that are to be compared and in terms of which the "$\bowtie$" notation is to be understood (the corresponding $\Diamond$-operators being F and P, taken as defined rather than primitive). The basic system K_t of tense logic (among many others) has the property that in it these two operators are identical twins in the sense of Definition 3.2(*ii*). In the tense-logical literature this property is often called the *mirror image* property, with $A^{\bowtie}$ being the 'mirror image' of the formula A. Steadfastly resisting any temptation to extend the biological metaphor by purloining talk of 'mirror twins' from that quarter, we note that this is a special case of a more general phenomenon:

> Let L be any normal bimodal logic determined by (i.e., sound and complete w.r.t.) a class $\mathbb{M}$ of models $\langle W, R, S, V \rangle$, R, S binary relations on $W \neq \varnothing$ the first listed being an accessibility relation for $\Box_1$, and the second, for $\Box_2$ (and V supplies subsets of W to the sentence letters, with the inductive definition of truth at a point proceeding as usual). Then if $\mathcal{M} = \langle W, R, S, V \rangle \in \mathbb{M}$ always implies $\langle W, S, R, V \rangle \in \mathbb{M}$, then $\Box_1$ and $\Box_2$ are identical twins in L. For the proof it is convenient to use the notation $\mathcal{M}^{\bowtie}$ for the result of interchanging the two accessibility relations; $A^{\bowtie}$ is the result of interchanging the corresponding non-Boolean primitives $\Box_1, \Box_2$ in A. One shows by induction on the complexity of A that for all $w \in W$, where $\mathcal{M} = \langle W, R, S, V \rangle$ that
>
> $$\mathcal{M} \models_w A \text{ if and only if } \mathcal{M}^{\bowtie} \models_w A^{\bowtie},$$
>
> and then argues that if $\vdash_L A$, we must have $\vdash_L A^{\bowtie}$ by appeal to the result inset above and the fact that L is determined by $\mathbb{M}$, which is closed under the $\bowtie$ operation on models.

In the case of K_t determined by the class of models in which $S = R^{-1}$, the desired condition of closure under the $\bowtie$ operation on models is evidently satisfied, as is also the case of S as the complement of R (relative to $W \times W$), considered in [21]. In both cases given W and either accessibility relation, the other accessibility relation is uniquely fixed (as the converse or the complement, respectively) but in general the $\bowtie$-closure condition does not require this. For example the class of models $\langle W, R, S, V \rangle$ such that for each $x \in W$ there exists $y \in W$ such that Rxy and Sxy, also satisfies the condition, though neither S nor R is uniquely fixed in terms of the other in this case. (The $\bowtie$-closure condition would most naturally be conducted in terms of frames rather than models, but one wants to avoid conveying the impression that these considerations bear only on the case of Kripke-complete normal modal logics.)

(iii) This example comes from deontic logic as a variation on the use in such papers as Anderson [3] of a sentential constant (or nullary connective) $\boldsymbol{s}$ informally read as "the sanction is applied" (or more accurately "there has been an infringement of the moral code") added to the vocabulary of a suitable alethically interpreted modal logic and governed by the axiom $\Diamond\neg\boldsymbol{s}$ to yield a deontic logic in which "It is forbidden/would be wrong that A" could be represented by $\Box(A \to \boldsymbol{s})$. Humberstone [20] suggests the addition of a second sentential constant $\boldsymbol{r}$ – the letter chosen to suggest "reward" with a view to reading $\Box(A \to \boldsymbol{r})$ as saying "It would be supererogatory that A", and examining the interrelations between these different deontic notions after trading in the previously cited axiom for $\Diamond(\neg\boldsymbol{s} \wedge \neg\boldsymbol{r})$. Evidently $\boldsymbol{s}$ and $\boldsymbol{r}$ are identical twins

(though not equivalent formulas) in the resulting logic – something referred as the 'exchange property' in [20], *q.v.* for acknowledgment that the deontic notions in play here really deserve a more explicitly agent-relative treatment. (However, the description in [20] of $\boldsymbol{s}, \boldsymbol{r}$ as propositional constants rather than sentential constants is ill-advised for reasons not germane to our present concerns. [33], p. 275, gives a brief discussion of the issue, and further references.) ◀

Remarks 3.4. (*i*) In connection with Example 3.3(*ii*), it is important to recall the metalinguistic understanding of definitions mentioned at the end of the previous section. $P\top$ for instance, is none other than the formula $\neg H \neg \top$; similarly, references to F are just our abbreviated way of referring to $\neg G \neg$. Without this understanding in place, we could not have given K_t as a logic according to which G and H are (non-equivalent) identical twins, since the provable $Gp \leftrightarrow \neg F \neg p$ would have as its mirror image (its $G \bowtie H$-swap) the unprovable $Hp \leftrightarrow \neg F \neg p$, rather than, as intended, discerning the occurrence of "G" concealed by the "F" notation, the provable formula $Hp \leftrightarrow \neg P \neg p$ ($= Hp \leftrightarrow \neg\neg H \neg\neg p$, in primitive notation). Similarly if we had been considering the equivalences corresponding to the definitions that would have related distinct primitives on the object-linguistic conception of definition – $Fp \leftrightarrow \neg G \neg p$ in the future tense case. But the present point has nothing specifically to do with biconditionals: the addition of any new primitive stipulated to stand in some non-trivial logical relation[17] to G but not H or vice versa – for example adding a one-place connective O with just the axiom $Op \vee Gp$ would stop G and H from being twins (since $Op \vee Hp$) would not be provable. In the case of duals, if one wanted to treat H and P (and G and F) as separate primitives, one would have to re-work the account so as to accommodate the more complicated relation "G is to F as H is to P" – or that *pair* $\langle G, F\rangle$ is a twin (or identical twin) of the pair $\langle H, P\rangle$. (Compare the even more general idea of (esp. 'weak') duality under a permutation introduced on p. 159 of McKinsey and Tarski [47].) However, note that we may be dealing with a case in which G and H are both primitive and only one of F, P is, in which case this move would not be available; whichever of the latter two was primitive would then be an extraneous connective spoiling – or interfering with – the twin relationship, to use the terminology introduced in Example 3.3(*i*).

(*ii*) As an alternative to Example 3.3(*i*), it would be nice to present alongside Example 3.3(*ii*) a tense-logical version, with G and H being non-identical twins. For example, we might consider the normal extension of K_t by (one of) Hamblin's discreteness axiom(s) $(p \wedge Hp) \rightarrow FHp$, together with $P\top$. A simple semantic argument shows that this logic does not contain the mirror image of that discreteness axiom;

[17] More explicitly: in some *proper coercive logical relation*, as this is explained in Definition 2.7(*ii*) in Humberstone [35].

but since the given axiom has $F\top$ as a consequence we have added $P\top$ so as to keep our past and future operators twins even though not identical twins. If this logic has further unmixed theorems whose mirror images are not theorems, the latter would have to be added, and one would want some guarantee that this could be done without bringing in their wake $((p \wedge Hp) \rightarrow FHp)^{\bowtie}$ – so the author can only conjecture that an example of the kind desired can be produced in this way. (Of course we don't always have to add the mirror images of the unmixed theorems since they are often consequences – even just against the background of K_t; $P\top$ was not thus automatically forthcoming as a theorem, reflecting the fact that a relation's being serial does not imply that its converse is, contrasting in this respect with adding the 4 of axiom $Gp \rightarrow GGp$ which yields its mirror image as a theorem, reflecting the fact that the converse of a transitive relation is transitive. A semantically more nuanced presentation of these issues can be found in [33], p.185*ff.*)

(*iii*) The proof given in the course of the discussion in Example 3.3(*ii*) – concerning models $\langle W, R, S, V\rangle$ – that G and H are identical twins in K_t is matched by a different strategy for the case of logics given proof-theoretically rather than semantically, by picking an axiomatization concerning which one shows that the mirror images of the axioms are provable and that the rules preserve the property of having a provable mirror image. (Indeed the sanction-and-reward case as presented in Example 3.3(*iii*) is such a case.) But here we pause to note that a similar semantic argument can be used whenever we are dealing with two items in the models – not just accessibility relations – which can be interchanged, keeping us inside a class of models determining the logic of interest, to show that the connectives they respectively interpret are identical twins in the logic determined. One cannot, incidentally, help but notice a resemblance between pairs of non-equivalent identical twins (in a logic) and pairs of objects which are not pairs of individuals, as this terminology is explained in Caulton and Butterfield [8]; non-equivalence of connectives should be understood for in terms of the non-synonymy – in the sense of [63], p. 116 – of compounds formed from the same components with their aid. Łukasiewicz's reaction this this phenomenon, mentioned in the final paragraph of Section 4 is likewise reminiscent of the puzzlement sometimes raised by the the idea that the positive and negative square roots of -1 are distinct – touched on again in [8] and many references there cited. ◀

In all of Examples 3.3, the two connectives compared are surrounded by a bevy of Boolean connectives left unaffected by the passage from A to $A^{\bowtie}$, and in general there the risk of interference from extraneous connectives alluded to in Example 3.3(*i*), which becomes acute when, to avoid comparing derived connectives, one gives those to be compared primitive status and the inevitability of what would have been used in the a definition (had they been non-primitive) to interfere – as that example (or, originally, in the last paragraph of the preceding section). One accordingly often has occasion to study the logic of the connectives concerned in the absence of other – even all other – connectives. Indeed this happened in the case of Proposition 1.1. However, in the case of some such pairs of connectives, there are no theorems to be found in the fragments thus purified:

Example 3.5. As a case in point, consider what in Zolin-inspired notation and terminology might be called the two 1-ary 'modalities' $\underset{\cdot}{\nabla} p = p \to \bot$ and $\underset{\cdot}{\Delta} p = ((\bot \to p) \to p) \to \bot$.[18] To avoid unwanted interference it may desirable to study the pure logic of these two connectives and compare them with each other when they are taken not only as primitives in their own right – to be written respectively as $\neg$ and $\neg'$ – rather than defined, but as the *only* primitive connectives of the language considered. For a more interesting case study in which the two connectives are not equivalent, we need to drop below not only classical but even intuitionistic logic, to – this atheorematic fragment of – Minimal logic. Thus, roughly speaking:

$$\neg A = A \to \bot \qquad \text{and} \quad \neg' A = ((\bot \to A) \to A) \to \bot,$$

with $\to$ and $\bot$ as in Minimal Logic, have the same logic when each is taken as the sole connective and the logic is given by the consequence relation, the restriction of which to the negation fragment we call $\vdash_{\mathsf{ML}_\neg}$ and to the non-standardly interpreted negation fragment, $\vdash_{\mathsf{ML}_{\neg'}}$.[19] ◀

This case gives us three logics of possible interest, $\mathsf{ML}_\neg$, the standard pure negation fragment of Minimal Logic, $\mathsf{ML}_{\neg'}$, a version of Minimal logic with the 'deviant' negation $\neg'$ as the sole primitive connective, and, naturally of greatest interest here since we are concerned with twinhood, $\mathsf{ML}_{\neg,\neg'}$ which has the two primitive connectives $\neg$ and $\neg'$. But here there is a problem: if the logic is the set of provable formulas, there is no logic to speak of in any of these three cases. Accordingly for the further treatment of this issue in Example 3.7(i) we pass to the associated consequence relations $\vdash_{\mathsf{ML}_\neg}$ etc., as was done – though this particular ('logic without

[18] In this case, unlike Zolin's, these are 1-ary modalities (compositionally derived connectives) with no 1-ary primitive connective in the language to play the anchoring role of $\Box$ in the modal case. One could use a 1-ary dummy connective $*$ (see note 15) and adapt the mapping $\tau_{\underset{\cdot}{\nabla}}$ so that it now maps preformulas involving $*$ by acting as the identity translation on sentence letters and for the (non-dummy) connectives of the object language by themselves, coming to life with the inductive step $\tau_{\underset{\cdot}{\nabla}}(*B) = \underset{\cdot}{\nabla}(\tau_{\underset{\cdot}{\nabla}}(B))$, and correspondingly in the case of $\tau_{\underset{\cdot}{\Delta}}$, *mutatis mutandis.*

[19] "Roughly speaking" because the above equalities are not to be taken as definitions within the current language, in which $\to$ and $\bot$ are not available to do the defining; rather, we intend the two negations to be taken as primitive but with $\vdash_{\mathsf{ML}_\neg}$ coinciding with $\{\neg, \neg'\}$ fragment of $\vdash_{\mathsf{ML}}$, and similarly with the further subfragments involving only one or other of $\neg, \neg'$.

theorems') problem did not arise there – in note 1 for the case of BCSK. We need this move also for another example in this section (Example 3.7(i)) as well as for the discussion of Łukasiewicz below (Section 4).

We assume familiarity with the notion of a consequence relation but rather than the cumbersome notation of note 1 in which one reads such things as the following (where $\vdash$ was actually something called $\vdash^{glo}_{\mathsf{S5}}$):

$$C_1[\Rightarrow], \ldots, C_n[\Rightarrow] \vdash A[\Rightarrow] \text{ iff } C_1[\rightarrow], \ldots, C_n[\rightarrow] \vdash A[\rightarrow],$$

it is more convenient to think of the pairs $\langle\{C_1, \ldots, C_n\}, A\rangle$ which are elements of a consequence relation as provable objects – 'sequents' – in their own right and use the 'context for a connective' notation on a sequent itself, and where $\sigma = \langle\Theta, A\rangle$ write $\vdash_L \sigma$ in place of the explicit notation $\Theta \vdash_L A$. Let us focus on the case in which Θ is $\{C_1, \ldots, C_n\}$ – though there is no general restriction to finite sets here – and σ is $\langle\Theta, A\rangle$, for which a more suggestive notation is usually employed, such as $\Theta \succ A$ or $\Theta : A$, the former notation being preferred here.[20] Then we can write the above more succinctly as

$$\vdash \sigma[\Rightarrow] \text{ iff } \vdash \sigma[\rightarrow].$$

More to the point, since we want to include the case in which both of the connectives of interest are present together (in the same sequent, whether or not they co-occur in any of the formulas making up that sequent), we can similarly denote by $\sigma^{\bowtie}$ the result of interchanging those connectives in all the formulas of σ. That is, with $(\cdot)^{\bowtie}$ for formulas as above, if σ is $\Theta \succ A$, then $\sigma^{\bowtie}$ is $\{C^{\bowtie} | C \in \Theta\} \succ A^{\bowtie}$.

Just for the record, let us give here the obvious reformulation of Definitions 3.2 to apply to the case of consequence relations. With respect to a pair of primitive connectives, of the same arity, in the language of a consequence relation $\vdash$ we call a sequent *mixed* if among its constituent formulas, both connectives appear, otherwise *unmixed*. For good measure we include a definition of the subconnective relation.

Definitions 3.6. *Suppose that $\vdash$ is a consequence relation on a language whose primitive connectives include n-ary κ, κ', in terms of which, as for Definitions 3.2, the 'unmixed' terminology and the "$\bowtie$" notation is to be understood (i.e., $\bowtie$ means $\kappa \bowtie \kappa'$). Then:*

(i) *κ, κ' are twins according to $\vdash$ if for all unmixed sequents σ of that language $\vdash \sigma$ iff $\vdash \sigma^{\bowtie}$;*

[20] Sometimes instead, "$\rightarrow$" or "$\Rightarrow$" is used as a sequent separator, which would be very confusing in the present discussion, since those are both already in service here as sentence connectives. As to the use of "Θ", see note 6.

(ii) κ, κ' *are identical twins according to* $\vdash$ *if for all sequents* σ *of that language* $\vdash \sigma$ *iff* $\vdash \sigma^{\bowtie}$.

(iii) κ *is a subconnective of* κ' *according to* $\vdash$ *if for all sequents* $\sigma[\kappa]$ *in which* κ *and any additional primitive connectives (of the language of* $\vdash$*) with the exception of* κ' *may occur, if* $\vdash \sigma[\kappa]$ *then* $\vdash \sigma[\kappa']$.

Note that under (iii) we could equally well have made the definition be: for all σ composed of formulas not constructed with the aid of κ', $\vdash \sigma$ implies $\vdash \sigma^{\bowtie}$, and also that the "iff"s in $(i), (ii)$ add nothing to what we would have with "only if" instead.

The following pair of examples concern twins in subclassical logics. The first concerns the case of $\mathsf{ML}_{\neg,\neg'}$, introduced after Example 3.5 above;the second is borrowed from [30].

Examples 3.7. (i) The connectives $\neg$ and $\neg'$ from Example 3.5 are twins according to $\mathsf{ML}_{\neg,\neg'}$, though not identical twins. A proof of the first assertion is to be found in the Appendix (see Proposition 5.2 there). For the second, note that for the mixed sequent $\sigma = \neg' p \succ \neg p$, we have $\vdash_{\mathsf{ML}_{\neg,\neg'}} \sigma$ while $\nvdash_{\mathsf{ML}_{\neg,\neg'}} \sigma^{\bowtie}$. That is, $\neg p$ is a consequence of $\neg' p$ according to the current logic, though not conversely (much as in Example 3.3(i)). To see this note that $((q \to p) \to p) \to q$ has $p \to q$ an intuitionistic (or 'positive logical' or indeed 'Minimal') consequence, so we can just substitute $\bot$ for q; on the other hand, since $\bot$ has no special logical powers in Minimal Logic, we could only have $p \to \bot$ as an ML-consequence of $((\bot \to p) \to p) \to \bot$ if $p \to q$ had $((q \to p) \to p) \to q$ as an ML-consequence – or, equivalently, an intuitionistic consequence, since only $\to$ is involved – which is easily checked not to be the case, e.g., using the Kripke semantics. (Alternatively, much as in Exercise 4.22.11 of [30]: observe that if this were so, then $(p \to q) \to (((q \to p) \to p) \to q)$ would be intuitionistically provable – an IL-consequence of the empty set, that is – and so, therefore, permuting antecedents, would

$$((q \to p) \to p) \to ((p \to q) \to q)$$

be. But on substituting $q \to p$ for all occurrences of p, we get a conditional that would then deliver Peirce's Law from Contraction by Modus Ponens.)

(ii) In the setting of intuitionistic logic (IL), consider the binary connectives (a) of *alternative denial* (to use Quine's phrase, though he did not take an interest in the present setting), $\downarrow$, with $A \downarrow B = \neg A \vee \neg B$ and (b) *nand* (or negated conjunction) $\bar{\wedge}$ with $A \,\bar{\wedge}\, B = \neg(A \wedge B)$. As with the two minimal negations of (i), we consider the fragment (now, of IL) in which only these two connectives figure, in which, as is familiar, they do not yield intuitionistically equivalent compounds. Specifically, for $\sigma = p \downarrow q \succ p \,\bar{\wedge}\, q$ we have $\vdash_{\mathsf{IL}} \sigma$ while $\nvdash_{\mathsf{IL}} \sigma^{\bowtie}$, so these two connectives are out of the running for being *identical* twins according to the current fragmentary subrelation of $\vdash_{\mathsf{IL}}$ – $\vdash_{\mathsf{IL}\downarrow,\bar{\wedge}}$, let's call it – which still leaves open the possibility that they are nevertheless, *twins* according to that consequence relation, which would then match the situation described under (i). But no, it is shown in [30] (Example 8.24.7 and Remark 8.24.8, p. 1245, where $\downarrow$ is written as $*$), that according to $\vdash_{\mathsf{IL}\downarrow,\bar{\wedge}}$, $\downarrow$ is a proper subconnective of $\bar{\wedge}$. (That is, $\downarrow$ is a subconnective of $\bar{\wedge}$ but not conversely.) ◀

Remarks 3.8. (i) Concerning Example 3.7(ii): As is also pointed out in [30], p. 397, the situation is quite different from that illustrated by the example in the case of *classical* logic ($\vdash_{\mathsf{CL}}$, as in Section 1), where no connective can be a subconnective of a non-equivalent connective (κ, κ' being said to be equivalent according to $\vdash$ when $\kappa(A_1, \ldots, A_n) \dashv\vdash \kappa'(A_1, \ldots, A_n)$ for all $A_1, \ldots, A_n$, where $ar(\kappa) = ar(\kappa') = n$).
(ii) Both the observation under (i) here and the proof in [30] of the '(proper) subconnective' claim alluded to in Example 3.7(ii) rely heavily on results established by W. Rautenberg. (The relevant references can be found in [30].) ◀

In this final paragraph of the present section, which can be skipped without jeopardizing the intelligibility of the remainder of the paper, the author confesses to being puzzled as to exactly how the present account of twins (Definitions 3.2(i) and 3.6(i)) is related to Zolin's account of analogousness. Zolin had no difficulty describing derived connectives ('modalities') as analogous in a given (monomodal) logic, as for example $\Box\Box$ and $\Box\Box\Box$ – or $\Box^2$ and $\Box^3$ for short – in K, whereas the $\bowtie$-based Definition 3.2(i), was seen (Remark 3.1) to be highly problematic for the case in which non-primitive logical vocabulary was under discussion. We might well have expected trouble with the 'identical twins' notion introduced in part (ii) of 3.2(i) (or 3.6) since it is not clear that there is any intuitive idea (for a generalization to non-primitive connectives of the $\bowtie$ operation to capture) of "interchanging $\Box^2$ and $\Box^3$" in a formula such as, for instance, the formula we might write for brevity as $\Box^8 p$. In the interests of not privileging any particular connective of the object language in the way in which $\Box$ is privileged in Zolin's treatment we might recruit the idea of a preformula with a dummy connective of the appropriate arity, here taken for simplicity to be 1. Thus where $A = A[\ast]$ is such a preformula, we might enquire as the equi-provability of $A[\underset{\cdot}{\nabla}]$ and $A[\underset{\cdot}{\Delta}]$ for the derived $\underset{\cdot}{\nabla}$ and $\underset{\cdot}{\Delta}$ of interest, where $A[\underset{\cdot}{\nabla}]$ is $\tau_{\underset{\cdot}{\nabla}}(\ast A)$ from note 18, and correspondingly for $A[\underset{\cdot}{\Delta}]$. But, for which preformulas $A[\ast]$ should one demand equi-provability of $A[\underset{\cdot}{\nabla}]$ and $A[\underset{\cdot}{\Delta}]$? The account should at least coincide with that given for twinhood of primitive connectives, extending it to cover non-primitives, so since we want primitive G and H in a formulation of K_t (with no other non-Boolean primitives) to count as twins – having seen them even to constitute *identical* twins in Example 3.3(ii), though here we are not worrying about the 'identical' aspect of this case.[21] For the preformula $A[\ast] = Hp \to \ast p$ we will have $A[H]$ ($= Hp \to Hp$) provable without $A[G]$ ($= Hp \to Gp$) being provable, which we do not want to count against the twin status of G, H, and can easily guard against by requiring that G, H do not themselves occur in the preformula $A[\ast]$.[22] This makes perfect sense because G, H happen to primitive connectives, and what would be needed is to make sense of this generally, but what exactly is it for a *derived* connective to occur in a formula? (Where $\boxdot A$ is, as in Example 2.2(i) $\Box A \wedge A$, does $\boxdot$ 'occur in' the formula $(\Box p \wedge q) \wedge p$? Or in $q \wedge \Box q$?) Finally, while this discussion has supposed that we can separate the 'twins' and 'identical twins' issues for derived connectives, that distinction hung on the contrast between mixed and unmixed formulas, a distinction which is problematic when non-primitive connectives are involved: returning to the above case of $\Box^2$ and $\Box^3$, is $\Box^6$ a mixed formula or an unmixed formula? If we are comparing $\Box$ and $\boxdot$, isn't any formula constructed using $\boxdot$ automatically a mixed formula? (This was the 'minor criticism' of Byrd in Section 2, where these were written as $\boldsymbol{B}$ and $\dot{\boldsymbol{B}}$.)

4 Back to Łukasiewicz

So frequently does commentary on the Ł-modal system of Łukasiewicz go astray that it is with some trepidation that one ventures into this territory. But venture we must, given the topic under discussion. That begins in the following paragraph, after we pause to illustrate the perils that have afflicted otherwise impeccably credentialed logicians. Segerberg's [58] broaches the subject of the Ł-modal logic on p. 209, where we are told that it is the smallest modal logic extending C with (all instances of the schema) – called by Segerberg (as is the logic itself) **Ł**: $\Box\top \to (A \to \Box A)$. Segerberg remarks that this is itself a regular modal logic, something not immediately evident from the above description of it since one of the defining conditions ([58], p. 12) for regular modal logics is closure under the rule which takes us from $A \to B$ to $\Box A \to \Box B$, making regularity a condition not automatically passed from modal logics to their extensions.[23] The description given obviously does not suffice to pick

[21] Conceivably, the pessimistic attitude already taken to identical twinhood between derived connectives notwithstanding, something could be made of this using preformulas $A[\ast_1, \ast_2]$ with two dummy connectives of the relevant arity.

[22] This is like 'extraneous interference' *à la* Example 3.3(i), except that here the interfering material – the "H" in the preformula – is here far from extraneous. The present example is highly reminiscent of Examples 6.1(i) and (ii) urged as making trouble for a particular account of logical independence in Humberstone [35].

[23] Here of course we are thinking of modal logics as sets of formulas (extensions thus being supersets), and more specifically as such sets as contain all classical truth-functional tautologies

out the Ł-modal logic since Segerberg's **Ł** does not containing every formula of the form $\Box A \to A$ – or more simply put, does not contain the formula $\Box p \to p$ – as the Ł-modal logic does. (At least this will be obvious in view of the matrix-based description of the logic below. This mistake is noted at the base of p. 48 in [33], no doubt among several other places.) Erring in the other direction, over-axiomatizing rather than under-axiomatizing the logic, the generally invaluable Font and Hájek ([13], p. 168) give an axiomatization of the Ł-modal system, using Modus Ponens as its sole rule, along with any axioms sufficient for classical propositional logic, and the following special axioms involving $\Box$:

$$\Box(A \to B) \to (\Box A \to \Box B) \qquad \Box A \to A \qquad \Box B \to (A \to \Box A)$$

but the first of these axioms (or more accurately, axiom schemata) is easily seen to be redundant given the other two. To be fair to Font and Hájek, (1) this is an oversight rather than an error, since the authors do not claim to be providing an independent axiomatization, and (2) it's not (originally, at least) *their* oversight, since, as they mention, they are take this axiomatization straight from Lemmon [40].[24] Font and Hájek, incidentally, also discuss consequence relations associated with the current set-of-formulas logic, though for reasons of space we defer entering into that topic until a later occasion, save for a parenthetical comment under Remark 4.1(i). (Similarly deferred is any discussion of the reaction by others to Łukasiewicz's treatment of the twins issue.)

Finally, let us note that Gottwald [18], p. 693, misidentifies the Ł-modal logic entirely, writing:

> In contrast to the situation with three-valued systems, where there are a lot of approaches and interpretations, only a few approaches concern four-valued systems and give particular interpretations to the four truth degrees. One of these rare exceptions is Łukasiewicz [1953] [= our [42]] who, in his later years, preferred a four-valued approach via his system L_4 toward a modal reading of the truth degrees over his original three-valued one via L_3 in [Łukasiewicz, 1920].

From the fact that Gottwald calls the famous three-valued Łukasiewicz logic of the

and are closed under uniform substitution and Modus Ponens. The other condition defining regular modal logics, aside from the monotonicity rule above (confusingly called the regularity rule by Segerberg) is that every formula of the form $(\Box A \wedge \Box B) \to \Box(A \wedge B)$ should belong to the logic. The two conditions are often conveniently combined into one, requiring closure under the rule taking us from $(A_1 \wedge \ldots \wedge A_n) \to B$ to $(\Box A_1 \wedge \ldots \wedge \Box A_n) \to \Box B$ (for $n \geq 1$). See also note 6 on other notations for $\Box$ and $\Diamond$ in use – including later in the present section.

[24] The axiomatization appears at p. 214 of [40], with the labels for the various axioms explained on p. 192.

1930s L_3, and from sentence after the passage quoted, with its reference to "L_4 with its linearly ordered truth degree set," it is clear that Gottwald has isolated the wrong four-valued logic here: even restricting attention to the modal connectives, the four-valued logic in the 1930s sequence of finitely many-valued logics explored by Łukasiewicz (and Tarski) has nothing to do with the four-valued logic on offer in [42], the partial order underlying the matrix treatment of the lattice connectives $\wedge$ and $\vee$ in this case being anything but a linear order.[25]

The third schema listed above could be replaced by

$$(A \leftrightarrow B) \to (\Box A \leftrightarrow \Box B)$$

explicitly revealing $\Box$ to be an extensional connective in the Ł-modal logic (alternatively put: revealing this to be an extensional modal logic) according to one natural use of the term 'extensional' as applied in propositional logic.[26] Given the classical background for the non-modal connectives here, the second occurrence of "$\leftrightarrow$" in the above schema can be replaced (without loss) by "$\to$", and given the schema $\Box A \to A$, both occurrences can be so replaced – a formulation Łukasiewicz was especially fond of, though Łukasiewicz's own preferred definition of the extensionality of a context $C(p)$ – in the above case, with $C(p) = \Box p$ – was not that $A \leftrightarrow B$ should provably imply $C(A) \leftrightarrow C(B)$ but rather that $C(A) \wedge C(\neg A)$ should provably imply $C(B)$.[27] Of course, this extensionality is widely taken to be responsible for every-

[25]As Simons ([60], p. 120) helpfully explains, to ward off the current misconception about the Ł-modal logic, "Suffice it to say that the logic is very unlike Łukasiewicz's earlier multivalent systems and also very unlike other modal systems. It is unlike his own systems in that it is an extension of classical bivalent logic and includes all bivalent tautologies."

[26]This is the usage to be found in Humberstone [22], where the present logic is one of those discussed, as well as in [30], in §3.2. The subsection there (3.24 – mentioned above in Remark 2.3) called 'Hybrids and the subconnective relation' devotes a Digression to the Ł-modal logic, beginning on p. 470. It appears in that subsection because necessity in this logic is obtained by hybridizing – isolating, that is, the common logical properties of – the one-place constant false truth-function and the identity truth-function, as will be clarified in the paragraph which follows. The consistent extensional modal logics are called *prime* logics in Zolin [72], though they are defined rather differently: as the logics of 'prime' modalities, the 1-ary modalities induced by formulas in which the sentence letter concerned does not appear in the scope of a modal operator – such as the formulas $\neg p$, $p \to \Box\bot$. In the case of the Ł-modal logic, as Zolin ([72], p. 866) notes, we are dealing with the modal logic in which $\Box p$ is equivalent to the prime modality $p \wedge \Box\top$; this modality figures extensively in translations between the present logic and others in Font and Hájek [13].

[27]The relation between these two notions of extensionality – the second of which, as one might expect, fails for intuitionistic logic – is explored in §5 (= Appendix A) of Humberstone [32]. Łukasiewicz's preferred way of expressing the negation-involving notion of extensionality would use his (one-place) variable functor "δ" and read as follows – though here we use infix notation rather than Polish notation for the binary connectives (and $\neg$ in place of N): $(\delta p \wedge \delta\neg p) \to \delta q$, or, in

thing philosophically objectionable about the Ł-modal logic as a plausible alethic modal logic,[28] though our current concern is not with pressing such objections.[29]

Historically, what Łukasiewicz wanted was a propositional logic intended for alethic modal applications, he favoured taking a $\Diamond$-operator in its language rather than a $\Box$-operator as primitive, but as mentioned in note 6, Łukasiewicz wrote Δ for $\Diamond$ (and Γ for $\Box$), so we follow suit when quoting from or otherwise presenting Łukasiewicz's material. The passage also mentions something written as ∇ – which again (repeating from Section 1) has nothing to do with the use of this symbol as a contingency operator, or with the $\underset{\cdot}{\nabla}$ introduced in Convention 2.1. The semantic description Łukasiewicz provides is four-valued: we take the direct product of the usual two-element matrix with itself for the Boolean connectives ($\to$ and $\neg$ in the case [42]), and interpret Δ by using the product of the identity and the (1-ary) constant true truth-functions. That is, denoting these functions by **I** and **V** respectively, and their product (in that order) by $\mathbf{I} \cdot \mathbf{V}$, the latter maps $\langle x, y\rangle$ ($x, y \in \{\mathrm{T}, \mathrm{F}\}$) to $\langle \mathbf{I}(x), \mathbf{V}(y)\rangle$.[30] The new idea occurring to Łukasiewicz in Section 7 ("The twin possibilities") of [42] is that we might also consider the product of these two two-valued truth-functions in the reverse order – $\mathbf{V} \cdot \mathbf{I}$ – and regard this as the interpretation of another possibility-like operator, namely the above ∇. For the language with both Δ and ∇, writing 1, 2, 3, 4, for $\langle \mathrm{T}, \mathrm{T}\rangle$, $\langle \mathrm{T}, \mathrm{F}\rangle$, $\langle \mathrm{F}, \mathrm{T}\rangle$ and $\langle \mathrm{F}, \mathrm{F}\rangle$, respectively we have the matrix depicted in Figure 1, with tables for the Boolean connectives and each of Δ, ∇. (As usual the designated value(s) – just one such in the present case – are indicated by an asterisk at first occurrence.)

Remarks 4.1. (i) We may take the Ł-modal logic to be the set of formulas valid in this matrix, since Smiley [62] showed (and Łukasiewicz seemed to have already known or at least presumed, in [42]) that the formulas provable on the basis of his

'exported' form $\delta p \to (\delta\neg p \to \delta q)$. It is interesting to read Łukasiewicz, in [44], p. 163, commenting, concerning several principles formulated with the aid of "δ", such as $\delta(p \to q) \to (p \to \delta q)$ (or, as he writes this: $C\delta CpqCp\delta q$), that these principles are "all very important but unknown to almost all logicians." Still correct today, on both counts – but the principles in question can be formulated as metalinguistic claims about contexts, without having to have a variable functor in the object language: see Humberstone [32] and (especially) [31] for this way of proceeding.

[28] See the end of §5.5 in Simons [61].

[29] On the subject of alternative axiomatization, let us note that of Tkaczyk [67] provides an interesting further example, intimately connected with Porte's representation (below) using Ω.

[30] This – along with **F** and **N** – for the constant false and negation truth-functions – is the notation used, e.g., in [28], p. 471; Łukasiewicz writes S, V, for **I**, **V**, and has no explicit product notation; to avoid the unintended (non)contingency associations of Łukasiewicz's Δ/∇ notation, these are written in [28] as $\Diamond$ and $\blacklozenge$ respectively. And naturally, Łukasiewicz uses Polish notation – $NCpNq$ for $\neg(p \to \neg q)$ etc. See [28], p. 469 for the rationale behind interpreting possibility as a product of the identity and constant true ('Verum') truth-functions. For much further background and information on the Ł-modal logic, see Font and Hájek [13] and references there given.

$\to$	1	2	3	4	$\neg$	Δ	∇
*1	1	2	3	4	4	1	1
2	1	1	3	3	3	1	2
3	1	2	1	2	2	3	1
4	1	1	1	1	1	3	2

Figure 1: Łukasiewicz's Matrix with Both Possibility Operators

axiomatization were precisely those valid in the matrix. It is clear that the twin possibility operators Δ and ∇ are indeed twins according to this logic, since the set of formulas valid in the product of two matrices is just the intersection of the sets of formulas valid in the respective factor matrices, the order of the factor matrices therefore being irrelevant. Whether a symbol is interpreted as $\mathbf{V} \cdot \mathbf{I}$ or as $\mathbf{I} \cdot \mathbf{V}$ it still exhibits just the logical behaviour it would exhibit both when interpreted as the identity truth-function and when interpreted as the constant true truth-function. As is well known, however, when logics are thought of as consequence relations, we lose the guarantee that the consequence relation determined by the product of two matrices coincides with (though it always includes) the intersection of the consequence relations determined by the factor matrices. (With generalized or multiple conclusion consequence relations, we lose even that. See Observation 2.12.6 in Humberstone [30] and the preceding discussion, for examples, proofs and references to the literature. Slogan version in the terminology of that discussion and of the already mentioned subsection 3.24 of the same work: in the framework Fmla, products of connectives are hybrids, but in Set-Fmla and Set-Set this is not guaranteed to be so.) We return to the question, raised in Section 1, of whether Δ and ∇ are not only twins but identical twins in the (formula) logic under consideration here in due course (Prop.4.4), after some further terrain familiarization. Readers for whom the answer that question is already obvious should still be able to enjoy the ride.

(*ii*) We may take the remaining Boolean connectives as defined in terms of $\to$ and $\neg$ in any of the usual ways, which will give them tables like those familiar from discussions of the product of (the matrix operation interpreting) a connective with itself (e.g., Bolc and Borowik [4], p. 21, or Rescher [55], pp. 96–98); this is just the usual notion of taking direct products of algebras, coupled with taken a value as designated in the product matrix iff each coordinate is designated in the respective factor matrix.[31] Note that this has nothing to do with the use of phrases like 'product connective' in the fuzzy logic subgenre of many-valued logic, where the reference is

[31] This, the standard approach, is Rescher's 'Policy 1' on p. 97 of [55]; ignore what he calls 'Policy 2' which replaces "each' with "at least one".

to arithmetical multiplication. (See the index entries starting with 'Product' in Metcalfe et al. [48].) Łukasiewicz [42] also observed that in this logic ∇ could be defined in terms of $\to$ and Δ: $\nabla A = \Delta A \to A$. If we are taking both of these as primitive, as suits the present discussion better (see Convention 4.2), then to secure the interreplaceability of what would otherwise be *definiens* and *definiendum*, it suffices to add as a further axiom schema $\nabla A \leftrightarrow (\Delta A \to A)$ (or the corresponding equivalence for the dual operators). ◀

Since modal matters are often discussed with $\Box$ rather than $\Diamond$ taken as primitive (cf. the opening paragraph of this section), and Łukasiewicz endorses the familiar equivalences of $\Box$ with $\neg\Diamond\neg$ and $\Diamond$ with $\neg\Box\neg$, we should note that the columns under a table for $\Box$ defined by the former equivalence when $\Diamond$ is taken as Δ would be (reading downward) 2, 2, 4, 4, while if $\Diamond$ is taken as ∇, for the defined $\Box$ we should have, instead: 3, 4, 3, 4. The associated truth-function products are (using, as well as the pair-forming $\cdot$ above, notation from note 30) $\mathbf{I} \cdot \mathbf{F}$ and $\mathbf{F} \cdot \mathbf{I}$, respectively.

Łukasiewicz concentrates on possibility rather than necessity, introducing no special notation for the $\Box$-operator dual to ∇ in the way that his Γ is dual to Δ, and writes:

> Δ and ∇ are indistinguishable when they occur separately, but their difference appears at once when they occur in the same formula. They are like twins who cannot be distinguished when met separately, but are instantly recognised as two when seen together. Take, for instance, the formulae $\Delta\Delta p$, $\nabla\nabla p$, $\Delta\nabla p$ and $\nabla\Delta p$. $\Delta\Delta p$ is equivalent to Δp which is rejected, and likewise $\nabla\nabla p$ is equivalent to ∇p which is rejected too. But $\Delta\nabla p$ and $\nabla\Delta p$ must be asserted (...) We cannot, therefore, replace in the two last formulae Δ by ∇ or vice versa (...).[32]

The talk of accepted and rejected formulas in the quoted passage may be understood in terms of the usual notion of validity and invalidity in matrix semantics, though Łukasiewicz also seeks to parallel these two with matching syntactic characterizations using counter-axioms and rules of rejection alongside the usual axioms and rules of proof – an aspect of his methodology of no concern to us here (and famously eliminated from the streamlined early presentation in Smiley [62][33]), except to note a possible confusion that can arise because of it. The risk in using

[32]This passage is from p. 370 of [42]; I have changed the word "undistinguishable" in the opening sentence to the more idiomatically suitable "indistinguishable". It would perhaps have been clearer to add after the "vice versa" the words "unless both replacements are made at once".

[33]Smiley begins by showing that the axioms and non-rejection-involving rules yield proofs of exactly the formulas valid in the matrix of Fig. 1 – the Ł-*matrix* for short; later he goes on to show that the formulas invalid in the matrix are derivable as rejected formulas in the combined proof + rejection system. Thus the rejected formulas end up being exactly those that are not provable.

terminology such as "the Ł-modal system" is that this may be taken to refer to Łukasiewicz's modal logic, understood as the phrase 'modal logic' usually is when logics are thought of as sets of (provable) formulas, and it may be taken to refer to the pair comprising the logic in that sense together with the set of explicitly rejected formulas, which may or may not coincide with the set of formulas not provable.[34] To see the need for this distinction, let us recall the discussion in Prior ([54] p. 126) where it is remarked that not only does the Ł-modal system lack theorems of the form $\Box A$ (of the form $L\alpha$, as Prior says – or $\Gamma\alpha$ as Łukasiewicz himself would put it), but that it would become inconsistent if any such formulas were added as further axioms; here he is, saying that and a bit more, and using Łukasiewicz's notation (from [44]) of L and M in place of Γ and Δ (or $\Box$ and $\Diamond$):

> Like S1–S3 it has neither the rule to infer $L\alpha$ from α nor the law MMp, but unlike them it is not merely consistent with both without implying either but positively inconsistent with both.

This may come as a surprise to someone whose first encounter with the Ł-modal logic was as in the opening paragraph of the present section where we listed three axioms (well, strictly, three axiom-schemata) deliberately including the first, redundant one not only to comment on its redundancy, but also because of its fame as the K-axiom often used as part of the characterization of normality among modal logics, and in the axiomatization of the smallest such logic, K itself. In these last two roles, one needs also closure under the rule of necessitation, conspicuously missing in the present case – and apparently according to Prior, not a closure condition that one could consistently impose. And this claim of inconsistency will seem surprising since obviously the consistent normal modal logic $\mathsf{KT!}$ (in the nomenclature of Chellas [9]), often called the Trivial modal logic, satisfies all these conditions (and only slightly less obviously, is the least normal modal logic to do so).

The resolution of the puzzle presented by these conflicting considerations is that Prior is not discussing the Ł-modal *logic* in the 'logic as a set of (provable) formulas' sense, but the Ł-modal *system* understood in the combined assertion + rejection sense. ($\mathsf{KT!}$ proves $\Diamond p \to p$, which is one of Łukasiewicz's counter-axioms. Note that this does not mean that all formulas of the form $\Diamond A \to A$ are rejected – many such formulas are provable – since Uniform Substitution moves in reverse for

[34] This contrast cross-cuts the distinction between logics – with or without the rejective component – and particular axiomatizations of them (also sometimes terminologically marked by the 'logic'/'system' distinction). When the rejective component is present, we have counter-axioms or anti-axioms (the initially rejected formulas) as well as rules involving both rejected and asserted (or provable) formulas. By 'explicitly rejected' in the text to which this footnote is appended is meant those formulas which are initially rejected or whose rejection follows from the axioms and the rules (rejection-involving or otherwise) of the proof system.

rejected formulas, from a rejected formula to the rejection of any formula *of which it is* a substitution instance.[35]) Let us denote the Ł-modal logic, without any of the rejective apparatus of the full Ł-modal system, and with its language shorn of the variable functor δ (see note 27), by Ł.

Convention 4.2. *More specifically, since when asking about the potential twinhood of pairs of connectives, we presume both connectives are primitive, the language of* Ł *should have both* Δ *and* ∇ *as its non-Boolean primitives, or else both of their duals,* Γ *and* L *– inspired by Simons' use of L in Simons [59] rather than Łukasiewicz's own use of the latter symbol (see note 6).*

The above-mentioned non-normality – in view of the failure of necessitation to preserve provability – of Ł has naturally led to its semantic treatment using the ideas introduced in Kripke [38], at the hands of Lemmon [40], as recapitulated in Font and Hájek [13] and also Tkaczyk [67]. A *frame* in this setting with universe W is equipped with a special subset N of W as well as an accessibility relation $R \subseteq W \times W$ whose elements are called *normal* worlds, and is converted into a model by the addition of a V assigning subsets of W to the sentence letters, and a formula is valid on a frame if it is true at every point (not just the normal worlds – this is feature introduced by Lemmon) in every model on the frame. $\Box A$ is deemed to be true at $w \in W$ in such a model if $w \in N$ and for every $y \in W$ with Rxy, we have A true at y. For certain purposes it is useful to break this up into its two components and introduce a propositional constant stipulated to be true in any model at precisely the normal worlds of that model, so that $\Box A$ as just defined amounts to the conjunction of that constant with $\Box' A$ where $\Box'$ is interpreted as in the semantics for normal modal logics, as just quantifying universally over accessible points. Cresswell [11] wrote a for such a constant, as did Aanderaa [1], though he took this to be a sentence letter or propositional variable, specified as one not occurring in a formula one was using the constant to translate in various embeddings between normal and non-normal modal logics. That was also Cresswell's concern, for which reason in neither case does a supplementary notation such as $\Box'$ above need to put in an appearance, each logic being cast in a language with a single necessity operator. For reasons that will

[35]As well as $\Delta p \to p$, just mentioned in the more familiar $\Diamond$ notation, Łukasiewicz has Δp as a counter-axiom, and from either of these by the rejective form of Uniform Substitution, we can derived p as a rejected formula – a counter-axiom used by Łukasiewicz when presenting non-modal propositional logic. There is also a rejective formulation of Modus Ponens: from asserted $A \to B$ and rejected B, to rejected A. This looks a lot like Modus Tollens but rejection in the present sense is not to be confused with rejection in the sense of denial (cf. [24]), and counter-axioms are not doing the work that presenting the negation of the formula concerned as an axiom would do – as the example of those counter-axioms just cited shows.

become evident presently, however, we will write this normality constant as Ω. Thus the equivalence mentioned above would be written as

$$\Box A \leftrightarrow (\Omega \wedge \Box' A).$$

Let us put this to the back of our minds for the moment as we complete this summary of the Kripke-inspired background for a range of non-normal modal logics before we get specifically the application of current interest.

Sotirov [64] notes that for the case of Ł we can discard the accessibility relation and use the simple 'accessibility-free' clauses below in the definition of truth at a point $x \in W$ in a model $\mathcal{M} = \langle W, N, V\rangle$; We write $\Box$ as Γ here so that it matches Δ, as we write the corresponding dual operator so that we can compare it with Łukasiewicz's ∇, and give the induced clause for this as (4.2):

$$\mathcal{M} \models_x \Gamma A \;\Leftrightarrow\; (x \in N \; and \; x \models A) \tag{4.1}$$

$$\mathcal{M} \models_x \Delta A \;\Leftrightarrow\; (x \notin N \; or \; x \models A) \tag{4.2}$$

Also noted in [64] is the fact that we can actually focus attention on a single frame, with one element in N and one in $W \smallsetminus N$, which validates precisely the formulas valid on all such frames. And these formulas are exactly the Ł-theorems, since at the normal point Γ is interpreted as **I** and, at the non-normal point, as **F**.

Observe that we could equally have used a metalinguistic material conditional to write the right-hand side of (4.2) as "$x \in N \Rightarrow x \models A$," which is a metalinguistic way of bringing out what Prior had in mind in writing the following ([53], p. 189), in the course of his description of a summer logic workshop in the Oxford of 1956:

> D. P. Henry (...) told me just before Lemmon's lecture that he had defined a modal logic within the propositional calculus by defining Mp as Czp, where z is a variable not put to any other purpose. (As the answer to "p?", "Possibly" means "Yes, if —".) From this information alone it would seem that this system would boil down to Łukasiewicz's, for $CpMp$ follows by substitution (q/z) in $CpCqp$, and $CMpCMNpNq$ by substitution in $CCrpCCrNpCrq$.[36]

[36]As Smiley [63], note 6, does in quoting this same passage, I have changed "L" to "Ł." The parenthetical sentence is a bit slick, adding a "Yes" to the "If —". By itself "If A then B" has no tendency to convey that it is possible that B unless it is presumed that it is possible that A. This is a point associated with Chisholm ([10], p. 5f.) in connection with the supposed equivalence, in some presentations of compatibilism, between "X could have done otherwise" and "If X had chosen to do otherwise, then X would have done otherwise," an equivalence undermined by cases in which the conditional is true but X could not have chosen to do otherwise. Note, incidentally, that the conditionals here are subjunctive, and with indicative conditionals there would be trouble with Prior's "Yes, if" line, on a straight material implication account (as the use of Polish notation's "C" suggests here): even if it is possible that A, from the additional information that it is (contingently)

Now this "z" in Henry's suggestion is an early appearance of the "a" of Aanderaa and Cresswell, which was written as Ω above because that is how Jean Porte wrote it, and it was Porte who first noted the light it shed on Ł.[37] As already remarked, while the Kripke-semantical route to this logic helps locate it relative to other (regular) non-normal modal logics, so little of the semantic apparatus ends up being exploited that we might as well simply start (as indeed Porte might have[38]) with the binary division into the normal points, at which Γ ends up expressing the truth-function $\mathbf{I}$ (and Δ, $\mathbf{I}$ also), and the non-normal points, at which it expresses $\mathbf{F}$ (and Δ, $\mathbf{V}$). Since validity requires us to take both into account we are successfully hybridizing these truth-functions with the above treatment. And we could just as well have begun with the truth-table on the left of Figure 2, focusing on the Γ case for $\Omega \wedge p$, treating Ω as though it were just another sentence letter.

Ω	$\wedge$	p
T	T	T
T	F	F
F	F	T
F	F	F

Ω	$\rightarrow$	p
T	T	T
T	F	F
F	T	T
F	T	F

Figure 2: Porte's perspective on Ł-modal necessity and possibility

Concentrating on the first table in Figure 2, we see just what we expect we for the conjunction of two formulas that can assume together all combinations of the values T and F. Focus first on the cases in which Ω gets the value T (the top two lines), thinking of this as a single operator $\Omega \wedge \_$ we apply to p, putting p into the blank indicated by underlining. What 1-ary (bivalent) truth-function of p's truth-value does this operator represent? In these two lines, we see that the whole formula has the same truth-valued as p itself, so the truth-function in question is

false that A it would follow that (material conditionally) *if A then B*, for any B, with, again, nothing to make one think on this basis that it is possible that B.

[37] An interesting coincidence, especially as all four suggestions were independent of one another: Aanderaa, Cresswell, Henry and Porte all select the first or last letters from the alphabets they are drawn from in this capacity. In the course of their re-discovery of this approach, Font and H'ajek [13] use L to play this role. For reasons of space the details are omitted here of the differences among these authors as to whether the symbol chosen should be regarded as a propositional variable (sentence letter) or as a nullary connective (sentential constant), as is any discussion of the comparative merits of these alternatives.

[38] ... but in fact did not, at least not in *quite* the form in which Lemmon presents the material, crediting work in the 1940s by Marcel Boll: [49], p. 918 base; for more detail, including on the relations between the Boll–Reinhart logics and Lemmon's E-systems see Porte [50].

the identity truth function, **I**. Similarly, looking at the bottom two lines, in which Ω has the value F, the whole conjunction has the value F regardless of p's truth-value, so in these cases $\Omega \wedge \_$ is delivering the constant false truth-function, **F**. So, if we count a formula involving Ω as valid when regardless of whether Ω is assigned T or F, the formula will have to have the value T when $\Omega \wedge \_$ expresses the identity truth function and also when $\Omega \wedge \_$ expresses the constant false function. But that is exactly what validity in the Ł-matrix demands of $\Box\_$ (writing in the blank just for the sake of a parallel notation here). Likewise in the case of Δ, the first two lines of the table on the right of Figure 2 present us with **I**, and in this case the bottom two lines, with **V**.

To see what light this throws on the 'twins' issue, recall from Remark 4.1(ii) that ∇A is Ł-equivalent to $\Delta A \to A$, which means that ∇A amounts to $(\Omega \to A) \to A$, or more briefly put, $\Omega \vee A$, or perhaps more usefully for present purpose, $\neg\Omega \to A$: more usefully because we can see that in passing to Γ's twin, L, what we have done is moved from Porte's constant to its negation, keeping us well inside the garden of modal operators (in the broadest sense) "hidden in classical logic", to quote from the title of Sotirov [64], where Vakarelov is credited alongside – indeed ahead of[39] – Porte for his horticultural endeavours. Figure 3 gives a fuller picture of the 1-ary operators O derived by applying the 16 binary truth-functions to a Porte-style Ω and another formula (represented by the sentence letter p). Each binary truth-functional connective $\#$ – for which we use infix notation here – yields an ordered pair $\langle f, g\rangle$, of 1-ary truth-functions satisfying the condition that for any Boolean (bivalent) valuation v, when $v(\Omega) = T$, $v(\Omega \# p)$ is $f(v(p))$ and when $v(\Omega) = F$, $v(\Omega \# p)$ is $g(v(p))$. Thus each such pair renders O as a product connective whose interpretation is $f \cdot g$. To save space we use overlining for negation, and write (connectives for) the constant true and constant false binary truth-functions as Ⓣ and Ⓕ, and the projections to the first and second coordinate as ①, ②, respectively.

Thus in particular, the top entry on the left with Op as $\Omega \wedge p$ gives us Łukasiewi-

[39] This claim of priority is not clearly correct: see note 40 of Humberstone [34]. The relevant Vakarelov references are can be found in Sotirov [64]; see also p. 171*f.* in Font and Hájek [13]. Sotirov also discusses Vakarelov's use of the $\langle \mathbf{V}, \mathbf{N}\rangle$ entry in Figure 3 ($p \to \Omega$) for handling a subclassical negation operator, so the for the record we include also Porte [51] in our bibliography. The fact that Ł can be treated in this way was already known to Prior in 1956, as we see from the earlier quotation from [53] concerning D. P. Henry. (See also p. 172 of Font and Hájek [13], for discussion of a similar idea in work by H. B. Curry.) The possibility of such a presentation is implicit in Aanderaa [1], disguised in his faithful embedding of Ł in the 'trivial modal logic' KT!, so it is not actually mentioned that we can throw away the 'modal' part of this description. Just to make the relevant observation even harder to extract, Aanderaa introduces KT! as TM at the start of [1], but when the embedding result is stated in its final line it is referred to as LT. (At least, that is the best sense I can make of Aanderaa's discussion.)

Ω compound	Hybridized pair	Ω compound	Hybridized pair
$\Omega \wedge p$	$\langle \mathbf{I}, \mathbf{F} \rangle$	$\Omega \vee p$	$\langle \mathbf{V}, \mathbf{I} \rangle$
$\Omega \wedge \overline{p}$	$\langle \mathbf{N}, \mathbf{F} \rangle$	$\Omega \vee \overline{p}$	$\langle \mathbf{V}, \mathbf{N} \rangle$
$\overline{\Omega} \wedge p$	$\langle \mathbf{F}, \mathbf{I} \rangle$	$\overline{\Omega} \vee p$	$\langle \mathbf{I}, \mathbf{V} \rangle$
$\overline{\Omega} \wedge \overline{p}$	$\langle \mathbf{F}, \mathbf{N} \rangle$	$\overline{\Omega} \vee \overline{p}$	$\langle \mathbf{N}, \mathbf{V} \rangle$
$\Omega \leftrightarrow p$	$\langle \mathbf{I}, \mathbf{N} \rangle$	Ω ① p	$\langle \mathbf{V}, \mathbf{F} \rangle$
$\overline{\Omega} \leftrightarrow p$	$\langle \mathbf{N}, \mathbf{I} \rangle$	$\overline{\Omega}$ ① p	$\langle \mathbf{F}, \mathbf{V} \rangle$
Ω Ⓣ p	$\langle \mathbf{V}, \mathbf{V} \rangle$	Ω ② p	$\langle \mathbf{I}, \mathbf{I} \rangle$
Ω Ⓕ p	$\langle \mathbf{F}, \mathbf{F} \rangle$	Ω ② $\overline{p}$	$\langle \mathbf{N}, \mathbf{N} \rangle$

Figure 3: Porte–Vakarelov Constant-induced Operators

cz's initially introduced $\Box$ connective Γ with interpretation $\mathbf{I} \cdot \mathbf{F}$, while the third entry down on the right represents his first stab at $\Diamond$ with Δ as $\mathbf{I} \cdot \mathbf{V}$. The twin possibility, ∇, appears at the top of that column, with interpretation $\mathbf{V} \cdot \mathbf{I}$, as mentioned in the vicinity of Figure 1, though now we see how the interaction with Ω gives rise to these variations. The dual operator, L, is the third entry in the first column (interpretation: $\mathbf{F} \cdot \mathbf{I}$).

Attending to $\neg\Omega$ no less than to Ω throws light on some of the Δ/∇ interactions commented on in the earlier quotation in this section from Łukasiewicz [42], in particular the remark that unlike the invalid $\Delta\Delta p$ and $\nabla\nabla p$, $\Delta\nabla p$ and $\nabla\Delta p$ were valid. (Invalid and valid in the Ł-matrix of Figure 1, that is; of course what Lukasiewicz actually said was "rejected" and "asserted".) $\Delta\nabla p$ becomes, when expressed in our current terms, $\Omega \to (\neg\Omega \to p)$ (or $\Omega \to (\Omega \vee p)$), while its Δ/∇ $\bowtie$-switch just interchanges Ω with its negation and so again is a truth-functional tautology (treating Ω as though it were formula of CL). For our current agenda, however, more significant is the fact that since Ω was a way of coding 'normality' in the $\langle W, N, V \rangle$ models in play for (4.1) and(4.2) above, its negation serves as a marker for non-normality, so we can now treat L and ∇ in such models by corresponding clauses in the definition of truth:

$$\models_x \mathrm{L}A \;\Leftrightarrow\; (x \notin N \textit{ and } x \models A) \tag{4.3}$$

$$\models_x \nabla A \;\Leftrightarrow\; (x \in N \textit{ or } x \models A) \tag{4.4}$$

Remark 4.3. Porte [49] closes with the following observation, credited to D. Lacombe, in which Porte's use of "N" (for necessity) has been replaced by "Γ":

> It is possible to generalize the Ω-system by introducing (in the propositional calculus) any number of constants similar to Ω: $\Omega_1, \ldots, \Omega_n$. We will eventu-

> ally get a characteristic matrix with 2^{n+1} elements, and we can define n 1-ary connectives similar to Γ.

What is missing here is the observation that already without adding any further Ω-style constants, we already have a second endogenous 1-ary connective 'similar to' Γ – namely the dual of ∇ (rather than of Δ) – and that, correspondingly, passing to the general case of $\Omega_1, \ldots, \Omega_n$ gives $2n$ such connectives rather than n, since in each case we can use either Ω_i or its negation to conjoin with a formula A to produce an Ł-style necessitation of A. ◀

Recall also that since the right-hand sides of these clauses do not direct us away from the point x of evaluation, we can (following Sotirov [64]) conduct the discussion entirely in terms of the characteristic two-element frame mentioned above – two elements so that the case of normal and nonnormal points are both covered. So we have a simple proof in the style of Example 3.3(*ii*) for the expected strengthening of Remark 4.1, inserting "identical":

Proposition 4.4. Γ *and* L *are identical twins in* **Ł**. *Alternatively, if* Δ *and* ∇ *are taken as the non-Boolean primitives: these two connectives are identical twins in* **Ł**.

Proof. For a model $\mathcal{M} = \langle W, N, V\rangle$ let $\mathcal{M}^{\bowtie}$ be the model $\mathcal{M} = \langle W, W \smallsetminus N, V\rangle$. Then induction on the complexity of arbitrary A (mixed or otherwise) shows that for w in W, $\mathcal{M} \models_w A$ if and only if $\mathcal{M}^{\bowtie} \models_w A^{\bowtie}$, from which we conclude that A and $A^{\bowtie}$ are equi-provable in **Ł**. □

This is not meant to be a new result, except insofar as it clarifies the issue of identical as opposed to 'merely fraternal' twinhood in its formulation, and places it in the context of other such results. It is not even a new proof, but a model-theoretic formulation of Łukasiewicz's matrix-theoretic proof in [42] (p. 372), which consists in observing that the function mapping 2 to 3 and 3 to 2 is a matrix isomorphism of the Ł-matrix of Figure 1: that is, an isomorphism of the algebras which preserves and reflects designation. This is just the matrix corresponding to the two-element characteristic frame in the familiar way, with 2 corresponding to (the one-element subset) N and 3 to $W \smallsetminus N$. Remark 3.4(*iii*) applies again here: one could equally well argue the case by an induction on the length of proofs in a suitable axiomatization, whether that for which Smiley [62] showed the Ł-matrix to be characteristic, or any of the alternatives suggested at the start of the present section. In view of Convention 4.2, one should bear in mind that for the latter $\Box$-based axiomatizations, in which we envisage $\Box$ re-written as Γ, the comment at the end of Remark 4.1(*ii*) about the "corresponding equivalence for the dual operators," which needs to be counted as one of the axioms, the definition-replacing equivalence in question is the perhaps surprising-looking: $\mathrm{L}A \leftrightarrow (A \wedge \neg \Gamma A)$, whose $(\Gamma \bowtie \mathrm{L})$-switch therefore needs checking in the basis part of that induction, and which incidentally reveals rather starkly the fact that the twin necessity Γ and L notions are mutually exclusive – as indeed is evident from the Porte–Vakarelov representations of ΓA and $\mathrm{L}A$ as having respectively Ω and $\neg\Omega$ as conjuncts. This last consideration shows that the situation is more extreme than that: not only are ΓA and $\mathrm{L}A$ incompatible according to Ł for all A – that is, the negation of their conjunction is Ł-provable – but so are ΓA and $\mathrm{L}B$ for any A, B. This is really just a re-phrasing of something already familiar as exhibiting the Halldén incompleteness of Ł: $\Delta A \vee \nabla B$ is always provable – so choose A, B to be variable-disjoint (e.g., to be p, q).[40]

In Section 1, it was noted out that Łukasiewicz did not – the present author's loose summaries of the status of Δ and ∇ as cited there – ever refer to these operators in [42] as being like identical twins. This provoked the present current enquiry as to whether one might, picking up on aspects of Łukasiewicz's discussion, distinguish a logical analogue of twins in general from a logical analogue of identical twins in particular. We have indeed found such a distinction to be sustainable, and of Łukasiewicz's Δ and ∇ to stand in the narrower relation of identical twinhood in his favoured logic – something we have now seen Łukasiewicz to be aware of, even though that exact terminology is not employed in [42]. It is, accordingly, gratifying to see that the passage, quoted at greater length above, in [42] (p. 37), reading:

> They are like twins who cannot be distinguished when met separately, but are instantly recognized as two when seen together

is subtly altered for its subsequent appearance on p. 173 of [44]:

> They are like identical twins who cannot be distinguished when met separately, but are instantly recognised as two when seen together.

Whether the alteration was intended to be anything more than stylistic is hard to say, but, as we have seen, the upshot can retrospectively be construed as providing a more precise description of the situation. On the other hand, in retrospect, we have also found (in effect) that Łukasiewicz's reaction – mentioned at the end of Remark 3.4(*iii*) – to the presence of twin connectives which yield non-equivalent formulas as

[40] Tkaczyk [67], p. 226, mentions this as though it were a new discovery; the observation can be found in Anderson [2]. The explanation Tkaczyk then proceeds to give – basically, that any hybrid of distinct truth-functional connectives engenders a Halldén-unreasonable disjunction with the disjuncts corresponding to the different truth-functions hybridized – was given in Humberstone [22], p. 33*f*. However, Tkaczyk does go on after that to formulate the observation in terms of the Kripke–Lemmon semantic treatment of Ł.

representing a "logical paradox" is certainly an overreaction,[41] the phenomenon in question being thoroughly commonplace – as, for instance, tense logics (like K_t, in Example 3.3(ii)) with the mirror image property.[42]

5 Appendix: Postscript to Section 3 on the Two Minimal Negations

Reminder: the wording "the two" in the title here is a reference to the $\neg$ and $\neg'$ introduced in Example 3.5 and last seen in Example 3.7(i); it is not intended to suggest that these are the only two negation-like connectives available in $\vdash_{\mathsf{ML}}$.[43] We recall from the discussion after Remarks 3.4 the two consequence relations $\vdash_{\mathsf{ML}\neg}$ and $\vdash_{\mathsf{ML}\neg'}$ and their common (conservative) extension $\vdash_{\mathsf{ML}\neg,\neg'}$. Our task here is to show that $\neg$ and $\neg'$ are twins according to this third consequence relation, which amounts to showing that the first two consequence relations coincide, modulo the notational shift between $\neg$ and $\neg'$. We begin by noting that in the present expressively impoverished setting, there is little opportunity for premisses to combine to yield conclusions:

Lemma 5.1. (i) *If* $\Theta \vdash_{\mathsf{ML}\neg} C$ *then there is* $\Theta_0 \subseteq \Theta$ *with* $|\Theta_0| \leq 2$ *with* $\Theta_0 \vdash_{\mathsf{ML}\neg} C$*, and* ($ii$) *if* $\Theta \vdash_{\mathsf{ML}\neg'} C$ *then there is* $\Theta_0 \subseteq \Theta$ *with* $|\Theta_0| \leq 2$ *with* $\Theta_0 \vdash_{\mathsf{ML}\neg'} C$*, and further,* Θ_0 *can be chosen in such a way that there are at most two sentence letters occurring in the sequent* $\Theta_0 \succ C$.

Proof. Note that every formula of the language of $\vdash_{\mathsf{ML}\neg}$ or $\vdash_{\mathsf{ML}\neg'}$ is the result of

[41] From just before the previous quotation on p. 370 of [42]: "We encounter here a logical paradox: although Δ and ∇ can be defined by the same matrix, they are not identical." Here the talk of identity is an allusion to the matrix isomorphism mentioned in our previous paragraph.

[42] At the semantic level, the situation with K_t is that, adapting the mirror image operation on models to one (similarly notated) on their underlying frames, it suffices for the mirror image property that the class of frames determining our tense logic is closed under the operation taking us from $\mathcal{F} = \langle W, R, S\rangle$ to $\mathcal{F}^{\bowtie} = \mathcal{F} = \langle W, S, R\rangle$ (from $\langle W, R\rangle$ to $\langle W, R^{-1}\rangle$, on the more familiar way of presenting such frames). Since a matrix corresponds to a single frame, the closer tense-logical analogue would be to have a single point-generated frame $\mathcal{F}$ which is isomorphic to $\mathcal{F}^{\bowtie}$. In the terminology of note 78 of Humberstone [33], we want a frame which is symmetrical but not symmetric – the latter because we want G and H to be identical twins without being equivalent. The logic determined by the two-element frame in which one element bears R (with converse S) to the other, or the reflexive closure of this frame, would do nicely, especially as the induced matrix has four values, as with the Ł-matrix.

[43] Numerous 1-ary connectives of interest are ML-definable which are equivalent to $\neg$ in IL though not in ML; one such, definable in ML as $A \leftrightarrow \bot$ is given some attention in [30], Observation 8.33.10, which shows that neither this nor the standard ML negation is a subconnective of the other (according to $\vdash_{\mathsf{ML}}$).

applying $\neg$ or $\neg'$, respectively, to some sentence letter p_i, and we refer to this sentence letter as the core of the formula in question. We begin with the case of $\vdash_{\mathsf{ML}\neg}$, in which $\neg(\cdot)$ is interpreted as $\cdot \to \bot$, with $\bot$ as in ML, and with the reminder that any formulas A_0, A_1 differing at most in the replacement of a subformula $\neg\neg\neg D$ with $\neg D$ are equivalent ($A_0 \dashv\vdash_{\mathsf{ML}\neg} A_1$). With this in mind, it is evident that $\Theta \vdash_{\mathsf{ML}\neg} C$ if and only if one of the following three situations obtains:

(1) "Shared core across $\vdash$, equivalence case" – C is in the same equivalence class as some formula in Θ, where formulas are equivalent if they are identical or if they have the form $\neg^m p_i$ and $\neg^n p_i$ where m, n differ by an even number and neither m nor n is 0;

(2) "Shared core across $\vdash$, one-way consequence case" – some formula in Θ properly implies C, which happens when the formula in Θ is a sentence letter and C prefixes an even number of negations to that sentence letter;

(3) "Shared core on the left case" – there are formulas $A, B \in \Theta$ with $A = \neg^m p_i$, $B = \neg^n p_i$, m, n differ by an odd number, and C is of the form $\neg D$.

Thus when $\Theta \vdash_{\mathsf{ML}\neg} C$ if we are in cases (1) or (2), there is a one-element subset Θ_0 of Θ for which $\Theta_0 \vdash_{\mathsf{ML}\neg} C$, while in case (3) there is a two-element subset Θ_0 for which this holds. (The situation with intuitionistic logic here would be identical except that in this last case there is no condition on the form of C.) Further in the sequent $\Theta_0 \succ C$ chosen on the basis of the formulas mentioned under (1)–(3), i.e., with no additional weakening in cases (1)–(2), we have only one sentence letter occurring (as the 'shared core') or else only two (one as the common core of the two formulas on the left, and one as the core of the formula on the right).

Turning to the case of $\neg'$, for which we recall that $\neg(\cdot)$ is interpreted as $((\bot \to \cdot) \to \cdot) \to \bot$, we find that exactly the same reasoning applies: we have the law of Triple Negation (for which it may help to note that, using both negations at once and taking $\vdash$ as $\vdash_{\mathsf{ML}\neg,\neg'}$, we have

$$\neg'\neg'\ A \dashv\vdash \neg\neg' A \dashv\vdash \neg'\neg A$$

for all A) and also precisely the same cases (1)–(3), with $\neg$ replaced by $\neg'$ as the alternative possibilities when $\Theta \vdash_{\mathsf{ML}\neg'} C$. □

Next we adopt the perspective taken in Example 3.7(i), where we recall that $\bot$ in ML behaves as though it were nothing but a further sentence letter. This simplifies the calculation of all consequence relationships among the formulas arising from Lemma 5.1. First, since whenever $\Theta \vdash C$, where $\vdash$ is either $\vdash_{\mathsf{ML}\neg}$ or $\vdash_{\mathsf{ML}\neg'}$, $A, B \in \Theta$

with $A, B \vdash C$ with $A = B$ (the 1-element Θ_0 case), in terms of the proof of Lemma 5.1 or else $A \neq B$ and neither $A \vdash C$ or $B \vdash C$ (the irredundant 2-element Θ_0 case). Since only two sentence letters are involved, we can assume without loss of generality that they are p, q, so that the formulas A, B, C are all the result of negating these sentence letters zero or more times. Reducing all triple to single negations – and here we write $\neg$, though the same applies if we are negating with $\neg'$ instead – A, B, C can be taken to be drawn from the following list: $p, \neg p, \neg\neg p, q, \neg q, \neg\neg q$. Now we write out the $\neg$ in terms of $\to$ and the first unused sentence letter (for $\bot$), which is r, since the ML logical relations among our six formulas are given by the IL logical relations among these translations. (We address the different translations we get for $\neg'$ in place of $\neg$ below.) This gives us the following six formulas, to contend with:

$$p, p \to r, (p \to r) \to r, q, q \to r, (q \to r) \to r. \tag{†}$$

Let us now recall C. A. Meredith's faithful embedding of the implicational fragment of classical logic – let us call it $\vdash_{\mathsf{CL}\to}$ – into the corresponding fragment of intuitionistic logic $\vdash_{\mathsf{IL}\to}$, with "$\to$" serving as the material implication connective, by contrast with the notational choices made for Proposition 1.1 and the discussion leading up to it, where "$\supset$" served in that capacity, the latter being used in this Postscript as a derived connective of the language of $\vdash_{\mathsf{IL}\to}$:

$$A \supset B \;=\; ((B \to A) \to A) \to B.$$

Note that this permits us to abbreviate the $\bot, \to$ definition of $\neg'$ to

$$\neg' A \;=\; A \supset \bot.$$

With the aid of $\supset$, so defined, Meredith's translation for taking us from classical to intuitionistic implication, call it τ_{Mer}, is defined inductively as follows:

- $\tau_{\mathsf{Mer}}(p_i) \;=\; p_i \;(i = 1, 2 \ldots)$;
- $\tau_{\mathsf{Mer}}(A \to B) \;=\; \tau_{\mathsf{Mer}}(A) \supset \tau_{\mathsf{Mer}}(B)$.

Lifting τ_{Mer} from formulas to sequents in the obvious way, then, one has for all (pure implicational) sequents σ:

$$\vdash_{\mathsf{CL}_\to} \sigma \text{ if and only if } \vdash_{\mathsf{IL}_\to} \tau_{\mathsf{Mer}}(\sigma). \tag{††}$$

(See [30], pp. 1081–1088 for further information and references.) We are now in a position to wrap things up; although our two negations are not identical twins according to the present consequence relation (see Example 3.7(i)), we do have:

Proposition 5.2. $\neg$ *and* $\neg'$ *are twins according to* $\vdash_{\mathsf{ML}\neg,\neg'}$.

Proof. It suffices to show that for any sequents σ, σ' constructed using only $\neg, \neg'$ respectively, $\vdash_{\mathsf{ML}\neg} \sigma[\neg]$ if and only if $\vdash_{\mathsf{ML}\neg'} \sigma[\neg']$. By Lemma 5.2 and the subsequent discussion it suffices to show that we have this equivalence all for $\sigma[\neg]$ constructed from the formulas $p, \neg p, \neg\neg p, q, \neg q, \neg\neg q$ and $\sigma[\neg']$ constructed correspondingly from $p, \neg' p, \neg'\neg' p, q, \neg' q, \neg'\neg' q$, for which it in turn suffices to show that for all sequents $\sigma[\to]$, constructed from the six formulas listed as (†), and the sequents $\sigma[\supset]$ constructed corresponding from the formulas replacing $\to$ by $\supset$ in those six formulas, we have

$$\vdash_{\mathsf{IL}} \sigma[\to] \text{ if and only if } \vdash_{\mathsf{IL}} \sigma[\supset].$$

(We do not need to add $\to$ to the "IL" subscript here because the sequents in play are all pure implicational sequents.) Since $\sigma[\supset]$ is $\tau_{\mathsf{Mer}}(\sigma[\to])$, this is equivalent by (††) to the claim that for the sequents in question

$$\vdash_{\mathsf{IL}} \sigma[\to] \text{ if and only if } \vdash_{\mathsf{CL}} \sigma[\to],$$

where we could have equally well written "$\vdash_{\mathsf{CL}} \sigma[\supset]$" on the right, since $\to$ and $\supset$ are classically equivalent, and indeed we might just as well omit the "$[\to]$ given that the formulation given uses the same connective on both sides. Since $\vdash_{\mathsf{IL}} \subseteq \vdash_{\mathsf{CL}}$ we have he "only if" direction automatically, so for the "if" direction we need only check that the additional strength of $\vdash_{\mathsf{CL}}$, or more particularly $\vdash_{\mathsf{CL}\to}$, does not show up when attention is restricted to the consequence relationships among the formulas (†). But this is easily done. Setting aside the $\vdash$-statements among these six formulas whose correctness is ensured simply because of the definition of a consequence relation or because of weakening an earlier such statement (adding more formulas to the left, that is), we have for $\vdash = \vdash_{\mathsf{CL}}$ just the following cases:

$p \vdash (p \to r) \to r$	$q \vdash (q \to r) \to r$
$p, p \to r \vdash q \to r$	$q, q \to r \vdash p \to r$
$p, p \to r \vdash (q \to r) \to r$	$q, q \to r \vdash (p \to r) \to r$
$(p \to r) \to r, p \to r \vdash q \to r$	$(q \to r) \to r, q \to r \vdash p \to r$
$(p \to r) \to r, p \to r \vdash (q \to r) \to r$	$(q \to r) \to r, q \to r \vdash (p \to r) \to r$

(In fact there is still some redundancy here, in that the fourth entry in the left (resp. right) column follows for arbitrary $\vdash$ the first and second entries in the left (resp. right) column.) And all of these relationships hold for $\vdash = \vdash_{\mathsf{IL}}$ no less than for $\vdash = \vdash_{\mathsf{CL}}$. $\square$

The above proof may seem unnecessarily indirect in getting rid of negation (whether $\neg$ or $\neg'$) and then of $\bot$ so as to end up in the pure implicational fragment of IL: why not work in the $\{\to, \bot\}$ fragment of IL, for instance? The answer is that Meredith's embedding does not work with $\bot$ (or $\neg$) present: the subscripts on the turnstiles in (††) if thus enlarged turn something true into something false. Tokarz and Wójcicki [68] show that there is no definitional translation at all that embeds $\vdash_{\mathsf{CL}\to,\bot}$ faithfully in $\vdash_{\mathsf{IL}\to,\bot}$: it is not just that for the particular case of τ_{Mer} that we have a failure of the envisaged variation on (††). And indeed this remains the case even if we drop the word "faithfully" – which amounts to dropping the "if" direction of (††) – as is shown in pp. 467–469 of Humberstone [26], where the discussion is couched in terms of $\to$ and $\neg$ rather than $\to$ and $\bot$ (an inconsequential difference for present purposes). The "definitional", however, cannot be dropped: the various 'negative translations' Troelstra and van Dalen ([69], §2.3) all fail to translate the (non-logical) atomic formulas by themselves or fail to be compositional, or both.

References

[1] Stal Aanderaa *Relation between different systems of modal logic*, Notices of the American Math. Soc. 13 (3) 1966, p. 391.

[2] Alan Ross Anderson *On the interpretation of a modal system of Łukasiewicz*, Journal of Computing Systems 1 (4) 1954, pp. 209–210.

[3] — *The formal analysis of normative systems*, N. Rescher (ed.), The Logic of Decision and Action, University of Pittsburgh Press, Pittsburgh PA, 1967, pp. 147–213.

[4] Leonard Bolc and Piotr Borowik Many-Valued Logics 1: Theoretical Foundations, Springer-Verlag, Berlin 1992

[5] Michael Byrd *Knowledge and true belief in Hintikka's epistemic logic*, Journal of Philosophical Logic 2 (2) 1973, pp. 181–192.

[6] John Thomas Canty and Thomas Scharle *Note on the singularies of S5*, Notre Dame Journal of Formal Logic 7 (1) 1966, p. 108.

[7] Rudolf Carnap *Modalities and quantification*, Journal of Symbolic Logic 11 (2) 1946, 33–66.

[8] Adam Caulton and Jeremy Butterfield *On kinds of indiscernibility in logic and metaphysics*, British Journal for the Philosophy of Science 63 (1) 2012, pp. 27–84.

[9] Brian Chellas Modal Logic, *An Introduction*, Cambridge University Press, Cambridge 1980.

[10] Roderick Chisholm *Human freedom and the self*, The Lindley Lecture, University of Kansas, April 23, 1964; Department of Philosophy, University of Kansas, 1964.

[11] Maxwell John Cresswell *The interpretation of some Lewis systems of modal logic*, Australasian Journal of Philosophy 45 (2) 1967, pp. 198–206.

[12] — *Necessity and contingency*, Studia Logica 47 (2) 1988, pp. 145–149.

[13] Josep Maria Font and Petr Hájek *On Łukasiewicz's four-valued modal logic*, Studia Logica 70 (2) 2002, pp. 157–182.

[14] Rohan French *A simplified embedding of E into monomodal K*, Logic Journal of the IGPL 17 (4) 2009, 421–28.

[15] — and Lloyd Humberstone *Partial confirmation of a conjecture on the boxdot translation in modal logic*', Australasian Journal of Logic 7 2009, pp. 56–61.

[16] — Translational Embeddings in Modal Logic, PhD Thesis, Monash University 2010.

[17] Peter Thomas Geach, review of Hintikka [19], Philosophical Books 4 (2) 1963, pp. 7–8.

[18] Siegfried Gottwald *Many-Valued Logics*, D. Jacquette (ed.), Handbook of the Philosophy of Science: Philosophy of Logic (series editors: D. M. Gabbay, P. Thagard and J. Woods), Elsevier, Amsterdam 2007, pp. 675–722.

[19] Jaakko Hintikka, Knowledge and Belief, *An Introduction to the Logic of the Two Notions*, Cornell University Press, Ithaca, NY 1962.

[20] Lloyd Humberstone *Logic for saints and heroes*, Ratio 16 (1) 1974, pp. 103–114.

[21] — *Inaccessible worlds*, Notre Dame Journal of Formal Logic 24 (3) 1983, pp. 346–352.

[22] — *Extensionality in sentence position*, Journal of Philosophical Logic 15 (1) 1986, pp. 27–54; also *ibid.* 17 (3) 1988, pp. 221–223, *The lattice of extensional connectives: a correction.*

[23] — *Singulary extensional connectives: a closer look*, Journal of Philosophical Logic 26 (3) 1997, pp. 341–356.

[24] — *The revival of rejective negation*, Journal of Philosophical Logic 29 (4) 2000, pp. 331–381.

[25] — *An intriguing logic with two implicational connectives*, Notre Dame Journal of Formal Logic 41 (1) 2000, pp. 1–40.

[26] — *Contra-classical logics*, Australasian Journal of Philosophy 78 (4) 2000, 437–474.

[27] — *Modality*, F. C. Jackson and M. Smith (eds.), The Oxford Handbook of Contemporary Philosophy, Oxford University Press, Oxford and New York, 2005, pp. 534–614 (= Chapter 20).

[28] — *Identical twins, deduction theorems, and pattern functions: exploring the implicative BCSK fragment of S5*, Journal of Philosophical Logic 35 (5) 2006, pp. 435–487.

[29] — *Weaker-to-stronger translational embeddings in modal logic*, G. Governatori, I. Hodkinson and Y. Venema (eds.), Advances in Modal Logic, Vol. 6, College Publications, London, 2006, pp. 279–297.

[30] — The Connectives, MIT Press, Cambridge MA, 2011.

[31] — *Replacement in logic*, Journal of Philosophical Logic 42 (1) 2013, pp. 49–89.

[32] — *Aggregation and idempotence*, Review of Symbolic Logic 6 (4) 2013, pp. 680–708.

[33] — Philosophical Applications of Modal Logic, College Publications, London 2016.

[34] — *Semantics without toil? Brady and Rush meet Halldén*, Organon F 26 (3) 2019, pp. 340–404.

[35] — *Explicating logical independence*, Journal of Philosophical Logic 49 (1) 2020, pp. 135–

218.

[36] Emil Jeřábek *Cluster expansion and the boxdot conjecture*, Mathematical Logic Quarterly 62 (6) 2016, pp. 608–614.

[37] Kathleen Johnson Wu *Hintikka and defensibility*, Ajatus 32 1970, pp. 25–31.

[38] Saul Kripke *Semantical Analysis of Modal Logic II: Non-Normal Modal Propositional Calculi*, J. W. Addison, L. Henkin, and A. Tarski (eds.), The Theory of Models, North-Holland, Amsterdam 1965, pp. 206–220.

[39] Edward John Lemmon *Review of Hintikka* [19], Philosophical Review 74 (3) 1965, pp. 381–384.

[40] — *Algebraic semantics for modal logics: II*, Journal of Symbolic Logic 31 (2) 1966, 191–218.

[41] Clarence Irving Lewis and Cooper Harold Langford Symbolic Logic, Dover Publications, New York NY, 1959. (Originally publ. 1932, without Appendix III.)

[42] Jan Łukasiewicz *A system of modal logic*, in L. Borkowski (ed.), Jan Łukasiewicz: Selected Works, North-Holland 1970, pp. 352–390; first appeared in Journal of Computing Systems 1 (3) 1953, 111–149.

[43] — *A system of modal logic*, Procs. Eleventh International Congress of Philosophy, Brussels 1953, Vol. XIV ('Additional Volume and Contributions to the Symposium on Logic'), North-Holland, Amsterdam 1953, pp. 82–87.

[44] — Aristotle's Syllogistic, Second (enlarged) edition, Oxford University Press, Oxford 1957.

[45] David Makinson *Intelim rules for classical connectives*, S. O. Hansson (ed.), David Makinson on Classical Methods for Non-Classical Problems, Springer, Dordrecht 2014, pp. 359–382.

[46] Gerald Massey *Normal form generation of S5 functions via truth functions*, Notre Dame Journal of Formal Logic 9 (1) 1968, pp. 81–85.

[47] John C. C. McKinsey and Alfred Tarski *On closed elements in closure algebras*, Annals of Mathematics 47 (1) 1946, pp. 122–162.

[48] George Metcalfe, Nicola Olivetti, and Dov Gabbay Proof Theory for Fuzzy Logics, Springer International, 2009.

[49] Jean Porte *The Ω-system and the Ł-system of modal logic*, Notre Dame Journal of Formal Logic 20 (4) 1979, pp. 915–920.

[50] — *Boll–Reinhart Modal Logic*, Logique et Analyse 25 (98) 1982, pp. 181–190.

[51] — *Łukasiewicz's Ł-modal system and classical refutability*, Logique et Analyse 27 (105) 1984, pp. 87–92.

[52] Arthur Norman Prior *The interpretation of two systems of modal logic*, Journal of Computing Systems 1 (4) 1954, pp. 201–208.

[53] — *Logicians at play; or Syll, Simp and Hilbert*, Australasian Journal of Philosophy 34 (3) 1956, pp. 182–192.

[54] — Time and Modality, Oxford University Press, Oxford 1957.

[55] Nicholas Rescher Many-Valued Logic, McGraw-Hill, New York NY, 1969.

[56] Rolf Schock Quasi-connectives Definable in Concept Theory, C. W. K. Gleerup, Lund 1971.

[57] Kurt Schütte Proof Theory, Springer-Verlag, Berlin 1977. (Translated by J. N. Crossley; German original: 1960.)

[58] Krister Segerberg An Essay in Classical Modal Logic, Filosofiska Studier, Uppsala 1971.

[59] Peter Simons *Łukasiewicz, Meinong and many-valued logic*, K. Szaniawski (ed.), The Vienna Circle and the Lvov-Warsaw School, Kluwer, Dordrecht 1989, pp. 249–291. Reprinted (with a re-punctuated title) as Chapter 8 of P. Simons, Philosophy and Logic in Central Europe from Bolzano to Tarski, Springer, Dordrecht 1992.

[60] — *Łukasiewicz and the several senses of possibility*, European Review 23 (1) 2015, pp. 114–124.

[61] — *Jan Łukasiewicz*, Stanford Encyclopedia of Philosophy (Spring 2017 Edition), Edward N. Zalta (ed.), URL = `<https://plato.stanford.edu/archives/spr2017/entries/lukasiewicz/>`

[62] Timothy John Smiley *On Łukasiewicz's Ł-modal system*, Notre Dame Journal of Formal Logic 2 (3) 1961, pp. 149–153.

[63] — *Relative necessity*, Journal of Symbolic Logic 28 (2) 1963, pp. 113–134.

[64] Vladimir Sotirov *Non-classical operations hidden in classical logic*, Journal of Applied Non-Classical Logics 18 (2–3) 2008, pp. 309–324.

[65] Phillip Staines *Some formal aspects of the argument-symbolisation relation*, Australian Logic Teachers' Journal 5 1981, pp. 1–15.

[66] Steven Karl Thomason *Independent propositional modal logics*, Studia Logica 39 (2–3) 1980, pp. 143–144.

[67] Marcin Tkaczyk *On axiomatization of Łukasiewicz's four-valued modal logic*, Logic and Logical Philosophy 20 (3) 2011, pp. 215–232.

[68] Marek Tokarz and Ryszard Wójcicki *The problem of reconstructability of propositional calculi*, Studia Logica 28 (1) 1971, pp. 119–127.

[69] Anne S. Troelstra and Dirk van Dalen Constructivism in Mathematics: an Introduction, Vol. I, North-Holland, Amsterdam 1988.

[70] Robert Veroff and Matthew Spinks *Axiomatizing the skew Boolean propositional calculus*, Journal of Automated Reasoning 37 (1–2) 2006, pp. 3–20.

[71] Timothy Williamson *Iterated attitudes*, T. J. Smiley (ed.), Philosophical Logic (Procs. British Academy 95), Oxford University Press, Oxford 1998, pp. 85–133.

[72] Evgeni E. Zolin *Embeddings of propositional monomodal logics*, Logic Journal of the IGPL 8 (6) 2000, pp. 861–882.

[73] — *Infinitary expressibility of necessity in terms of contingency*, K. Striegnitz (ed.), Proceedings of the Sixth ESSLLI Student Session, 2001, pp. 325–334.

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Fundamentals of Computability Logic 2020

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1 Introduction

Not to be confused with the generic term "computational logic", computability logic (CoL) is the proper name of a philosophical platform and mathematical framework for developing ever more expressive computationally meaningful extensions of traditional logic. The main pursuit of this ongoing long-term project is to offer a convenient language for specifying computational tasks and relations between them in a systematic way, and to provide a deductive apparatus for systematically telling what can be computed and how. This line of research was officially introduced in [14] and developed in a series [3]-[4],[14]-[43],[45],[47]-[48],[51]-[55] of subsequent papers.

Under the approach of CoL, formulas represent computational problems, logical operators stand for operations on such entities, and "truth" is seen as computability. Computational problems, in turn, are understood in their most general, interactive sense, and are mathematically construed as games played by a machine against its environment, with computability meaning existence of a machine (algorithmic strategy) that always wins.

CoL understands propositions or predicates of traditional logic as games with no moves, automatically won by the machine when true and lost when false. This naturally makes the classical concept of truth a special case of computability — computability by doing nothing. Further, all operators of classical logic are conservatively generalized from moveless games to all games, which eventually makes classical logic a conservative fragment of the otherwise much more expressive CoL. Based on the overall philosophy and intuitions associated with intuitionistic and linear logics, the latter can also be seen as special fragments of CoL, even though, unlike classical logic, "not quite" conservative ones.

A long list of related or unrelated game semantics can be found in the literature proposed by various authors. Out of those, Blass's [6] game semantics, which in turn is a refinement of Lorenzen's [46] dialogue semantics, is the closest precursor of the semantics of CoL, alongside with Hintikka's [13] game-theoretic semantics. More often than not, the motivation for studying games in logic has been to achieve a better understanding of some already existing systems, such as intuitionistic ([5, 9, 46]), classical ([13]) or linear ([1, 6]) logics. In contrast, CoL's motto is that logic should serve games rather than the other way around. For logic is meant to be the most general and universal intellectual tool for navigating real life; and it is games that offer the most adequate mathematical models for the very essence of all "navigational" activities of agents: their interactions with the surrounding world. An agent and its environment translate into game-theoretic terms as two players; their actions as moves; situations arising in the course of interaction as positions; and successes or failures as wins or losses.

This chapter is a semitutorial-style introduction to the basics of CoL, containing many definitions, illustrations, claims and even exercises but no technical proofs whatsoever. It is primarily focused on the language of CoL and its semantics, paying considerably less attention to the associated proof theory or applications. A more detailed and continuously updated survey of the subject is maintained online at [44].

2 Games

Computability is a property of *computational problems* and, before attempting to talk about the former, we need to agree on the precise meaning of the latter. According to the mainstream understanding going back to Church [8] and Turing [50], a computational problem is a *function*—more precisely, the task of systematically generating the values of that function at different arguments. Such a view, however, as more and more researchers have been acknowledging [11], is too narrow. Most tasks performed by computers are *interactive*, far from being as simple as functional transformations from inputs to outputs. Think of the work of a network server for instance, where the task is to maintain a certain infinite process, with incoming ("input") and outgoing ("output") signals interleaved in some complex and probably unregulated fashion, depending on not only immediately preceding signals but also various events taken place in the past. In an attempt to advocate for the conventional view of computational problems, one might suggest to understand an interactive computational task as the task of repeatedly computing the value of a function whose argument is not just the latest input but the whole preceding interaction. This is hardly a good solution though, which becomes especially evident with computational complexity considerations in mind. If the task performed by your personal computer was like that, then you would have noticed its performance worsening after every use due to the need to read the ever longer history of interaction with you.

Instead, CoL postulates that a computational problem is a *game* between two agents: a machine and its environment, symbolically named $\top$ and $\bot$, respectively. $\top$ is a mechanical device only capable of following algorithmic strategies, while there are no similar assumptions about $\bot$ whose behavior can be arbitrary. Computational tasks in the traditional sense now become special cases of games with only two moves, where the first move ("input") is by $\bot$ and the second move ("output") by $\top$.

The following notational and terminological conventions are adopted. A **move** is any finite string over the standard keyboard alphabet. A **labeled move** is a move prefixed with $\top$ or $\bot$, with such a prefix (**label**) indicating which player is the author of the move. We will not always be very strict about differentiating between moves and labeled moves, sometimes saying "move" where, strictly speaking, "labeled move" is meant. A **run** is a (finite or infinite) sequence of labeled moves, and a **position** is a finite run. We usually use lowercase Greek letters as metavariables for moves, and uppercase Greek letters for runs. We will be writing runs and positions as $\langle\alpha,\beta,\gamma\rangle$, $\langle\Theta,\Gamma\rangle$, $\langle\Theta,\alpha,\Gamma\rangle$, etc. The meanings of such expressions should be clear. For instance, $\langle\Theta,\alpha,\Gamma\rangle$ is the run consisting of the (labeled) moves of the position Θ, followed by the move α, and then by the moves of the run Γ.

A set S of runs is said to be **prefix-closed** iff, whenever a run is in S, so are all of its initial segments. The **limit-closure** of a set S of runs is the result of adding to S every infinite run Γ such that all finite initial segments of Γ are in S.

Definition 2.1. A **game** is a pair $G = (\mathbf{Lp}^G, \mathbf{Wn}^G)$, where:

1. $\mathbf{Lp}^G$ is a nonempty, prefix-closed set of positions. We write $\mathbf{Lr}^G$ for the limit-closure of $\mathbf{Lp}^G$.
2. $\mathbf{Wn}^G$ is a mapping from $\mathbf{Lr}^G$ to $\{\top,\bot\}$.

Intuitively, in the context of a given game G, $\mathbf{Lp}^G$ is the set of **legal positions** and $\mathbf{Lr}^G$ is the set of **legal runs**. Note that, since $\mathbf{Lp}^G$ is required to be nonempty and prefix-closed, the **empty position** $\langle\rangle$, being an initial segment of all runs, is always legal. With $\wp$ here and elsewhere standing for either player and $\overline{\wp}$ for its adversary $\wp \neq \overline{\wp} \in \{\top,\bot\}$, a **legal move** by $\wp$ in a position Θ is a move α such that $\langle\Theta,\wp\alpha\rangle \in \mathbf{Lp}^G$. We say that a run Γ is $\wp$**-legal** iff either Γ is legal, or else, where Θ is the shortest illegal initial segment of Γ, the last move of Θ is $\overline{\wp}$-labeled. Intuitively, such a Γ is a run where $\wp$ has not made any illegal moves unless its adversary $\overline{\wp}$ has done so first. In all cases, we shall say "**illegal**" for "not legal" and "lost" for "not won". For each legal run, $\mathbf{Wn}^G$ tells us which of the two players $\wp \in \{\top,\bot\}$ has won the run. The following definition extends this meaning of the word "won" from legal runs to all runs by stipulating that an illegal run is always lost by the player that has made the first illegal move:

Definition 2.2. For a game G, run Γ and player $\wp$, we say that Γ is a **$\wp$-won** (or won by $\wp$) run of G iff Γ is

1. either a legal run of G with $\mathbf{Wn}^G\langle\Gamma\rangle = \wp$, or
2. a $\overline{\wp}$-illegal run of G.

Games—at least when they are finite—can be visualized as trees in the style of Figure 1. Each complete or incomplete branch of such a tree represents a legal run, namely, the sequence of the labels of the edges of the branch. The nodes represent positions, where the label $\top$ or $\bot$ of a node indicates which player is the winner if the play ends in the corresponding position.

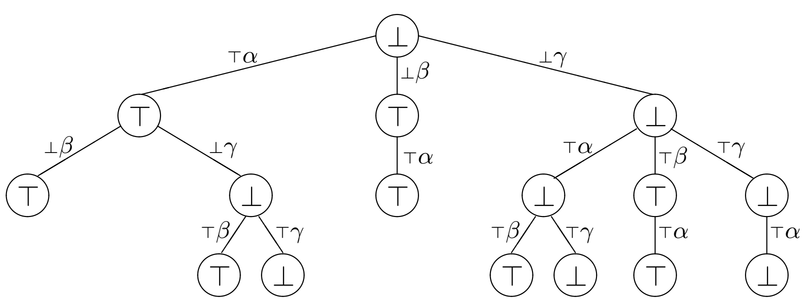


Figure 1: A game

A distinguishing feature of CoL games is the absence of rules governing the order in which (legal) moves can or should be made. In some situations, such as in the root position of the game of Figure 1, both players may have legal moves, and which if any player moves first depends on which one wants or can act faster. Imagine a simultaneous play of chess on two boards, where you play white on both boards. At the beginning, only you have legal moves. But once you make an opening move—say, on board #1—the situation changes. Now both you and your environment have legal moves: the environment may respond on board #1, while you can make another opening move on board #2. It would be unnatural to impose rules determining the next player to move in this case, especially if your environment consists of two independent and non-communicating adversaries. The relaxed nature of our games makes them more direct and adequate tools for modeling real-life interactions like this and beyond than stricter games would be.

But how are such loose games played and, most importantly, what does an algorithmic winning strategy mean? Below is an example of such a strategy. It is left to the reader to convince himself or herself that following it guarantees $\top$ a win in the game of Figure 1:

> *Regardless of what the adversary is doing or has done, go ahead and make move α; make β as your second move if and when you see that the adversary has made move γ, no matter whether this happened before or after your first move.*

Formally, $\top$'s algorithmic strategies can be understood as what CoL, for historical reasons, calls **HPM**s ("hard-play machines"). An HPM is a Turing machine with the capability of making moves. This is just like the capability of generating an output, with the only difference that, while an ordinary Turing machine halts after generating an output, an HPM generally does not halt after making a move, so it can continue its work and make more moves later. Also, an HPM is equipped with an additional, read-only tape called the *run tape*, initially empty. Every time the HPM makes a move α, the string $\top\alpha$ is automatically appended to the content of this tape. At any time, any $\bot$-labeled move $\bot\beta$ may also be nondeterministically appended to the content of the run tape. This event is interpreted as that the environment has just made move β. This way, at any step of the process, the run tape spells the current position of the play. It is hardly necessary to define HPMs in full detail here, for the Church-Turing thesis extends from ordinary Turing machines to HPMs, according to which HPMs adequately correspond to what we intuitively perceive as algorithmic strategies. So, rather than attempting to formally describe an HPM playing a given game, we can simply describe its work in relaxed, informal terms in the style of the earlier-displayed strategy for the game of Figure 1. There is also no need to anyhow define $\bot$'s strategies: all possible behaviors by $\bot$ are accounted for by the above-mentioned different nondeterministic updates of the run tape, including $\bot$'s relative speed because there are no restrictions on when or with what frequency the updates can take place.

Depending on what nondeterministic events ($\bot$'s moves) occur in the course of the work of an HPM $\mathcal{M}$ and when, different runs will be eventually (in the limit) spelled on $\mathcal{M}$'s run tape. We call any such run a **run generated by $\mathcal{M}$**.

Definition 2.3. We say that an HPM $\mathcal{M}$ **computes** a game G iff every run generated by $\mathcal{M}$ is $\top$-won run of G. Such an $\mathcal{M}$ is said to be a **solution** of (or an **algorithmic winning strategy** for) G.

By the **depth** of a game we mean the (possibly infinite) length of its longest legal run. Computational problems in the traditional sense, i.e. functions, are games of depth 2 of the kind seen in Figure 2. In such a game, the upper level edges represent possible inputs provided by the environment. This explains why their labels are $\perp$-prefixed. The lower level edges represent possible outputs generated by the machine, so their labels are $\top$-prefixed. The root is $\top$-labeled because it corresponds to the situation where nothing happened, namely, no input was provided by the environment. The machine has nothing to answer for in this case, so it wins. The middle level nodes are $\perp$-labeled because they correspond to situations where there was an input but the machine failed to generate an output, so the machine loses. Each group of the bottom level nodes has exactly one $\top$-labeled node, because a function has exactly one (correct) value at each argument.

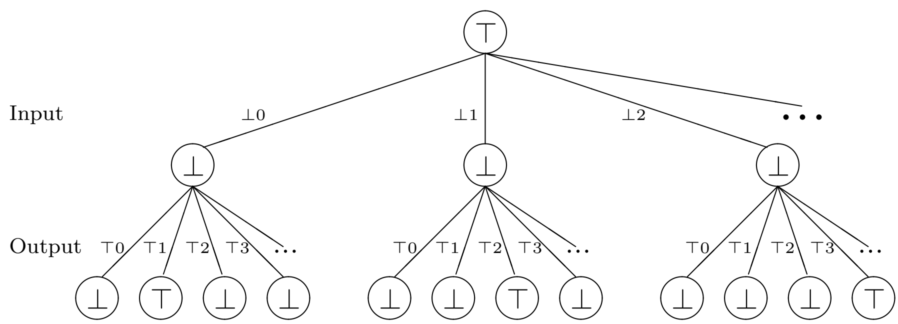


Figure 2: The successor function as a game

But why limit ourselves only to trees of the above sort? First of all, we may want to allow branches to be longer than 2, or even infinite to be able to model long or infinite tasks performed by computers. And why not allow any other sorts of arrangements of $\top$ and $\perp$ in nodes or on edges? For instance, consider the task of computing the function $5/x$. It would be natural to make the node to which the input 0 takes us not $\perp$-labeled, but $\top$-labeled. For the function is not defined at 0, and the machine cannot be held responsible for failing to generate an output on such an input.

It makes sense to generalize computational problems not only in the direction of increasing their depths, but also decreasing. Games of depth 0, i.e., games that have no nonempty legal runs, are said to be **elementary**. There are exactly two elementary games, for which we use the same symbols $\top, \perp$ as for the two players. Namely, $\top$ is the elementary game G with $\mathbf{Wn}^G\langle\rangle = \top$, and $\perp$ is the elementary game G with $\mathbf{Wn}^G\langle\rangle = \perp$. Intuitively, $\top$ and $\perp$ are moveless games, with (the only legal run $\langle\rangle$ of) $\top$ automatically won by the machine and $\perp$ won by the environment. While the game $\perp$ has no solution, the "do nothing" strategy is a solution of $\top$.

Extensionally, true propositions of classical logic such as "snow is white" are understood in CoL as the game $\top$, and false propositions such as "$2 + 2 = 5$" as the game $\bot$. Propositions are thus special—elementary—cases of our games. This allows us to say that games are generalized propositions.

3 Gameframes

This section is devoted to the basic concepts necessary for lifting CoL from the propositional level to the first-order level. What we call *gameframes* are generalized predicates in the same sense as games are generalized propositions.

We fix an infinite set $\mathit{Variables} = \{var_1, var_2, var_3, \cdots\}$ of **variables**. As usual, lowercase letters near the end of the Latin alphabet will be used as metavariables for variables. We further fix the set $\mathit{Constants} = \{0, 1, 2, 3, \cdots\}$ of decimal numerals, and call its elements **constants**.

A **universe** (of discourse) is a pair $U = (Dm, Dn)$, where Dm, called the **domain** of U, is a nonempty set, and Dn, called the **denotator** of U, is a total function of the type $\mathit{Constants} \rightarrow Dm$. Elements of Dm will be referred to as **individuals**. The intuitive meaning of $d = Dn(c)$ is that the individual d is the **denotat** of the constant c and thus c is a **name** of d.

A nice natural example of a universe is the **arithmetical universe**, whose domain is the set of natural numbers and whose denotator is the bijective function that sends each constant to the number it represents in standard decimal notation. Generally, however, the denotator is required to be neither injective nor surjective, meaning that some individuals may have multiple names, and some no names at all. For instance, in the informal universe of astronomy, most individuals—celestial bodies—have no names while some have several names (Morning Star = Evening Star = Venus). A natural example of a mathematical universe with an intrinsically non-surjective denotator would be one whose domain is the set of real numbers. Even if the set of constants was not fixed, the denotator here could not be surjective for the simple reason that, while there are uncountably many real numbers, there can only be countably many names. This is so because names, by their very nature and purpose, have to be finite objects.

Many properties of common interest, such as computability or decidability, are sensitive with respect to how objects (individuals) are named, as they deal with the names of those objects rather than the objects themselves. For instance, strictly speaking, computing a function $f(x)$ means the ability to tell, after seeing a (the) name of an arbitrary individual a, a (the) name of the individual b with $b = f(a)$. Similarly, an algorithm deciding a predicate $p(x)$ on a set S, strictly speaking, takes as inputs not elements of S—which may be abstract objects such as numbers or graphs—but rather names of those elements, such as decimal codes. It is not hard to come up with a nonstandard naming of the natural numbers via decimal numerals where the predicate "x is even" is undecidable. On the other hand, for any undecidable arithmetical predicate $p(x)$, one can come up with a naming such that $p(x)$ becomes decidable—for instance, one that assigns even-length names to all a satisfying $p(a)$ and assigns odd-length names to all a with $\neg p(a)$. Classical logic exclusively deals with individuals of a universe without a need to also consider names for them, as it is not concerned with decidability or computability. CoL, on the other hand, with its computational semantics, inherently calls for being more careful about differentiating between individuals and their names, and hence for explicitly considering universes in the form (Dm, Dn) rather than just Dm as classical logic does.

For a set Vr of variables and a domain Dm, by a (Vr, Dm)**-valuation** we mean a total function e of the type $Vr \to Dm$. When Vr is finite, such a valuation e can be simply written as an n-tuple $(a_1, \cdots, a_n)$ of individuals, meaning that $e(x_1) = a_1, \cdots, e(x_n) = a_n$, where $x_1, \cdots, x_n$ are the variables of Vr listed lexicographically.

Definition 3.1. Let n be a natural number. An n-ary **gameframe** is a quadruple (Dm, Dn, Vr, G), where (Dm, Dn) is a universe, Vr is a set of n distinct variables, and G is a mapping that sends every (Vr, Dm)-valuation e to a game $G(e)$.

Given a gameframe $\mathcal{G} = (Dm, Dn, Vr, G)$, we refer to Dm as the *domain of* $\mathcal{G}$, to Dn as the *denotator of* $\mathcal{G}$, to (Dm, Dn) as the *universe of* $\mathcal{G}$, to the elements of Vr as the *variables on which* $\mathcal{G}$ *depends* (or simply the *variables of* $\mathcal{G}$), and to G as the **extension of** $\mathcal{G}$. For a gameframe (Dm, Dn, Vr, G) we customarily use the same name G as for its extension. This never causes ambiguity, as it is usually clear from the context whether G refers to the gameframe itself or just its extension. In informal contexts where a universe is either fixed or irrelevant, we think of games as special—nullary—cases of gameframes. Namely, a nullary gameframe $(Dm, Dn, \emptyset, G)$ will be understood as the game $G()$, usually simply written as G.

In classical logic, under an intensional (variable-sensitive) understanding, the definition of the concept of an n-ary predicate would look exactly like our definition of an n-ary gameframe after omitting the redundant denotator component, with the only difference that there the extension function would return propositions rather than games. And, just like propositions are nothing but 0-ary predicates, games are nothing but 0-ary gameframes. Thus, gameframes generalize games in the same way as predicates generalize propositions.

In formal contexts, we choose a similar intensional approach to functions. The definition of a function given below is literally the same as our definition of a gameframe, with the only difference that the extension component now maps valuations to individuals rather than games.

Definition 3.2. Let n be a natural number. An n-ary **function** is a tuple (Dm, Dn, Vr, f), where (Dm, Dn) is a universe, Vr is a set of n distinct variables, and f is a mapping that sends every (Vr, Dm)-valuation to an element $f(e)$ of Dm.

Just as in the case of gameframes, we customarily use the same name f for a function (Dm, Dn, Vr, f) as for its last component. We refer to the elements of Vr as the variables on which the function f depends, refer to Dm as the domain of f, etc.

Given a gameframe (Dm, Dn, Vr, G), a set X of variables with $Vr \subseteq X$ and an (X, Dm)-valuation e, we write $G(e)$ to mean the game $G(e')$, where e' is the restriction of e to Vr (i.e., the (Vr, Dm)-valuation that agrees with e on all variables from Vr). Such a game $G(e)$ is said to be an **instance** of G, and the operation that generates $G(e)$ from G and e is said to be the **instantiation** operation. For readability, we usually write $\mathbf{Lp}^G_e$, $\mathbf{Lr}^G_e$ and $\mathbf{Wn}^G_e$ instead of $\mathbf{Lp}^{G(e)}$, $\mathbf{Lr}^{G(e)}$ and $\mathbf{Wn}^{G(e)}$. Similarly, given a function (Dm, Dn, Vr, f), a set X of variables with $Vr \subseteq X$ and an (X, Dm)-valuation e, we write $f(e)$ to denote the individual $f(e')$ to which f maps e', where e' is the restriction of e to Vr.

We say that a gameframe is **elementary** iff so are all of its instances. Thus, gameframes generalize elementary gameframes in the same sense as games generalize elementary games. In turn, elementary gameframes generalize elementary games in the same sense as predicates generalize propositions in classical logic. So, just as we identify elementary games with propositions, we will identify elementary gameframes with predicates. Specifically, in the context of a given universe (Dm, Dn), we understand a predicate p on Dm as the elementary gameframe (Dm, Dn, Vr, G), where Vr is the set of variables on which p depends, and G is such that, for any (Vr, Dm)-valuation e, $\mathbf{Wn}^G_e\langle\rangle = \top$ iff p is true at e. And vice versa: an elementary gameframe G will be understood as the predicate p that depends on the same variables as G does and is true at a given valuation e iff $\mathbf{Wn}^G_e\langle\rangle = \top$.

Convention 3.3. Assume $U = (Dm, Dn)$ is a universe, $a \in Dm$, $c \in$ *Constants*, and $x \in$ *Variables*. We shall write a^U to mean the nullary (constant) function $(Dm, Dn, \emptyset, f)$ such that $f() = a$. We shall write c^U to mean the nullary function $(Dm, Dn, \emptyset, f)$ such that $f() = Dn(c)$. And we shall write x^U to mean the unary function $(Dm, Dn, \{x\}, f)$ such that, for any $a \in Dm$, $f(a) = a$.

Convention 3.4. Assume $K = (Dm, Dn, Vr, K)$ is a function (resp. gameframe). Following the standard readability-improving practice established in the literature for functions and predicates, we may fix a tuple $(x_1, \cdots, x_n)$ of pairwise distinct variables for K when first mentioning it, and write K as $K(x_1, \cdots, x_n)$. When doing so, we do not necessarily mean that $\{x_1, \cdots, x_n\} = Vr$. Representing K as $K(x_1, \cdots, x_n)$ sets a context in which, for whatever functions $f_1 = (Dm, Dn, Vr_1, f_1), \cdots, f_n = (Dm, Dn, Vr_n, f_n)$, we can write $K(f_1, \cdots, f_n)$ to mean the function (resp. gameframe) (Dm, Dn, Vr', K') such that:

- $Vr' = (Vr - \{x_1, \cdots, x_n\}) \cup Vr_1 \cup \cdots \cup Vr_n$.
- For any (Vr', Dm)-valuation e', $K'(e') = K(e)$, where e is the (Vr, Dm)-valuation such that $e(x_1) = f_1(e'), \cdots, e(x_n) = f_n(e')$ and e agrees with e' on all other variables from Vr.

Further, we allow for any of the above f_i to be (written as) just an individual a, just a constant c or just a variable x. In such cases, f_i should be correspondingly understood as the function a^U, c^U or x^U, where $U = (Dm, Dn)$. So, for instance, $K(0, x)$ is our lazy way to write $K(0^U, x^U)$.

4 The operator zoo of computability logic

Logical operators in CoL stand for operations on gameframes. With games seen as nullary gameframes, such operations are automatically also operations on games. There is an open-ended pool of operations of potential interest, and which of those to study may depend on particular needs and taste. Below is an incomplete list of the operations that have been officially introduced so far.

- Negation: $\neg$.
- Conjunctions: $\wedge$ (parallel); $\sqcap$ (choice); $\triangle$ (sequential); $⩚$ (toggling).
- Disjunctions: $\vee$ (parallel); $\sqcup$ (choice); $\triangledown$ (sequential); $⩛$ (toggling).
- Implications: $\rightarrow$ (parallel); $\sqsupset$ (choice); $\triangleright$ (sequential); $⩾$ (toggling).
- Universal quantifiers: $\forall$ (blind); $\bigwedge$ (parallel); $\bigsqcap$ (choice); $\bigtriangleup$ (sequential); $⩚$ (toggling).
- Existential quantifiers: $\exists$ (blind); $\bigvee$ (parallel); $\bigsqcup$ (choice); $\bigtriangledown$ (sequential); $⩛$ (toggling).
- Recurrences: $⫰$ (branching); $⅄$ (parallel); $\underline{⅄}$ (sequential); $⅄$ (toggling).
- Corecurrences: $⫯$ (branching); Y (parallel); $\overline{Y}$ (sequential); Y (toggling).
- Rimplications: $\circ\!-$ (branching); $\succ\!-$ (parallel); $\triangleright\!-$ (sequential); $⩾\!-$ (toggling).
- Repudiations: $\circ\!\neg$ (branching); $\succ\!\neg$ (parallel); $\triangleright\!\neg$ (sequential); $⩾\!\neg$ (toggling).

Among the symbolic names for the above operations we see all operators of classical logic, and our choice of the classical notation for them is no accident: classical first-order logic is nothing but the result of discarding all other operators in CoL and forbidding all but elementary gameframes. Indeed, after analyzing the relevant definitions, each of the classically-shaped operators, when restricted to elementary gameframes, can be easily seen to be virtually the same as the corresponding operator of classical logic. For instance, if A and B are elementary games or gameframes, then so is $A \wedge B$, and the latter is exactly the classical conjunction of A and B understood as propositions or predicates. In the nonelementary case, however, the logical behavior of $\neg$, $\wedge$, $\vee$, $\rightarrow$ becomes more reminiscent of—yet not the same as—that of the corresponding operators of multiplicative linear logic.

This section contains formal definitions of all of the above-listed operations. We agree that, throughout those definitions, Φ ranges over positions, Γ over runs and e over (Vr, Dm)-valuations, where Dm is the domain of the gameframe G that is being defined and Vr is the set of variables on which that gameframe depends. All such metavariables should be considered universally quantified in the corresponding clause(s) of the definition unless otherwise implied by the context. Each definition has two clauses, one defining $\mathbf{Lp}^G$ and the other $\mathbf{Wn}^G$. The second clause, telling us who wins a given run of $G(e)$, always implicitly assumes that such a run is in $\mathbf{Lr}^G_e$.

This section also contains many examples and informal explanations. For clarity let us agree that in all such cases, unless otherwise implied by the context, we have the arithmetical universe (cf. Section 3) in mind. This is so even if we talk about seemingly non-number individuals such as people, Turing machines, etc. The latter should simply be understood as the natural numbers that encode the corresponding objects in some fixed encoding, and the (non-numeral) names of such objects understood as the corresponding decimal numerals. Fixing the universe allows us to understand games as nullary gameframes as explained in Section 3. The informal discussions found in this section sometimes use the word "valid", which, intuitively, should be understood as "always computable". The precise meaning(s) of this concept will only be defined later in Section 6. When describing machine's winning strategies, we usually assume implicitly that the environment never makes illegal moves, for, if it does, the machine automatically wins regardless of what happens afterwards.

From our formal definitions of propositional (non-quantifier) operations it can be seen immediately that instantiation commutes with all such operations: $(\neg A)(e) = \neg(A(e))$, $(A \wedge B)(e) = A(e) \wedge B(e)$, etc. So, in order to understand the meanings of the propositional operations, it would be sufficient to just understand how they modify nullary gameframes. For this reason, in the corresponding informal explanations we always implicitly assume that the gameframes that we talk about are nullary, and call them simply "games". For similar reasons, when informally explaining the meaning of $\mathbb{Q}xA$ where $\mathbb{Q}$ is one of our quantifiers, it will be implicitly assumed that A is a unary gameframe that only depends on x and (hence) $\mathbb{Q}xA$ is nullary.

When omitting parentheses in compound expressions, we assume that all unary operators (negation, repudiations, recurrences, corecurrences and quantifiers) take precedence over all binary operators (conjunctions, disjunctions, implications, rimplications), among which implications and rimplications have the lowest precedence. So, for instance, $A \to \neg B \vee C$ should be understood as $A \to \big((\neg B) \vee C\big)$.

4.1 Prefixation and negation

Unlike the operations listed in the preceding outline, the operation of prefixation is not meant here to act as a logical operator in the formal language of CoL. Yet, it is very useful in characterizing and analyzing games, and we want to start our tour of the zoo with it.

Definition 4.1. Assume $A = (Dm, Dn, Vr, A)$ is a gameframe and Ψ is a legal position of every instance of A (otherwise the operation is undefined). The **Ψ-prefixation** of A, denoted $\langle\Psi\rangle A$, is defined as the gameframe $G = (Dm, Dn, Vr, G)$ such that:

- $\mathbf{Lp}^G_e = \{\Phi \mid \langle\Psi, \Phi\rangle \in \mathbf{Lp}^A_e\}$;
- $\mathbf{Wn}^G_e\langle\Gamma\rangle = \mathbf{Wn}^A_e\langle\Psi, \Gamma\rangle$.

Intuitively, $\langle\Psi\rangle A$ is a game playing which means playing A starting (continuing) from position Ψ. That is, $\langle\Psi\rangle A$ is the game to which A **evolves** (is **brought down**) after the moves of Ψ have been made. Visualized as a tree, $\langle\Psi\rangle A$ is nothing but the subtree of A rooted at the node corresponding to position Ψ.

To define the **negation** operation $\neg$, read as "*not*", let us agree that, for a run Γ, $\overline{\Gamma}$ means the result of changing the label $\top$ to $\bot$ and vice versa in each move of Γ.

Definition 4.2. Assume $A = (Dm, Dn, Vr, A)$ is a gameframe. $\neg A$ is defined as the gameframe $G = (Dm, Dn, Vr, G)$ such that:

- $\mathbf{Lp}^G_e = \{\overline{\Phi} \mid \Phi \in \mathbf{Lp}^A_e\}$;
- $\mathbf{Wn}^G_e\langle\Gamma\rangle = \top$ iff $\mathbf{Wn}^A_e\langle\overline{\Gamma}\rangle = \bot$.

Intuitively, $\neg A$ is A with the roles of the two players interchanged: $\top$'s (legal) moves and wins become $\bot$'s moves and wins, and vice versa. Let *Chess*, here and later, be the game of chess from the point of view of White, with draws ruled out (say, by declaring them to be wins for Black). Then $\neg$*Chess* is the same game but as seen by Black.

Obviously the double negation principle $\neg\neg A = A$ holds: interchanging the players' roles twice restores the original roles of the players. It is also easy to see that we always have $\neg\langle\Psi\rangle A = \langle\overline{\Psi}\rangle\neg A$. So, for instance, if α is $\top$'s legal move in the empty position of A that brings A down to B, then the same α is $\bot$'s legal move in the empty position of $\neg A$, and it brings $\neg A$ down to $\neg B$.

4.2 Choice operations

This group of operations consists of $\sqcap$ (**choice conjunction**, read as "*chand*"), $\sqcup$ (**choice disjunction**, read as "*chor*"), $\sqsupset$ (**choice implication**, read as "*chimplication*"), $\bigsqcap$ (**choice universal quantifier**, read as "*chall*") and $\bigsqcup$ (**choice existential quantifier**, read as "*chexists*").

$A \sqcap B$ is a game where, in the initial (empty) position, only the environment has legal moves. Such a move should be either "0" or "1". If the environment moves 0, the game continues as A, meaning that $\langle\bot 0\rangle(A \sqcap B) = A$; if it moves 1, then the game continues as B, meaning that$\langle\bot 1\rangle(A \sqcap B) = B$; and if it fails to make either move ("choice"), then it loses. $A \sqcup B$ is similar, with the difference that here it is the machine who has initial moves and who loses if no such move is made. Formally, we have:

Definition 4.3. Assume $A_0 = (Dm, Dn, Vr_0, A_0)$ and $A_1 = (Dm, Dn, Vr_1, A_1)$ are gameframes.

(a) $A_0 \sqcap A_1$ is defined as the gameframe $G = (Dm, Dn, Vr_0 \cup Vr_1, G)$ such that:

- $\mathbf{Lp}_e^G = \{\langle\rangle\} \cup \{\langle\bot i, \Phi\rangle \mid i \in \{0,1\},\ \Phi \in \mathbf{Lp}_e^{A_i}\}$.
- $\mathbf{Wn}_e^G\langle\rangle = \top$; $\mathbf{Wn}_e^G\langle\bot i, \Gamma\rangle = \mathbf{Wn}_e^{A_i}\langle\Gamma\rangle$.

(b) $A_0 \sqcup A_1$ is defined as the gameframe $G = (Dm, Dn, Vr_0 \cup Vr_1, G)$ such that:

- $\mathbf{Lp}_e^G = \{\langle\rangle\} \cup \{\langle\top i, \Phi\rangle \mid i \in \{0,1\},\ \Phi \in \mathbf{Lp}_e^{A_i}\}$.
- $\mathbf{Wn}_e^G\langle\rangle = \bot$; $\mathbf{Wn}_e^G\langle\top i, \Gamma\rangle = \mathbf{Wn}_e^{A_i}\langle\Gamma\rangle$.

(c) $A_0 \sqsupset A_1 \ =_{def}\ \neg A_0 \sqcup A_1$.

The symbol $\sqsupset$ is seldom used in this chapter: instead of $A \sqsupset B$, we often prefer to write the intuitively more transparent $\neg A \sqcup B$.

Note the perfect symmetry between the first two clauses of the above definition: clause (b) is nothing but clause (a) with $\top$ and $\bot$ interchanged everywhere, and vice versa. Such symmetry is called *duality*:

Terminology 4.4. We say that a concept $\mathbb{B}$ is **dual** to a concept $\mathbb{A}$ iff the definition of $\mathbb{B}$ can be obtained from the definition of $\mathbb{A}$ by interchanging $\top$ and $\bot$. For instance, $\sqcup$ (or $A \sqcup B$) is dual to $\sqcap$ (or $A \sqcap B$), and vice versa.

It is not hard to see that, due to duality, the De Morgan laws go through for $\sqcap, \sqcup$: we always have $\neg(A \sqcap B) = \neg A \sqcup \neg B$ and $\neg(A \sqcup B) = \neg A \sqcap \neg B$. Together with the earlier observed double negation principle, this means that $A \sqcup B = \neg(\neg A \sqcap \neg B)$ and $A \sqcap B = \neg(\neg A \sqcup \neg B)$. Similarly for the quantifier counterparts $\bigsqcap$ and $\bigsqcup$ of $\sqcap$ and $\sqcup$. And similarly for all other sorts of conjunctions, disjunctions, recurrences, corecurrences and quantifiers defined in this section.

$\bigsqcap x A(x)$ can be understood as the infinite conjunction $A(0) \sqcap A(1) \sqcap A(2) \sqcap \cdots$, and $\bigsqcup x A(x)$ as the infinite disjunction $A(0) \sqcup A(1) \sqcup A(2) \sqcup \cdots$. Specifically, $\bigsqcap x A(x)$ is a game where, in the initial position, only the environment has legal moves, and such a move should be one of the constants. If the environment moves c, then the game continues as $A(c)$, and if the environment fails to make an initial move/choice, then it loses. $\bigsqcup x A(x)$ is similar, with the difference that here it is the machine who has initial moves and who loses if no such move is made. So, we always have $\langle \bot c \rangle \bigsqcap x A(x) = A(c)$ and $\langle \top c \rangle \bigsqcup x A(x) = A(c)$. Below is a formal definition of the choice quantifiers:

Definition 4.5. Assume $A(x) = (Dm, Dn, Vr, A)$ is a gameframe.

(a) $\bigsqcap x A(x)$ is defined as the gameframe $G = (Dm, Dn, Vr - \{x\}, G)$ such that:

- $\mathbf{Lp}_e^G = \{\langle\rangle\} \cup \{\langle \bot c, \Phi \rangle \mid c \in \mathit{Constants},\ \Phi \in \mathbf{Lp}_e^{A(c)}\}$.
- $\mathbf{Wn}_e^G \langle\rangle = \top$; $\mathbf{Wn}_e^G \langle \bot c, \Gamma \rangle = \mathbf{Wn}_e^{A(c)} \langle \Gamma \rangle$.

(b) $\bigsqcup x A(x)$ is dual to $\bigsqcap x A(x)$.

With choice operators we can easily express the most common sorts of computational problems, such as the problem of computing a function f or the problem of deciding a predicate p. The former can be written as $\bigsqcap x \bigsqcup y (y = f(x))$, and the latter as $\bigsqcap x (p(x) \sqsupset p(x))$. That is, f is computable in the standard sense iff $\bigsqcap x \bigsqcup y (y = f(x))$ is computable in our sense, and p is decidable in the standard sense iff $\bigsqcap x (p(x) \sqsupset p(x))$ is computable in our sense. So, the game of Figure 2 is nothing but $\sqcap x \sqcup y(y = x + 1)$. Every run of this game can be seen as a short dialogue between the machine and its environment. The first move—say, 2—is by $\perp$, and intuitively it amounts to asking "what is the successor of 2?". It brings the game down to $\sqcup y(y = 2 + 1)$. In order win, $\top$ has to make the move 3, amounting to saying that 3 is the successor of 2. Any other move, or no move at all, would be a loss for $\top$.

Classical logic has been repeatedly criticized for its operators not being constructive. Consider, for example, $\forall x \exists y(y = f(x))$. It is always true in the classical sense (as long as f is a total function). Yet its truth has no practical import, for "$\exists y$" merely signifies existence of y, without implying that such a y can actually be found. And, indeed, if f is an incomputable function, there is no method for finding y. On the other hand, the choice operations of CoL are constructive. Computability ("truth") of $\sqcap x \sqcup y(y = f(x))$ means more than just existence of y; it means the possibility to actually find (compute, construct) the corresponding y for every x.

Similarly, let $\mathit{Halts}(x, y)$ be the predicate "Turing machine x halts on input y". Consider the statement $\forall x \forall y(\neg \mathit{Halts}(x, y) \vee \mathit{Halts}(x, y))$. It is true in classical logic, yet not in a constructive sense. Its truth means that, for all x and y, either $\neg \mathit{Halts}(x, y)$ or $\mathit{Halts}(x, y)$ is true, but it does not imply existence of an actual way to tell which of these two is true after all. And such a way does not really exist, as the halting problem is undecidable. This means that $\sqcap x \sqcap y(\neg \mathit{Halts}(x, y) \sqcup \mathit{Halts}(x, y))$ is not computable. Generally, the law of excluded middle $\neg A$ OR A, validated by classical logic and causing the indignation of the constructivistically-minded, is not valid in computability logic with OR understood as choice disjunction. The following is an example of a game of the form $\neg A \sqcup A$ with no algorithmic solution (why, by the way?):

$$\neg \sqcap x \sqcap y(\neg \mathit{Halts}(x, y) \sqcup \mathit{Halts}(x, y)) \sqcup \sqcap x \sqcap y(\neg \mathit{Halts}(x, y) \sqcup \mathit{Halts}(x, y)).$$

4.3 Parallel operations

This group of operations consists of $\wedge$ (**parallel conjunction**, read as "*pand*"), $\vee$ (**parallel disjunction**, read as "*por*"), $\rightarrow$ (**parallel implication**, read as "*pimplication*"), $\bigwedge$ (**parallel universal quantifier**, read as "*pall*"), $\bigvee$ (**parallel existential quantifier**, read as "*pexists*"), $\curlywedge$ (**parallel recurrence**, read as "*precurrence*"), $\curlyvee$ (**parallel corecurrence**, read as "*coprecurrence*"), $\succ\!-$ (**parallel rimplication**, read as "*primplication*") and $\succ\!\neg$ (**parallel repudiation**, read as "*prepudiation*").

$A \wedge B$ and $A \vee B$ are games playing which means playing the two games simultaneously. In order to win in $A \wedge B$ (resp. $A \vee B$), $\top$ needs to win in both (resp. at least one) of the components A, B. For instance, $\neg \mathit{Chess} \vee \mathit{Chess}$ is a two-board game, where $\top$ plays black on the left board and white on the right board, and where it needs to win in at least one of the two parallel sessions of chess. A win can be easily achieved here by just mimicking in *Chess* the moves that the adversary is making in $\neg$*Chess*, and vice versa. This **copycat strategy** guarantees that the positions on the two boards always remain symmetric ("synchronized"), and thus $\top$ eventually loses on one board but wins on the other. This is very different from $\neg Chess \sqcup Chess$. In the latter $\top$ needs to choose between the two components and then win the chosen one-board game, which makes $\neg Chess \sqcup Chess$ essentially as hard to win as either $\neg$*Chess* or *Chess*. A game of the form $A \vee B$ is generally easier (at least, not harder) to win than $A \sqcup B$, the latter is easier to win than $A \sqcap B$, and the latter in turn is easier to win than $A \wedge B$.

Technically, a move α in the left (resp. right) $\wedge$-conjunct or $\vee$-disjunct is made by prefixing α with "0." (resp. "1."). For instance, in the initial position of $(A \sqcup B) \vee (C \sqcap D)$, the move "1.0" is legal for $\perp$, meaning choosing the left $\sqcap$-conjunct in the right $\vee$-disjunct of the game. If such a move is made, the game continues as $(A \sqcup B) \vee C$. The player $\top$, too, has initial legal moves in $(A \sqcup B) \vee (C \sqcap D)$, which are "0.0" and "0.1".

The rest of this chapter will rely on the following important notational convention:

Notation 4.6. For a run Γ and a string α, Γ^{α} means the result of removing from Γ all moves except those of the form $\alpha\beta$, and then deleting the prefix "α" in the remaining moves. For instance, $\langle \top 0.1, \perp 3.1, \perp 0.0\rangle^{0.} = \langle \top 1, \perp 0\rangle$.

Definition 4.7. Assume $A_0 = (Dm, Dn, Vr_0, A_0)$ and $A_1 = (Dm, Dn, Vr_1, A_1)$ are gameframes.

(a) $A_0 \wedge A_1$ is defined as the gameframe $G = (Dm, Dn, Vr_0 \cup Vr_1, G)$ such that:

- $\Phi \in \mathbf{Lp}_e^G$ iff every move of Φ has the prefix "0." or "1." and, for both $i \in \{0, 1\}$, $\Phi^{i.} \in \mathbf{Lp}_e^{A_i}$.
- $\mathbf{Wn}_e^G\langle\Gamma\rangle = \top$ iff, for both $i \in \{0, 1\}$, $\mathbf{Wn}_e^{A_i}\langle\Gamma^{i.}\rangle = \top$.

(b) $A_0 \vee A_1$ is dual to $A_0 \wedge A_1$.

(c) $A_0 \rightarrow A_1 \ =_{def}\ \neg A_0 \vee A_1$.

Example 4.8. $\Gamma = \langle \perp 1.5, \top 0.5, \perp 0.25, \top 1.25\rangle$ is a legal run of the game $A = \sqcup x \sqcap y(y \neq x^2) \vee \sqcap x \sqcup y(y = x^2)$. It induces the following what we call *evolution sequence*, showing how things evolve as Γ runs, i.e., how the moves of Γ affect/modify the game that is being played:

$$
\begin{array}{ll}
\bigsqcup x \bigsqcap y(y \neq x^2) \vee \bigsqcap x \bigsqcup y(y = x^2) & \text{i.e. } A \\
\bigsqcup x \bigsqcap y(y \neq x^2) \vee \bigsqcup y(y = 5^2) & \text{i.e. } \langle \bot 1.5 \rangle A \\
\bigsqcap y(y \neq 5^2) \vee \bigsqcup y(y = 5^2) & \text{i.e. } \langle \bot 1.5, \top 0.5 \rangle A \\
25 \neq 5^2 \vee \bigsqcup y(y = 5^2) & \text{i.e. } \langle \bot 1.5, \top 0.5, \bot 0.25 \rangle A \\
25 \neq 5^2 \vee 25 = 5^2 & \text{i.e. } \langle \bot 1.5, \top 0.5, \bot 0.25, \top 1.25 \rangle A
\end{array}
$$

The run hits the true proposition $25 \neq 5^2 \vee 25 = 5^2$, and hence is won by the machine.

As one may guess, $\bigwedge x A(x)$ is nothing but $A(0) \wedge A(1) \wedge A(2) \wedge \cdots$, and $\bigvee x A(x)$ is nothing but $A(0) \vee A(1) \vee A(2) \vee \cdots$. Formally these two quantifiers are defined as follows:

Definition 4.9. Assume $A(x) = (Dm, Dn, Vr, A)$ is a gameframe.

(a) $\bigwedge x A(x)$ is defined as the gameframe $G = (Dm, Dn, Vr - \{x\}, G)$ such that:

- $\Phi \in \mathbf{Lp}_e^G$ iff every move of Φ has the prefix "$c.$" for some $c \in \mathit{Constants}$ and, for all such c, $\Phi^{c.} \in \mathbf{Lp}_e^{A(c)}$.
- $\mathbf{Wn}_e^G\langle\Gamma\rangle = \top$ iff, for all $c \in \mathit{Constants}$, $\mathbf{Wn}_e^{A(c)}\langle\Gamma^{c.}\rangle = \top$.

(b) $\bigvee x A(x)$ is dual to $\bigwedge x A(x)$.

The next group of parallel operators are $\curlywedge$ and its dual $\curlyvee$. Intuitively, playing $\curlywedge A$ means simultaneously playing in infinitely many "copies" of A, and $\top$ is the winner iff it wins A in all copies. $\curlyvee A$ is similar, with the only difference that here winning in just one copy is sufficient. So, $\curlywedge A$ is nothing but the infinite parallel conjunction $A \wedge A \wedge A \wedge \cdots$, and $\curlyvee A$ is $A \vee A \wedge A \vee \cdots$. Equivalently, $\curlywedge A$ and $\curlyvee A$ can be respectively understood as $\bigwedge x A$ and $\bigvee x A$, where x is a dummy variable on which A does not depend. The following definition formalizes these intuitions:

Definition 4.10. Assume $A = (Dm, Dn, Vr, A)$ is a gameframe.

(a) $\curlywedge A$ is defined as the gameframe $G = (Dm, Dn, Vr, G)$ such that:

- $\Phi \in \mathbf{Lp}_e^G$ iff every move of Φ has the prefix "$c.$" for some $c \in \mathit{Constants}$ and, for all such c, $\Phi^{c.} \in \mathbf{Lp}_e^{A}$.
- $\mathbf{Wn}_e^G\langle\Gamma\rangle = \top$ iff, for all $c \in \mathit{Constants}$, $\mathbf{Wn}_e^{A}\langle\Gamma^{c.}\rangle = \top$.

(b) $\curlyvee A$ is dual to $\curlywedge A$.

The prefix "r" in the qualification "rimplication" stands for "recurrence". Generally, a rimplication of one or another sort is a weak (recurrence-based) implication, and a repudiation is a weak negation. The parallel versions $\succ\!\!-$ and $\succ\!\neg$ of such operations are defined as follows.

Definition 4.11. **(a)** $A \succ\!\!- B \ =_{def}\ \curlywedge A \to B$. **(b)** $\succ\!\neg A \ =_{def}\ \curlyvee\neg A$.

Just like negation and unlike choice operations, parallel operations preserve the elementary property of games. When restricted to elementary games, the meanings of $\wedge$, $\vee$ and $\to$ coincide with those of classical conjunction, disjunction and implication. Further, as long as all individuals of the universe have naming constants, the meanings of $\bigwedge$ and $\bigvee$ coincide with those of classical universal quantifier and existential quantifier. The same conservation of classical meaning (but without any conditions on the universe) is going to be the case with the blind quantifiers $\forall, \exists$ defined later; so, at the elementary level, when all individuals of the universe have naming constants, $\bigwedge$ $\bigvee$ and are indistinguishable from and $\forall$ and $\exists$, respectively. As for the parallel recurrence and corecurrence, for an elementary A we simply have $A = \curlywedge A = \curlyvee A$.

While all classical tautologies automatically remain valid when parallel operators are applied to elementary games, in the general case the class of valid (in the strict sense of either sort of validity defined in Section 6) principles shrinks. For example, $P \to P \wedge P$, i.e. $\neg P \vee (P \wedge P)$, is not valid. Back to our chess example, one can see that the earlier copycat strategy successful for $\neg \mathit{Chess} \vee \mathit{Chess}$ would be inapplicable to $\neg \mathit{Chess} \vee (\mathit{Chess} \wedge \mathit{Chess})$. The best that $\top$ can do in this three-board game is to synchronize $\neg \mathit{Chess}$ with one of the two conjuncts of $\mathit{Chess} \wedge \mathit{Chess}$. It is possible that then $\neg \mathit{Chess}$ and the unmatched Chess are both lost, in which case the whole game will be lost as well.

The principle $P \to P \wedge P$ is valid in classical logic because the latter sees no difference between P and $P \wedge P$. On the other hand, in virtue of its semantics, CoL is resource-conscious, and in it P is by no means the same as $P \wedge P$ or $P \vee P$. Unlike $P \to P \wedge P$, $P \succ\!\!- P \wedge P$ is a valid principle. Here, in the antecedent, we have infinitely many "copies" of P. Pick any two copies and, via copycat, synchronize them with the two conjuncts of the consequent. A win is guaranteed. The principle $P \to P \sqcap P$ can also be seen to be valid.

This talk about resource-consciousness immediately reminds us of linear logic [10]. The latter, for instance, also rejects $P \to P \wedge P$ while accepting both $P \to P \sqcap P$ and $\curlywedge P \to P \wedge P$ with $\wedge, \vee, \to$ understood as multiplicatives, $\sqcap, \sqcup$ as additives and $\curlywedge, \curlyvee$ as exponentials. Together with similarities, there are also considerable discrepancies though. The class of principles provable in linear logic or even its extension known as affine logic forms a proper subclass of the principles validated by the semantics of CoL. An example of a purely multiplicative formula separating the two classes is Blass's [6] principle

$$(P \wedge P) \vee (P \wedge P) \rightarrow (P \vee P) \wedge (P \vee P).$$

Other examples include $\curlyvee\curlywedge P \rightarrow \curlywedge\curlyvee P$. On the other hand, it is believed (but never has been officially proven) that the class of CoL's valid principles in the signature $\{\neg, \wedge, \vee, \sqcap, \sqcup, \bigsqcap, \bigsqcup\}$ is indistinguishable from the class of principles validated by Blass's [6] game semantics. This, however, stops being the case if $\curlywedge$—or any later-defined sort of recurrence for that matter—is added to the signature as a purported counterpart of Blass's *repetition* operator. For instance, $\curlywedge(P \sqcup Q) \rightarrow \curlywedge P \sqcup \curlywedge Q$ is valid in Blass's sense but it is not a valid principle of CoL; on the other hand, CoL validates $P \wedge \curlywedge(P \rightarrow Q \wedge P) \rightarrow \curlywedge P$ (cf. [33]) which is not valid in Blass's sense.

4.4 Reduction

The operator $\rightarrow$ deserves a separate subsection. The intuition associated with $A \rightarrow B$ is that this is the problem of *reducing* B to A: solving $A \rightarrow B$ means solving B while having A as a computational resource. Specifically, $\top$ may observe how A is being solved by its adversary, and utilize this information in its own solving B. Resources are symmetric to problems: what is a problem to solve for one player is a resource that the other player can use, and vice versa. Since A is negated in $A \rightarrow B = \neg A \vee B$ and negation means switching the players' roles, A (as opposed to $\neg A$) comes as a resource rather than problem to $\top$ in $A \rightarrow B$. Our copycat strategy for $\neg \mathit{Chess} \vee \mathit{Chess}$ was an example of reducing *Chess* to *Chess*. The same strategy was underlying Example 4.8, where $\bigsqcap x \bigsqcup y(y = x^2)$ was reduced to itself.

Let us look at a more meaningful example: reducing the acceptance problem to the halting problem. The former, as a decision problem, will be written as $\bigsqcap x \bigsqcap y\big(\neg \mathit{Accepts}(x,y) \sqcup \mathit{Accepts}(x,y)\big)$, where $\mathit{Accepts}(x,y)$ is the predicate "Turing machine x accepts input y". Similarly, as we already agreed, the halting problem is written as $\bigsqcap x \bigsqcap y\big(\neg \mathit{Halts}(x,y) \sqcup \mathit{Halts}(x,y)\big)$. Neither problem has an algorithmic solution, yet the following implication does:

$$\bigsqcap x \bigsqcap y\big(\neg \mathit{Halts}(x,y) \sqcup \mathit{Halts}(x,y)\big) \rightarrow \bigsqcap x \bigsqcap y\big(\neg \mathit{Accepts}(x,y) \sqcup \mathit{Accepts}(x,y)\big). \quad (1)$$

Here is $\top$'s winning strategy for (1). Wait till $\bot$ makes the moves $1.m$ and $1.n$ for some m and n. Making these moves essentially means asking the question "Does machine m accept input n?". If such moves are never made, you (the machine) win. Otherwise, the moves bring the game down to

$$\bigsqcap x \bigsqcap y\big(\neg \mathit{Halts}(x,y) \sqcup \mathit{Halts}(x,y)\big) \rightarrow \neg \mathit{Accepts}(m,n) \sqcup \mathit{Accepts}(m,n).$$

Make the moves $0.m$ and $0.n$, thus asking the counterquestion "Does machine m halt on input n?". Your moves further bring the game down to

$$\neg \mathit{Halts}(m,n) \sqcup \mathit{Halts}(m,n) \rightarrow \neg \mathit{Accepts}(m,n) \sqcup \mathit{Accepts}(m,n).$$

$\perp$ will have to answer this counterquestion, or else it loses (why?). If it answers by 0.0 ("No, m does not halt on n"), you make the move 1.0 (say "m does not accept n"). The game will be brought down to $\neg \mathit{Halts}(m,n) \rightarrow \neg \mathit{Accepts}(m,n)$. You win, because this is a true proposition: if m does not halt on n, then it does not accept n, either. Otherwise, if $\perp$ answers by 0.1 ("Yes, m halts on n"), start simulating m on n until m halts. If you see that m accepted n, make the move 1.1 (say "m accepts n"); otherwise make the move 1.0 (say "m does not accept n"). Of course, it is a possibility that this simulation goes on forever. But then $\perp$ has lied when saying "m halts on n"; in other words, the antecedent is false, and you win regardless of what happens in the consequent. Note that what the machine did when following this strategy was indeed reducing the acceptance problem to the halting problem: it solved the former using an external (environment-provided) solution of the latter.

There are many natural concepts of reduction, and a strong case can be made that **pimplicative reduction**, i.e. the reduction captured by $\rightarrow$, is the most basic one. For this reason we agree that, if we simply say "reduction", it always means pimplicative reduction. A great variety of other reasonable concepts of reduction is expressible in terms of $\rightarrow$. Among those is Turing reduction. Remember that a predicate $q(x)$ is said to be *Turing reducible* to a predicate $p(x)$ if $q(x)$ can be decided by a Turing machine equipped with an oracle for $p(x)$. For a positive integer n, *n-bounded Turing reducibility* is defined the same way, with the only difference that here the oracle is allowed to be used only n times. It turns out that $\succ$ is a conservative generalization of Turing reduction. Namely, when $p(x)$ and $q(x)$ are elementary games (i.e. predicates), $q(x)$ is Turing reducible to $p(x)$ if and only if the problem $\sqcap x(\neg p(x) \sqcup p(x)) \succ \sqcap x(\neg q(x) \sqcup q(x))$ has an algorithmic solution. If here we change $\succ$ back to $\rightarrow$, we get the same result for 1-bounded Turing reducibility. More generally, as one might guess, n-bounded Turing reduction will be captured by

$$\sqcap x_1(\neg p(x_1) \sqcup p(x_1)) \wedge \cdots \wedge \sqcap x_n(\neg p(x_n) \sqcup p(x_n)) \rightarrow \sqcap x(\neg q(x) \sqcup q(x)).$$

If, instead, we write

$$\sqcap x_1 \cdots \sqcap x_n\Big((\neg p(x_1) \sqcup p(x_1)) \wedge \cdots \wedge (\neg p(x_n) \sqcup p(x_n))\Big) \rightarrow \sqcap x(\neg q(x) \sqcup q(x)),$$

then we get what is called *n-bounded weak truth-table reduction*. The latter differs from n-bounded Turing reduction in that here all n oracle queries should be made at once, before seeing responses to any of those queries. What is called *mapping* (or *many-one*) *reducibility* of $q(x)$ to $p(x)$ is nothing but computability of $\sqcap x \sqcup y(q(x) \leftrightarrow p(y))$, where $A \leftrightarrow B$ abbreviates $(A \to B) \wedge (B \to A)$. One could go on and on with this list.

And yet many other natural concepts of reduction expressible in the language of CoL may have no established names in the literature. For example, from the previous discussion it can be seen that a certain reducibility-style relation holds between the predicates $Accepts(x, y)$ and $Halts(x, y)$ in an even stronger sense than computability of (1). In fact, not only (1) has an algorithmic solution, but also the generally harder-to-solve problem

$$\sqcap x \sqcap y\big(\neg Halts(x, y) \sqcup Halts(x, y) \to \neg Accepts(x, y) \sqcup Accepts(x, y)\big).$$

Among the merits of CoL is that it offers a formalism and deductive machinery for systematically expressing and studying computation-theoretic relations such as reducibility, decidability, enumerability, etc., and all kinds of variations of such concepts.

Back to reducibility, while the standard approaches only allow us to talk about (a whatever sort of) reducibility as a *relation* between problems, in our approach reduction becomes an *operation* on problems, with reducibility as a relation simply meaning computability of the corresponding combination of games, such as $A \to B$ for pimplicative reducibility. Similarly for other relations or properties such as the property of *decidability*. The latter becomes the operation of *deciding* if we define the problem of deciding a predicate $p(x)$ as the game $\sqcap x\big(\neg p(x) \sqcup p(x)\big)$. So, now we can meaningfully ask questions such as "*Is the reduction of the problem of deciding $q(x)$ to the problem of deciding $p(x)$ always reducible to the mapping reduction of $q(x)$ to $p(x)$?*". This question would be equivalent to whether the following formula is valid in CoL:

$$\sqcap x \sqcup y(q(x) \leftrightarrow p(y)) \to \Big(\sqcap x\big(\neg p(x) \sqcup p(x)\big) \to \sqcap x\big(\neg q(x) \sqcup q(x)\big)\Big). \tag{2}$$

The answer turns out to be "Yes", meaning that mapping reduction is at least as strong as pimplicative reduction. Here is a strategy that wins this game no matter what particular predicates $p(x)$ and $q(x)$ are. At first, wait till, for some m, the environment brings the game down to

$$\sqcap x \sqcup y(q(x) \leftrightarrow p(y)) \to \Big(\sqcap x\big(\neg p(x) \sqcup p(x)\big) \to \neg q(m) \sqcup q(m)\Big).$$

Respond by bringing the game down to

$$\sqcup y(q(m) \leftrightarrow p(y)) \to \Big(\sqcap x\big(\neg p(x) \sqcup p(x)\big) \to \neg q(m) \sqcup q(m)\Big).$$

Wait again till, for some n, the environment further brings the above game down to

$$\big(q(m) \leftrightarrow p(n)\big) \to \Big(\sqcap x\big(\neg p(x) \sqcup p(x)\big) \to \neg q(m) \sqcup q(m)\Big).$$

Bring this game down to $\big(q(m) \leftrightarrow p(n)\big) \to \big(\neg p(n) \sqcup p(n) \to \neg q(m) \sqcup q(m)\big)$, after which wait till the environment further brings the game down to either $\big(q(m) \leftrightarrow p(n)\big) \to \big(\neg p(n) \to \neg q(m) \sqcup q(m)\big)$ or $\big(q(m) \leftrightarrow p(n)\big) \to \big(p(n) \to \neg q(m) \sqcup q(m)\big)$. In the former case, bring the game down to $\big(q(m) \leftrightarrow p(n)\big) \to \big(\neg p(n) \to \neg q(m)\big)$, and you have won; in the latter case, bring the game down to $\big(q(m) \leftrightarrow p(n)\big) \to \big(p(n) \to q(m)\big)$, and you have won, again.

One could also ask: "*Is the mapping reduction of $q(x)$ to $p(x)$ always reducible to the reduction of the problem of deciding $q(x)$ to the problem of deciding $p(x)$?*". This question would be equivalent to whether the following formula is valid:

$$\Big(\sqcap x\big(\neg p(x) \sqcup p(x)\big) \to \sqcap x\big(\neg q(x) \sqcup q(x)\big)\Big) \to \sqcap x \bigsqcup y\big(q(x) \leftrightarrow p(y)\big). \qquad (3)$$

The answer here turns out to be "No", meaning that mapping reduction is properly stronger than pimplicative reduction. This negative answer can be obtained by showing that the above formula is not provable in one of the sound and complete deductive systems for CoL whose language allows us to write (3), such as system **CL12** found later in Section 7.3. Similarly, had our ad hoc attempt to come up with a strategy for (2) failed, its validity could have been easily established by finding a proof of it in such a system.

To summarize, CoL offers not only a convenient language for specifying computational problems and relations or operations on them, but also a systematic tool for asking and answering questions in the above style and beyond.

4.5 Blind operations

This group only includes $\forall$ (**blind universal quantifier**, read as "*blall*") and $\exists$ (**blind existential quantifier**, read as "*blexists*"), with no propositional counterparts. Our definition of $\forall x A(x)$ and $\exists x A(x)$ below assumes that the gameframe $A(x)$ is "unistructural" in x. Intuitively, unistructurality in x means that the **Lp** component of the gameframe does not depend on the (value of the) variable x. Formally, we say that a gameframe $A(x) = (Dm, Dn, Vr, A)$ is **unistructural in** x iff, for any (Vr, Dm)-valuation e and any $a, b \in Dm$, we have $\mathbf{Lp}_e^{A(a)} = \mathbf{Lp}_e^{A(b)}$. All nullary or elementary gameframes are unistructural in (whatever variable) x. And all operations of CoL are known to preserve this property.

Definition 4.12. Assume $A(x) = (Dm, Dn, Vr, A)$ is a gameframe unistructural in x.

(a) $\forall x A(x)$ is defined as the gameframe $G = (Dm, Dn, Vr - \{x\}, G)$ such that:

- $\mathbf{Lp}_e^G = \mathbf{Lp}_e^{A(x)}$.
- $\mathbf{Wn}_e^G\langle\Gamma\rangle = \top$ iff, for all $a \in Dm$, $\mathbf{Wn}_e^{A(a)}\langle\Gamma\rangle = \top$.

(b) $\exists x A(x)$ is dual to $\forall x A(x)$.

Intuitively, playing $\forall x A(x)$ or $\exists x A(x)$ means playing $A(x)$ "blindly", without knowing the value of x. In $\forall x A(x)$, the machine wins iff the play it generates is successful for every possible value of x from the domain, while in $\exists x A(x)$ being successful for just one value is sufficient. When applied to elementary games, the blind quantifiers act exactly like the corresponding quantifiers of classical logic.

Unlike $\bigwedge x A(x)$ which is a game on infinitely many boards, both $\forall x A(x)$ and $\sqcap x A(x)$ are one-board games. Yet, they are very different from each other. To see this difference, compare the problems $\sqcap x\big(\mathit{Even}(x) \sqcup \mathit{Odd}(x)\big)$ and $\forall x\big(\mathit{Even}(x) \sqcup \mathit{Odd}(x)\big)$. The former is an easily winnable game of depth 2: the environment selects a number, and the machine tells whether that number is even or odd. The latter, on the other hand, is a game which is impossible to win. This is a game of depth 1, where the value of x is not specified by either player, and only the machine moves—tells whether (the unknown) x is even or odd. Whatever the machine says, it loses, because there is always a value for x that makes the answer wrong.

This should not suggest that nontrivial $\forall$-games can never be won. For instance, the problem

$$\forall x\Big(\mathit{Even}(x) \sqcup \mathit{Odd}(x) \to \sqcap y\big(\mathit{Even}(x+y) \sqcup \mathit{Odd}(x+y)\big)\Big)$$

has an easy solution. The idea of a winning strategy here is that, for any given y, in order to tell the parity of $x + y$, it is not really necessary to know the value of x. Rather, just knowing the parity of x is sufficient. And such knowledge can be obtained from the antecedent. In other words, for any known y and unknown x, the problem of telling whether $x + y$ is even or odd is reducible to the problem of telling whether x is even or odd. Specifically, if both x and y are even or both are odd, then $x + y$ is even; otherwise $x + y$ is odd. Below is the evolution sequence (cf. Exercise 4.8) induced by the run $\langle \bot 1.7, \bot 0.0, \top 1.1\rangle$ where the machine has used such a strategy.

$$\begin{array}{l}
\forall x\Big(\mathit{Even}(x) \sqcup \mathit{Odd}(x) \to \sqcap y\big(\mathit{Even}(x+y) \sqcup \mathit{Odd}(x+y)\big)\Big)\\
\forall x\big(\mathit{Even}(x) \sqcup \mathit{Odd}(x) \to \mathit{Even}(x+7) \sqcup \mathit{Odd}(x+7)\big)\\
\forall x\big(\mathit{Even}(x) \to \mathit{Even}(x+7) \sqcup \mathit{Odd}(x+7)\big)\\
\forall x\big(\mathit{Even}(x) \to \mathit{Odd}(x+7)\big)
\end{array}$$

The machine won because the play hit the true $\forall x(\mathit{Even}(x) \to \mathit{Odd}(x+7))$. Notice how $\forall x$ persisted throughout the sequence. Generally, the $(\forall, \exists)$-structure of a game will remain unchanged in such sequences. The same is the case with parallel operations such as $\to$ in the present case.

To help us appreciate the contrast between the logical behaviors of $\forall$, $\sqcap$ and $\bigwedge$, the following list shows some valid ($\Vdash$) and invalid ($\nVdash$) principles of CoL, where validity ("always computability") can be understood in either sense defined later in Section 6.

1. $\Vdash \forall x P(x) \to \sqcap x P(x)$
2. $\nVdash \sqcap x P(x) \to \forall x P(x)$
3. $\nVdash \forall x P(x) \to \bigwedge x P(x)$
4. $\nVdash \bigwedge x P(x) \to \forall x P(x)$
5. $\Vdash \bigwedge x P(x) \to \sqcap x P(x)$
6. $\nVdash \sqcap x P(x) \to \bigwedge x P(x)$
7. $\Vdash \mathbb{Q} x P(x) \wedge \mathbb{Q} x R(x) \to \mathbb{Q} x(P(x) \wedge R(x))$ for all three $\mathbb{Q} \in \{\forall, \sqcap, \bigwedge\}$
8. $\Vdash \forall x(P(x) \wedge R(x)) \to \forall x P(x) \wedge \forall x R(x)$
9. $\nVdash \sqcap x(P(x) \wedge R(x)) \to \sqcap x P(x) \wedge \sqcap x R(x)$
10. $\Vdash \bigwedge x(P(x) \wedge R(x)) \to \bigwedge x P(x) \wedge \bigwedge x R(x)$

4.6 Branching operations

This group consists of ⫰ (**branching recurrence**, read as "*brecurrence*"), ⫯ (**branching corecurrence**, read as "*cobrecurrence*"), ⟜ (**branching rimplication**, read as "*brimplication*") and $\circ\!\neg$ (**branching repudiation**, read as "*brepudiation*"). Let us talk about ⫰ first, as all other branching operations are definable in terms of it.

What is common for the members of the family of game operations called recurrences is that, when applied to a game A, they turn it into a game playing which means repeatedly playing A. In terms of resources, recurrence operations generate multiple "copies" of A, thus making A a reusable/recyclable resource. In classical logic, recurrence-style operations would be meaningless, because classical logic is resource-blind and thus sees no difference between one and multiple copies of A. In the resource-conscious CoL, however, recurrence operations are not only meaningful, but also necessary to achieve a satisfactory level of expressiveness and realize its potential and ambitions. Hardly any computer program is used only once; rather, it is run over and over again. Loops within such programs also assume multiple repetitions of the same subroutine. In general, the tasks performed in real life by computers, robots or humans are typically recurring ones or involve recurring subtasks.

There is more than one naturally emerging recurrence operation. The differences between various recurrence operations are related to how "repetition" or "reusage" is exactly understood. Imagine a computer with a chess-playing program. The resource that such a computer provides is obviously something stronger than just our old friend *Chess* (as long as it always wins), for it permits to play *Chess* as many times as the user wishes, while *Chess*, as such, only assumes one play. The simplest operating system would allow to start a session of *Chess*, then—after finishing or abandoning and destroying it—start a new play again, and so on. The game that such a system plays—i.e. the resource that it supports/provides—is $\underline{⅄}Chess$, which assumes an unbounded number of plays of *Chess* in a sequential fashion. A formal definition of the operation $\underline{⅄}$, called *sequential recurrence*, will be given later is Section 4.7.

A more advanced operating system, however, would not require to destroy the old sessions before starting new ones; rather, it would allow to run as many parallel sessions as the user wants. This is what is captured by $⅄Chess$, meaning nothing but the infinite parallel conjunction $Chess \wedge Chess \wedge Chess \wedge \cdots$. As we remember from Section 4.3, $⅄$ is called *parallel recurrence.*

Yet a really good operating system would not only allow the user to start new sessions of *Chess* without destroying old ones; it would also make it possible to branch/replicate any particular stage of any particular session, i.e., create any number of "copies" of any already reached position of the multiple parallel plays of *Chess*, thus giving the user the possibility to try different continuations from the same position. What corresponds to this intuition is the branching recurrence $⫰Chess$ of *Chess.*

At the intuitive level, the difference between $⫰$ and $⅄$ is that in $⫰A$, unlike $⅄A$, the environment does not have to restart A from the very beginning every time it wants to reuse it (as a resource); rather, it is allowed to backtrack to any of the previous—not necessarily starting—positions and try a new continuation from there, thus depriving the adversary of the possibility to reconsider the moves it has already made in that position. This is in fact the type of reusage every purely software resource allows or would allow in the presence of an advanced operating system and unlimited memory: one can start running a process (task, game); then fork it at any stage thus creating two threads that have a common past but possibly diverging futures (with the possibility to treat one of the threads as a "backup copy" and preserve it for backtracking purposes); then further fork any of the branches at any time; and so on.

The less flexible type of reusage of A assumed by $⅄A$, on the other hand, is closer to what infinitely many autonomous physical resources would naturally offer, such as an unlimited number of independently acting robots each performing task A, or an unlimited number of computers with limited memories, each one only capable of and responsible for running a single thread of process A. Here the effect of forking an advanced stage of A cannot be achieved unless, by good luck, there are two identical copies of the stage, meaning that the corresponding two robots or computers have so far acted in precisely the same ways.

In early papers [14, 26] on CoL, the formal definitions of ⫰ and its dual ⫯ were direct formalizations of the above intuitions, with an explicit presence of "replicative" moves used by players to fork a given thread of A and create two threads out of one. Later, in [32], another definition was found which was proven to be equivalent to the old one in the sense of mutual reducibility of the old and the new versions of $⫰A$. The new definition less directly corresponds to the above intuitions, but is technically simpler, and we choose it as our "canonical" definition of branching (co)recurrence. To be able to state it, we agree on the following:

Notation 4.13. Where Γ is a run and w is a **bitstring** (finite or infinite sequence of 0s and 1s), $\Gamma^{\preceq w}$ means the result of deleting from Γ all moves except those that look like $u.\alpha$ for some initial segment u of w, and then further deleting the prefix "$u.$" from such moves. E.g., $\langle \bot 00.77, \top 01.88, \top 0.66\rangle^{\preceq 00} = \langle \bot 77, \top 66\rangle$.

Definition 4.14. Assume $A = (Dm, Dn, Vr, A)$ is a gameframe.

(a) $⫰A$ is defined as the gameframe $G = (Dm, Dn, Vr, G)$ such that:

- $\Phi \in \mathbf{Lp}^G_e$ iff every move of Φ has the prefix "$u.$" for some finite bitstring u and, for every infinite bitstring w, $\Phi^{\preceq w} \in \mathbf{Lp}^A_e$;
- $\mathbf{Wn}^G_e\langle\Gamma\rangle = \top$ iff, for every infinite bitstring w, $\mathbf{Wn}^A_e\langle\Gamma^{\preceq w}\rangle = \top$.

(b) $⫯A$ is dual to $⫰A$.

The direct intuitions underlying this definition are as follows. To play $⫰A$ or $⫯A$ means to simultaneously play in multiple parallel copies/threads of A. Each infinite bitstring w denotes one such thread (so, there are in fact uncountably many threads, even if some of them coincide). Every legal move by either player looks like $u.\alpha$ for some finite bitstring u, and the effect/meaning of such a move is simultaneously making the move α in all threads w such that u is an initial segment of w. So, where Γ is the overall run of $⫰A$ or $⫯A$, the run in a given thread w of A is $\Gamma^{\preceq w}$. In order to win $⫰A$, the machine needs to win A in all threads, while for winning $⫯A$ it is sufficient to win in just one thread.

⫰ can be shown to be stronger than its parallel counterpart ⅄, in the sense that the principle $⫰P \rightarrow ⅄P$ is valid while $⅄P \rightarrow ⫰P$ is not. The two operators, in isolation from each other, also validate different principles. For instance, $P \wedge ⅄(P \rightarrow Q \wedge P) \rightarrow$

$⅄Q$ is valid while $P \wedge ⫰(P \rightarrow Q \wedge P) \rightarrow ⫰Q$ is not; $⫰(P \sqcup Q) \rightarrow ⫰P \sqcup ⫰Q$ is valid while $⅄(P \sqcup Q) \rightarrow ⅄P \sqcup ⅄Q$ is not. In its overall spirit, the earlier mentioned Blass's repetition operator $\mathfrak{R}$ is much closer to $⫰$ than $⅄$, yet Blass's semantics validates a different set of principles with $\mathfrak{R}$ than CoL does with $⫰$. For instance, the following formula is invalid [33] in CoL but valid in Blass's semantics with $\mathfrak{R}$ in the role of $⫰$: $P \wedge ⫰(P \rightarrow Q \wedge P) \wedge ⫰(R \vee Q \rightarrow R) \rightarrow ⫰R$.

The branching sorts of rimplication and repudiation are defined in terms of $\neg, \rightarrow$ and $⫰, ⫯$ the same way as the parallel sorts of rimplication and repudiation are defined in terms of $\neg, \rightarrow$ and $⅄, Y$:

Definition 4.15. **(a)** $A \circ\!\!- B \ =_{def} \ ⫰A \rightarrow B$. **(b)** $\circ\!\neg A \ =_{def} \ ⫯\neg A$.

Similarly to the earlier defined pimplicative reducibility, for games A, B we say that B is **brimplicatively** (resp. primplicatively, etc.) **reducible** to A iff $A \circ\!\!- B$ (resp. $A \succ\!\!- B$, etc.) is computable.

Exercise 4.16. The *Kolmogorov complexity* $k(x)$ of a natural number x is the size of a smallest Turing machine that outputs x on input 0. The Kolmogorov complexity problem $\sqcap x \sqcup y(y = k(x))$ has no algorithmic solution. Nor is it pimplicatively reducible to the halting problem. It, however, is reducible to the halting problem in the weaker sense of brimplicative reducibility, meaning that $\top$ has a winning strategy for $\sqcap x \sqcap y(\neg \mathit{Halts}(x,y) \sqcup \mathit{Halts}(x,y)) \circ\!\!- \sqcap x \sqcup y(y = k(x))$. Describe such a strategy, informally.

Both brimplicative and primplicative reducibilities are conservative generalizations of Turing reducibility: for any predicates $p(x)$ and $q(x)$, $\sqcap x(\neg p(x) \sqcup p(x)) \circ\!\!- \sqcap x(\neg q(x) \sqcup q(x))$ is computable iff $q(x)$ is Turing reducible to $p(x)$ iff $\sqcap x(\neg p(x) \sqcup p(x)) \succ\!\!- \sqcap x(\neg q(x) \sqcup q(x))$ is computable. Generally, when restricted to traditional sorts of problems such as problems of deciding a predicate or computing a function as in Exercise 4.16, $\circ\!\!-$ and $\succ\!\!-$ are extensionally indistinguishable. This, however, stops being the case when these operators are applied to problems with higher degrees of interactivity. For instance, the following problem is computable, but becomes incomputable with $\succ\!\!-$ instead of $\circ\!\!-$:

$$\begin{array}{l} \sqcup y \sqcap x(\neg \mathit{Halts}(x,y) \sqcup \mathit{Halts}(x,y)) \circ\!\!- \\ \sqcup y\Big(\sqcap x(\neg \mathit{Halts}(x,y) \sqcup \mathit{Halts}(x,y)) \wedge \sqcap x(\neg \mathit{Halts}(x,y) \sqcup \mathit{Halts}(x,y))\Big). \end{array}$$

Generally, $(P \succ\!\!- Q) \rightarrow (P \circ\!\!- Q)$ is valid but $(P \circ\!\!- Q) \rightarrow (P \succ\!\!- Q)$ is not.

While both $\succ\!\!-$ and $\circ\!\!-$ are weaker than $\rightarrow$ and hence more general than the latter, $\circ\!\!-$ is still a more interesting operation of weak reduction than $\succ\!\!-$. What makes it special is the belief stated in Thesis 4.17 below. The latter, in turn, is based on the belief that ⫰ (and by no means ⅄) is the operation allowing to reuse its argument in the strongest algorithmic sense possible.

Thesis 4.17. Brimplicative reducibility is an adequate mathematical counterpart of our intuition of reducibility in the weakest—and thus most general—algorithmic sense possible. Specifically:

(a) Whenever a problem B is brimplicatively reducible to a problem A, B is also algorithmically reducible to A according to anyone's reasonable intuition.

(b) Whenever a problem B is algorithmically reducible to a problem A according to anyone's reasonable intuition, B is also brimplicatively reducible to A.

The above is pretty much in the same sense as, by the Church-Turing thesis, a function f is computable by a Turing machine iff f has an algorithmic solution according to anyone's reasonable intuition.

Understanding the intuitionistic negation, implication, conjunction, disjunction and quantifiers as $\circ\!\neg$, $\circ\!-$, $\sqcap$, $\sqcup$, $\bigsqcap$, $\bigsqcup$, respectively, Heyting's system for intuitionistic logic has been shown [23] to be sound with respect to the semantics of CoL. It is also "almost complete", as the following formula of an imposing length is among the shortest known propositional formulas valid in CoL but unprovable in Heyting's calculus:

$$(\circ\!\neg P \circ\!- Q \sqcup R) \sqcap (\circ\!\neg\circ\!\neg P \circ\!- Q \sqcup R) \circ\!-$$
$$(\circ\!\neg P \circ\!- Q) \sqcup (\circ\!\neg P \circ\!- R) \sqcup (\circ\!\neg\circ\!\neg P \circ\!- Q) \sqcup (\circ\!\neg\circ\!\neg P \circ\!- R).$$

4.7 Sequential operations

This group consists of $\triangle$ (**sequential conjunction**, read as "*sand*"), $\triangledown$ (**sequential disjunction**, read as "*sor*"), $\triangleright$ (**sequential implication**, read as "*simplication*"), $\bigtriangleup$ (**sequential universal quantifier**, read as "*sall*"), $\bigtriangledown$ (**sequential existential quantifier**, read as "*sexists*"), $\triangle\!\!\!\mid$ (**sequential recurrence**, read as "*srecurrence*"), $\triangledown\!\!\!\mid$ (**sequential corecurrence**, read as "*cosrecurrence*"), $\triangleright\!\!-$ (**sequential rimplication**, read as "*srimplication*") and $\triangleright\!\!\neg$ (**sequential repudiation**, read as "*srepudiation*").

The game $A \triangle B$ starts and proceeds as A. It will also end as A unless, at some point, the environment decides to switch to the next component, in which case A is abandoned, and the game restarts, continues and ends as B. $A \triangledown B$ is similar, with the difference that here it is the machine who decides whether and when to switch from A to B.

The original formal definition of $A \triangle B$ and $A \triangledown B$ found in [25] was a direct formalization of the above description. Definition 4.18 given below, while less direct, still faithfully formalizes the above intuitions, and we opt for it because it is technically simpler. Specifically, Definition 4.18 allows either player to continue making moves in A even after a switch takes place; such moves are meaningless but harmless. Similarly, it allows either player to make moves in B without waiting for a switch to take place, even though a smart player would only start making such moves if and when a switch happens.

Definition 4.18. Assume $A_0 = (Dm, Dn, Vr_0, A_0)$ and $A_1 = (Dm, Dn, Vr_1, A_1)$ are gameframes.

(a) $A_0 \bigtriangleup A_1$ is defined as the gameframe $G = (Dm, Dn, Vr_0 \cup Vr_1, G)$ such that:

- $\Phi \in \mathbf{Lp}_e^G$ iff Φ has the form $\langle \Psi, \Theta \rangle$ or $\langle \Psi, \bot 1, \Theta \rangle$, where every move of $\langle \Psi, \Theta \rangle$ has the prefix "0." or "1." and, for both $i \in \{0, 1\}$, $\langle \Psi, \Theta \rangle^{i.} \in \mathbf{Lp}_e^{A_i}$.
- If Γ does not contain a ("*switch*") move $\bot 1$, $\mathbf{Wn}_e^G\langle \Gamma \rangle = \mathbf{Wn}_e^{A_0}\langle \Gamma^{0.} \rangle$; otherwise $\mathbf{Wn}_e^G\langle \Gamma \rangle = \mathbf{Wn}_e^{A_1}\langle \Gamma^{1.} \rangle$

(b) $A_0 \bigtriangledown A_1$ is dual to $A_0 \bigtriangleup A_1$.

(c) $A_0 \rhd A_1 \ =_{def} \ \neg A_0 \bigtriangledown A_1$.

Recall that, for a predicate $p(x)$, $\sqcap x(p(x) \sqsupset p(x))$ is the problem of deciding $p(x)$. The similar-looking $\sqcap x(p(x) \rhd p(x))$, on the other hand, can be seen to be the problem of *semideciding* $p(x)$: the machine has a winning strategy in this game if and only if $p(x)$ is semidecidable, i.e., recursively enumerable. Indeed, if $p(x)$ is recursively enumerable, a winning strategy by $\top$ is to wait until $\bot$ brings the game down to $p(n) \rhd p(n)$, i.e., $\neg p(n) \bigtriangledown p(n)$, for some particular n. After that, $\top$ starts looking for a certificate of $p(n)$'s being true. If and when such a certificate is found (meaning that $p(n)$ is indeed true), $\top$ makes a switch move turning $\neg p(n) \bigtriangledown p(n)$ into the true $p(n)$; and if no certificate exists (meaning that $p(n)$ is false), then $\top$ keeps looking for a non-existent certificate forever and thus never makes any moves, so the game ends as $\neg p(n)$, which, again, is true. And vice versa: any effective winning strategy for $\sqcap x(\neg p(x) \bigtriangledown p(x))$ can obviously be seen as a semidecision procedure for $p(x)$, which accepts an input n iff the strategy ever makes a switch move in the scenario where $\bot$'s initial choice of a value for x is n.

As we remember from Section 4.4, Turing reducibility of a predicate $p(x)$ to a predicate $q(x)$ means nothing but computability of $\sqcap x(q(x) \sqsupset q(x)) \succ\!\!- \sqcap x(p(x) \sqsupset p(x))$ (the same holds with $\circ\!-$ instead of $\succ\!\!-$). One can show that changing $\sqsupset$ to $\rhd$ here yields another known concept of reducibility, called *enumeration reducibility* (cf. [49]). That is, $p(x)$ is enumeration reducible to $q(x)$ iff $\sqcap x(q(x) \rhd q(x)) \succ\!\!- \sqcap x(p(x) \rhd p(x))$ is computable in our sense. Similarly, the formula $\sqcap x(q(x) \sqsupset q(x))$

$\succ\!\!-\sqcap x(p(x) \rhd p(x))$ captures *relative computable enumerability* (again, cf. [49]). And so on and so forth.

Existence of effective winning strategies for games is known [14] to be closed under "*from $A \to B$ and A conclude B*", "*from A and B conclude $A \wedge B$*", "*from A conclude $\sqcap xA$*", "*from A conclude $\circ\!\!\mid A$*" and similar rules. In view of such closures, the validity of the principles discussed below implies certain known facts from the theory of computation. Those examples once again demonstrate how CoL can be used as a systematic tool for defining new interesting properties and relations between computational problems, and not only reproducing already known theorems but also discovering an infinite variety of new facts.

The valid formula $\sqcap x(p(x) \rhd p(x)) \wedge \sqcap x(\neg p(x) \rhd \neg p(x)) \to \sqcap x(p(x) \sqsupset p(x))$ "expresses" the well known fact that, if a predicate $p(x)$ and its complement $\neg p(x)$ are both recursively enumerable, then $p(x)$ is decidable. Actually, the validity of this formula means something more: it means that the problem of deciding $p(x)$ is reducible to (the $\wedge$-conjunction of) the problems of semideciding $p(x)$ and $\neg p(x)$. In fact, reducibility in an even stronger sense—a sense that has no name—holds, expressed by the formula $\sqcap x\Big((p(x) \rhd p(x)) \wedge (\neg p(x) \rhd \neg p(x)) \to (p(x) \sqsupset p(x))\Big)$.

The formula $\sqcap x \sqcup y(q(x) \leftrightarrow p(y)) \wedge \sqcap x(p(x) \rhd p(x)) \to \sqcap x(q(x) \rhd q(x))$ is also valid, which implies the known fact that, if a predicate $q(x)$ is mapping reducible to a predicate $p(x)$ and $p(x)$ is recursively enumerable, then $q(x)$ is also recursively enumerable. Again, the validity of this formula, in fact, means something even more: it means that the problem of semideciding $q(x)$ is reducible to the problems of mapping reducing $q(x)$ to $p(x)$ and semideciding $p(x)$.

Certain other reducibilities hold only in the sense of rimplications rather than implications. Here is an example. Two Turing machines are said to be equivalent iff they accept exactly the same inputs. Let $Neq(x, y)$ be the predicate "Turing machines x and y are not equivalent". This predicate is neither semidecidable nor co-semidecidable. However, the problem of its semideciding primplicatively (and hence also brimplicatively) reduces to the halting problem. Specifically, $\top$ has an effective winning strategy for the game

$$\sqcap z \sqcap t(\neg Halts(z, t) \sqcup Halts(z, t)) \succ\!\!-\sqcap x \sqcap y(\neg Neq(x, y) \bigtriangledown Neq(x, y)),$$

in terms of [49] meaning that $Neq(x, y)$ is computably enumerable relative to $Halts(z, t)$. The strategy is to wait till the environment specifies some values m and n for x and y. Then, create a variable i, initialize it to 1 and do the following. Specify z and t as m and i in one yet-unused copy of the antecedent, and as n and i in another yet-unused copy. That is, ask the environment whether m halts on input i and whether n halts on the same input. The environment will have to provide the correct pair of answers, or else it loses. (1) If the answers are "No,No", increment i to $i+1$ and repeat the step. (2) If the answers are "Yes,Yes", simulate both m and n on input i until they halt. If both machines accept or both reject, increment i to $i+1$ and repeat the step. Otherwise, if one accepts and one rejects, make a switch move in the consequent and celebrate victory. (3) If the answers are "Yes,No", simulate m on i until it halts. If m rejects i, increment i to $i+1$ and repeat the step. Otherwise, if m accepts i, make a switch move in the consequent and you win. (4) Finally, if the answers are "No,Yes", simulate n on i until it halts. If n rejects i, increment i to $i+1$ and repeat the step. Otherwise, if n accepts i, make a switch move in the consequent and you win.

As expected, $\bigtriangleup xA(x)$ is essentially the infinite sequential conjunction $A(0) \bigtriangleup A(1) \bigtriangleup A(2) \bigtriangleup \cdots$, $\bigtriangledown xA(x)$ is $A(0) \bigtriangledown A(1) \bigtriangledown A(2) \bigtriangledown \cdots$, $\curlywedge A$ is $A \bigtriangleup A \bigtriangleup A \bigtriangleup \cdots$ and $\curlyvee A$ is $A \bigtriangledown A \bigtriangledown A \bigtriangledown \cdots$. Formally, we have:

Definition 4.19. Assume $A(x) = (Dm, Dn, Vr, A)$ is a gameframe.

(a) $\bigtriangleup xA(x)$ is defined as the gameframe $G = (Dm, Dn, Vr - \{x\}, G)$ such that:

- $\Phi \in \mathbf{Lp}_e^G$ iff Φ has the form $\langle \Psi_0, \bot 1, \Psi_1, \cdots, \bot n, \Psi_n \rangle$ $(n \geq 0)$, where every move of $\langle \Psi_0, \cdots, \Psi_n \rangle$ has the prefix "$c.$" for some $c \in \mathit{Constants}$ and, for every such c, $\langle \Psi_0, \cdots, \Psi_n \rangle^{c.} \in \mathbf{Lp}_e^{A(c)}$.
- Call $\bot 1, \bot 2, \cdots$ *switch moves.* If Γ does not contain a switch move, then $\mathbf{Wn}_e^G\langle\Gamma\rangle = \mathbf{Wn}_e^{A(0)}\langle\Gamma^{0.}\rangle$; if Γ contains infinitely many switch moves, then $\mathbf{Wn}_e^G\langle\Gamma\rangle = \top$; otherwise, where $\bot n$ is the last switch move of Γ, $\mathbf{Wn}_e^G\langle\Gamma\rangle = \mathbf{Wn}_e^{A(n)}\langle\Gamma^{n.}\rangle$.

(b) $\bigtriangledown xA(x)$ is dual to $\bigtriangleup xA(x)$.

Definition 4.20. Assume $A = (Dm, Dn, Vr, A)$ is a gameframe.

(a) $\curlywedge A$ is defined as the gameframe $G = (Dm, Dn, Vr, G)$ such that:

- $\Phi \in \mathbf{Lp}_e^G$ iff Φ has the form $\langle \Psi_0, \bot 1, \Psi_1, \cdots, \bot n, \Psi_n \rangle$ $(n \geq 0)$, where every move of $\langle \Psi_0, \cdots, \Psi_n \rangle$ has the prefix "$c.$" for some $c \in \mathit{Constants}$ and, for every such c, $\langle \Psi_0, \cdots, \Psi_n \rangle^{c.} \in \mathbf{Lp}_e^{A}$.
- Call $\bot 1, \bot 2, \cdots$ *switch moves.* If Γ does not contain a switch move, then $\mathbf{Wn}_e^G\langle\Gamma\rangle = \mathbf{Wn}_e^{A}\langle\Gamma^{0.}\rangle$; if Γ contains infinitely many switch moves, then $\mathbf{Wn}_e^G\langle\Gamma\rangle = \top$; otherwise, where $\bot n$ is the last switch move of Γ, $\mathbf{Wn}_e^G\langle\Gamma\rangle = \mathbf{Wn}_e^{A}\langle\Gamma^{n.}\rangle$.

(b) $\curlyvee A$ is dual to $\curlywedge A$.

For insights into the above-defined operations, remember the Kolmogorov complexity function $k(x)$ from Exercise 4.16. It is known that the value of $k(x)$ is always smaller than x (in fact, logarithmically smaller). While $\sqcap x \sqcup y(y = k(x))$ is not computable, $\top$ does have an algorithmic winning strategy for the problem $\sqcap x Ɏ \sqcup y(y = k(x))$. It goes like this: Wait till $\bot$ specifies a value m for x, thus asking "what is the Kolmogorov complexity of m?" and bringing the game down to $Ɏ \sqcup y(y = k(m))$. Answer (generously) that the complexity is m, i.e. specify y as m. After that, start simulating, in parallel, all machines n of sizes smaller than m on input 0. Whenever you find a machine n that returns m on input 0 and is smaller than any of the previously found such machines, make a switch move and, in the new copy of $\sqcup y(y = k(m))$, specify y as the size $|n|$ of n. This obviously guarantees success: sooner or later the real Kolmogorov complexity c of m will be reached and named; and, even though the strategy will never be sure that $k(m)$ is not something yet smaller than c, it will never really find a reason to further reconsider its latest claim that $c = k(m)$.

Exercise 4.21. Describe a winning strategy for $\sqcap x \bigtriangledown y(k(x) = x - y)$.

Definition 4.22. **(a)** $A \rhd\!\!- B \ =_{def}\ ⅄A \rightarrow B$. **(b)** $\rhd\!\!-\!\neg A \ =_{def}\ Ɏ\neg A$.

4.8 Toggling operations

This group consists of ⩚ (**toggling conjunction**, read as "*tand*"), ⩛ (**toggling disjunction**, read as "*tor*"), ⩥ (**toggling implication**, read as "*timplication*"), ⩚ (**toggling universal quantifier**, read as "*tall*"), ⩛ (**toggling existential quantifier**, read as "*texists*"), ⅄ (**toggling recurrence**, read as "*trecurrence*"), Ɏ (**toggling corecurrence**, read as "*cotrecurrence*"), ⤚ (**toggling rimplication**, read as "*trimplication*") and ⤚¬ (**toggling repudiation**, read as "*trepudiation*").

Let us for now focus on ⩛. One of the ways to characterize $A ⩛ B$ is the following. This game starts and proceeds as a play of A. It will also end as an ordinary play of A unless, at some point, $\top$ decides to switch to B, after which the game becomes B and continues as such. It will also end as B unless, at some point, $\top$ "changes its mind" and switches back to A. In such a case the game again becomes A, where A resumes from the position in which it was abandoned (rather than from its start position, as would be the case, e.g., in $A \bigtriangledown B \bigtriangledown A$). Later $\top$ may again switch to the abandoned position of B, and so on. $\top$ wins the overall play iff it switches from one component to another at most finitely many times and wins in its final choice, i.e., in the component which was chosen last to switch to.

An alternative characterization $A ⩛ B$, on which our formal definition of ⩛ is directly based, is to say that it is played just like $A \sqcup B$, with the only difference that $\top$ is allowed to make a "choose A" or "choose B" move any number of times. If infinitely many choices are made, $\top$ loses. Otherwise, the winner in the play will be the player who wins in the component that was chosen last ("the eventual choice"). The case of $\top$ having made no choices at all is treated as if it had chosen A. Thus, as in $A \bigtriangledown B$, the left component is the "default", or "automatically made", initial choice. It is important to note that $\top$'s adversary—or perhaps even $\top$ itself—never knows whether a given choice of a component of $A \curlyvee B$ is the last choice or not.

What would happen if we did not require that $\top$ can change its mind only finitely many times? There would be no "final choice" in this case. So, the only natural winning condition in the case of infinitely many choices would be to say that $\top$ wins iff it simply wins in one of the components. But then the resulting operation would be essentially the same as $\vee$, as a smart $\top$ would always opt for keeping switching between components forever. That is, allowing infinitely many choices would amount to not requiring any choices at all, as is the case with $A \vee B$.

The very weak sort of choice captured by $\curlyvee$ is the kind of choice that, in real life, one would ordinarily call choice after trial and error. Indeed, a problem is generally considered to be solved after trial and error (a correct choice/solution/answer found) if, after perhaps coming up with several wrong solutions, a true solution is eventually found. That is, mistakes are tolerated and forgotten as long as they are eventually corrected. It is however necessary that new solutions stop coming at some point, so that there is a last solution whose correctness determines the success of the effort. Otherwise, if answers have kept changing all the time, no answer has really been given after all.

As we remember, for a predicate $p(x)$, $\sqcap x(p(x) \sqsupset p(x))$ is the problem of deciding $p(x)$, and $\sqcap x(p(x) \triangleright p(x))$ is the weaker (easier to solve) problem of semideciding $p(x)$. Not surprisingly, $\sqcap x(p(x) \gtrdot p(x))$—which abbreviates $\sqcap x(\neg p(x) \curlyvee p(x))$—is also a decision-style problem, but still weaker than the problem of semideciding $p(x)$. This problem has been studied in the literature under several names, the most common of which is *recursively approximating* $p(x)$. It means telling whether $p(x)$ is true or not, but doing so in the same style as semideciding does in negative cases: by correctly saying "Yes" or "No" at some point (after perhaps taking back previous answers several times) and never reconsidering this answer afterwards. In similar terms, semideciding $p(x)$ can be seen as always saying (the default) "No" at the beginning and then, if this answer is incorrect, changing it to "Yes" at some later time; so, when the answer is negative, this will be expressed by saying "No" and never taking back this answer, yet without ever indicating that the answer is final and will not change. Thus, the difference between semideciding and recursively approximating is that, unlike a semidecision procedure, a recursive approximation procedure can reconsider both negative and positive answers, and do so several times rather than only once.

As an example of a predicate which is recursively approximable but neither semidecidable nor co-semidecidable, consider the predicate $k(x)<k(y)$, saying that number x is simpler than number y in the sense of Kolmogorov complexity. As noted earlier, $k(z)$ (the Kolmogorov complexity of z) is always smaller than z. Here is an algorithm that recursively approximates the predicate $k(x)<k(y)$, i.e., solves the problem $\sqcap x \sqcap y(k(x)\geq k(y) \veemidvert k(x)<k(y))$. Wait till the environment brings the game down to $k(m)\geq k(n) \veemidvert k(m)<k(n)$ for some m and n. Then start simulating, in parallel, all Turing machines t of sizes less than $max(m,n)$ on input 0. Whenever you see that a machine t returns m and the size of t is smaller than that of any other previously found machine that returns m or n on input 0, choose $k(m)<k(n)$. Quite similarly, whenever you see that a machine t returns n and the size of t is smaller than that of any other previously found machine that returns n on input 0, as well as smaller or equal to the size of any other previously found machine that returns m on input 0, choose $k(m)\geq k(n)$. Obviously, the correct choice between $k(m)\geq k(n)$ and $k(m)<k(n)$ will be made sooner or later and never reconsidered afterwards. This will happen when the procedure hits a smallest-size machine t that returns either m or n on input 0.

Anyway, here is our formal definition of $\wedgemidvert$, $\veemidvert$ and $\succ$:

Definition 4.23. Assume $A_0 = (Dm, Dn, Vr_0, A_0)$ and $A_1 = (Dm, Dn, Vr_1, A_1)$ are gameframes.

(a) $A_0 \wedgemidvert A_1$ is defined as the gameframe $G = (Dm, Dn, Vr_0 \cup Vr_1, G)$ such that:

- $\Phi \in \mathbf{Lp}_e^G$ iff Φ has the form $\langle \Psi_0, \bot i_1, \Psi_1, \cdots, \bot i_n, \Psi_n \rangle$ $(n \geq 0)$, where $i_1, \cdots, i_n \in \{0,1\}$, every move of $\langle \Psi_0, \cdots, \Psi_n \rangle$ has the prefix "0." or "1." and, for both $i \in \{0,1\}$, $\langle \Psi_0, \cdots, \Psi_n \rangle^{i.} \in \mathbf{Lp}_e^{A_i}$.
- Call $\bot 0$ and $\bot 1$ *switch moves*. If Γ does not contain a switch move, then $\mathbf{Wn}_e^G\langle \Gamma \rangle = \mathbf{Wn}_e^{A_0}\langle \Gamma^{0.} \rangle$; if Γ contains infinitely many switch moves, then $\mathbf{Wn}_e^G\langle \Gamma \rangle = \top$; otherwise, where $\bot i$ is the last switch move of Γ, $\mathbf{Wn}_e^G\langle \Gamma \rangle = \mathbf{Wn}_e^{A_i}\langle \Gamma^{i.} \rangle$.

(b) $A_0 \veemidvert A_1$ is dual to $A_0 \wedgemidvert A_1$.

(c) $A_0 \succ A_1 \ =_{def} \ \neg A_0 \veemidvert A_1$.

From the formal definitions that follow one can see that, as expected, $\mathop{\wedgemidvert} x A(x)$ is essentially $A(0) \wedgemidvert A(1) \wedgemidvert A(2) \wedgemidvert \cdots$, $\mathop{\veemidvert} x A(x)$ is $A(0) \veemidvert A(1) \veemidvert A(2) \veemidvert \cdots$, $\curlywedge A$ is $A \wedgemidvert A \wedgemidvert A \wedgemidvert \cdots$ and $\curlyvee x A(x)$ is $A \veemidvert A \veemidvert A \veemidvert \cdots$.

Definition 4.24. Assume $A(x) = (Dm, Dn, Vr, A)$ is a gameframe.

(a) $⩓ xA(x)$ is defined as the gameframe $G = (Dm, Dn, Vr - \{x\}, G)$ such that:

- $\Phi \in \mathbf{Lp}_e^G$ iff Φ has the form $\langle \Psi_0, \bot c_1, \Psi_1, \cdots, \bot c_n, \Psi_n \rangle$ $(n \geq 0)$, where $c_1, \cdots, c_n \in$ *Constants*, every move of $\langle \Psi_0, \cdots, \Psi_n \rangle$ has the prefix "$c.$" for some $c \in$ *Constants* and, for every such c, $\langle \Psi_0, \cdots, \Psi_n \rangle^{c.} \in \mathbf{Lp}_e^{A(c)}$.
- Call the above $\bot c_1, \bot c_2, \cdots$ *switch moves.* If Γ does not contain a switch move, then $\mathbf{Wn}_e^G \langle \Gamma \rangle = \mathbf{Wn}_e^{A(0)} \langle \Gamma^{0.} \rangle$; if Γ contains infinitely many switch moves, then $\mathbf{Wn}_e^G \langle \Gamma \rangle = \top$; otherwise, where $\bot c$ is the last switch move of Γ, $\mathbf{Wn}_e^G \langle \Gamma \rangle = \mathbf{Wn}_e^{A(c)} \langle \Gamma^{c.} \rangle$.

(b) $⩔ xA(x)$ is dual to $⩓ xA(x)$.

Definition 4.25. Assume $A = (Dm, Dn, Vr, A)$ is a gameframe.

(a) $\curlywedge A$ is defined as the gameframe $G = (Dm, Dn, Vr, G)$ such that:

- $\Phi \in \mathbf{Lp}_e^G$ iff Φ has the form $\langle \Psi_0, \bot c_1, \Psi_1, \cdots, \bot c_n, \Psi_n \rangle$ $(n \geq 0)$, where $c_1, \cdots, c_n \in$ *Constants*, every move of $\langle \Psi_0, \cdots, \Psi_n \rangle$ has the prefix "$c.$" for some $c \in$ *Constants* and, for every such c, $\langle \Psi_0, \cdots, \Psi_n \rangle^{c.} \in \mathbf{Lp}_e^A$.
- Call the above $\bot c_1, \bot c_2, \cdots$ *switch moves.* If Γ does not contain a switch move, then $\mathbf{Wn}_e^G \langle \Gamma \rangle = \mathbf{Wn}_e^A \langle \Gamma^{0.} \rangle$; if Γ contains infinitely many switch moves, then $\mathbf{Wn}_e^G \langle \Gamma \rangle = \top$; otherwise, where $\bot c$ is the last switch move of Γ, $\mathbf{Wn}_e^G \langle \Gamma \rangle = \mathbf{Wn}_e^A \langle \Gamma^{c.} \rangle$.

(b) $\curlyvee A$ is dual to $\curlywedge A$.

To see toggling quantifiers at work, remember that Kolmogorov complexity $k(x)$ is not a computable function, i.e., the problem $\sqcap x \sqcup y(y = k(x))$ has no algorithmic solution. However, replacing $\sqcup y$ with $⩔ y$ in it yields an algorithmically solvable problem. A solution for $\sqcap x ⩔ y(y = k(x))$ goes like this. Wait till the environment chooses a number m for x, thus bringing the game down to $⩔ y(y = k(m))$, which is essentially nothing but $0 = k(m) ⩔ 1 = k(m) ⩔ 2 = k(m) ⩔ \cdots$. Create a variable i initialized to m, and perform the following routine: Switch to the disjunct $i = k(m)$ of $0 = k(m) ⩔ 1 = k(m) ⩔ 2 = k(m) ⩔ \cdots$ and then start simulating on input 0, in parallel, all Turing machines whose sizes are smaller than i; if and when you see that one of such machines returns m, update i to the size of that machine, and repeat the present routine.

Definition 4.26. **(a)** $A \succ\!\!- B \ =_{def} \ \curlywedge A \to B$. **(b)** $\succ\!\!\neg A \ =_{def} \ \curlyvee \neg A$.

4.9 Cirquents

The constructs called **cirquents** take the expressive power of CoL to a qualitatively higher level, allowing us to form, in a systematic way, an infinite variety of game operations. Each cirquent is—or can be seen as—an independent operation on games, generally not expressible via composing operations taken from some fixed finite pool of primitives, such as the operations seen in the preceding subsections of the present section.

Cirquents come in a variety of versions, but common to all them is having mechanisms for explicitly accounting for possible *sharing* of subcomponents between different components. Sharing is the main distinguishing feature of cirquents from more traditional means of expression such as formulas, sequents, hypersequents [2], or structures of the calculus of structures [12]. While the latter can be drawn as (their parse) trees, cirquents more naturally call for circuit- or graph-style constructs. The earliest cirquents [17] were intuitively conceived as collections of one-sided sequents (sequences of formulas) that could share some formulas and, as such, could be drawn like circuits rather than linear expressions. This explains the etimology of the word: CIRcuit+seQUENT. All Boolean circuits are cirquents, but not all cirquents are Boolean circuits. Firstly, because cirquents may have various additional sorts of gates ($\sqcap$-gates, $\triangle$-gates, $\curlywedge$-gates, etc.). Secondly, because cirquents may often have more evolved sharing mechanisms than just child- (input-) sharing between different gates. For instance, a "cluster" [29] of $\sqcup$-gates may share choices associated with $\sqcup$ in game-playing: if the machine chooses the left or the right child for one gate of the cluster, then the same left or right choice automatically extends to all gates of the cluster regardless of whether they share children or not.

We are not going to introduce cirquents and their semantics in full generality or formal detail here. For intuitive insights, let us only focus on cirquents that look like Boolean circuits with $\wedge$- and $\vee$-gates. Every such cirquent C can be seen as an n-ary parallel operation on games, where n is the number of inputs of C.

Figure 3: The two-out-of-three combination of resources

The left cirquent of Figure 3 represents the 3-ary game operation $\heartsuit$ informally defined as follows. Playing $\heartsuit(P, Q, R)$, as is the case with all parallel operations, means playing simultaneously in all components of it. In order to win, $\top$ needs to win in at least two out of the three components. Any attempt to express this operation in terms of $\wedge, \vee$ or other already defined operations is going to fail. For instance, the natural candidate $(P \wedge Q) \vee (P \wedge R) \vee (Q \wedge R)$ is very far from being adequate. The latter is a game on six rather than three boards, with P played on boards #1 and #3, Q on boards #2 and #5, and R on boards #4 and #6. Similarly, the formula $(P \wedge P) \vee (P \wedge P) \vee (P \wedge P)$ is not an adequate representation of the right cirquent of Figure 3. It fails to indicate for instance that the 1st and the 3rd occurrences of P stand for the same copy of P while the 2nd occurrence for a different copy in which a different run can be generated.

Cirquents are thus properly more expressive than formulas even at the most basic $(\wedge, \vee)$ level. It is this added expressiveness and flexibility that, for some fragments of CoL, makes a difference between axiomatizability and unaxiomatizability: even if one is only trying to set up a deductive system for proving valid formulas, intermediate steps in proofs of such formulas still inherently require using cirquents that cannot be written as formulas. An example is the system **CL15** found in Section 7.1.

The present article is exclusively focused on the formula-based version of CoL, seeing cirquents (in Section 7.1) only as technical servants to formulas. This explains why we do not attempt to define the semantics of cirquents formally. It should however be noted that cirquents are naturally called for not only within the specific formal framework of CoL, but also in the framework of all resource-sensitive approaches in logic, like linear logic. Such approaches may intrinsically require the ability to account for the ubiquitous phenomenon of resource sharing. The insufficient expressiveness of linear logic is due to the inability of formulas to explicitly show (sub)resource sharing or the absence thereof. The right cirquent of Figure 3 stands for a multiplicative-style disjunction of three resources, with each disjunct, in turn, being a conjunction of two subresources of type P. However, altogether there are three rather than six such subresources, each one being shared between two different disjuncts of the main resource. From the abstract resource-philosophical point of view of cirquent-based CoL, classical logic and linear logic are two imperfect extremes. In the former, all occurrences of a same subformula mean the same (represent the same resource), i.e., *everything is shared that can be shared*; and in the latter, each occurrence stands for a separate resource, i.e., *nothing is shared at all*. Neither approach does thus permit to account for mixed cases where certain occurrences are meant to represent the same resource while some other occurrences stand for different resources of the same type.

5 Static games

While games in the sense of Definition 2.1 are apparently general enough to model anything one would call an interactive computational problem, they are a little bit too general. Consider a game *Bad* whose only nonempty legal runs are $\langle \top\alpha \rangle$, won by $\top$, and $\langle \bot\alpha \rangle$, won by $\bot$. Whichever player is fast enough to make the move α first will thus be the winner. Since there are no natural, robust assumptions regarding the relative speeds of the players, obviously *Bad* is not something that could qualify as a meaningful computational problem. For such reasons, CoL limits its focus on a natural proper subclass of all games in the sense of Definition 2.1 called *static*. Intuitively, static games are games where speed is irrelevant because, using Blass's words, "it never hurts a player to postpone making moves".

In order to define static games, recall that, for a player $\wp$, $\overline{\wp}$ means "the other player". Further recall the concepts of a $\wp$-legal and $\wp$-won runs from Section 2. Given a run Γ, we let $\Gamma^\top$ denote the subsequence of (all and only) $\top$-labeled moves of Γ; similarly for $\Gamma^\bot$. We say that a run Ω is a **$\wp$-delay** of a run Γ iff the following two conditions are satisfied:

- $\Omega^\top = \Gamma^\top$ and $\Omega^\bot = \Gamma^\bot$;
- For any $n, k \geq 1$, if the kth $\overline{\wp}$-labeled move is made earlier than the nth $\wp$-labeled move in Γ, then so is it in Ω.

The above Ω, in other words, is the result of possibly shifting to the right ("delaying") some $\wp$-labeled moves in Γ without otherwise violating the order of moves by either player.

Definition 5.1. We say that a game G is **static** iff, for either player $\wp \in \{\top, \bot\}$ and for any runs Γ, Ω where Ω is a $\wp$-delay of Γ, the following conditions are satisfied:

1. If Γ is a $\wp$-legal run of G, then so is Ω.
2. If Γ is a $\wp$-won run of G, then so is Ω.

A gameframe is said to be **static** iff so are all of its instances.

Exercise 5.2. Verify that the game of Figure 1 is static.

The class of static games or gameframes is very broad. Suffice it to say that all elementary gameframes are static, and that all operations defined in the preceding long section preserve the static property of gameframes. Thus, the closure of elementary gameframes under those operations is one natural subclass of the class of all static games.

6 The formal language of computability logic and its semantics

It is not quite accurate to say "the language" of CoL because, as pointed out earlier, CoL has an open-ended formalism. Yet, in the present article, by "the language of CoL" we will mean the particular language defined below. It extends the language of first-order classical logic by adding to it all operators defined in Section 4, and differentiating between two—elementary and general—sorts of atoms.

The set *Variables* of **variables** and the set *Constants* of **constants** of the language are those fixed in Section 3. Per each natural number n, we also have infinitely many n-ary *extralogical* **function letters**, **elementary gameframe letters** and **general gameframe letters**. We usually use $f, g, h, \cdots$ as metavariables for function letters, $p, q, r, \cdots$ for elementary gameframe letters, and $P, Q, R, \cdots$ for general gameframe letters. Other than these extralogical letters, there are three *logical* gameframe letters, all elementary: $\top$ (nullary), $\bot$ (nullary) and $=$ (binary).

Terms are defined inductively as follows:

- All variables and constants are terms.
- If $t_1, \cdots, t_n$ are terms ($n \geq 0$) and f is an n-ary function letter, then $f(t_1, \cdots, t_n)$ is a term.

Atoms are defined by:

- $\top$ and $\bot$ are atoms. These two atoms are said to be *logical*, and all other atoms *extralogical*.
- If t_1 and t_2 are terms, then $t_1 = t_2$ is an atom.
- If $t_1, \cdots, t_n$ are terms ($n \geq 0$) and L is an extralogical n-ary gameframe letter, then $L(t_1, \cdots, t_n)$ is an atom. Such an extralogical atom is said to be *elementary* or *general* iff L is so.

Finally, **formulas** are defined by:

- All atoms are formulas.
- If E is a formula, then so are $\neg(E)$, $⫰(E)$, $⫯(E)$, $\curlywedge(E)$, $\curlyvee(E)$, $\underset{\triangle}{\curlywedge}(E)$, $\overset{\triangledown}{\curlyvee}(E)$, $\curlywedge\!\!\!\mid(E)$, $\curlyvee\!\!\!\mid(E)$, $\circ\!\!\neg(E)$, $\succ\!\!\neg(E)$, $\triangleright\!\!\neg(E)$, $\succcurlyeq\!\!\neg(E)$.
- If E and F are formulas, then so are $(E) \wedge (F)$, $(E) \vee (F)$, $(E) \sqcap (F)$, $(E) \sqcup (F)$, $(E) \bigtriangleup (F)$, $(E) \bigtriangledown (F)$, $(E) ⩕ (F)$, $(E) ⩖ (F)$, $(E) \rightarrow (F)$, $(E) \sqsupset (F)$, $(E) \triangleright (F)$, $(E) \succcurlyeq (F)$, $(E) \circ\!\!- (F)$, $(E) \succ\!\!- (F)$, $(E) \triangleright\!\!- (F)$, $(E) \succcurlyeq\!\!- (F)$.
- If E is a formula and x is a variable, then $\forall x(E)$, $\exists x(E)$, $\bigwedge x(E)$, $\bigvee x(E)$, $\bigsqcap x(E)$, $\bigsqcup x(E)$, $\bigtriangleup x(E)$, $\bigtriangledown x(E)$, $⩕ x(E)$, $⩖ x(E)$ are formulas.

Unnecessary parentheses will be usually omitted in formulas according to the standard conventions, with partial precedence order as agreed upon earlier for the corresponding game operations. The notions of *free* and *bound* occurrences of variables are also standard, with the only adjustment that now we have eight rather than two quantifiers. A **sentence**, or a **closed formula**, is a formula with no free occurrences of variables. While officially $\wedge$ is a binary operator, we may still write $E_1 \wedge \cdots \wedge E_n$ for a possibly unspecified $n \geq 0$. This should be understood as $E_1 \wedge (E_2 \wedge \cdots (E_{n-1} \wedge E_n) \cdots)$ when $n > 2$, as just E_1 when $n = 1$, and as $\top$ when $n = 0$. Similarly for all other sorts of conjunctions. And similarly for all disjunctions, with the difference that an empty disjunction of whatever sort is understood as $\bot$ rather than $\top$.

For the following definitions, recall Conventions 3.3 and 3.4. Also recall that $var_1, \cdots, var_n$ are the first n variables from the lexicographic list of all variables.

Definition 6.1. An **interpretation** is a mapping * such that, for some fixed universe U called the **universe of** *, we have:

- * sends every n-ary function letter f to an n-ary function $f^*(var_1, \cdots, var_n)$ whose universe is U and whose variables are the first n variables of *Variables.*
- * sends every n-ary extralogical game letter L to an n-ary static gameframe $L^*(var_1, \cdots, var_n)$ whose universe is U and whose variables are the first n variables of *Variables*; besides, if the letter L is elementary, then so is the gameframe $L^*(var_1, \cdots, var_n)$.

Such a * is said to be **admissible for** a formula E (or E**-admissible**) iff, whenever E has an occurrence of a general atom $P(t_1, \cdots, t_n)$ in the scope of $\forall x$ or $\exists x$ and one of the terms t_i $(1 \leq i \leq n)$ contains the variable x, P^* is unistructural in var_i. We uniquely extend * to a mapping that sends each term t to a function t^*, and each formula E for which it is admissible to a game E^*, by stipulating the following:

- Where c is a constant, c^* is (the nullary function) c^U.
- Where x is a variable, x^* is (the unary function) x^U.
- Where f is an n-ary function letter and $t_1, \cdots, t_n$ are terms, $\big(f(t_1, \cdots, t_n)\big)^*$ is $f^*(t_1^*, \cdots, t_n^*)$.
- $\top^*$ is $\top$ and $\bot^*$ is $\bot$.
- Where t_1 and t_2 are terms, $(t_1 = t_2)^*$ is $t_1^* = t_2^*$.
- Where L is an n-ary gameframe letter and $t_1, \cdots, t_n$ are terms, $\big(L(t_1, \cdots, t_n)\big)^*$ is $L^*(t_1^*, \cdots, t_n^*)$.

- $*$ commutes with all logical operators, seeing them as the corresponding game operations: $(\neg E)^*$ is $\neg(E^*)$, $(\text{⫰}E)^*$ is $\text{⫰}(E^*)$, $(E \wedge F)^*$ is $(E^*) \wedge (F^*)$, $(\sqcap x E)^*$ is $\sqcap x(E^*)$, etc.

When O is a function letter, gameframe letter, term or formula and $O^* = W$, we refer to W as "O **under** interpretation $*$".

Definition 6.2. For a sentence S we say that:

1. S is **logically valid** iff there is an HPM $\mathcal{M}$ such that, for every S-admissible interpretation $*$, $\mathcal{M}$ computes S^*. Such an $\mathcal{M}$ is said to be a **logical solution** of S.
2. S is **extralogically valid** iff for every S-admissible interpretation $*$ there is an HPM $\mathcal{M}$ such that $\mathcal{M}$ computes S^*.

Convention 6.3. When S is a formula but not a sentence, its validity is understood as that of the $\sqcap$-closure of S, i.e., of the sentence $\sqcap x_1 \cdots \sqcap x_n S$, where $x_1, \cdots, x_n$ are all free variables of S listed lexicographically.

Every logically valid formula is, of course, also extralogically valid. But some extralogically valid formulas may not necessarily be also logically valid. For instance, where p is a 0-ary elementary gameframe letter, the formula $\neg p \sqcup p$ is valid extralogically but not logically. It is extralogically valid for a trivial reason: given an interpretation $*$, either $\neg p$ or p is true under $*$. If $\neg p$ is true, then the strategy that chooses the first disjunct wins; and if p is true, then the strategy that chooses the second disjunct wins. The trouble is that, even though we know that one of these two strategies succeeds, generally we have no way to tell which one does. And this is why $\neg p \sqcup p$ fails to be logically valid.

Extralogical validity is not only a non-constructive, but also a fragile sort of validity: this property, unlike logical validity, is not closed under substitution of extralogical atoms. For instance, where p is as before and q is a unary extralogical elementary gameframe letter, the formula $\neg q(x) \sqcup q(x)$, while having the same form as $\neg p \sqcup p$, is no longer extralogically valid. The papers on CoL written prior to 2016 had a more relaxed understanding of interpretations than our present understanding. Namely, there was no requirement that an interpretation should respect the arity of a gameframe letter. In such a case, as it turns out, the extensional difference between logical and extralogical validity disappears: while the class of logically valid principles remains the same, the class of extralogically valid principles shrinks down to that of logically valid ones.

Intuitively, a logical solution $\mathcal{M}$ for a sentence S is an interpretation-independent winning strategy: since the intended interpretation is not known to the machine, $\mathcal{M}$ has to play in some standard, uniform way that would be successful for any possible interpretation of S. It is logical rather than extralogical validity that is of interest in all applied systems based on CoL (cf. Section 8). In such applications we want a logic that could be built into a universal problem-solving machine. Such a machine should be able to solve problems represented by logical formulas without any specific knowledge of the meanings of their atoms, other than the knowledge explicitly provided in the knowledgebase (extralogical axioms) of the system. Otherwise the machine would be special-purpose rather than universal. For such reasons, in the subsequent sections we will only be focused on the logical sort of validity, which will be the default meaning of the word "valid".

Definition 6.4. We say that a sentence F is a **logical consequence** of a set $\mathbb{B}$ of sentences iff, for some $E_1, \cdots, E_n \in \mathbb{B}$, the sentence $E_1 \wedge \cdots \wedge E_n \circ\!\!- F$ is logically valid.

Remember the symmetry between computational resources and computational problems: a problem for one player is a resource for the other. Having a problem A as a computational resource intuitively means having the (perhaps externally provided) ability to successfully solve/win A. For instance, as a resource, $\sqcap x \sqcup y(y = x^2)$ means the ability to tell the square of any number. According to Thesis 6.5 below, the relation of logical consequence lives up to its name. The main utility of this thesis, as will be illustrated in Section 7.3, is that it allows us to rely on informal, intuitive arguments instead of formal proofs when reasoning within CoL-based applied theories.

Thesis 6.5. Consider sentences $E_1, \cdots, E_n, F$ ($n \geq 0$) and an admissible interpretation * for them. Assume there is a winning strategy for F^* that relies on availability and "recyclability"—in the strongest sense possible—of $E_1^*, \cdots, E_n^*$ as computational resources but no other knowledge or assumptions about * (see Example 7.10 for an instance of such a strategy). Then F is a logical consequence of $E_1, \cdots, E_n$.

7 Axiomatizations

While the semantical setup for CoL in the language of Section 6 is complete, a corresponding proof theory is still at earlier stages of development. Due to the inordinate expressive power of the language, successful axiomatization attempts have only been made for various fragments of CoL obtained by moderating its language in one way or another. It should be pointed out that every conceivable application of CoL will only need some fragment of CoL rather than the "whole" CoL anyway.

As of 2020 there are seventeen deductive systems for various fragments of CoL, named **CL1** through **CL17**. Based on their languages, these systems can be divided into three groups: **elementary-base**, **general-base** and **mixed-base**. Of the extralogical gameframe letters, the languages of elementary-base systems only allow elementary ones, the languages of general-base systems only allow general ones, and the languages of mixed-base systems allow both sorts of letters. Based on the style of the underlying proof theory, the systems can be further subdivided into two groups: **cirquent calculus systems** and **brute force systems**. Either sort is rather unusual, not seen elsewhere in proof theory. The cirquent calculus systems operate with cirquents rather than formulas, with formulas understood as special cases of cirquents. The brute force systems operate with formulas (sometimes referred to as sequents for technical reasons), but in an unusual way, with their inference rules being relatively directly derived from the underlying game semantics and hence somewhat resembling games themselves.

Theorem 7.1. *Each of the above-mentioned systems* $\mathbf{S} \in \{\mathbf{CL1}, \cdots, \mathbf{CL17}\}$ *is* **adequate** *in the sense that, for any sentence F of the language of* $\mathbf{S}$*, we have:*

(a) Soundness: *If F is provable in* $\mathbf{S}$*, then it is logically valid and, furthermore, a logical solution for F can be automatically extracted from a proof of F.*

(b) Completeness: *If F is logically valid, then it is provable in* $\mathbf{S}$*.*

In this article we shall take a look at only three of the systems: the general-base cirquent calculus system **CL15** in the logical signature $\{\neg, \wedge, \vee, ⫰, ⫯\}$, the mixed-base brute force system **CL13** in the signature $\{\neg, \wedge, \vee, \sqcap, \sqcup, \triangle, \triangledown, ⩚, ⩛\}$, and the elementary-base brute force system **CL12** in the signature $\{\neg, \wedge, \vee, \sqcap, \sqcup, \forall, \exists, \bigsqcap, \bigsqcup, \circ\!\!- \}$ (with $\circ\!\!-$ only allowed to be applied externally). Their adequacy proofs can respectively be found in [34, 35], [28] and [37].

7.1 The cirquent calculus system CL15

CL15-formulas—or just *formulas* in this subsection—are formulas of the language of CoL that do not contain any function letters, do not contain any gameframe letters other than 0-ary general gameframe letters, and do not contain any operators other than $\neg, \wedge, \vee, ⫰, ⫯$. Besides, $\neg$ is only allowed to be applied to atoms. Shall we still write $\neg E$ for a nonatomic E, it is to be understood as the standard *DeMorgan abbreviation* defined by $\neg\neg F = F$, $\neg(F \wedge G) = \neg F \vee \neg G$, $\neg(F \vee G) = \neg F \wedge \neg G$, $\neg ⫰ F = ⫯ \neg F$, $\neg ⫯ F = ⫰ \neg F$. Similarly, $F \to G$, $F \circ\!\!- G$ and $\circ\!\!\neg F$ should be understood as $\neg F \vee G$, $⫯ \neg F \vee G$ and $⫯ \neg F$, respectively.

Definition 7.2. A **CL15-cirquent** (henceforth simply "cirquent") is a triple $C = (\vec{F}, \vec{U}, \vec{O})$ where:

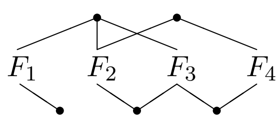
1. $\vec{F}$ is a nonempty finite sequence of **CL15**-formulas, whose elements are said to be the **oformulas** of C. Here the prefix "o" is for "occurrence", and is used to mean a formula together with a particular occurrence of it in $\vec{F}$. So, for instance, if $\vec{F} = \langle E, G, E \rangle$, then the cirquent has three oformulas even if only two formulas.

2. Both $\vec{U}$ and $\vec{O}$ are nonempty finite sequences of nonempty sets of oformulas of C. The elements of $\vec{U}$ are said to be the **undergroups** of C, and the elements of $\vec{O}$ are said to be the **overgroups** of C. As in the case of oformulas, it is possible that two undergroups or two overgroups are identical as sets (have identical **contents**), yet they count as different undergroups or overgroups because they occur at different places in the sequence $\vec{U}$ or $\vec{O}$. Simply "group" will be used as a common name for undergroups and overgroups.

3. Additionally, every oformula is required to be in at least one undergroup and at least one overgroup.

While oformulas are not the same as formulas, we may often identify an oformula with the corresponding formula and, for instance, say "the oformula E" if it is clear from the context which of the possibly many occurrences of E is meant. Similarly, we may not always be very careful about differentiating between groups and their contents.

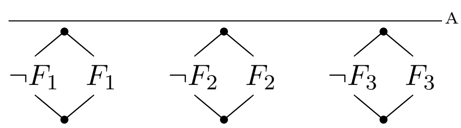
We represent cirquents using three-level diagrams such as the one shown below:



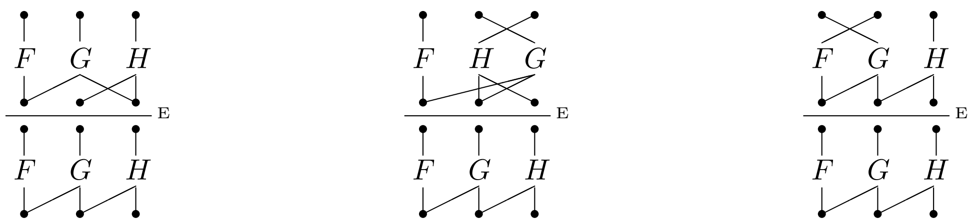
This diagram represents the cirquent with four oformulas F_1, F_2, F_3, F_4, three undergroups $\{F_1\}, \{F_2, F_3\}, \{F_3, F_4\}$ and two overgroups $\{F_1, F_2, F_3\}$, $\{F_2, F_4\}$. Each (under- or over-) group is represented by a $\bullet$, where the **arcs** (lines connecting the $\bullet$'s with oformulas) are pointing to the oformulas that the given group contains.

CL15 has ten rules of inference. The first one takes no premises, which qualifies it as an axiom. All other rules take a single premise. Below we explain them in a relaxed fashion, in terms of deleting arcs, swapping oformulas, etc. Such explanations are rather clear, and translating them into rigorous formulations in the style and terms of Definition 7.2, while possible, is hardly necessary.

Axiom (A): The conclusion of this premiseless rule looks like an array of n ($n \geq 1$) "diamonds" as seen below for the case of $n = 3$, where the oformulas within each diamond are $\neg F$ and F for some formula F.

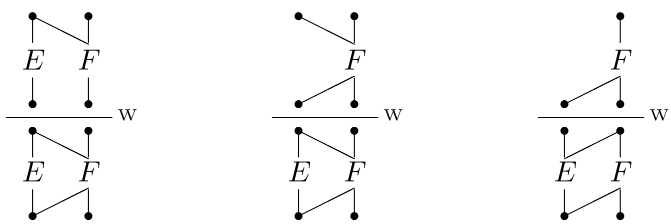


Exchange (E): This rule comes in three flavors: Undergroup Exchange, Oformula Exchange and Overgroup Exchange. Each one allows us to swap any two adjacent objects (undergroups, oformulas or overgroups) of a cirquent, otherwise preserving all oformulas, groups and arcs. Below we see three examples, one per each sort of Exchange. In all cases, of course, the upper cirquent is the premise and the lower cirquent is the conclusion of an application of the rule.

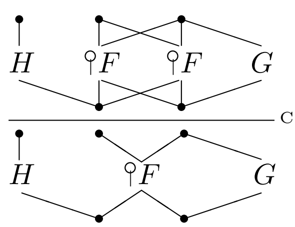


The presence of Exchange essentially allows us to treat all three components ($\vec{F}, \vec{U}$, $\vec{O}$) of a cirquent as multisets rather than sequences.

Weakening (W): The premise of this rule is obtained from the conclusion by deleting an arc between some undergroup U with ≥ 2 elements and some oformula F; if U was the only undergroup containing F, then F should also be deleted (to satisfy condition 3 of Definition 7.2), together with all arcs between F and overgroups; if such a deletion makes some overgroups empty, then they should also be deleted (to satisfy condition 2 of Definition 7.2). Below are three examples:

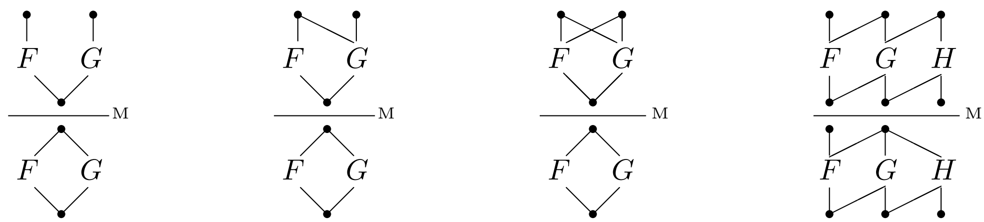


Contraction (C): The premise of this rule is obtained from the conclusion through replacing an oformula $⫯F$ by two adjacent oformulas $⫯F, ⫯F$, and including them in exactly the same undergroups and overgroups in which the original oformula was contained. Example:



Duplication (D): This rule comes in two versions: Undergroup Duplication and Overgroup Duplication. The conclusion of Undergroup Duplication is the result of replacing, in the premise, some undergroup U with two adjacent undergroups whose contents are identical to that of U. Similarly for Overgroup Duplication. Examples:

Merging (M): In the top-down view, this rule merges any two adjacent overgroups, as illustrated below.



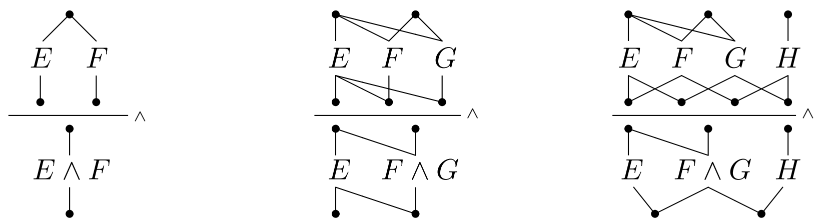
Disjunction Introduction (∨): The premise of this rule is obtained from the conclusion through replacing an oformula $F \vee G$ by two adjacent oformulas F, G, and including both of them in exactly the same undergroups and overgroups in which the original oformula was contained, as illustrated below:

Conjunction Introduction (∧): The premise of this rule is obtained from the conclusion by picking an arbitrary oformula $F \wedge G$ and applying the following two steps:

- Replace $F \wedge G$ by two adjacent oformulas F, G, and include both of them in exactly the same undergroups and overgroups in which the original oformula was contained.

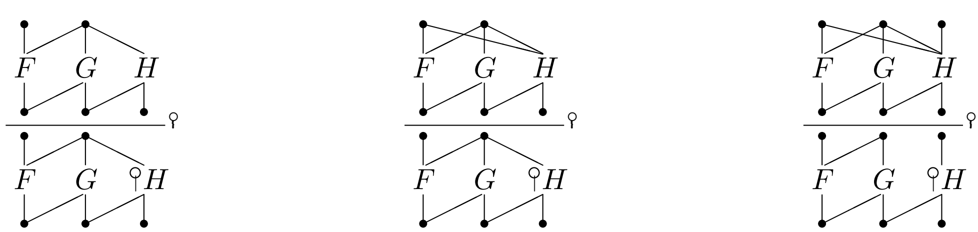
- Replace each undergroup U originally containing $F \wedge G$ (and now containing F, G instead) by the two adjacent undergroups $U - \{G\}$ and $U - \{F\}$.

Below we see three examples.



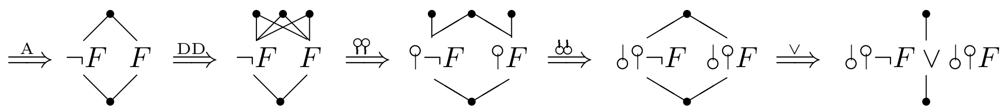
Recurrence Introduction (⫰): The premise of this rule is obtained from the conclusion through replacing an oformula ⫰F by F (while preserving all arcs), and inserting, anywhere in the cirquent, a new overgroup that contains F as its only oformula. Examples:

Corecurrence Introduction (⫯): The premise of this rule is obtained from the conclusion through replacing an oformula ⫯F by F, and including F in any (possibly zero) number of the already existing overgroups in addition to those in which the original oformula ⫯F was already present. Examples:



A **proof** (in **CL15**) of a cirquent C is a sequence of cirquents ending in C such that the first cirquent is the conclusion of (an instance of) Axiom, and every subsequent cirquent follows from the immediately preceding cirquent by one of the rules of **CL15**. A proof of a formula F is understood as a proof of the cirquent $(\langle F\rangle, \{F\}, \{F\})$.

As an example, below is a proof of the formula $⫯⫰F \rightarrow ⫰⫯F$, i.e., $⫰⫯\neg F \vee ⫰⫯F$. To save space, the cirquents in it have been arranged horizontally, separated with $\Longrightarrow$'s together with the symbolic names of the rules used; if such a name is duplicated as in DD, it means that the rule was applied twice rather than once.



Exercise 7.3. Prove the following formulas in **CL15**:

- $F \circ\!\!- F$ (i.e. $\circ\!\!\neg F \vee F$, i.e. $⫯\neg F \vee F$).
- $F \wedge F \rightarrow F$.
- $F \circ\!\!- ⫰F \wedge ⫰F$.
- $F \circ\!\!- ⫰⫰F$.
- $⫰E \vee ⫰F \rightarrow ⫰(E \vee F)$.
- $(E \wedge F) \vee (G \wedge H) \rightarrow (E \vee G) \wedge (F \vee H)$.

W. Xu and S. Liu [53] showed that **CL15** remains sound with ⅄, Y instead of ⫰, ⫯. Completeness, however, is lost in this case because, for instance, as shown in [33], the formula $F \wedge ⅄(F \rightarrow F \wedge F) \rightarrow ⅄F$ is logically valid while $F \wedge ⫰(F \rightarrow F \wedge F) \rightarrow ⫰F$ is not.

Open Problem 7.4.

1. Is (the problem of provability in) **CL15** decidable?
2. Extend the language of **CL15** by including $\sqcap, \sqcup$ and axiomatize (if possible) the set of logically valid formulas in this extended language.
3. Replace ⫰, ⫯ with ⅄, Y in the language of **CL15**. Is the set of logically valid formulas in this new language axiomatizable and, if yes, how?
4. Does **CL15** remain complete with respect to extralogical (as opposed to logical) validity?

7.2 The brute force system CL13

CL13-formulas—or just *formulas* in this subsection—are formulas of the language of CoL that do not contain any function letters or non-nullary gameframe letters, and do not contain any operators other than $\neg, \wedge, \vee, \sqcap, \sqcup$, △, ▽, ⩓, ⩔. As in the case of **CL15**, officially $\neg$ is only allowed to be applied to extralogical atoms, otherwise understood as the corresponding DeMorgan abbreviation, including understanding

$\neg\top$ as $\bot$ and $\neg\bot$ as $\top$. Each of the implication operators $\rightarrow, \sqsupset, \vartriangleright, \succ$ should also be understood as an abbreviation of its standard meaning in terms of negation and the corresponding sort of disjunction. To define the system axiomatically, we need certain terminological conventions.

- A **literal** means an atom A with or without negation $\neg$. Such a literal is said to be elementary or general iff A is so.
- As in Section 7.1, we often need to differentiate between *subformulas* as such, and particular *occurrences* of subformulas. We will be using the term **osubformula** to mean a subformula together with a particular occurrence. The prefix "o" will be used with a similar meaning in terms such as **oatom**, **oliteral**, etc.
- An osubformula is **positive** iff it is not in the scope of $\neg$. Otherwise it is **negative**. According to our conventions regarding the usage of $\neg$, only oatoms may be negative.
- A **politeral** is a positive oliteral.
- A $\wedge$-(sub)formula is a (sub)formula of the form $E \wedge F$. Similarly for the other connectives.
- A **sequential (sub)formula** is one of the form $E \bigtriangleup F$ or $E \bigtriangledown F$. We say that E is the **head** of such a (sub)formula, and F is its **tail**.
- Similarly, a **parallel (sub)formula** is one of the form $E \wedge F$ or $E \vee F$, a **choice (sub)formula** is one of the form $E \sqcap F$ or $E \sqcup F$, and a **toggling (sub)formula** is one of the form $E \curlywedge F$ or $E \curlyvee F$.
- A formula is said to be **quasielementary** iff it contains no general atoms and no operators other than $\neg, \wedge, \vee, \curlywedge, \curlyvee$.
- A formula is said to be **elementary** iff it is a formula of classical propositional logic, i.e., contains no general atoms and no operators other than $\neg, \wedge, \vee$.
- A **semisurface osubformula** (or occurrence) is an osubformula (or occurrence) which is not in the scope of a choice connective.
- A **surface osubformula** (or occurrence) is an osubformula (or occurrence) which is not in the scope of any connectives other than $\neg, \wedge, \vee$.
- The **quasielementarization** of a formula F, denoted by $|F|$, is the result of replacing in F every sequential osubformula by its head, every $\sqcap$-osubformula by $\top$, every $\sqcup$-osubformula by $\bot$, and every general politeral by $\bot$ (the order of these replacements does not matter). For instance, the quasielementarization of $\big((P \curlyvee q) \vee ((p \wedge \neg P) \bigtriangleup (Q \wedge R))\big) \curlywedge \big(q \sqcap (r \sqcup s)\big)$ is $\big((\bot \curlyvee q) \vee (p \wedge \bot)\big) \curlywedge \top$.

- The **elementarization** of a quasielementary formula F, denoted by $\|F\|$, is the result of replacing in F every $\wedge\!\!\!\wedge$-osubformula by $\top$ and every $\vee\!\!\!\vee$-osubformula by $\bot$ (again, the order of these replacements does not matter). For instance, $\|(s \wedge (p \wedge\!\!\!\wedge (q \vee\!\!\!\vee r))) \vee (\neg s \vee (p \vee\!\!\!\vee r))\| = (s \wedge \top) \vee (\neg s \vee \bot)$.
- A quasielementary formula F is said to be **stable** iff its elementarization $\|F\|$ is a tautology of classical logic.

We now define **CL13** by the following six rules of inference, where $\vec{H} \Longrightarrow F$ means "from premise(s) $\vec{H}$ conclude F". Axioms are not explicitly stated, but the set of premises of the ($\wedge\!\!\!\wedge$) rule can be empty, in which case (the conclusion of) this rule acts like an axiom.

Rule ($\wedge\!\!\!\wedge$): $\vec{H} \Longrightarrow F$, where F is a stable quasielementary formula, and $\vec{H}$ is the smallest set of formulas satisfying the following condition:

- Whenever F has a surface osubformula $E_0 \wedge\!\!\!\wedge E_1$, for both $i \in \{0, 1\}$, $\vec{H}$ contains the result of replacing in F that osubformula by E_i.

Rule ($\vee\!\!\!\vee$): $H \Longrightarrow F$, where F is a quasielementary formula, and H is the result of replacing in F a surface osubformula $E \vee\!\!\!\vee G$ by E or G.

Rule ($\triangle \sqcap$): $|F|, \vec{H} \Longrightarrow F$, where F is a non-quasielementary formula (note that otherwise $F = |F|$), and $\vec{H}$ is the smallest set of formulas satisfying the following two conditions:

- Whenever F has a semisurface osubformula $G_0 \sqcap G_1$, for both $i \in \{0, 1\}$, $\vec{H}$ contains the result of replacing in F that osubformula by G_i.
- Whenever F has a semisurface osubformula $E \triangle G$, $\vec{H}$ contains the result of replacing in F that osubformula by G.

Rule ($\sqcup$): $H \Longrightarrow F$, where H is the result of replacing in F a semisurface osubformula $E \sqcup G$ by E or G.

Rule ($\triangledown$): $H \Longrightarrow F$, where H is the result of replacing in F a semisurface osubformula $E \triangledown G$ by G.

Rule (M): $H \Longrightarrow F$, where H is the result of replacing in F two—one positive and one negative—semisurface occurrences of some general atom P by an extralogical elementary atom p that does not occur in F.

A **proof** (in **CL13**) of a formula F is a sequence of formulas ending in F such that every formula follows from some (possibly empty) set of earlier formulas by one of the rules of the system.

Example 7.5. Pick any two distinct connectives $\&_1$ and $\&_2$ from the list $\wedge$, $\Wedge$, $\triangle$, $\sqcap$. Then **CL13** proves the formula $P\&_1Q \to P\&_2Q$ if and only if $\&_1$ is to the left of $\&_2$ in the list. Similarly for the list $\sqcup$, $\bigtriangledown$, $\Vee$, $\vee$. Here we verify this fact only for the case $\{\&_1, \&_2\} = \{\Wedge, \triangle\}$. The reader may want to try some other combinations as exercises. Below is a proof of $P \Wedge Q \to P \triangle Q$ together with step justifications:

1. $\neg p \vee p$ From no premises by ($\Wedge$).
2. $(\neg p \Vee \bot) \vee p$ From 1 by ($\Vee$)
3. $\neg q \vee q$ From no premises by ($\Wedge$)
4. $(\neg p \Vee \neg q) \vee q$ From 3 by ($\Vee$)
5. $(\neg p \Vee \neg Q) \vee Q$ From 4 by (M)
6. $(\neg p \Vee \neg Q) \vee (p \triangle Q)$ From 2,5 by ($\triangle\sqcap$)
7. $(\neg P \Vee \neg Q) \vee (P \triangle Q)$ From 6 by (M)

On the other hand, the formula $P \triangle Q \to P \Wedge Q$, i.e. $(\neg P \bigtriangledown \neg Q) \vee (P \Wedge Q)$, has no proof in **CL13**. This can be shown through attempting and failing to construct, bottom-up, a purported proof of the formula. Here we explore one of the branches of a proof-search tree. $(\neg P \bigtriangledown \neg Q) \vee (P \Wedge Q)$ is not quasielementary, so it could not be derived by (be the conclusion of) the ($\Vee$) or ($\Wedge$) rule. The ($\sqcup$) rule does not apply either, as there is no $\sqcup$ in the formula. This leaves us with one of the rules ($\bigtriangledown$), ($\triangle\sqcap$) and (M). Let us see what happens if our target formula is derived by ($\bigtriangledown$). In this case the premise should be $\neg Q \vee (P \Wedge Q)$. The latter can be derived only by ($\triangle\sqcap$) or (M). Again, let us try (M). The premise in this case should be $\neg q \vee (P \Wedge q)$ for some elementary atom q. But the only way $\neg q \vee (P \Wedge q)$ can be derived is by ($\triangle\sqcap$) from the premise $\neg q \vee (\bot \Wedge q)$. This formula, in turn, could only be derived by ($\Wedge$), in which case $\neg q \vee \bot$ is one of the premises. Now we are obviously stuck, as $\neg q \vee \bot$ is not the conclusion of any of the rules of the system. We thus hit a dead end. All remaining possibilities can be checked in a similar routine/analytic way, and the outcome in each case will be a dead end.

Exercise 7.6.

1. Construct a proof of $(P \rhd P) \wedge (\neg P \rhd \neg P) \to P \sqsupset P$.

2. For which of the four disjunctions $\cup \in \{\vee, \sqcup, \bigtriangledown, \Vee\}$ are the following formulas provable and for which are not? (a) $\neg P \cup P$; (b) $P \cup Q \to Q \cup P$; (c) $P \cup P \to P$; (d) $p \cup p \to p$.

Open Problem 7.7.

1. Consider the first-order version of the language of **CL13** with choice (to start with) quantifiers. Adequately axiomatize the set of logically valid formulas in this language.

2. Consider the set of the theorems of **CL13** that do not contain extralogical elementary letters. Does this set remain complete with respect to extralogical validity?

7.3 The brute force system CL12

CL12-formulas—or just *formulas* in this subsection—are formulas of the language of CoL that do not contain any general gameframe letters, and do not contain any operators other than $\neg, \wedge, \vee, \sqcap, \sqcup, \bigsqcap, \bigsqcup, \forall, \exists$. As in the preceding two sections, $\neg$ applied to formulas other than extralogical atoms is understood as the corresponding DeMorgan abbreviation, $E \to F$ is understood as an abbreviation of $\neg E \vee F$, and $E \sqsupset F$ as an abbreviation of $\neg E \sqcup F$.

CL12-sequents—or just *sequents* in this subsection—are expressions of the form $E_1, \cdots, E_n \circ\!\!- F$, where $E_1, \cdots, E_n$ $(n \geq 0)$ and F are **CL12**-formulas; for simplicity and safety, we require that no variable has both free and bound occurrences in the (not necessarily the same) formulas of the sequent. The sequence $E_1, \cdots, E_n$ is said to be the *antecedent* of the sequent, and F its *succedent*. Semantically, such a sequent is identified with the (non-**CL12**) formula $E_1 \wedge \cdots \wedge E_n \circ\!\!- F$. So for instance, when we say that the former is logically valid, we mean that so is the latter, and a logical solution of the former means a logical solution of the latter. Each **CL12**-formula F, in turn, can be identified with the empty-antecedent sequent $\circ\!\!- F$. A **CL12**-sequent is **closed** iff so is its succedent as well as all formulas of the antecedent. When applied to **CL12**, the word "sentence" in Theorem 7.1 should be interpreted as "closed **CL12**-sequent" rather than (merely) "closed **CL12**-formula".

Note that the language of **CL12** is an extension of the full language of classical first-order logic. Due to this fact, together with the presence of $\sqcap, \sqcup, \bigsqcap, \bigsqcup, \circ\!\!-$ in the language, **CL12** is a very powerful tool for constructing CoL-based applied theories (see Section 8), and has been repeatedly [27, 30, 36, 37, 38, 39, 40, 41] used as such with significant advantages over the less expressive and computationally less meaningful classical logic. Below is some terminology employed in our axiomatization of **CL12**.

- A **surface occurrence** of a subformula is an occurrence that is not in the scope of any choice operators.
- A formula not containing choice operators—i.e., a formula of the language of classical first order logic—is said to be **elementary**. A sequent is **elementary** iff all of its formulas are so. The **elementarization** $\|F\|$ of a formula F is the result of replacing in F all surface occurrences of $\sqcup$- and $\bigsqcup$-subformulas by $\bot$, and all surface occurrences of $\sqcap$- and $\bigsqcap$-subformulas by $\top$. Note that $\|F\|$ is

(indeed) an elementary formula. The **elementarization** $\|E_1, \cdots, E_n \circ\!\!- F\|$ of a sequent $E_1, \cdots, E_n \circ\!\!- F$ is the elementary formula $\|E_1\| \wedge \cdots \wedge \|E_n\| \rightarrow \|F\|$.

- A sequent is said to be **stable** iff its elementarization is classically valid (i.e., provable in some standard version of classical first-order calculus with constants, function letters and =).
- We will be using the notation $F[E]$ to mean a formula F together with some fixed surface occurrence of a subformula E. Using this notation sets a context, in which $F[H]$ will mean the result of replacing in $F[E]$ that occurrence of E by H.
- $\vec{G}, \vec{K}, \vec{L}, \vec{M}$, stand for finite sequences of formulas.

We now define **CL12** by the following six rules of inference, where $S_1, \cdots, S_m \Longrightarrow S$ means "from premise(s) $S_1, \cdots, S_n$ conclude S". Axioms are not explicitly stated, but the set of premises of the Wait rule can be empty, in which case (the conclusion of) this rule acts like an axiom. In each rule, i is assumed to be either 0 or 1, t is either a constant or a variable with no bound occurrences in the premise, and y is a variable not occurring in the conclusion; $H(t)$ (resp. $H(y)$) is the result of replacing in the formula $H(x)$ all free occurrences of the variable x by t (resp. by y).

$\sqcup$-Choose: $\vec{G} \circ\!\!- F[H_i] \Longrightarrow \vec{G} \circ\!\!- F[H_0 \sqcup H_1]$, for either i.

$\sqcap$-Choose: $\vec{G}, E[H_i] \circ\!\!- F \Longrightarrow \vec{G}, E[H_0 \sqcap H_1] \circ\!\!- F$, for either i.

$\bigsqcup$-Choose: $\vec{G} \circ\!\!- F[H(t)] \Longrightarrow \vec{G} \circ\!\!- F[\bigsqcup x H(x)]$, for any t.

$\bigsqcap$-Choose: $\vec{G}, E[H(t)], \vec{K} \circ\!\!- F \Longrightarrow \vec{G}, E[\bigsqcap x H(x)], \vec{K} \circ\!\!- F$, for any t.

Replicate: $\vec{G}, E, \vec{K}, E \circ\!\!- F \Longrightarrow \vec{G}, E, \vec{K} \circ\!\!- F$.

Wait: $S_1, \cdots, S_n \Longrightarrow S$ ($n \geq 0$), where S is stable and the following four conditions are satisfied:

- Whenever S has the form $\vec{K} \circ\!\!- E[H_0 \sqcap H_1]$, both sequents $\vec{K} \circ\!\!- E[H_0]$ and $\vec{K} \circ\!\!- E[H_1]$ are among $S_1, \cdots, S_n$.
- Whenever S has the form $\vec{L}, J[H_0 \sqcup H_1], \vec{M} \circ\!\!- E$, both sequents $\vec{L}, J[H_0], \vec{M} \circ\!\!- E$ and $\vec{L}, J[H_1], \vec{M} \circ\!\!- E$ are among $S_1, \cdots, S_n$.
- Whenever S has the form $\vec{K} \circ\!\!- E[\bigsqcap x H(x)]$, for some y, the sequent $\vec{K} \circ\!\!- E[H(y)]$ is among $S_1, \cdots, S_n$.

- Whenever S has the form $\vec{L}, J[\sqcup xH(x)], \vec{M} \circ\!- E$, for some y, the sequent $\vec{L}, J[H(y)], \vec{M} \circ\!- E$ is among $S_1, \cdots, S_n$.

Each rule—seen bottom-up—encodes an action that a winning strategy should take in a corresponding situation, and the name of each rule is suggestive of that action. For instance, Wait (indeed) prescribes the strategy to wait till the adversary moves. This explains why we use the name "Replicate" for one of the rules rather than the more standard "Contraction".

A **proof** (in **CL12**) of a sequent S is a sequence $S_1, \cdots, S_n$ of sequents, with $S_n = S$, such that each S_i follows by one of the rules of **CL12** from some (possibly empty in the case of Wait, and certainly empty in the case of $i = 1$) set $\vec{P}$ of premises such that $\vec{P} \subseteq \{S_1, \cdots, S_{i-1}\}$. A proof of a formula F is understood as a proof of the empty-antecedent sequent $\circ\!- F$.

Example 7.8. Here $\times$ is a binary function letter and ^{3} is a unary function letter. We write $x \times y$ and x^3 instead of $\times(x, y)$ and $^3(x)$. The following sequence is a proof of its last sequent.

1. $\forall x(x^3 = (x \times x) \times x), t = s \times s, r = t \times s \circ\!- r = s^3$
(by Wait from no premises)
2. $\forall x(x^3 = (x \times x) \times x), t = s \times s, r = t \times s \circ\!- \sqcup y(y = s^3)$
(by $\sqcup$-Choose from 1)
3. $\forall x(x^3 = (x \times x) \times x), t = s \times s, \sqcup z(z = t \times s) \circ\!- \sqcup y(y = s^3)$
(by Wait from 2)
4. $\forall x(x^3 = (x \times x) \times x), t = s \times s, \sqcap y \sqcup z(z = t \times y) \circ\!- \sqcup y(y = s^3)$
(by $\sqcap$-Choose from 3)
5. $\forall x(x^3 = (x \times x) \times x), t = s \times s, \sqcap x \sqcap y \sqcup z(z = x \times y) \circ\!- \sqcup y(y = s^3)$
(by $\sqcap$-Choose from 4)
6. $\forall x(x^3 = (x \times x) \times x), \sqcup z(z = s \times s), \sqcap x \sqcap y \sqcup z(z = x \times y) \circ\!- \sqcup y(y = s^3)$
(by Wait from 5)
7. $\forall x(x^3 = (x \times x) \times x), \sqcap y \sqcup z(z = s \times y), \sqcap x \sqcap y \sqcup z(z = x \times y) \circ\!- \sqcup y(y = s^3)$
(by $\sqcap$-Choose from 6)
8. $\forall x(x^3 = (x \times x) \times x),\ \sqcap x \sqcap y \sqcup z(z = x \times y),\ \sqcap x \sqcap y \sqcup z(z = x \times y) \circ\!- \sqcup y(y = s^3)$
(by $\sqcap$-Choose from 7)
9. $\forall x(x^3 = (x \times x) \times x), \sqcap x \sqcap y \sqcup z(z = x \times y) \circ\!- \sqcup y(y = s^3)$
(by Replicate from 8)

10. $\forall x(x^3 = (x \times x) \times x), \sqcap x \sqcap y \sqcup z(z = x \times y) \circ\!- \sqcap x \sqcup y(y = x^3)$
(by Wait from 9)

Exercise 7.9. To see the resource-consciousness of **CL12**, show that it does not prove $p \sqcap q \to (p \sqcap q) \wedge (p \sqcap q)$, even though this formula has the form $F \to F \wedge F$ of a classical tautology. Then show that, in contrast, **CL12** proves the sequent $p \sqcap q \circ\!- (p \sqcap q) \wedge (p \sqcap q)$ because, unlike the antecedent of a pimplication, the antecedent of a brimplication is reusable (trough Replicate).

For any closed sequent $E_1, \cdots, E_n \circ\!- F$, following holds due to the adequacy Theorem 7.1:

$$\textbf{CL12} \textit{ proves } E_1, \cdots, E_n \circ\!- F \textit{ if and only if} \\ F \textit{ is a logical consequence of } E_1, \cdots, E_n. \tag{4}$$

This explains why we call the following rule of inference **Logical Consequence**:

$$E_1, \cdots, E_n \implies F, \textit{ where } \textbf{CL12} \textit{ proves the sequent } E_1, \cdots, E_n \circ\!- F.$$

Logical Consequence is the only logical rule of inference in all **CL12**-based applied theories briefly discussed in Section 8. To appreciate the convenience that Thesis 6.5 offers when reasoning in such theories, let us look at the following example.

Example 7.10. Imagine a **CL12**-based applied formal theory, in which we have proven or postulated $\forall x(x^3 = (x \times x) \times x)$ (the meaning of "cube" in terms of multiplication) and $\sqcap x \sqcap y \sqcup z(z = x \times y)$ (the computability of multiplication), and now we want to derive $\sqcap x \sqcup y(y = x^3)$ (the computability of "cube"). This is how we can reason to justify $\sqcap x \sqcup y(y = x^3)$:

> *Consider any s (selected by the environment for x in $\sqcap x \sqcup y(y = x^3)$). We need to find s^3. Using the resource $\sqcap x \sqcap y \sqcup z(z = x \times y)$ twice, we first find the value t of $s \times s$, and then the value r of $t \times s$. According to $\forall x(x^3 = (x \times x) \times x)$, such an r is the sought s^3.*

Thesis 6.5, in view of (4), promises that the above intuitive argument will be translatable into a proof of

$$\forall x(x^3 = (x \times x) \times x), \sqcap x \sqcap y \sqcup z(z = x \times y) \circ\!- \sqcap x \sqcup y(y = x^3)$$

in **CL12**, and hence the succedent $\sqcap x \sqcup y(y = x^3)$ will be derivable in the theory by Logical Consequence as the formulas of the antecedent are already proven. Such a proof indeed exists—see Example 7.8.

Open Problem 7.11.

1. Add the sequential connectives $\triangle$, $\triangledown$ to the language of **CL12** and adequately axiomatize the corresponding logic.
2. Axiomatize (if possible) the set of extralogically valid **CL12**-sequents.
3. Along with elementary gameframe letters, allow also general gameframe letters in the language of **CL12**, and axiomatize (if possible) the set of valid sequents of this extended language.

8 Applied systems based on computability logic

The main utility of CoL, actual or potential, is related to the benefits of using it as a logical basis for applied systems, such as axiomatic theories or knowledgebase systems.

The most common logical basis for applied systems is classical first-order logic (CFOL). This is due to the fact that CFOL is universal: its language allows one to say anything one could say, and its proof system allows one to justify anything one could justify logically. But when it comes to expressing *tasks* (as opposed to *facts*) and reasoning about them, such as the task/problem expressed by $\sqcap x \sqcup y(y = x^2)$, using CFOL can be an extremely circuitous and awkward way. Asking why we need CoL if everything can be done with CFOL is akin to asking, for instance, why we study modal logics if anything one can express or reason about in modal logic can just as well be expressed or reasoned about using CFOL.

For specificity, let us imagine what a typical applied system $\mathbb{S}$ based on the already axiomatized fragment **CL12** of CoL would look like. The construction of such a system would start from building its extralogical basis $\mathbb{B}$, with some fixed interpretation * in mind. In what follows, for readability, we omit explicit references to this * and, terminologically, identify each sentence E with the game E^*. Depending on the context or traditions, $\mathbb{B}$ would generally be referred to as the **knowledgebase**, or the set of **axioms**, of $\mathbb{S}$. It would be a collection of relevant (to the purposes of the system) sentences expressing computational problems with already known, fixed solutions. Those can be atomic elementary sentences expressing true facts such as $Jane = MotherOf(Bob)$ or $0 \neq 1$; nonatomic elementary sentences expressing general or conceptual knowledge such as $\forall x \forall y(y = MotherOf(x) \to Female(y))$ or $\forall x(x^2 = x \times x)$; nonatomic nonelementary sentences such as $\sqcap x \sqcup y(y = DateOfBirth(x))$ expressing the ability to tell any person's date of birth (perhaps due to having access to an external database), $\sqcap x \sqcap y \sqcup z(z = x \times y)$ expressing the ability to compute multiplication, or

$$\sqcap x \sqcap y(\neg Halts(x, y) \sqcup Halts(x, y)) \to \sqcap x \sqcap y(\neg Accepts(x, y) \sqcup Accepts(x, y))$$

expressing the ability to reduce the acceptance problem to the halting problem. The only logical rule of inference in $\mathbb{S}$ would be Logical Consequence defined in Section 7.3.

A **proof** $\mathcal{P}$ of a sentence F in such a system $\mathbb{S}$ will be defined in a standard way, with the elements of $\mathbb{B}$ acting as axioms. The rule of Logical Consequence preserves computability in the sense that, as long as all sentences of the antecedent of a **CL12**-sequent are computable under a given interpretation, so is the succedent and, furthermore, a solution of the latter can be extracted from solutions of the sentences of the antecedent. Since solutions of all axioms are already known, $\mathcal{P}$ thus automatically translates into a solution $\mathcal{H}$ of F. Think of $\mathbb{S}$ as a declarative programming language, $\mathcal{P}$ as a program written in that language, the sentence F as a specification of (the goal of) such a program, the mechanism extracting solutions from proofs as a compiler, and the above $\mathcal{H}$ as a machine-language-level translation of the high-level program $\mathcal{P}$. Note that the notoriously hard problem of program verification is fully neutralized in this paradigm: being a proof of the sentence F, $\mathcal{P}$ automatically also serves as a formal verification of the fact that the program $\mathcal{P}$ meets its specification F. Further, $\mathcal{P}$ is a program commented in an extreme sense, with every line/sentence in this program being its own, best possible, comment.

A relevant question, of course, is how efficient the above solution $\mathcal{H}$ would be in terms of computational complexity. Here come more pieces of positive news. The traditional complexity-theoretic concepts such as time or space complexities find in CoL natural and conservative generalizations from the traditional sorts of problems to all games (cf. [37]). The Logical Consequence rule is complexity-theoretically well behaved, with the time (resp. space) complexity of its conclusion guaranteed to be at most linearly (resp. logarithmically) different from the time (resp. space) complexities of the solutions of the premises. So, how efficient the solutions extracted from proofs in $\mathbb{S}$ are, is eventually determined by how efficient the solutions of the axioms comprising $\mathbb{B}$ are. If, for instance, all axioms have linear time and/or logarithmic space solutions, then so do all theorems of $\mathbb{S}$ as well.

In some cases, the extralogical postulates of $\mathbb{S}$ would consist of not only the axioms $\mathbb{B}$, but also certain extralogical rules of inference such as some versions of induction or comprehension. Depending on the version, such a rule may less closely preserve computational complexity than Logical Consequence does. For instance, each application of induction may increase the time complexity quadratically rather than linearly. By limiting the number of such applications or imposing certain other restrictions, we can still get a system all of whose theorems are problems with low-order polynomial time complexities as long as all axioms are so.

By now CoL has found applications in a series **CLA1-CLA11** of formal number theories dubbed "**clarithmetics**". All of these theories are based on **CL12**, and differ between each other only in their extralogical postulates. The language of each theory has the same extralogical vocabulary $\{0, +, \times, '\}$ (where x' means "the successor of x") as the language of first-order Peano arithmetic. Unlike other approaches with similar aspirations such as that of bounded arithmetic [7], this approach avoids a need for adding more extralogical primitives to the language as, due to extending rather than restricting traditional Peano arithmetic, all arithmetical functions or predicates remain expressible in standard ways. The extralogical postulates of clarithmetical systems are also remarkably simple, with their sets of axioms consisting of all axioms of Peano arithmetic plus the single nonelementary sentence $\sqcap x \sqcup y(y = x')$ or just a few similarly innocuous-looking axioms, and the set of extralogical inference rules consisting of induction and perhaps one more rule such as comprehension. Different clarithmetics serve different computational complexity classes, which explains their multiplicity. Each system has been proven to be sound and complete with respect to its target complexity class C. Sound in the sense that every theorem T of the system expresses an arithmetical problem A with a C complexity solution and, furthermore, such a solution can be automatically obtained from the proof of T. And complete in the sense that every arithmetical problem A with a C complexity solution is expressed by some theorem T of the system. Furthermore, if one adds all true sentences of Peano arithmetic to the set of axioms, then this *extensional* completeness result strengthens to *intensional* completeness, according to which every (rather than just some) sentence F expressing such an A is a theorem of the system.

Among **CLA1-CLA11**, the system **CLA11** stands out in that it is a scheme of clarithmetical theories rather than a particular theory, taking three parameters $\mathfrak{a}, \mathfrak{s}, \mathfrak{t}$ and correspondingly written as $\mathbf{CLA11}(\mathfrak{a}, \mathfrak{s}, \mathfrak{t})$. These parameters range over sets of terms or pseudoterms used as bounds for $\sqcap, \sqcup$ in certain postulates. $\mathfrak{t}$ determines the time complexity of all theorems of the system, $\mathfrak{s}$ determines space complexity and $\mathfrak{a}$ the so called *amplitude complexity* (the complexity measure concerned with the sizes of $\top$'s moves relative to the sizes of $\bot$'s moves). By tuning these three parameters in an essentially mechanical, brute force fashion, one immediately gets a system sound and complete with respect to one or another combination of time, space and amplitude complexities. For instance, for *Linear amplitude + Logarithmic space + Polynomial time*, it is sufficient to choose $\mathfrak{a}$ to be the canonical set of terms expressing all linear functions (i.e. terms built from variables, 0, + and $'$), $\mathfrak{t}$ the canonical set of terms for all polynomial functions (namely, terms built from variables, 0, +, $\times$ and $'$), and $\mathfrak{s}$ the set of canonical pseudoterms for all logarithmic functions. This way one can obtain a system for essentially all natural (whatever this means) combinations of time, space and amplitude complexities. See the introductory section of [39] for a more detailed account.

References

[1] Samson Abramsky, Radha Jagadeesan *Games and full completeness of multiplicative linear logic*, The Journal of Symbolic Logic 59 (2) 1994, pp. 543-574.

[2] Arnon Avron *A constructive analysis of RM*, The Journal of Symbolic Logic 52 (4) 1987, pp. 939-951.

[3] Matthew Bauer *A PSPACE-complete first order fragment of computability logic*, ACM Transactions on Computational Logic 15 (1:1), 2014, pp. 1-12.

[4] Matthew Bauer *The computational complexity of propositional cirquent calculus*, Logical Methods is Computer Science 11 (1:12), 2015, pp. 1-16.

[5] Andreas Blass *Degrees of indeterminacy of games*, Fundamenta Mathematicae 77 (2), 1972, pp. 151-166.

[6] Andreas Blass *A game semantics for linear logic*, Annals of Pure and Applied Logic 56, (1-3) 1992, pp. 183-220.

[7] Samuel Buss Bounded Arithmetic, Bibliopolis, Naples 1986. (Revised version of PhD thesis.)

[8] Alonzo Church *An unsolvable problem of elementary number theory*, American Journal of Mathematics 58 (2), 1936, pp. 345-363.

[9] Walter Felscher *Dialogues, strategies and intuitionistic provability*, Annals of Pure and Applied Logic 28 (3), 1985, pp. 217-254.

[10] Jean-Yves Girard *Linear logic*, Theoretical Computer Science 50 (1), 1987, pp. 1-102.

[11] Dina Goldin, Scott Smolka, and Peter Wegner (eds.), Interactive Computation: The New Paradigm, Springer, Berlin 2006.

[12] Alessio Guglielmi *A system of interaction and structure*, ACM Transactions on Computational Logic 8 (1), 2007, pp. 1-64.

[13] Jaakko Hintikka Logic, Language-Games and Information: Kantian Themes in the Philosophy of Logic, Clarendon Press, Oxford 1973.

[14] Giorgi Japaridze *Introduction to computability logic*, Annals of Pure and Applied Logic 123 (1-3), 2003, pp. 1-99.

[15] – *Propositional computability logic I*, ACM Transactions on Computational Logic 7 (2), 2006, pp. 302-330.

[16] – *Propositional computability logic II*, ACM Transactions on Computational Logic 7 (2), 2006, pp. 331-362.

[17] – *Introduction to cirquent calculus and abstract resource semantics*, Journal of Logic and Computation 16 (4), 2006, pp. 489-532.

[18] – *Computability logic: a formal theory of interaction*, in: Dina Goldin, Scott Smolka, and Peter Wegner (eds.), Interactive Computation: The New Paradigm, Springer, Berlin 2006, pp. 183-223.

[19] – *From truth to computability I*, Theoretical Computer Science 357 (1-3), 2006, pp. 100-135.

[20] – *From truth to computability II*, Theoretical Computer Science 379 (1-2), 2007, pp.

20-52.

[21] – *The logic of interactive Turing reduction*, The Journal of Symbolic Logic 72 (1), 2007, pp. 243-276.

[22] – *The intuitionistic fragment of computability logic at the propositional level*, Annals of Pure and Applied Logic 147 (3), 2007, pp. 187-227.

[23] – *Intuitionistic computability logic*, Acta Cybernetica, 18 (1), 2007, pp. 77-113.

[24] – *Cirquent calculus deepened*, Journal of Logic and Computation 18 (6), 2008, pp. 983-1028.

[25] – *Sequential operators in computability logic*, Information and Computation 206 (12), 2008, pp. 1443-1475.

[26] – *In the beginning was game semantics*, in: Ondrej Majer, Ahti-Veikko Pietarinen, and Tero Tulenheimo (eds.), Games: Unifying Logic, Language, and Philosophy, Springer, Berlin 2009, pp. 249-350.

[27] – *Towards applied theories based on computability logic*, The Journal of Symbolic Logic 75 (2), 2010, pp. 565-601.

[28] – *Toggling operators in computability logic*, Theoretical Computer Science 412 (11), 2011, pp. 971-1004.

[29] – *From formulas to cirquents in computability logic*, Logical Methods is Computer Science 7 (2:1), 2011, pp. 1-55.

[30] – *Introduction to clarithmetic I*, Information and Computation 209 (10), 2011, pp. 1312-1354.

[31] – *A logical basis for constructive systems*, Journal of Logic and Computation 22 (3), 2012, pp. 605-642.

[32] – *A new face of the branching recurrence of computability logic*, Applied Mathematics Letters 25 (11), 2012, pp. 1585-1589.

[33] – *Separating the basic logics of the basic recurrences*, Annals of Pure and Applied Logic 163 (3), 2012, pp. 377-389.

[34] – *The taming of recurrences in computability logic through cirquent calculus, Part I*, Archive for Mathematical Logic 52 (1-2), 2013, pp. 173-212.

[35] – *The taming of recurrences in computability logic through cirquent calculus, Part II*, Archive for Mathematical Logic 52 (1-2), 2013, pp. 213-259.

[36] – *Introduction to clarithmetic III*, Annals of Pure and Applied Logic 165 (1), 2014, pp. 241-252.

[37] – *On the system CL12 of computability logic*, Logical Methods is Computer Science 11 (3:1), 2015, pp. 1-71.

[38] – *Introduction to clarithmetic II*, Information and Computation 247, [April] 2016, pp. 290-312.

[39] – *Build your own clarithmetic I: Setup and completeness*, Logical Methods in Computer Science 12 (3:8), 2016, pp. 1-59.

[40] – *Build your own clarithmetic II: Soundness*, Logical Methods in Computer Science 12

(3:12), 2016, pp. 1-62.

[41] – *Elementary-base cirquent calculus I: Parallel and choice connectives*, Journal of Applied Logics - IfCoLoG Journal of Logics and their Applications 5 (1), 2018, pp. 367-388.

[42] – *Computability logic: Giving Caesar what belongs to Caesar*, Logical Investigations 25 (1), 2019, pp. 100-119.

[43] – *Elementary-base cirquent calculus II: Choice quantifiers*, Logic Journal of the IGPL, jzaa022, 2020, https://doi.org/10.1093/jigpal/jzaa022

[44] – *Computability Logic Homepage*. An Online Survey of Computability Logic, 2020. [www.csc.villanova.edu/∼japaridz/CL/]

[45] Keehang Kwon *Expressing algorithms as concise as possible via computability logic*, IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences, E97-A 2014, pp. 1385-1387.

[46] Paul Lorenzen *Ein dialogisches Konstruktivitätskriterium*, in: Infinitistic Methods (Proc. Sympos. Foundations of Math., Warsaw 1959), Pergamon, Oxford 1961, pp. 193-200.

[47] Ilya Mezhirov, Nikolay Vereshchagin *On abstract resource semantics and computability logic*, Journal of Computer and Systems Sciences 76 (5), 2010, pp. 356-372.

[48] Meixia Qu, Junfeng Luan, Daming Zhu, Meng Du *On the toggling-branching recurrence of computability logic*, Journal of Computer Science and Technology 28 (2), 2013, pp. 278-284.

[49] Mariya Soskova *The theory of the enumeration degrees, definability, and automorphisms*, in: Melvin Fitting, Massoud Pourmahdian, Adrian Rezus, and Ali Sadegh Daghighi (eds.), Research Trends in Contemporary Logic (to appear).

[50] Alan Turing *On Computable numbers with an application to the entsheidungsproblem*, Proceedings of the London Mathematical Society 2:42 (3), 1936, pp. 230-265.

[51] Wenyan Xu, Sanyang Liu *Soundness and completeness of the cirquent calculus system CL6 for computability logic*, Logic Journal of the IGPL 20 (1), 2012, pp. 317-330.

[52] – *The countable versus uncountable branching recurrences in computability logic*, Journal of Applied Logic 10 (4), 2012, pp. 431-446.

[53] – *The parallel versus branching recurrences in computability logic*, Notre Dame Journal of Formal Logic 54 (1), 2013, pp. 61-78.

[54] Wenyan Xu *A propositional system induced by Japaridze's approach to IF logic*, Logic Journal of the IGPL 22 (6), 2014, pp. 982-991.

[55] – *A cirquent calculus system with clustering and ranking*, Journal of Applied Logic 16 (July) 2016, pp. 37-49.

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Relative Necessity Extended

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Abstract

When we talk about possibility and necessity we might mean any of a wide range of different kinds of modality, for example, logical modality, physical modality, or epistemic modality, to name just a few. One natural way to understand the relation between different kinds of possibility and necessity is that some are relativizations of others. Hale and Leech (2017) [4] propose a new formalization of relative necessities, but that proposal is restricted to *alethic, non-epistemic* necessities. In this paper I explore the prospects for extending the account to non-alethic and epistemic necessities.

1 Introduction

When we talk about possibility and necessity – when we say that, for example, something *must be* the case, or something *could have been so* – we might mean any of a wide range of different kinds of modality, for example, logical modality, physical or natural modality, or epistemic modality, to name just a few. One natural way to understand the relation between different kinds of possibility and necessity is that some are relativizations of others. For example, one might think that whatever is physically necessary is also what follows as a matter of logical necessity from the laws of physics. In Hale and Leech (2017) [4] we made a proposal (detailed below) for how to formalize relative necessities, taking logical necessity to be our base necessity, and other necessities as relative necessities. But that proposal was offered with a clear restriction to *alethic, non-epistemic* necessities, that is, necessities which are factive (such that: if it is necessary that p, then p) and not epistemic (not defined in epistemic terms, in relation to knowledge or evidence etc.). There were good reasons for such a restriction for, as we shall see, extending such an account beyond the alethic raises significant problems. However, as I shall argue, there are also reasons for wanting to treat all necessities – alethic or not, epistemic or not – in the same way. Hence, in

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this paper I explore the prospects for such an extended relative necessity account; one might also describe it as an attempt to offer a *unifying* account of different kinds of necessity.[1]

I proceed as follows. First, I introduce the account of relative necessity offered in Hale and Leech (2017) [4]. I then present what I call "the similarity argument", motivating a uniform treatment of a wider range of modalities. I then introduce a major problem for this project: a set of difficulties that arise from non-alethic modalities such as epistemic and doxastic (belief-based) modalities, and modalities that appear to tolerate contradiction, such as legal modalities. I will review leading treatments of such cases developed by Kratzer [8, 9, 10, 11], before introducing the main proposal of this paper: a theory of relative necessity that draws on relevant logic. I will indicate some potential advantages that might be gained by exploring this approach.

2 Relative Necessity Reformulated

The basic starting idea of a relative necessity in general is that it is relatively necessary that p, in some specific sense, just when p follows logically (as a matter of logical necessity) from some specific set or conjunction of propositions – call them Φ-propositions. So, for example, it is physically necessary that p just when p follows logically from laws of physics, or it is London-necessary that p just when p follows logically from true propositions about London. (Note: there is no reason, on the face of it, why relative necessities should be the familiar and non-arbitrary kinds of necessities that we usually list, such as physical necessity.)[2] In the past, it was thought that this idea could be captured formally simply by a necessitated (material) conditional,[3] or equivalently a strict conditional, such as

$$\Box(\phi \supset p)$$

where, for example, 'ϕ' stands for a conjunction of the laws of physics. However, such an approach leads to trouble.

I shall briefly review one of the key problems, as it will become relevant later: If $\Box$ in this formulation satisfies the S4 axiom, which is plausible for logical necessity,[4] then it follows that all relative necessities thus defined *also* satisfy the S4 axiom. Put briefly, the S4 axiom for a relative necessity thus defined

$$\Box(\phi \supset p) \supset \Box(\phi \supset \Box(\phi \supset p))$$

[1] This is my attempt to extend the account of [4]. I can't say whether or not Bob would endorse the following. In his other work, he endorsed a negative free logic (e.g., [5, ch.9]), which I think means he would not be in favour of using a relevant logic. I hope at least he would find it interesting.

[2] See [12]

[3] I shall use '$\supset$' throughout to signify the material conditional.

[4] E.g., see [18, p.76] for an argument that the logic of the necessity of the classical consequence relation is S5.

is itself a theorem of S4 for $\Box$. But one might think that we should be able to capture the idea that some necessities may be both relative and have a logic weaker than S4. Humberstone [7] draws out a model-theoretic diagnosis of the problem: if we interpret relative modality in terms of a range restriction of the accessibility relation over possible worlds (say to all worlds with the same laws of physics), then in crucial cases we do not thereby change the properties of the accessibility relation. In particular, if it is transitive, then it remains transitive, and so S4 will hold for whatever modality is defined, whatever the range restriction. Humberstone offers a proposed solution in terms of two-dimensional semantics [7, 6]. In [4] Hale and I raise some concerns about this approach, and propose the following alternative.

The initial informal gloss on relative necessity made an important assumption, resulting in the omission of crucial information. The formula '$\Box(\phi \supset p)$' says nothing about what ϕ is, but that is a crucial part of relative necessity: not just that p follows logically from *something,* but that it follows logically from propositions of a certain kind. We proposed that this information be put back into the formalization, in the form of the explicit assumption that *there are propositions of a certain kind* from which p follows. Hence we have, in general:

RN It is Φ-necessary that p iff $\exists q(\Phi(q) \land \Box(q \supset p))$[5]

Where "$\Phi(q)$" should be understood as "it is a Φ-proposition that q" (q may be a conjunction). To give a specific example:

RNp It is physically necessary that p iff $\exists q(\pi(q) \land \Box(q \supset p))$

Where "$\pi(q)$" should be understood as "it is a law (or laws) of physics that q". Such a formalization no longer falls foul of the S4 problem, as the S4 rendering of a relative necessity is clearly no longer a theorem of S4 for $\Box$.[6] There is no other way that we could see to generate the same problematic result.

One notable feature of the proposal that is important for present purposes is that if there is at least one Φ-proposition, then logical necessity ($\Box p$) implies relative necessity ($\Box_\Phi p$). Finally, to reiterate, this proposal was offered for alethic, non-epistemic necessity.

[5] If we want to ensure closure under logical consequence for relative modalities, i.e., such that if $A_1, ..., A_n \vdash B$ then $\Box_\Phi A_1, ..., \Box_\Phi A_n \vdash \Box_\Phi B$ for $n \geq 1$, one option is to amend the definition to: $\Box_\Phi A =_{def} \exists q_1, ..., \exists q_n(\Phi(q_1) \land ... \land \Phi(q_n) \land \Box(q_1 \land ... \land q_n \supset A))$. In [4] this is what we opted for. I reserve judgment for now whether the extended view under discussion in this paper should also have this closure property. It is also worth noting that all three formulations so far – the standard account, RN, and the schema in this note – may face a problem if one wants to define a necessity relative to an *infinite* set of propositions. Although the problem is not fatal. See e.g. [17], p.711.

[6] $\exists q(\Phi(q) \land \Box(q \supset p)) \supset \exists r(\Phi(r) \land \Box(r \supset \exists q(\Phi(q) \land \Box(q \supset p))))$

3 The similarity argument

Why should we hope to treat different kinds of possibility and necessity together in a single framework?

Different kinds of modality in the world seem to have something important in common (just as on many views in linguistics all the "must"s and "can"s have a shared core meaning).[7] For example, metaphysical necessity and natural necessity can be taken to be distinct kinds of necessity, subject to different principles. Even so, they are fundamentally alike. They are both *necessities* for a start. That may sound obvious, but if the two phenomena are given entirely different accounts, it may seem a mystery why they seem to have something so distinctive in common, that they both concern a way in which things *must* be so. That different necessities and possibilities have something in common demands explanation. One plausible explanation that meets this need, is that all modalities can be expressed as relativizations of one fundamental kind of modality in the world (or as that fundamental kind itself). In particular, all non-logical necessities can be expressed as relativizations of logical necessity.

This argument is not quite an argument to the best explanation, but rather an argument to a plausible explanation: there is a phenomenon to be explained, and treating modalities as relative provides a fairly simple and plausible explanation. Let us briefly consider one alternative that might seem just as good. The core idea is that we can explain what different kinds of modality have in common by the fact that they can be expressed as *derivative of* a shared kind of necessity. Relativization is indeed one way to derive one modality from another. But what about other modes of derivation, such as restriction? Perhaps one could account for similarity in at least some cases because the relevant kind of modality is a *restriction* of some other modality, i.e. the physical possibilities might be a subset of the metaphysical possibilities, or the logical necessities a subset of the metaphysical necessites. Let me offer one reason why relativization may be more appealing than restriction.

In order to explain important similarities between different kinds of *necessity* in terms of restriction, one will need a very wide notion of necessity to restrict. For example, one might plausibly take the realm of physical necessity to be narrower than the realm of human necessity (there are human necessities not physically necessary, such as not running 100m in less than 5 seconds, but no physical impossibilities are humanly possible). If one were

[7] See especially Kratzer [8]. Vetter and Viebahn [20] argue, to the contrary, that modals do not share a single core meaning with differences accounted for in terms of *context-sensitivity*, but rather that modals are *polysemous*: they have related meanings, such as, for example, 'healthy' in the sense that a salad is healthy and in the sense that a dog is healthy. To properly engage with their arguments here would take me beyond the scope of the present essay, but it is worth noting that Vetter and Viebahn do allow for context-sensitivity *within* what they call different 'modal flavours'. Moreover, even if the *linguistic* story is polysemous, the argument from similarity, understood as concerning modalities themselves, and not necessarily our words for them, may still hold water.

to apply a restrictive strategy here, the physical necessities would simply be a subclass of the human necessities. In order to accommodate all the different kinds, the widest kind of necessity is going to be something very weak; hardly an ideal paradigm from which to explain how it is that, e.g., physical necessities are necessary. In contrast, relativization does not need the fundamental kind of modality to somehow "contain" all the others. This allows us to take a stronger and more intuitively robust kind of modality as the basic kind. One might reply: there is no such problem if we proceed by restriction of *possibility,* for the widest and weakest possibilities are often taken to be of a plausibly fundamental kind, such as logical possibilities. Even so, relativization has the advantage of being able to take the same fundamental kind of modality, whether starting with necessity or possibility. For example, we might start instead with a general account of relative possibility:

RP It is Φ-possible that p iff $\neg\exists q(\Phi(q) \wedge \neg\Diamond(q \wedge p))$

That is: it is Φ-possible that p just when there's no Φ-proposition that rules out p. This is just the dual of RN, and thus takes the same kind of modality – just logical possibility ($\Diamond$) rather than logical necessity. The fact that taking necessity or possibility first makes no difference to the variety of base modality suggests that this kind of derivation is more robust and significant than restriction.

So, we have some motivation for treating modalities together. But what about epistemic and non-alethic modalities? Can the relative necessity view be extended to necessities more widely construed?

4 The Problem with Other Modalities

Let me first clarify what is at stake. The aim is to offer a formal way to capture various notions of necessity as relative necessities. But what would it take to successfully capture such a necessity? A primary constraint will be to honour intuitions about the truth of various claims made using the associated modal term. So, for example, if a sentence such as 'Joe must be the killer' seems plausibly true, in a given context, for an epistemic reading of 'must', then the formalization should be able to accommodate that. And similarly, if there is a salient context where 'Jasmine must be the killer' would be false, it should also be possible to accommodate that. That said, my aim here is not to offer a comprehensive semantics for modal words, nor is it to precisely capture every such case that we might think of. Perhaps the framework I propose could be developed into something like this, but I do not propose to do so here. Rather, the notions I aim to capture should be understood as pseudo-technical notions of modality, which bear enough resemblance to our untutored notions of various modalities to deserve names such as 'epistemic modality', and which honour the strongest intuitions concerning the truth of various modal statements, but which also have some leeway for regimentation and idealization. The aim is to strike a happy medium between the kind of wholly implausible formalization we get by heavy-handedly applying the Hale-Leech relative necessity formulation to non-alethic and epistemic cases, and an entirely linguistically and psychologically adequate formalization.

4.1 Epistemic Modality

The notions of epistemic necessity and possibility concern the "must"s and "might"s in sentences such as:

> (Given all the evidence) Joe *must* be the killer.
>
> (For all I know) Jane *might* be at home.

Epistemic modality is typically linked to sets of known propositions, or sets of evidential propositions, which might further be indexed to a particular epistemic subject. As such, it would be natural to formulate epistemic necessity as a relative necessity.

RNe It is epistemically necessary (for S) that $p =_{\text{df.}} \exists q(K_S(q) \wedge \Box(q \supset p))$

Where "$K_S(q)$" should be understood as "S knows that q". That is, it is epistemically necessary for some epistemic subject S that p just when p follows as a matter of logical necessity from some propositions known by S. For example, if S knows that if Jasmine was out of town, then Joe was the killer, and if S learns also that Jasmine was out of town, it follows that Joe was the killer. Given our definition, Joe *must* – in the epistemic sense – be the killer.

However, there is a significant problem with this formulation. As long as at least one proposition is known (that is, *if S knows anything*), then all logical truths will come out as epistemically necessary. For all logical truths follow logically from any proposition (at least assuming a classical logic). So, for any logical truth L, it would be true to say that it *must, for all S knows,* be the case that L. But what if S is utterly ignorant of logic? Then it would seem strange at best, incorrect at worst, to say that for S it is epistemically necessary that L. Or put it this way. When S first attends their logic class, they haven't yet learned any propositional logic, and certainly haven't yet learned that it is a logical truth that for all p, q, r, s, $(p \supset q) \supset ((q \supset (r \supset s)) \supset ((p \wedge r) \supset s))$. So it would seem more appropriate to say that, for all S knows, it's possible – it's *epistemically* possible – that for some p, q, r, s, $\neg[(p \supset q) \supset ((q \supset (r \supset s)) \supset ((p \wedge r) \supset s))]$, and hence the logical truth itself is not epistemically necessary.

To clarify, the problem is not one of S's logical reasoning abilities; it may well be that if S thought long and hard enough about it, they would realise that L has to be true. The issue is that, intuitively, there is a point at which whether or not some logical truth L is the case seems to be genuinely open for S, which is supposed to be captured by saying that both L and $\neg L$ are epistemically possible for them (at that point). But according to RNe, that is ruled out: L is epistemically necessary, and $\neg L$ is epistemically impossible.

This is sometimes known as the problem of logical omniscience, although, as we shall see, it extends beyond the knowledge case to other, non-alethic modalities. Not every logical necessity is, plausibly, epistemically necessary; not every logical impossibility is epistemically impossible.

4.2 Propositional Attitudes

Knowledge is often taken to be a kind of propositional attitude. The problem besetting epistemic modality carries over to other kinds of modality based on propositional attitudes, such as *doxastic* modality (belief) and *boulomaic* (or *bouletic*) modality (desire). We might informally introduce doxastic necessity as what is required by someone's beliefs. So, for example,

> (Given everything Columbo believes) Joe *must* be the killer.

We might then naturally define doxastic necessity as relative to a subject's beliefs.

RNd It is doxastically necessary that $p =_{\text{df.}} \exists q(B_S(q) \wedge \Box(q \supset p))$

Where "$B_S(q)$" should be understood as "S believes that q". But the problem from above is replicated: If Columbo, say, has at least one belief – if he believes anything at all – then all logical truths will turn out to be doxastically necessary for him. But what if Columbo has no beliefs about logic at all? It would seem strange to say that, for him, it *must* be that for all p, q, r, s, $(p \supset q) \supset ((q \supset (r \supset s)) \supset ((p \wedge r) \supset s))$. Moreover, suppose that for some logical truth L, Columbo believes neither L nor $\neg L$. We would like to be able to say that, for all Columbo believes, it could be that L, or it could be that $\neg L$, such that, given his current state of belief, both are doxastically possible for him. But if L is in fact true, and logically necessarily so, then according to RNd it is doxastically necessary that L, and doxastically impossible that $\neg L$. Again, we have a problem, that we might term the problem of logical omnicredence.

The same goes for modalities defined in terms of desire. We might informally introduce boulomaic necessity as what is required by someone's desires. So, for example,

> (Given Rob Gordon's desires for music) he *must* have that album.

We might then define boulomaic necessity as relative to a subject's desires.

RNb It is boulomaically necessary that $p =_{\text{df.}} \exists q(D_S(q) \wedge \Box(q \supset p))$

Where "$D_S(q)$" should be understood as "S desires that q". But again, if Rob has at least one desire, then all logical truths will be boulomaically necessary. But that's bizarre: why should one hanker for logical truth, just because one desires music? We might call this the problem of logical omnicupiscence.

As in the epistemic case, it seems implausible that all logical impossibilities should for that reason be doxastically and boulomaically *impossible*, i.e. in some sense unbelievable and undesirable. For a bad logician may believe a logical falsehood, and a long-suffering logician with an unwelcome end to their proof may desire that this logical falsehood be true. (And this is all without yet considering that we often have contradictory desires and beliefs.)

What seems to lie at the heart of these problems? Intuitively, in the problematic cases, the propositions that are known, or believed, or desired, don't appear to have anything to do with some of the logical truths. For example, if I believe *that Torquay is in Devon*, this, intuitively speaking, has implications for the proposition that *Torquay is not in Devon*: given what I believe, the latter can't be true. But my belief that Torquay is in Devon doesn't appear to have anything to do with the proposition *that 2+2=4 or it's not the case that 2+2=4*. That latter might be necessarily true for other reasons, but it seems to be left open by my belief, in an important sense. And it is this that we wish to capture. One might say: my belief that Torquay is in Devon simply isn't *relevant* to whether *2+2=4 or it's not the case that 2+2=4,* in the way that my belief *is* relevant to whether Torquay is in Cornwall. This is all informal and intuitive, but something in this vicinity has been captured in attempts to develop *relevant logics,* a family of logics that, amongst other things, try to do justice to the idea that entailments hold only between propositions that have some kind of otherwise related content. But before I say more about relevance, I shall introduce a further problem case.

4.3 Inconsistent Conditions

Another problem case arises where the propositions to which a kind of modality is supposed to be relative are *inconsistent*. There are recognizable such kinds of modality. For example, it does not seem unlikely that someone might hold inconsistent beliefs (affecting doxastic modality), or that they might hold inconsistent desires (affecting boulomaic modality), or that a complicated legal system built up over many centuries might contain inconsistent laws (affecting legal necessity), and so on.

The problem is as follows. If the Φ-propositions to which Φ-modality is relative are inconsistent, then every proposition will turn out to be Φ-necessary — assuming a logical consequence relation which conforms to the explosion rule of inference, *ex falso quodlibet*, such as the classical consequence relation. For, in such a logic, everything follows from a contradiction. But we would not expect, for example, some inconsistency in the legislature to result in *everything* being legally necessary, including murder, arson, and completing the Times crossword.[8]

Conversely, and perversely, nothing would be Φ-possible. Take again the proposed formulation of relative possibility:

RP It is Φ-possible that p iff $\neg\exists q(\Phi(q) \wedge \neg\Diamond(q \wedge p))$

(equivalently: It is Φ-possible that p iff $\neg\exists q(\Phi(q) \wedge \Box(q \supset \neg p))$). Suppose that there is a proposition $\bot$ which satisfies condition Φ but which is inconsistent. Suppose also that something is Φ-possible, say p. Since it's Φ-possible that p, no Φ-proposition rules out p, i.e., $\neg\exists q(\Phi(q) \wedge \neg\Diamond(q \wedge p))$, hence $\forall q(\Phi q \supset \neg\Box(q \supset \neg p))$. And in particular, since $\Phi(\bot)$, $\neg\Box(\bot \supset \neg p)$. But this is equivalent to $\Diamond(\bot \wedge p)$, from which it follows that $\Diamond\bot$. But $\bot$ is inconsistent and of course not logically possible. Contradiction! Holding fixed the inconsistency and RP, we must deny that anything is Φ-possible. But again, we would not expect inconsistencies in the law to result in *everything* being illegal, including giving to charity, stopping at red lights, and completing the Times crossword. Something has gone badly wrong.

5 Kratzer on Relative Modality

Perhaps the most well-known and influential accounts of relative necessity have been developed by Angelika Kratzer. This paper would be incomplete without at least a brief review of her work. One can discern two kinds of approach to the issues raises above: what I will call her "early inconsistency" account, developed in [8] in response to problems arising from inconsistent conditions, and what I will call her "three-dimensional account" (first presented in [9], which I will be taking from [10, 11]), which offers a more nuanced solution.

5.1 Early Inconsistency

Kratzer (1977) [8] offers an analysis of modal phrases as sharing a common structure: (1) a 'relative modal phrase', such as *can in view of* or *must in view of*; (2) a first argument for that modal phrase: *that in view of which* something is possible or necessary; and (3) a second argument for the modal phrase, which is the proposition in scope of the modal. So, for example, "Rob must have that album" could be analysed as: 'In view of Rob's desires,

[8]The question of logical closure is relevant here (see footnote 5). For it is not clear whether if $\Phi\phi$ and $\Phi\psi$ then $\Phi(\phi \wedge \psi)$, and in particular, whether if $\Phi\phi$ and $\Phi\neg\phi$ then $\Phi(\phi \wedge \neg\phi)$. For example, if S believes that p, and S believes that $\neg p$, does S believe that $p \wedge \neg p$? Note that if we opt for the amended definition – $\Box_\Phi A =_{def} \exists q_1, ..., \exists q_n(\Phi(q_1) \wedge ... \wedge \Phi(q_n) \wedge \Box(q_1 \wedge ... \wedge q_n \supset A))$ – then inconsistent Φ-propositions generate the problem even without this conjunction property. I shall proceed here in terms of the simpler formulation, under the assumption that there are at least some relevant cases where there is an inconsistent Φ-proposition.

he must have that album', where (1) is *must in view of*, (2) is Rob's desires, and (3) is Rob's having that album.

Kratzer offers a first attempt at a precise definition of the relative modal phrases (I'll focus just on necessity here):

> DEFINITION 5. The meaning of 'must in view of' is that function ζ, which fulfils the following conditions:
>
> (i) If p is a proposition and f a function which assigns a set of propositions to every $w \in W$, then (f, p) is in the domain of ζ.
>
> (ii) For any f and p such that (f, p) is in the domain of ζ, $\zeta(f, p)$ is that proposition which is true in exactly those $w \in W$ for which the following holds: p follows (logically) from $f(w)$.
>
> [8, p.346]

p is necessary relative to the propositions selected by f just when p follows logically from those propositions.

Kratzer introduces a case to motivate and illustrate the problem of inconsistency.

> Let us imagine a country where the only source of law is the judgements which are handed down. There are no hierarchies of judges, and all judgements have equal weight. There are no majorities to be considered. It does not matter whether one judgement has a hundred judgements against it; it does not have less importance for all that. Let New Zealand be such a country.
>
> There is one judgement in New Zealand which provides that murder is a crime. Never in the whole history of the country has anyone dared to attack this judgement. No judgement in the whole history of New Zealand suggests that murder is not a crime. There are other judgements, however. Some judges did not quite agree, and there were even judges who disagreed so much that they did not talk to each other any more.
>
> Here is an example of such a disagreement. In Wellington a judgement was handed down which provided that deer are not personally responsible for damage they inflict on young trees. In Auckland a judgement was handed down which provided that deer *are* personally responsible for damage they inflict on young trees. This means that the set of propositions which the New Zealand judgements provide is an inconsistent set of propositions. [8, pp.347–8]

The problem is that, given Definition 5, both of the following sentences are true, for in this case $f(w)$ is the set of New Zealand judgements, and given that that set is inconsistent, any proposition will logically follow, including both that murder is a crime, and that murder is not a crime.

(6) In view of what the New Zealand judgements provide, murder must be a crime.

(7) In view of what the New Zealand judgements provide, it must be that murder is not a crime.

[8, p.348, Kratzer's numbering]

Such a result is unacceptable. Even if the laws are confusing when it comes to trees and deer, there's no disagreement at all over the criminality of murder.

Kratzer's repair here is to eliminate the inconsistency by adding structure to the sets of propositions to which a relative necessity is relative.

DEFINITION 7. The meaning of 'must in view of' is that function ζ which fulfils the following conditions:

(i) As in Definition 5.

(ii) For any f and p such that (f, p) is in the domain of ζ, $\zeta(f, p)$ is that proposition which is true in exactly those $w \in W$ for which the following holds: if X is the set of all consistent subsets of $f(w)$, then there is for every set in X a superset in X from which p follows (logically).

[8, p.351]

This solves our problem admirably. The set of New Zealand judgments consists of one about murder, call it m, and two inconsistent ones about deer, call them d and $\neg d$. Call this set Z: $\{m, d, \neg d\}$, and the set of all consistent subsets of Z, Y: $\{\{m\}, \{d\}, \{\neg d\}, \{m, d\}, \{m, \neg d\}, \varnothing\}$. Then for every set in Y, we can see that either m follows logically, or Y contains a consistent extension of that set from which m follows logically. By contrast, things are not so for d, as, for example, there is no consistent extension of $\{\neg d\}$ in from which d follows. The proposal allows us to 'overlook the inconsistences' [8, p.351] of the set of New Zealand judgments, by looking at its consistent subsets and consistent extensions of those subsets.

If we wanted to apply this to the Hale-Leech account, we would need a formulation according to which a relative necessity is no longer logically necessary *relative to the* Φ-*propositions*, but rather logically necessary *relative to the supersets of the consistent subsets of the set of* Φ-*propositions.* However that might be worked out, we should first pause, for there are some shortcomings of the proposal.

Kratzer's proposal still appears to suffer from what we might in general call "the logical-omni problem": all logical truths follow (classically) logically from all of the consistent sets of propositions.[9] We may wish to genuinely capture the fact that it is legally necessary that d and legally necessary that $\neg d$, but not legally necessary that $\neg m$. If a legislature really is

[9] Note: Kratzer does not claim to solve problems of logical omniscience in this paper.

inconsistent, isn't that better captured by recognising that we are presented with inconsistent legal necessities – it's both a crime and not a crime for a deer to damage a young tree – rather than by insisting that, by some transformation, the inconsistent code generates only a consistent set of legal necessities? The problem with an inconsistent law might be thought to be that we are required by law to perform inconsistent actions, *not* that we have no legal requirements at all where inconsistency is to be found.[10]

It might be preferable, for this reason, if there was a way to avoid explosion – everything being legally necessary – without entirely avoiding the inconsistency. One thing that Kratzer's analysis takes for granted is what it takes to follow logically. And it is this that, arguably, causes the problems. For, given a classical consequence relation, a logical truth follows from anything whatsoever (causing the logical-omni problems), and anything whatsoever follows from a contradiction (causing the inconsistency problem). Rather than accepting that these consequences must hold, and attempting to solve the problems another way, an alternative way forward – which I shall explore below – is to change the consequence relation.

5.2 The Three-Dimensional Account

In more recent work, Kratzer [9, 10, 11] offers an account of modals which identifies three distinct elements: *modal force,* i.e., possibility, necessity, and so on (roughly equivalent to the *relative modal phrase* earlier); *modal base,* the set of accessible worlds which our modality is 'in view of'; and an *ordering source,* an element which induces an ordering on the modal base, such as desires or values.

> [D]ifferences between modal expressions in different languages can be capture in terms of three dimensions:
>
> Dimension 1: **modal force:** necessity, weak necessity, good possibility, possibility, slight possibility, at least as good a possibility, better possibility, maybe others.
>
> Dimension 2: **modal base:** circumstantial versus epistemic ...
>
> Dimension 3: **ordering source:** deontic, bouletic, stereotypical etc.
>
> [10, p.649]

[10]One might compare these considerations to Ruth Barcan Marcus's claim that moral dilemmas are genuine and do not erase moral obligations, even if they cannot be fulfilled in the same circumstances. She writes, for example, that 'wherever circumstances are such that an obligation to do x and an obligation to do y cannot as a matter of circumstance be fulfilled, the obligations to do each are not erased, even though they are unfulfillable'. [2, pp.131–2]

The ordering source is the mechanism by which inconsistencies can be tolerated. Kratzer offers the following definition of necessity.

> *Definition 6.*
>
> A proposition p is a **necessity** in a world w with respect to a modal base f and an ordering source g iff the following condition is satisfied:
>
> For all $u \in \bigcap f(w)$ there is a $v \in \bigcap f(w)$ such that $v \leq_{g(w)} u$ and for all $z \in \bigcap f(w)$: if $z \leq_{g(w)} v$ then $z \in p$.
>
> ... Roughly, it says that a proposition is a necessity if and only if it is true in all accessible worlds which come closest to the ideal established by the ordering source.
>
> ([10, p. 644]. See also [11, p.40].)

Let us consider, for example, *what I must do to best satisfy my desires.* The modal force is necessity. The modal base is circumstantial: we are interested a modal base that is constrained by how the world could be, as I need to satisfy my desires in the world, rather than a modal base that is constrained by information. The ordering source is my desires. Now, suppose that I want to A, to B, and to not-C, but suppose also that one As if and only if one Cs. For example,

> I want to become popular, I don't want to go to the pub (more precisely: I want not to go to the pub), and I want to hike in the mountains. ... As a matter of fact, I live in a world where it is an unalterable fact that I will become popular if and only if I go to the pub. [10, p.647]

The only worlds in the modal base are consistent worlds – all possible circumstances are consistent. These worlds will then be ordered with those that do most to maximise my desire satisfaction uppermost. No world is such that A, B, and not-C; the worlds closest to my ideal are those which manage to satisfy 2 out of 3, namely, those in which A, B and C, and those in which not-A, B, and not-C. Homing in on that set of worlds – the set of those accessible worlds which come closest to what I want – we then have that B is boulomaically necessary (it is true in all of those worlds), and A, not-A, C, not-C, are all possible (optional). (I must hike in the mountains.)

In effect, inconsistency is now dealt with in two steps: the modal base contains only consistent worlds, but then the ordering gives us a set of worlds as close as we can get to the ideal without inconsistency. This is arguably an advance on the 1977 view, insofar as the way that inconsistency is removed from the picture is more clearly motivated: it's not just a way to find consistent sets of conditions, but rather we understand that all of the worlds are consistent because they belong to a certain modal base – e.g. circumstantial –

and there is a sensible motivation for privileging some of those consistent worlds, because they are 'closest to the ideal' determined by the ordering source. Kratzer also argues that this account is able to accommodate graded and comparative modal notions, but I won't have space to pursue these aspects of the view here.

However, as before, the view does not obviously capture the genuine inconsistency we might expect to find in the modal statements. Inconsistent desires are not translated into inconsistent musts, but are dissolved by the combination of modal base and ordering. The logical-omni problem also remains. So long as the logical necessities are true in all worlds, they will be relatively necessary, raising the problems discussed in section 4. One might wonder if the appeal to modal base could help here. For example, in the epistemic case, if the modal base is a set of epistemically possible worlds, then if it is open, for all I know, that some logical necessity L is false, there will be an epistemically possible world at which not-L. However, the modal base itself is just defined in terms of accessibility relations, e.g., worlds that are epistemically accessible from the actual world [10, p.644]. So if our domain of possible worlds contains, as standard, only consistent, logically possible worlds, then the modal base is always going to be constrained by logical possibility.

Another problem also threatens. We can, effectively, still read the proposal as boiling down to a restriction on possible worlds. A claim of relative necessity, for example, is a claim to truth in all of a restricted class of worlds, where this class is restricted first with respect to base and second with respect to ordering source. But still, ultimately, we end up with a set of worlds. As Kratzer herself puts it:

> In an ordering semantics for modals, ordering sources are used as domain restrictions for the set of accessible worlds: not all, but only the "closest" accessible worlds matter for what is possible or necessary. [11, p.45]

But this approach to relative necessity is precisely what led to the S4 problem sketched in section 2. Recall: the proposed definition of necessity says, roughly, that 'a proposition is a necessity if and only if it is true in all accessible worlds which come closest to the ideal established by the ordering source.' [10, p.644] But if we interpret relative modality in terms of a range restriction of the accessibility relation over possible worlds – restricted to worlds of a certain modal base that come closest to the ideal – then in crucial cases we do not thereby change the properties of the accessibility relation. Again, we have the problem that the transitivity of the accessibility relation over the whole domain of worlds carries over to relative necessity, whether we want it to or not. This problem certainly afflicts simple modal cases where the ordering source is empty, for then relative necessity is just truth in all f-accessible worlds.[11] This is already in itself a serious problem, so long as we want

[11] Kratzer allows that 'the interpretations of modals depend on both a modal base and an ordering source, but either parameter can be filled by the empty conversational background' [11, p.49].

to be able to accommodate relative necessities for which an ordering source is empty, and with fairly weak logical properties (e.g., not S4). I believe we can understand the work of the ordering source as further restricting the accessibility relation to what we might call the "ideal-f-accessible" worlds. If this is right, then the problem spans further across the proposed analysis.

Again, we have motivation to consider an alternative that has the capacity to avoid the logical-omni problems, the inconsistency problem, *and* the S4 problem. In the remainder of this paper, I shall attempt to provide such an alternative. It has already been shown how the Hale-Leech proposal avoids the S4 problem, but it must now be modified to avoid the others.

6 Relevance

I noted in section 5.1 that we might understand the logical-omni problem and the inconsistency problem as sharing a common core: a consequence relation according to which a logical truth is a consequence of any proposition and anything is a consequence of an inconsistent (set of) proposition(s). It is this feature in particular which causes problems for applying the Hale-Leech view beyond alethic modalities. At the heart of that formulation is a strict conditional: a conditional bound by a logical necessity operator. Another way to view the same problem is as stemming from these strict conditionals. For the unwanted consequences are simply versions of the paradoxes of strict implication.[12]

The paradoxes of strict implication

$\Box B \vDash A \multimap B$

$\neg \Diamond A \vDash A \multimap B$

In the history of logic, paradoxes and problems arising from the conditional have often been met with new logics that are better able to tolerate the kinds of inferences we do – and don't – want to capture. Hence, the proposal here is to change the *logic* behind the core conditional. It is telling that in an early paper Alan Ross Anderson, one of the Godfathers of relevant logic, wrote:

> Defining deontic modalities in terms of "relevant implication," and using the necessity thereof (i.e., entailment) as the appropriate sense of "logical consequence," yields a system of deontic logic which satisfies all our dearest desires: it is faithful to the rigor loved by logicians, to the justice loved by all, and to our common discourse. [1, p.359]

[12] I shall use '$\multimap$' to signify the strict conditional.

It is precisely some of the kinds of problems that appear to be plaguing a universal analysis of relative necessity that also led to the development of a relevant logic. Perhaps the two should go hand in hand.

What is a relevant logic? Mares and Meyer [14, p.286] list three conditions that any relevant logic should satisfy:

> 1. They should avoid the paradoxes of implication and, in particular, give a way of dealing with contradictions and other impossibilities non-trivially.
> 2. They should satisfy the variable sharing constraint. [A logic, L, satisfies the variable sharing constraint iff whenever $A \leftrightarrow B$ is a theorem of L, A and B share at least one propositional variable.][13]
> 3. They should contain a deducibility relation that requires all premises in a valid deduction to be capable of being used in that deduction and they should satisfy a deduction theorem.[14]

We can also list some additional features of relevant logics that are important for our purposes (following Priest [15]).

4. Relevant logics are paraconsistent, in the sense that they can tolerate contradiction: the rule of explosion (that anything follows from a contradiction) fails.[15]

5. Relevant logics are non-normal logics, in the sense that they contain in their semantics non-normal worlds, that is, worlds where 'logic is not guaranteed to hold'. [15, p.69]

6. Validity is truth preservation over all normal worlds, that is, truth preservation over all worlds in which the laws of logic are guaranteed to hold.

7. The rule of necessitation (if $\vDash A$ then $\Box A$) fails in all non-normal logics, and so also in relevant logics. For if $\Box A$ is true just when A is true at all worlds, then even if some formula is valid, say $\vDash A$, this only guarantees that A is true at all *normal* worlds; it may not be that A at all of the non-normal worlds as well.[16]

The intuitive appeal of a relevant logic is the variable-sharing constraint. For it seems to make sense that the consequent of a true conditional should have *something or other* to

[13] I shall use the plain arrow '$\rightarrow$' and similar, i.e. '$\leftrightarrow$', to signify an unspecified conditional.

[14] A logic typically includes a derivation system that determines what is deducible from what. Mares and Meyer explain that a relevant logic needs a conception of deducibility according to which '$A_1, ..., A_n \vdash B$ is *relevantly valid* only if $A_1, ..., A_n$ may *all* be really used in the deduction of B.' [14, p.284]

[15] This is implied by condition 1, but it's worth emphasising.

[16] See [15, pp.68–69], in particular, 'The failure of the Rule of Necessitation is, perhaps, the most distinctive feature of non-normal systems.'

do with the antecedent. Compare, for example, "If Nancy is tall and Nancy is swift then Nancy is swift", and "If 1=0 then Nancy is swift". It is the variable-sharing constraint that intuitively captures this: that Nancy is tall and swift obviously has something to do with Nancy's being swift, but that 0=1 seems to have nothing whatsoever to do with Nancy. Schematically, we have "if P and Q, then Q", and "If R then Q". In the first case, the propositional variable Q is shared, whereas in the second, there is no propositional variable in common.

An intuitive semantics for relevant logic has been notoriously difficult to find. There are semantics available, but it is difficult to make sense of them as making clear a familiar meaning of the conditional in the way that, for example, standard possible world semantics for normal modal logics makes compelling semantic proxies for possibility and necessity. Nevertheless, let me briefly sketch some features of a semantics for relevant logic (after Priest [15]). We start with a structure $\langle W, N, R, *, v\rangle$, where W is the set of worlds, N such that $N \subseteq W$ is the set of normal worlds, R is a ternary relation on worlds, $*$ is a function from worlds to worlds such that $w{*}{*} = w$ and, roughly, if $\neg p$ is true at w, p is false at $w*$, and v a function which 'assigns a truth value to every parameter at every world, and to every formula of the form $A \rightarrow B$ at every non-normal world' [15, p.170].

The ternary relation comes into play when evaluating conditionals. The basic way to understand the conditional is as follows: $A \rightarrow B$ is true at world $w \in W$ just when, for all $x, y \in W$ such that $Rwxy$, if A is true at x then B is true at y [15, p.189]. One informal way of thinking about this is in terms of information flow [13, 15, 16]: if the link from A to B holds, then being at an A-world can get you to a B-world.

In practice, when a conditional is evaluated at a normal world, it behaves like a strict conditional (such that $Rwxy$ iff $x = y$ [15, p.189]): in all worlds, A gets you to B.

> w is normal: $v_w(A \rightarrow B) = 1$ iff for all $x \in W$ such that $v_x(A) = 1$, $v_x(B) = 1$.

However, if a conditional is evaluated at a non-normal world, we cannot assume that $x = y$.

> w is non-normal: $v_w(A \rightarrow B) = 1$ iff for all $x, y \in W$ such that $Rwxy$, $v_x(A) = 1$, $v_y(B) = 1$.

In a relevant logic as sketched, the two troublesome kinds of consequence are blocked: from any propositions to any logical truths, and from any contradiction to any conclusion. Syntactically speaking, there is no guarantee that a logical truth will share a propositional variable with some set of premises, and so it is not guaranteed to follow from them; and there is no guarantee that a contradiction will share a propositional variable with some potential conclusion, and so no guarantee that the conclusion will follow from the contradiction.

The proposed change to the relative necessity formulation is thus, at first blush, as follows: Relative necessities are *relevant* consequences of certain propositions (rather than strict). More precisely:

RN* It is Φ-necessary that p iff $\exists q(\Phi(q) \wedge (q \Rightarrow p))$[17]

Where "$\Rightarrow$" stands for relevant implication.

For example, we could now define epistemic necessity as follows.

RNe* It is epistemically necessary (for S) that $p =_{\text{df.}} \exists q(K_S(q) \wedge (q \Rightarrow p))$

According to this definition, it is not the case that whenever p is a logical truth, p is epistemically necessary. But the definition allows that some logical truths may be epistemically necessary, depending on what is known by S.

Take as a second example the following candidate definition of legal necessity, where the operator "L" can be read as "is a conjunction of NZ laws".

RNl* It is legally necessary that $p =_{\text{df.}} \exists q(L(q) \wedge (q \Rightarrow p))$

Suppose that this conjunction of laws is in fact contradictory. For example, let's take again Kratzer's example: murder is a crime; deer are not personally responsible for damage they inflict on young trees; and deer *are* personally responsible for damage they inflict on young trees. Each of these propositions is a relevant consequence of itself (in a sufficiently strong relevant logic), and so it is legally necessary that deer are not personally responsible for damage they inflict on young trees *and* legally necessary that deer are personally responsible for damage they inflict on young trees. There are inconsistent legal necessities, as one might expect. But it is *not* the case that any proposition whatsoever is a relevant consequence of these laws, so we avoid the problem of the legal necessity glut raised above. It is also not the case that all logical truths follow from these laws; only those which satisfy the constraints of the relevant logic, chiefly the variable-sharing constraint.

It is true, in the case of the logical-omni problem, that many of the problematic consequences are blocked. For example, it does not relevantly follow from the three laws, and so is not legally necessary, that *either Jasmine completes the Times crossword or it's not the case that Jasmine completes the Times crossword.* However, there are some relevant consequences that might still be seen to be troublesome. For example, in many relevant logics the following are relevant consequences:

$$p \Rightarrow (p \vee \neg p)$$

$$p \Rightarrow (p \vee q)$$

[17]Informally put, the property of closure under relevant consequence seems to be reasonable and desirable. For if $A_1, ..., A_n \vdash B$ is relevantly valid, and so all of $A_1, ...A_n$ are "relevant" to deducing B, then one might reasonably suppose that $\Box_\Phi A_1, ..., \Box_\Phi A_n \vdash \Box_\Phi B$ for $n \geq 1$ may also be relevantly valid, insofar as the only variation across the premises and conclusion are $A_1, ...A_n$ and B. If so, then we can amend the definition to: $\Box_\Phi A =_{def} \exists q_1, ..., \exists q_n(\Phi(q_1) \wedge ... \wedge \Phi(q_n) \wedge (q_1 \wedge ... \wedge q_n \Rightarrow A))$.

But don't these also lead to counterintuitive consequences? Consider: given Rob's desires (supposing Rob desires only the new Radiohead album), it must be that *either Rob gets the album or he doesn't*. Well, perhaps so. But it seems to misrepresent Rob's desires to say he *must* either get the album *or not*. He wants the album! Or, for all Columbo knows (supposing he knows only that Joe is the killer), it must be that *either Joe is the killer or it is raining*. But surely Columbo has no thoughts of rain at all – just the knowledge of Joe's crime.

At this point, it is important to reiterate the present aim. My aim is not to present a perfectly psychologically adequate account of various modals here, such as would accommodate Columbo's only having epistemic necessities, in the above case, about Joe's being the killer. The aim is to develop an account of how one can capture some pseudo-technical modal notions better than the result of simply applying the Hale-Leech account across the board. At least the necessities in these cases have *something* fairly obviously to do with the kinds of proposition to which each kind of necessity is relative. Rob's disjunctive boulomaic necessity is still about the album, and Columbo's disjunctive epistemic necessity is still in part about Joe's crime. It seems to me that this is certainly more acceptable than just *any* logical truth becoming relatively necessary.

To reiterate, the logical-omni problems raised above concern whether RN and its different versions render all logical truths relatively necessary, and all logical impossibilities relatively impossible, even where it is plausible that the falsity of some cases of logical truth, and the truth of some logical impossibilities, should be left open (as merely possible). These problems are due to the assumption of a particular behaviour for the consequence relation. If we introduce a relevant consequence relation instead, the worst of these problematic cases are solved. Moreover, because relevant consequence is tolerant of contradictions, we are also able to solve the inconsistency problem.

7 Relative Necessity Unified?

The proposal is to formulate relative non-alethic and epistemic necessities using a relevant logic. But how does this lend itself to unification with alethic necessities? How should we now integrate alethic and other modalities together? If we straightforwardly apply RN* to the familiar alethic modalities, we get counterintuitive results in the opposite direction. We are typically happy to accept that all logical necessities are also physically necessary, and so on, for it seems wrong to suppose that some logical impossibility might nevertheless be physically possible. In the alethic case, we were not only happy to allow the logical-omni results, we plausibly *require* them. By contrast, we do not require either that inconsistency is preserved in alethic modals, nor do we require explosion; because alethic modalities are factive, that means that the propositions from which alethic necessities follow will themselves be true, and hence there will be no inconsistent bases. The inconsistency problem simply doesn't arise.

How, then, can we unify two sets of modals, one of which should include the logical necessities as relative necessities too, and one of which should not guarantee the relative necessity of logical necessities?

One option would be to build the solution into the "Φ", such that, for example, the laws of physics include the laws of logic. Perhaps there is a way to make this example plausible: if the laws of physics concern all physical things, and the laws of logic concern all things whatsoever, then the latter laws should be included in the former, say. But it is less clear that such a strategy could work in the general case, for example, the necessity relative to truths about London. Do the truths about London really include the laws of logic? It's not as though London is obviously *illogical,*[18] but it seems a bit much to specify the laws of logic in, say, a guidebook to London.

Another, perhaps less heavy-handed, option would be to think of relevant consequence as a *restriction* of classical (or near classical) consequence. So non-alethic and epistemic necessities can still be understood as inheriting their necessity from (classical) logical necessity. It is just that it is not via relativization alone, but via an additional step of restriction to only the *relevant* relative necessities. This restriction—at least in the case of those necessities linked to propositional attitudes—can be motivated by the kinds of considerations offered above. However, this requires a certain way of thinking about relevant logic, as what has been called a "filter logic". Filter logics have some different features to the kinds of relevant logics that I have briefly introduced, and might be understood to have different background motivations. For example, Priest prefers to think of relevant logic as having its own distinct source.

> There are some approaches to relevant logic where a conditional is taken to be valid iff it is classically valid *and* satisfies some extra constraint, for example that antecedent and consequent share a parameter. (These are sometimes called *filter logics*, since the extra constraint filters out 'undesirables'.) ...
>
> In the present approach, relevance is not some extra condition imposed *on top* of classical validity. Rather, relevance, in the form of parameter sharing, falls out of something more fundamental, namely the taking into account of a suitably wide range of situations. [15, p.173–4]

For Priest, relevance isn't just a restriction on consequence, it is one possible result of taking into account non-normal worlds, that is, situations where logic may fail to hold. The filter logic conception may appear to fit my purposes fairly neatly here, and remains a live

[18] At most in a figurative sense.

option. But I shall explore one further option, which avoids – at least in the first instance – one having to take a stance on these issues.

The third and final option I shall consider is to take the logic to be a parameter that can be varied, just as the operator Φ can be varied. The thought would then be that the shared core of necessities, the answer to the similarity argument, is not a shared kind of *necessity* to which all of the necessities are relative, after all, but rather a shared *form.*

RN It is Φ-necessary that p iff $\exists q(\Phi(q) \wedge (q \rightarrow_L p))$

All relative necessities, so goes the thought, can be expressed using the same general formula, with three parameters: (1) an operator that specifies the kinds of propositions relative to which something is necessary, Φ; (2) a conditional that is typically either strict or relevant, $\rightarrow_L$; and (3) the proposition which is relatively necessary, p. As indicated, this proposal constitutes a slightly different answer to the similarity argument, but it has an answer nevertheless. The (to me) appealing metaphysical idea of a basic kind of necessity, logical necessity, of which all other necessities are relativizations, is diluted somewhat. But at the core of this proposal there remains some kind of logically valid conditional from which stems the relative necessity. Further unification – beyond this single shared form – might then be achieved by asking further questions about the relations between the different conditionals, perhaps taking us back to the considerations of the previous option (of filter logics). But I will leave these matters open here.

One final point to note is that, for both options 2 and 3, it is clear here that the logic is, in some sense, a parameter to be varied; in the first case, we take either classical logic or a restriction of classical logic, in the second we take either a strict conditional or a relevant conditional. This is a notably different approach to the kinds of parameters allowed for in Kratzer's proposal. It is for further study to what extent this additional parameter, and alternative approach, compares favourably or not with Kratzer's accounts.

References

[1] Alan Ross Anderson, Some Nasty Problems in the Formal Logic of Ethics, *Nous 1* (4), 1967, pp. 345–360.

[2] Ruth Barcan Marcus, Moral Dilemmas and Consistency, in: *Modalities: Philosophical Essays*, Oxford University Press, Oxford and New York 1993, pp. 125–141.

[3] Dorothy Edgington, Two Kinds of Possibility, *Aristotelian Society Supplementary Volume 78* (1), 2004, pp. 1–22.

[4] Bob Hale, and Jessica Leech, Relative Necessity Reformulated, *Journal of Philosophical Logic 46* (1), 2017, pp. 1–26.

[5] Bob Hale, *Necessary Beings: An Essay on Ontology, Modality, and the Relations Between Them*, Oxford University Press, Oxford 2013.

[6] Lloyd Humberstone, Two-Dimensional Adventures, *Philosophical Studies 118*(1–2), 2004, pp. 17–65.

[7] —Relative Necessity Revisited, *Reports on Mathematical Logic 13* 1981, pp. 33–42.

[8] Angelika Kratzer, What 'must' and 'can' must and can mean, *Linguistics and Philosophy 1* (3), 1977, pp. 337–355.

[9] —The Notional Category of Modality, in: Hans J. Eikmeyer and Hannes Rieser (eds.), *Words, Worlds, and Contexts*, De Gruyter, Berlin and New York 1981, pp. 38–74.

[10] —Modality, in: Arnim v. Stechow and Dieter Wunderlich (eds.), *Handbuch Semantik/Handbook Semantics*, De Gruyter, Berlin and New York 1991, pp. 639–650.

[11] —The Notional Category of Modality, in: *Modals and Conditionals*, Oxford University Press, Oxford 2012, pp. 27–69. (A revised and expanded version of Kratzer (1981).)

[12] Jessica Leech, The Varieties of (Relative) Modality, *Pacific Philosophical Quarterly 97* (2), 2016, pp. 158–180.

[13] Edwin D. Mares, Relevant Logic and the Theory of Information, *Synthese 109* (3), 1996, pp. 345–360.

[14] Edwin D. Mares, and Robert K. Meyer, Relevant Logics, in: Lou Goble (ed.), *The Blackwell Guide to Philosophical Logic*, Blackwell 2001, pp. 280–308.

[15] Graham Priest, *An Introduction to Non-Classical Logic: From If to Is*, Cambridge University Press 2008. (Second edition.)

[16] Greg Restall, Information Flow and Relevant Logic, in: Jeremy Seligman, and Dag Westerstahl (eds.) *Logic, Language and Computation: the 1994 Moraga Proceedings*, CSLI Press, Stanford 1995, pp. 463–477.

[17] Alexander Roberts, Relative Necessity and Propositional Quantification, *Journal of Philosophical Logic 49*, 2020, pp. 703–726.

[18] Ian Rumfitt, *The Boundary Stones of Thought: An Essay in the Philosophy of Logic*, Oxford University Press, Oxford 2015.

[19] Bas C. Van Fraassen, The Only Necessity is Verbal Necessity, *The Journal of Philosophy 74* (2), 1977, pp. 71–85.

[20] Barbara Vetter and Emanuel Viebahn, How many meanings for 'may'? The case for modal polysemy, *Philosophers' Imprint 16* (10), pp. 1–26.

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Potentiality and Indeterminacy in Mathematics

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Abstract

The purpose of this article is to explore the use of modal logic and/or intuitionistic logic to explicate potentiality and incomplete or indeterminate domains in mathematics. Our primary applications are the traditional notion of potential infinity, predicativity, a version of real analysis based on Brouwerian choice sequences, and a potentialist account of the iterative hierarchy in set theory.

The purpose of this article is to explore the use of modal logic and/or intuitionistic logic to explicate potentiality and incomplete or indeterminate domains in mathematics. Our primary applications are the traditional notion of potential infinity, predicativity, a version of real analysis based on Brouwerian choice sequences, and a potentialist account of the iterative hierarchy in set theory. We present a unified framework in which these phenomena can be described and studied. We then locate various views—historical as well as contemporary (including some developed by ourselves)—in this framework.

Section 1 is a brief presentation of the history and philosophy behind potentialism, with a focus on mathematics. We argue that modality provides the best (or, at any rate, a very good) framework to explicate potentialism. Section 2 develops the proper modal logics for various kinds of potentiality. One key issue is the proper background logic for this. Should it be classical or intuitionistic? We argue that this distinction turns on a central philosophical thesis that the potentialist might (or might not) adopt, concerning modal propositions. Section 3 provides sketches of different applications.

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1 Potential infinity

1.1 A modal analysis

Aristotle famously rejected the notion of the actual infinite—a complete, existing entity with infinitely many members. He argued that the only sensible notion is that of potential infinity. In *Physics* 3.6 (206a27-29), he wrote:

> For generally the infinite has this mode of existence: one thing is always being taken after another, and each thing that is taken is always finite, but always different.

As Richard Sorabji [41] (322-3) once put it, for Aristotle, "infinity is an extended finitude"(see also [24], [25]).

The attitude toward the infinite was echoed by the vast majority of mathematicians and philosophers at least until late in the nineteenth century. In 1831, for example, Gauss [10] wrote:

> I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics. Infinity is merely a way of speaking.

Aristotle did accept what is sometimes called *potential infinity*, against the ancient atomists (see [30]). He argued that this makes sense of the mathematics of his day.[1] Subsequent mathematicians followed this, and, indeed, made brilliant use of potential infinity. But what is potential infinity?

The notion can be motivated by considering *procedures* that can be repeated indefinitely.[2] A nice example is provided by Aristotle's claim, against the atomists, that matter is infinitely divisible. Consider a stick. However many times one has divided the stick, it is always possible to divide it again (or so it is assumed).

[1] Aristotle wrote (*Physics 207b27*):

> Our account does not rob the mathematicians of their science, by disproving the actual existence of the infinite in the direction of increase, in the sense of the untraversable. In point of fact they do not need the infinite and do not use it. They postulate only that a finite straight line may be produced as far as they wish. It is possible to have divided into the same ratio as the largest quantity another magnitude of any size you like. Hence, for the purposes of proof, it will make no difference to them to have such an infinite instead, while its existence will be in the sphere of real magnitudes.

See [21] for a detailed analysis of ancient Greek mathematics on these issues.

[2] There is some controversy over whether Aristotle took procedures like these to be central to the view, or whether he was more concerned with the structure of matter and space. See, for example, [19],[24], [25], and [5].

It is natural to explicate this in a modal way.[3] This yields the following analysis of the infinite divisibility of a stick s:

$$\Box\forall x(Pxs \rightarrow \Diamond\exists y\, Pyx), \tag{1}$$

where Pxy means that x is a *proper* part of y. If, by contrast, the parts of the stick formed an actual infinity, the following would hold:

$$\forall x(Pxs \rightarrow \exists y\, Pyx). \tag{2}$$

According to Aristotle, the stick does not have, and cannot have, infinitely many parts:

$$\neg\Diamond\forall x(Pxs \rightarrow \exists y\, Pyx). \tag{3}$$

By endorsing (1) and (3), one is asserting that the divisions of the stick are *merely potentially infinite*. We thus see that a potential infinity is not the same as the possibility of an actual infinity. This contrasts with many uses of the word 'potential'; e.g. to say that someone is a potential champion is to say that possibly he or she is a champion. On the Aristotelian view, there is no totality or collection that could become infinite (whatever that might mean).

As noted above, our present concern is with mathematics. According to Aristotle, the natural numbers are merely potentially infinite. We can represent this view as the conjunction of the following theses:

$$\Box\forall m\Diamond\exists n\, \textsc{Succ}(m,n) \tag{4}$$
$$\neg\Diamond\forall m\exists n\, \textsc{Succ}(m,n), \tag{5}$$

where $\textsc{Succ}(m,n)$ states that n comes right after m. The modal language thus provides a nice way to distinguish the merely potential infinite from the actual infinite.[4]

Like David Hilbert ([18]), we are not looking to leave Cantor's paradise. Following contemporary practice, we accept the existence of actually infinite collections. We suggest, however, that there is room for potentiality in contemporary mathematics, and in the philosophy of contemporary mathematics. The modal analysis helps explicate that. It provides a framework in which actual and potential infinity can live side by side, sometimes in the very same context (see [28]). In other words, we

[3]We make use here of contemporary modal notions. We make no attempt to recapitulate what Aristotle himself says about modality.

[4]For Aristotle, there *cannot* be an actual infinity. So, here, one might say $\Box\neg\Diamond\forall m\exists n\, \textsc{Succ}(m,n)$.

envisage cases where there are actual infinities, but other "collections" or "totalities" are merely potential.[5] We describe some examples below.

1.2 Three orientations towards the infinite

It is useful to distinguish different orientations towards a given infinite totality. *Actualism* unreservedly accepts actual infinities, of a given kind, and thus finds no use for modal notions—or at least no use that is specific to the analysis of the infinity in question. Actualists maintain that the non-modal language of ordinary mathematics is already fully explicit and thus deny that we need a translation into some modal language. Furthermore, actualists accept classical logic when reasoning about the infinite (or the infinite in question).

Potentialism is the orientation that stands opposed to actualism. According to it, the objects with which mathematics is concerned—or some of the objects with which mathematics is concerned—are generated successively, and at least some of these generative processes cannot be completed. So there is an inherent potentiality to (at least some) mathematical objects.

There is room for disagreement about *which* processes can be completed. As noted from the above passages, the traditional Aristotelian form of potentialism takes a very restrictive view, insisting that at any stage, there are never more than finitely many objects, but that we always (i.e., necessarily) have the ability to go on and generate more. Recall Sorabji's suggestion that, for Aristotle, infinity is "extended finitude". Generalized forms of potentialism take a more relaxed attitude. Potentialism about set theory provides an extreme example. According to this view, any generative process that is indexed by a set-theoretic ordinal can be completed, but it is impossible to complete the the entire process of forming sets. The so-called "relative predicativism" lies in between the traditional Aristotelian orientation and potentialized set theory. The view accepts the natural numbers as a complete infinity, but insists that sets of natural numbers are defined in stages, and there is no stage at which all sets of natural numbers exist together, so to speak. Other sorts of relative predicativity are possible as well. Brouwerian choice sequences can also be taken to fit the mold of potentialism. See §3 below for a bit more detail on these cases.

Potentialists also differ from each other with respect to a *qualitative* matter. As characterized above, potentialism is the view some or all of the *objects* with which mathematics is concerned are successively generated and that some of these

[5]Strictly speaking, there are no sets, collections, or totalities that are potentially infinite. But it is useful to use a count noun to talk about the kinds of "things" said to be potentially infinite. We will use "collections" for this, sometimes in scare quotes.

generative processes cannot be completed. What about *the truths* of mathematics? Of course, on any form of potentialism, these are modal truths concerned with certain generative processes. But how should these modal truths be understood?

Liberal potentialists regard the modal truths as unproblematic, adopting bivalence for the modal language. Consider Goldbach's conjecture. As potentialists interpret it, the conjecture says that necessarily any even natural number that is generated can be written as a sum of two primes. Liberal potentialists maintain that this modal statement has an unproblematic truth-value—it is either true or false. Their approach to modal theorizing in mathematics is thus much like a realist approach to modal theorizing in general: there are objective truths about the relevant modal aspects of reality, and this objectivity warrants the use of some classical form of modal logic.

Strict potentialists differ from their liberal cousins by requiring, not only that every object be generated at some stage of a process, but also that every truth be "made true" at some stage. Consider, again, the Goldbach conjecture. If there are counterexamples to the conjecture, then its negation will presumably be "made true" at the stage where the first counterexample is generated. But suppose there never will be any counterexamples. Since the conjecture is concerned with *all* the natural numbers, it is hard to see how it could be "made true" without completing the generation of natural numbers. This completion would, however, violate the strict potentialists' requirement that any truth be made true at some stage of the process.

We suggest that strict potentialists should adopt a modal logic whose underlying logic is intuitionistic (or intermediate between classical and intuitionistic logic); this allows them to adopt a conception of universal generality which does not presuppose that all the instances are available, thus overcoming the problem just identified. In particular, strict potentialists should not accept every instance of the law of excluded middle in the background modal language (see [28] for more details). This dovetails with a view that Solomon Feferman and others adopt towards predicative mathematics, and it has ramifications for the articulation of predicativism and the extent of the mathematics that it captures.

1.3 The modality

Here, as elsewhere, it is often useful to invoke the contemporary heuristic of possible worlds when discussing the modality in question. Here we insist that this is *only* heuristic, as a manner-of-speaking. Our official theory is formulated in the modal language, with (one or both of) the modal operators as primitive. The modal language is rock bottom, not explained or defined in terms of anything else.[6]

The potentialist does, of course, reject the now common thesis that mathematical objects exist of necessity (if they exist at all). To invoke the heuristic, the now common thesis is that all mathematical objects exist in all worlds. The potentialist gives that up. There is no world with all of the objects in question—all natural numbers for the Aristotelian, all sets of numbers for the relative predicativist, all sets for the set-theoretic potentialist, etc.

The potentialist does, however, maintain that once a mathematical object comes into existence—by being constructed—it continues to exist, of necessity. To paraphrase Aristotle (from another context), the potentialist accepts generation, but not corruption.

What about the philosophical nature of the modality invoked in the analysis of potentiality? For the Aristotelian, it can perhaps be the ordinary metaphysical modality invoked in contemporary philosophy (or perhaps defined from that notion). The idea is that mathematical objects are generated successively, in time. At any stage—in any world—there are finitely many natural numbers, but each such world has access to another where some more numbers have been generated. Given enough time, any given natural number can be generated.[7]

Charles Parsons [32] once argued that this sort of modality does not make sense for the richer potentialisms, where the "procedure" of generating mathematical objects extends into the transfinite. Intuitively, generation takes place in time, and the richer potentialisms stretch the the notion of time too far. Of course, the potentialist is not going to presuppose a totality of ordinals (or anything else) by which to make sense of the generation.

So perhaps the non-Aristotelian potentialist should simply sever any link between metaphysical modality and the modality invoked in explicating potential infinity. Instead, one might regard the latter as an altogether distinct kind of modality, say the logico-mathematical modality of [34] or [11] (though see also [32]), or the interpretational modality of [9], [26], or [42].

Here we remain neutral on the exact interpretation of the modal operators. What

[6] If a potentialist did make use of the explication of modality in terms of possible worlds, she would, presumably, think of the collection of worlds as itself potentially infinite. So it is not clear that there is much of a gain in understanding, analysis, or the like. Here we make use of the usual possible-worlds semantics to obtain results about what is, and is not, derivable in the formal systems. We do not directly address the interesting question concerning the extent to which a potentialist can accept our results. Thanks to two referees for pressing this matter.

[7] In terms of possible worlds, the relevant modality is the one that results from restricting the accessibility associated with metaphysical modality by imposing the additional requirement that domains don't ever decrease along the accessibility relation. This restriction can be captured proof-theoretically, using the resources of plural logic (see [28], p. 188, n. 15).

matters for us are the structural features of any plausible interpretation. That is, we are concerned to develop the right modal *logic*.

2 The logic of potentiality

2.1 The modal logic

For the time being, we will be neutral on the liberal vs. strict divide and thus also on whether the non-modal part of the logic should be classical or intuitionistic. To invoke the heuristic, the idea is that a "possible world" has access to other possible worlds that contain objects that have been constructed or generated from those in the first world. From the perspective of the earlier world, the "new" objects in the second exist only potentially.

Geometry provides a good illustration, if we take seriously the constructive language in, say, Euclid's *Elements*. One world might contain a line segment, and a "later" (or accessible) world might contain a bisect of that line segment. Another later world might contain an extension of that line segment. Other sorts of constructions are arithmetic: the later world might contain more natural numbers than those of the first, say the successor of the largest natural number in the first world. Or, for a third kind of example, the later world may contain a set whose members are all in the first world.

An Aristotelian (or Gauss, etc.) assumes that every possible world is finite, in the sense that it contains only finitely many objects. This, of course, just is the rejection of the actually infinite. As noted, we make no such assumption here. Our goal is to *contrast* the actually infinite and the potentially infinite, so we need a framework where both can occur (to speak loosely). An actual infinity—or, to be precise, the possibility of an actual infinity—is realized at a possible world if it contains infinitely many objects.

As noted, we also assume that objects are not destroyed in the process of construction or generation. So, to continue the heuristic, it follows from the foregoing that the domains of the possible worlds grow (or, better, are non-decreasing) along the accessibility relation. So we assume:

$$w_1 \leq w_2 \rightarrow D(w_1) \subseteq D(w_2) \tag{6}$$

where '$w_1 \leq w_2$' says that w_2 is accessible from w_1, and for each world w, $D(w)$ is the domain of w. As is well-known, the conditional (6) entails that the converse Barcan formula is valid. That is,

$$\exists x \Diamond \phi(x) \rightarrow \Diamond \exists x \varphi(x). \tag{CBF}$$

For present purposes, we can think of a possible world as determined completely by the mathematical objects—regions, numbers, sets, etc.—it contains. In other words, we assume the converse of (6). We will talk neutrally about the extra mathematical objects existing at a world w_2 but not at an "earlier" world w_1 which accesses w_2, as having been "constructed" or "generated". This motivates the following principle:

> **Partial ordering**: The accessibility relation $\leq$ is a partial order. That is, it is reflexive, transitive, and anti-symmetric.

So the underlying logic is at least S4. So far, then, we have S4 plus (CBF).[8]

At any stage in the process of construction, we generally have a choice of which objects to generate. For some types of construction, but not all, it makes sense to require that a license to generate objects is not revoked at accessible worlds. Intuitively geometric construction is like this. For example, we might have, at some stage, two intervals that don't yet have bisections. We can choose to bisect one or the other of them, or perhaps to bisect both simultaneously. Assume we are at a world w_0 where we can choose to generate objects, in different ways, so as to arrive at either w_1 or w_2. Say at w_1 we bisect an interval i and at w_2 we bisect another interval j. It seems plausible to require that the licence to bisect i can be executed at w_2 or any later world. In other words, nothing we do can prevent us from being able to bisect the other interval.

This corresponds to a requirement that any two worlds w_1 and w_2 accessible from a common world have a common extension w_3. This is a directedness property known as *convergence* and formalized as follows:

$$\forall w_0 \forall w_1 \forall w_2 (w_0 \leq w_1 \wedge w_0 \leq w_2 \rightarrow \exists w_3 (w_1 \leq w_3 \ \wedge \ w_2 \leq w_3))$$

For constructions that have this property, then, we adopt the following principle:

> **Convergence**: The accessibility relation $\leq$ is convergent.

This principle ensures that, whenever we have a choice of mathematical objects to generate, the order in which we choose to proceed is irrelevant. Whichever object(s) we choose to generate first, the other(s) can always be generated later. Unless $\leq$ is convergent, our choice whether to extend the ontology of w_0 to that of w_1 or that of w_2 might have an enduring effect.

[8] Recall that S4 and (non-free) first-order logic entails (CBF). We can also require the accessibility relation to be well-founded, on the grounds that all mathematical construction has to start somewhere. However, nothing of substance turns on this here.

It is well known that the convergence of $\leq$ ensures the soundness of the following principle:

$$\Diamond\Box p \to \Box\Diamond p. \tag{G}$$

The modal propositional logic that results from adding this principle to a complete axiomatization of S4 is known as S4.2. As noted, not all construction principles sanction this principle. We give an example below.

2.2 The logic of potential infinity

What is the correct logic when reasoning about the potentially infinite? Informal glosses aside, the language of contemporary mathematics is strictly non-modal. We thus need a translation to serve as a bridge connecting the non-modal language in which mathematics is ordinarily formulated with the modal language in which our analysis of potentiality is developed. Suppose we adopt a translation $*$ from a non-modal language $\mathcal{L}$ to a corresponding modal language $\mathcal{L}^{\Diamond}$. The question of the right logic of potential infinity is the question of which entailment relations obtain in $\mathcal{L}$.

To determine whether $\varphi_1, \dots, \varphi_n$ entail ψ, in the non-modal system, we need to (i) apply the translation and (ii) ask whether $\varphi_1^*, \dots, \varphi_n^*$ entail ψ^* in the modal system. This means that the right logic of potential infinity depends on several factors. First, the logic depends on the bridge that we choose to connect the non-modal language of ordinary mathematics with the modal language in which our analysis of potential infinity is given. Second, the logic obviously depends on our modal analysis of potential infinity; in particular, on the modal logic that is used in this analysis—including the underlying logic of the modal language, whether it is classical or intuitionistic. Let us now turn to the first factor.

The heart of potentialism, as we see it, is the idea that the existential quantifier of ordinary non-modal mathematics has an implicit modal aspect. Consider the statement that a given number has a successor. For the Aristotelian, this is a proposition that each number *potentially* has a successor—that it is *possible* to generate a successor. This suggests that the right translation of $\exists$ is $\Diamond\exists$.

Since we consider both classical and intuitionistic backgrounds, we treat the universal quantifier separately. But it is understood in a dual way. When a potentialist says that a given property holds of all objects (of a certain sort), he means that the property holds of all objects (of that sort) *whenever they are generated.* This suggests that $\forall$ be translated as $\Box\forall$.

Thus understood, the quantifiers of ordinary non-modal mathematics are understood as devices for generalizing over absolutely all objects, not only the ones available at some stage, but also any that we may go on to generate. In our modal language, these generalizations are effected by the strings $\Box\forall$ and $\Diamond\exists$. Although these strings are strictly speaking composites of a modal operator and a quantifier proper, they behave logically just like quantifiers ranging over all entities at all (future) worlds. We will therefore refer to the strings as *modalized quantifiers*.

Our proposal is thus that each quantifier of the non-modal language is translated as the corresponding modalized quantifier.Each connective is translated as itself. Let us call this the *potentialist translation*, and let $\varphi^\Diamond$ represent the translation of φ.[9] We say that a formula is *fully modalized* just in case all of its quantifiers are modalized. Clearly, the potentialist translation of any non-modal formula is fully modalized.

Say that a formula φ is *stable* if the necessitations of the universal closures of the following two conditionals hold:

$$\varphi \rightarrow \Box\varphi \qquad\qquad \neg\varphi \rightarrow \Box\neg\varphi$$

Intuitively, a formula is stable just in case it never "changes its mind", in the sense that, if the formula is true (or false) of certain objects at some world, it remains true (or false) of these objects at all "later" worlds as well.

We are now ready to state two key results, which answer the question about the correct logic for those kinds of potentiality that enjoy the above convergence property. Let $\vdash$ be the relation of classical deducibility in a non-modal first-order language $\mathcal{L}$. Let $\mathcal{L}^\Diamond$ be the corresponding modal language, and let $\vdash^\Diamond$ be deducibility, in this corresponding language, by $\vdash$, S4.2, and axioms asserting the stability of all atomic predicates of $\mathcal{L}$.

Theorem 1 (Classical potentialist mirroring)**.** *For any formulas $\varphi_1, \ldots, \varphi_n$, and ψ of $\mathcal{L}$, we have:*

$$\varphi_1, \ldots, \varphi_n \vdash \psi \quad \textit{iff} \quad \varphi_1^\Diamond, \ldots, \varphi_n^\Diamond \vdash^\Diamond \psi^\Diamond.$$

(See [26] for a proof.)

The theorem has a simple moral. Suppose we are interested in logical relations between formulas in the range of the potentialist translation, in a classical (first-order) modal theory that includes S4.2 and the stability axioms. Then we may delete all the modal operators and proceed by the ordinary non-modal logic underlying $\vdash$.[10] In particular, under the stated assumptions, the modalized quantifiers $\Box\forall$ and $\Diamond\exists$

[9] This is an alternative to the more familiar Gödel translation, which translates '$\forall$' as '$\Box\forall$' (as we do), '$\exists$' as itself, and also adopts a non-homophonic translation of negation and the conditional. This translation is poorly suited to explicating potentialism. For example, the translation of $\forall m \exists n \textsc{Succ}(m, n)$ is $\Box\forall m \exists n \Box \textsc{Succ}(m, n)$, which, as discussed in Section 1.1, a potentialist would reject. For more detail, see [28].

[10] There are interesting issues concerning comprehension axioms in higher-order frameworks. See [28], §7.

behave logically just as ordinary quantifiers, except that they generalize across all (accessible) possible worlds rather than a single world. This buttresses our choice of the potentialist translation as the bridge connecting actualist and potentialist theories. We will observe, as we go along, that the stability axioms on which the mirroring theorem relies are acceptable.

It is important to note, however, that our interest won't always be limited to formulas in the range of the potentialist translation. One can often use the extra expressive resources afforded by the modal language to engage in reasoning that takes us outside of this range. The modal language allows us to look at the subject matter under a finer resolution, which can be turned on or off, according to our needs.[11]

An important upshot of the theorem is that ordinary classical first-order logic is validated via this bridge. However, this response depends on the robustness of our grasp on the modality. We noted that our *liberal* potentialist accepts classical logic when it comes to the modality. Our first mirroring theorem fits in nicely with that perspective. As noted above, however, Linnebo and Shapiro [28] argue that a stricter form of potentialism pushes in the direction of intuitionistic logic. What to do then?

The answer is given by a second mirroring theorem, which we now explain. As usual, we say that a formula φ is *decidable* in a given (intuitionistic) theory if the universal closure of $\varphi \vee \neg\varphi$ is deducible in that theory. Let $\vdash_{\text{int}}$ be the relation of intuitionistic deducibility in a first-order language $\mathcal{L}$, and let $\vdash^{\Diamond}_{\text{int}}$ be deducibility in the modal language corresponding to $\mathcal{L}$, by $\vdash_{\text{int}}$, S4.2, the stability axioms for all atomic predicates of $\mathcal{L}$, and the decidability of all atomic formulas of $\mathcal{L}$.[12]

Theorem 2 (Intuitionistic potentialist mirroring)**.** *For any formulas $\varphi_1, \ldots, \varphi_n$, and ψ of $\mathcal{L}$, we have:*

$$\varphi_1, \ldots, \varphi_n \vdash_{int} \psi \quad \textit{iff} \quad \varphi_1^{\Diamond}, \ldots, \varphi_n^{\Diamond} \vdash^{\Diamond}_{int} \psi^{\Diamond}.$$

(See [28] for a proof.)

Together, the two mirroring theorems show how our analysis of quantification over a potentially infinite domain can be separated from the question of whether the

[11] A salient example is the Aristotelian statement, above, rejecting the actual infinity of the natural numbers:

$$\neg\Diamond\forall m\exists n\textsc{Succ}(m, n) \tag{5}$$

This is not in the range of the potentialist translation, and so has no counterpart in the non-modal framework. Moreover, the formula, $\neg\Box\forall m\Diamond\exists n\textsc{Succ}(m, n)$, which is in the range of the translation, is the contradictory opposite of (4)

[12] The intuitionistic modal predicate system must be formulated with some care, since the two modal operators are not inter-definable. See [40] for the details.

appropriate logic is classical or intuitionistic—at least for those kinds of potentiality that have the convergence property (and for which the underlying logic is first-order). Hold fixed our modal analysis of potential infinity, the propositional modal logic S4.2, and the potentialist bridge. Then the appropriate logic of potential infinity depends entirely on the (first-order) logic used in the modal system. Whichever logic we plug in on the modal end—classical or intuitionistic—we also get out on the non-modal end. Since liberal potentialists see no reason to plug in anything other than *classical* first-order logic, they can reasonably regard this as the correct logic for potential infinity, for the cases in question.

3 Applications

We will now describe some applications of the framework presented above.

3.1 Aristotelian potentialism

Let us begin with *Aristotelian potentialism*, that is, the view that even the natural numbers do not form a completed "collection", only a potential one. The view has two parts. First, there is the positive thesis that necessarily, given any natural number, it is possible for there to exist a successor of it. As before, let '$\text{Succ}(m, n)$' express that the immediate successor of m is n. The mentioned view can then be formalized as:

$$\Box\forall m\Diamond\exists n\text{Succ}(m, n) \tag{4}$$

Next, there is the negative thesis that it is impossible for all of the natural numbers to exist simultaneously:

$$\neg\Diamond\forall m\exists n\text{Succ}(m, n) \tag{5}$$

Once again, we can answer the vexed question of the correct logic for ordinary non-modal reasoning about the natural numbers when these are understood as merely potential. Provided our view of the modality is sufficiently robust to warrant the use of classical logic combined with a modal logic at least as strong as S4.2, the mentioned kind of reasoning is governed by classical first-order logic. This is the upshot of our first mirroring theorem. The second such theorem ensures that, if only intuitionistic logic can be assumed in the modal language, then only intuitionistic logic is warranted in the ordinary non-modal language.

It is also instructive to use our framework to locate some kindred views. Geoffrey Hellman's modal structuralism [11] provides an example. The subtle details of the view don't matter for present purposes. Hellman avoids asserting the existence of infinitely many objects. Instead, he asserts the *possible* existence of a model of second-order Dedekind-Peano arithmetic. In effect, this is to assert the contradictory opposite of (5), i.e. $\Diamond \forall m \exists n \textsc{Succ}(m, n)$. In present terms, this is to assert the *possibility of an actual infinity*. This is a strictly stronger modal commitment than that of the Aristotelian potentialist, though still a weaker one than the claim famously disputed by Hilbert [18], namely that there actually exists a completed infinity of objects.

3.2 Set-theoretic potentialism

Cantor famously rejected the Aristotelian ban on actual infinities, which had been the dominant view in mathematics and philosophy up until his time. At times, he appears to endorse the diametrically opposite view that for every potential infinity, there is a corresponding actual infinity:

> ... every potential infinite, if it is to be applicable in a rigorous mathematical way, presupposes an actual infinite ([3], 410–411).

At least at times of his career, however, Cantor retained traces of the old potentialist view, only now applied to the "multiplicity" of all sets rather than the "multiplicity" of natural numbers. In a much quoted letter to Dedekind, in 1899, he wrote:

> [I]t is necessary ... to distinguish two kinds of multiplicities (by this I always mean definite multiplicities). For a multiplicity can be such that the assumption that all of its elements 'are together' leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as 'one finished thing'. Such multiplicities I call *absolutely infinite* or *inconsistent multiplicities* ... If on the other hand the totality of the elements of a multiplicity can be thought of without contradiction as 'being together', so that they can be gathered together into 'one thing', I call it a *consistent multiplicity* or a 'set'. ([8], 931-932)

In other words, although it is possible for all the natural numbers to "exist together", or to form a completed totality, this cannot be said about the "collection" of all sets or the "collection" all ordinals. These are "inconsistent multiplicities" whose members cannot all "coexist". In present terms, the sets and the ordinals are potential collections.

Several more recent thinkers have been inspired by these ideas of Cantor's, inchoate though they may be. There are two main traditions. One is nicely encapsulated in the following passage by Charles Parsons:

> A multiplicity of objects that exist together *can* constitute a set, but it is not necessary that they *do*. ... However, the converse does hold and is expressed by the principle that the existence of a set implies that of all its elements. (pp. 293-4)

This requires some explanation. First, there is the idea that a set exists *potentially* relative to its elements. When the elements of some would-be set exist, we have all that it takes to define or specify the set in question: it is the set of precisely *these things*. Then, there is the related idea that the elements are ontologically prior to their set. The elements can exist although the set does not—much like a floor of a building can exist without the higher floors that it supports. But a higher floor cannot exist without the lower floors that support it. Likewise, a set cannot exist without its elements, which are prior to it and on which the set is therefore ontologically dependent.

This view suggests that any objects potentially form a set. In the language of plural logic:[13]

$$\Box\forall xx\Diamond\exists y\textsc{Set}(xx,y) \tag{7}$$

However, on pain of paradox, we cannot admit a corresponding completed totality; that is, we have:

$$\neg\Diamond\forall xx\exists y\textsc{Set}(xx,y) \tag{8}$$

Notice the parallel with the two theses, (4) and (5) of Aristotelian potentialism.

This "potentialist" view of set theory is interesting, philosophically as well as technically. Potentialist set theories have been developed with (7) at their heart ([42], [26], inspired by [32]). Moreover, by applying the potentialist translation described in Section 2.2, these theories validate either classical or intuitionistic logic, depending on whether the modal logic employed is classical or intuitionistic.

A second tradition takes its departure from Ernst Zermelo's famous 1930 article [46]. Studying models of second-order ZF set theory—henceforth ZF2—which includes a standard replacement axiom, Zermelo comes to the conclusion that the distinction between sets and proper classes is only a relative one: what is a proper class in one model is merely a set from the point of view of a larger model.

> But [the set-theoretic paradoxes] are only apparent 'contradictions', and depend solely on confusing set theory itself, which is not categorically determined by its axioms, with individual models representing it. What appears as an 'ultrafinite non- or super-set' in one model is, in the succeeding model, a perfectly good, valid set with both a cardinal number

[13] See Boolos [2] for the seminal contribution and [27] for a recent survey.

> and an ordinal type, and is itself a foundation stone for the construction of a new domain. ([46], 1233)

Let us spell things out. Consider a model of ZF2 based on a domain M and a membership relation $R \subseteq M \times M$, in terms of which the membership predicate $\in$ is interpreted. The model is said to be *standard* if (i) the membership relation R is well founded, (ii) the model has a maximality property akin to the axiom of separation:

> Consider any a in M. Let X be the collection of objects that bear R to a. Then, for any subcollection $Y \subseteq X$, there must be some b in M such that Y is the collection of objects that bear R to b,

and (iii) a similar clause for replacement holds. Letting M and M^+ range over standard models, Zermelo's idea can be formalized as the following extendability principle:[14]

$$\Box\forall M\Diamond\exists M^+(M^+ \text{ properly extends } M) \tag{EP}$$

This approach to set theory has been developed further by Putnam [34] and Hellman [11]. In particular, they show how this approach too enables us to interpret ordinary first-order set theoretic discourse. To do so, we need a translation from the language of ordinary set theory into the language that talks about possible models and their extensions. A simple example suffices to convey the idea, which is quite intuitive. Consider the claim that for every ordinal there is a greater ordinal: $\forall\alpha\exists\beta(\alpha < \beta)$. This claim is translated as:

> Necessarily, for every standard model and every object α that plays the role of an ordinal in this model, possibly there is an extended standard model containing an object β that also plays the role of an ordinal, and according to which α is smaller than β.

How much set theory does this validate? Sam Roberts [35] provides an answer by formulating a modal structuralist set theory in which a slight strengthening of Zermelo set theory is faithfully interpretable.

3.3 Predicativism

A third view in the foundations of mathematics where potentialist ideas naturally come up is predicativism. This may be surprising, given that predicativism is often

[14] Admittedly, Zermelo's language isn't consistently modal or potentialist. He does write, however, that "every categorically determined domain can also be interpreted as a set" (1232) and describes this step as "a creative advance" (1233).

seen as encapsulated in Russell's Vicious Circle principle, which instructs us that no entity can be defined in a way that quantifies over a totality to which this entity belongs, on the grounds that any such definition would be unacceptably circular. It is not immediately obvious what this non-circularity requirement has to do with potentialist ideas.

However, potentialist ideas figure centrally in other characterizations of predicativism. Some authors connect predicativism closely with the view that some totalities are inherently *potential.* Consider Solomon Feferman:[15]

> ... we can never speak sensibly (in the predicative conception) of the "totality"of all sets as a "completed totality" but only as a *potential totality* whose full content is never fully grasped but only *realized in stages.* ([12], p. 2)

The potentialist framework described above is useful for explicating these ideas.

To see how, consider predicativism relative to the natural numbers. This is the view that takes the natural numbers to be a completed infinity and then proceeds to generate sets of natural numbers in a predicative manner. We thus start with a base world containing all of the natural numbers. We now consider a system of possible worlds which add more and more sets of natural numbers. The essential constraint on this generative process is that the sets we add be given a stable definition, that is, a definition that isn't disrupted as more entities are generated and the domain thus expands.[16] To ensure this definitional stability, it suffices to restrict all quantifiers to sets available at the relevant world.

How might this restriction be effected? The crux is to observe that a little bit of "coding" enables us to use a single set of natural numbers to represent a countable collection of such sets. If X is a set of numbers and n is a number, we define the *n-section of X*, denoted X_n, as $\{x | \langle n, x\rangle \in X\}$. If X is a set-variable and φ is a formula without any occurrences of X, let $\varphi^{<X}$ be the result of restricting the set-quantifiers in φ to the sections of X. That is, we translate $\forall Y \psi(Y)$ as

$$\forall Y \forall z(Y = X_z \to \psi^{<X}(Y)),$$

where z is a new first-order variable. And $\exists Y\, \psi(Y)$ is translated in the obvious dual manner.

We contend that all of the sets that exist at any given world can be "coded up" as the sections of a single set X that exists at some other world. This means that all sets that are predicatively definable, relative to a certain world, are definable by

[15] Other examples can be found Poincaré [33], p. 463.

[16] In fact, this emphasis on stability of definitions goes back to [33].

a formula of the form $\varphi^{<X}$. The desired predicative set formation principle can thus be formulated as:

$$\Diamond\exists Y\Box\forall x(x \in Y \leftrightarrow \varphi^{<X}(x)) \tag{9}$$

where the formula φ does not contain X free, but may contain parameters. It is possible to formulate stronger principles which assert not only the possibility of generating a single, predicatively defined set but of simultaneously generating all sets that are predicatively definable relative to a certain possible world. Call this step a *predicative jump*.

Eventually, we lay down that, for any relation R which by predicatively acceptable means can be proved to be a well-order, it is possible to iterate the predicative jump along R. The exact analysis of this idea is subtle and somewhat controversial, so cannot be discussed here (see [29]).

3.4 Free choice sequenecs

L.E.J. Brouwer's approach to intuitionistic real analysis made crucial use of *free choice sequences*. Each such sequence can be thought of as generated by an ideal mathematician. At any one time, the mathematician has specified some finite initial segment of the sequence, but she always has the ability to go on and specify a larger initial segment. However, it is not in the mathematician's power to complete the specification of the entire sequence. Each choice sequence is thus a potentially infinite object: at each moment, it consists of some finite initial segment, and there is always a possibility of going on.

As realized by Saul Kripke [23] and Joan Moschovakis [31], the idea of choice sequences naturally admits of a modal explication. For instance, while there is no upper bound to how long a sequence α can be, it cannot be infinitely long:

$$\Box\forall n\Diamond\, l(\alpha) \geq n \tag{10}$$

$$\neg\Diamond\Box\forall n(l(\alpha) \geq n) \tag{11}$$

Consider a free choice sequence α and suppose a is an initial segment. Then, for any natural number n, it is possible that α should have n as its next entry:

$$\Box\forall a\forall n\Diamond\exists x(x = a^\frown n)$$

where '$a^\frown n$' is the result of appending n to the end of a.

This brings out a novel phenomenon not encountered in the forms of potentialism discussed above: the generation in question is *indeterministic*. Suppose α has length 10. While it is possible that the 11th entry should be 0, there are many other, incompatible possibilities: if the 11th entry turns out to be 0, it will always remain 0, which means that the possibility of this entry being 1—which existed when the sequence had only 10 entries—has been shut down forever.

This indeterminacy has some important consequences. Most immediately, it means that the convergence property discussed in Section 2.1 fails. And this failure has important knock-on effects. Without convergence, we lose the justification for the axiom G, $\Diamond\Box\phi \rightarrow \Box\Diamond\phi$, and the mirroring theorem is no longer available. There may, however, be other translations from the non-modal language of intuitionistic analysis into our classical modal language. A natural contender is the Gödel translation—although as explained in footnote 9, this is poorly suited to explicate potentiality. But the translation would at least have the effect of rendering the logic of choice sequences intuitionistic.

Much work still remains to be done. First, we need to provide a more complete theory of choice sequences in a classical modal language based on S4 or some related system. Second, it would be good to provide a translation from the non-modal language into the modal one that better captures potentialist ideas.

4 Conclusion

We have outlined a powerful and very general framework for analyzing a wide variety of potentialist ideas. We have made good progress applying this framework to various such ideas, although much work still remains.

References

[1] Aristotle, *The Basic Works of Aristotle*, R. McKeon, ed. Random House, New York, 1941.

[2] George Boolos, To be is to be a value of a variable (or to be some values of some variables), *Journal of Philosophy 81*, 1984, pp. 430-450.

[3] Georg Cantor, Mitteilungen zur Lehre vom Transfiniten 1, 2, *Zeitschrift für Philosophie und philosophische Kritik 91* 1887, pp, 81-125, and vol. 92, pp. 252-270, 250-265; reprinted in [4], 378-439.

[4] — *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*, edited by E. Zermelo, Springer, Berlin, 1932.

[5] Ursula Coope, Aristotle on the Infinite, *The Oxford Handbook of Aristotle*, edited by Christopher Shields, Oxford, Oxford University Press, pp. 267-286, 2012.

[6] Laura Maria Crosilla, *Constructivity and Predicativity: Philosophical Foundations*, University of Leeds, PhD dissertation, 2016.

[7] Michael Dummett, *Frege: Philosophy of Mathematics*, Harvard University Press, Cambridge MA, 1991.

[8] William Ewald, editor, *From Kant to Hilbert: A Sourcebook in the Foundations of Mathematics*, Oxford University Press, Oxford 1996.

[9] Kit Fine, Our knowledge of mathematical objects, in *Oxford Studies in Epistemology I*, edited by T. S. Gendler and J. Hawthorne, Oxford University Press, Oxford, 2005, pp. 89-110.

[10] Karl Friedrich Gauss, Briefwechsel mit Schumacher, *Werke*, Band 8, p. 216, 1831.

[11] Geoffrey Hellman, *Mathematics Without Numbers: Towards a Modal-Structural Interpretation*, Oxford University Press, Oxford, 1989.

[12] Solomon Feferman, Systems of predicative analysis, *Journal of Symbolic Logic 29*, pp. 1–30, 1964.

[13] — Predicativity, in *The Oxford Handbook of Philosophy of Mathematics and Logic*, edited by Stewart Shapiro, Oxford University Press, Oxford, pp. 590-624, 2006.

[14] — On the strength of some semi-constructive theories, in *Proofs, Categories and Computations: Essays in Honor of Grigori Mints*, edited by S. Feferman and W. Sieg, College Publications, London, pp. 109-129, 2010.

[15] — Is the continuum hypothesis a definite mathematical problem, `http://logic.harvard.edu/EFI_Feferman_IsCHdefinite.pdf`

[16] Solomon Feferman, and Geoffrey Hellman, Predicative foundations of arithmetic, *Journal of Philosophical Logic 24*, pp. 1-17, 1995.

[17] — Challenges to Predicative Foundations of Arithmetic, in *Between Logic and Intuition: Essays in Honor of Charles Parsons*, edited by Gila Sher and Richard Tieszen, Cambridge University Press, Cambridge, pp. 317-338, 2000.

[18] David Hilbert, Über das Unendliche, *Mathematische Annalen 95*, pp. 161-190, 1925; translated as On the infinite, in *Philosophy of Mathematics*, second edition, edited by Paul Benacerraf and Hilary Putnam, Cambridge, Cambridge University Press, pp. 183-201, 1983.

[19] Jaakko Hintikka, Aristotelian infinity, *The Philosophical Review 75*, pp. 197-218, 1966.

[20] Stephen Cole Kleene, On the interpretation of intuitionistic number theory, *Journal of Symbolic Logic 10*, pp. 109-124, 1945.

[21] Wilbur R. Knorr, Infinity and Continuity: The Interaction of Mathematics and Philosophy in Antiquity, in *Infinity and Continuity in Ancient and Medieval Thought*, edited by Norman Kretzmann, Cornell University Press, pp. 112–45, 1982.

[22] Georg Kreisel, La prédicativité, *Bulletin de la Societé Mathématique de France 88*, pp. 371-391, 1960.

[23] Saul Kripke, Free choice sequences: a temporal interpretation compatible with acceptance of classical mathematics, *Indagationes Mathematicae 30*, pp. 492-499, 2019.

[24] Jonathan Lear, Aristotelian infinity, *Proceedings of the Aristotelian Society new series 80*, pp. 187-210, 1980.

[25] — Aristotle's philosophy of mathematics, *Philosophical Review 41*, pp. 161-192, 1982.

[26] Øystein Linnebo, The potential hierarchy of sets, *Review of Symbolic Logic 6*, pp. 205-228, 2013.

[27] — Plural quantification, *Stanford Internet Encyclopedia of Philosophy*, `https://plato.stanford.edu/archives/sum2017/entries/plural-quant/`

[28] Øystein Linnebo, and Stewart Shapiro, Actual and potential infinity *Noûs 53*, pp. 160-191, 2019.

[29] — Predicativism and potential totalities, under review.

[30] Fred D. Miller, Aristotle against the atomists, in *Infinity and Continuity in Ancient and Medieval Thought*, edited by Normal Kretzmann, Cornell University Press, Ithaca NY, pp. 87-111, 2014.

[31] Joan Rand Moschovakis, Intuitionistic analysis at the end of time, *Bulletin of Symbolic Logic 23*, pp. 279-295, 2017.

[32] Charles Parsons, What is the iterative conception of set? in *Logic, Foundations of Mathematics and Computability Theory*, edited by R. Butts and J. Hintikka, Dordrecht, Holland, D. Reidel, pp. 335-367, 1977.

[33] Henri Poincaré, "La logique de l'infini", *Revue de Métaphysique et Morale*, pp. 461-482, 1908.

[34] Hilary Putnam, Mathematics without foundations, *Journal of Philosophy 64*, pp. 5-22, 1967.

[35] Sam Roberts, Model structuralism and reflection, *Review of Symbolic Logic* 12, pp. 823-860, 2019.

[36] Bertrand Russell, Mathematical logic as based on a theory of types, *American Journal of Mathematics 30*, pp. 222-262, 1908.

[37] — *Essays in analysis*, George Allen and Unwin Ltd., London, 1973.

[38] Kurt Schütte, *Proof Theory*, translated by J. N. Crossley, Springer-Verlag, Berlin, 1977.

[39] Stewart Shapiro, *Foundations Without Foundationalism: A Case for Second-Order Logic*, Oxford University Press, Oxford, 1991.

[40] Alex K. Simpson, *The Proof Theory and Semantics of Intuitionistic Modal Logic*, Ph.D. Dissertation, University of Edinburgh, 1994.

[41] Richard Sorabji, *Time, Creation and the Continuum: Theories in Antiquity and the Early Middle Ages*, University of Chicago Press, Chicago, 2006.

[42] James P. Studd, *Everything, More or Less: A Defence of Generality Relativism*, Oxford University Press, Oxford, 2019.

[43] Mark van Atten, Predicativity and parametric polymorphism of Brouwerian implication, `https://arxiv.org/pdf/1710.07704.pdf`

[44] Nik Weaver, Predicativity beyond Γ_0, `https://www.math.wustl.edu/~nweaver/gamma0.pdf`.

[45] Timothy Williamson, *Modal Logic as Metaphysics*, Oxford University Press, Oxford, 2013.

[46] Ernst Zermelo, Über Grenzzahlen und Mengenbereiche: Neue Untersuchungen über die

Grundlagen der Mengenlehre, *Fundamenta Mathematicae 16*, pp. 29-47, 1930; translated as, On boundary numbers and domains of sets: New investigations in the foundations of set theory, in [8].

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First-Degree Entailment and Binary Consequence Systems

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Abstract

This paper presents an account of Belnap and Dunn's logic of first-degree entailment and some related logics based on a proof-theoretic machinery of binary (Fmla-Fmla) consequence systems. It is shown how the logic of first-degree entailment can be represented by various deductively equivalent systems, up to a purely structural system with transitivity as the only inference rule. A family of possible extensions of this later system is represented in a systematic manner. This is a review article, which recapitulates certain recent advances in investigating the Belnap-Dunn logic, and organizes the corresponding material from a genuinely first-degree entailment perspective.

Keywords: First-degree entailment, consequence system, binary consequence, logical frameworks, structural reasoning, super-Belnap logics

1 The idea of first-degree entailment

The logic of *first-degree entailment* occupies an important place among modern non-classical logics. As Hitoshi Omori and Heinrich Wansing put it:

> There is a continuum of nonclassical logics, but some systems have emerged as particularly interesting and useful. Among these distinguished nonclassical logics is a system of propositional logic that has become well-known as Belnap and Dunn's *useful four-valued logic* or *first-degree entailment logic*, **FDE**. [25, p. 1021]

The notion of first-degree entailment has been introduced by Nuel Belnap in his doctoral dissertation [5] (see also [25, p. 1021]), and put into circulation in a short abstract of his talk at the twenty-fourth annual meeting of the Association for Symbolic Logic held on Monday, December 28, 1959 at Columbia University in New York [6]. This notion was elaborated then in detail in [2] as a tool for "finding plausible criteria" for valid entailments of the form $\varphi \to \psi$, where φ and ψ are purely truth-functional.

To this effect Belnap defines: "$\varphi \to \psi$ is a *first degree entailment* iff φ and ψ are both written solely in terms of propositional variables, $\wedge$, $\vee$, and $\sim$ (other truth-functional connectives being treated as defined by these)" [6, notation adjusted]. Belnap explains that a first-degree entailment is *valid* if and only if it is *tautological*, which means that it is of the form $\varphi_1 \vee \ldots \vee \varphi_m \to \psi_1 \wedge \ldots \wedge \psi_n$ (or reducible to them by special replacement rules), where every $\varphi_i \to \psi_j$ is an "explicitly tautological entailment". A first-degree entailment is *explicitly tautological* iff it is of the form $\chi_1 \wedge \ldots \wedge \chi_m \to \xi_1 \vee \ldots \vee \xi_n$, where $\chi_1, \ldots, \chi_m, \xi_1, \ldots, \xi_n$ are all atoms (i.e. propositional variables or the negates thereof),[1] and with some atom χ_i being the same as some atom ξ_j. Belnap remarks that tautological entailmenthood is effectively decidable, and observes strong equality between the set of first-degree theorems of the system **E** (of entailment) and the set of tautological entailments, see also [2].

J. Michael Dunn in [10] initiated a highly innovative research program for semantic justification of the first-degree entailments, culminating in his paper [11]. The main point of the program consists in allowing underdetermined and overdetermined valuations that can in certain situations falsify logical laws or verify contradictions, see [31]. One way to achieve this is to treat valuation as a function from the set of sentences of a language to the *subsets* of classical truth values $\{t, f\}$, cf. [11, p. 156].

Consider sentential language $\mathcal{CDN}$ defined as follows:[2]

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid {\sim}\varphi.$$

A generalized valuation is a map v from the set of sentential variables into $\mathcal{P}(\{t, f\})$. This valuation is extended to the whole language by the following conditions:

Definition 1.

(1) $t \in v(\varphi \wedge \psi) \Leftrightarrow t \in v(\varphi)$ *and* $t \in v(\psi)$, $f \in v(\varphi \wedge \psi) \Leftrightarrow f \in v(\varphi)$ *or* $f \in v(\psi)$;

(2) $t \in v(\varphi \vee \psi) \Leftrightarrow t \in v(\varphi)$ *or* $t \in v(\psi)$, $f \in v(\varphi \vee \psi) \Leftrightarrow f \in v(\varphi)$ *and* $f \in v(\psi)$;

(3) $t \in v({\sim}\varphi) \Leftrightarrow f \in v(\varphi)$, $f \in v({\sim}\varphi) \Leftrightarrow t \in v(\varphi)$.

A truth-value function, so defined, produces exactly *four* possible assignments that can be ascribed to a formula φ:

[1] "Atom" is the original term used by Belnap for what nowadays is usually called "literal".

[2] $\mathcal{CDN}$ stands for "conjunction, disjunction, negation".

1. $v(\varphi) = \{t\}$, i.e., $t \in v(\varphi)$ and $f \notin v(\varphi)$;
2. $v(\varphi) = \{f\}$, i.e., $t \notin v(\varphi)$ and $f \in v(\varphi)$;
3. $v(\varphi) = \{t, f\}$, i.e., $t \in v(\varphi)$ and $f \in v(\varphi)$;
4. $v(\varphi) = \{\ \}$, i.e., $t \notin v(\varphi)$ and $f \notin v(\varphi)$.

In this way one arrives at a certain *reinterpretation* of classical truth and falsity on a semantic level. Indeed, if we consider a classical valuation v^c to be a usual map from the set of formulas into the set $\{t, f\}$, then "φ is true" is explicated as $v^c(\varphi) = t$, and "φ is false" is explicated as $v^c(\varphi) = f$. In Dunn's semantics alternatively "φ is true" is explicated as $t \in v(\varphi)$, and "φ is false" is explicated as $f \in v(\varphi)$. As a result, the following principles (for any φ):

$$\varphi \text{ is true, or } \varphi \text{ is false} \tag{\textit{Bivalence}}$$

$$\varphi \text{ is not true, or } \varphi \text{ is not false} \tag{\textit{Univocality}}$$

do *not* generally hold in Dunn's semantics, as they do in classical.

Belnap in his seminal papers [7] and [8] (reproduced in [4] as § 81, see also [26]) famously explicated assignments in Dunn's semantics as new *truth values* (which can be called "generalized truth values", see [34, p. 763]): $N = \varnothing$, $F = \{f\}$, $T = \{t\}$ and $B = \{f, t\}$, thus obtaining a "useful four-valued logic" for a "computer-based reasoning". Truth values for compound formulas are determined by the following matrices:

$\sim$	
T	F
B	B
N	N
F	T

$\wedge$	T	B	N	F
T	T	B	N	F
B	B	B	F	F
N	N	F	N	F
F	F	F	F	F

$\vee$	T	B	N	F
T	T	T	T	T
B	T	B	T	B
N	T	T	N	N
F	T	B	N	F

Moreover, Belnap's four truth values (being ascribed to a sentence φ) can be explicated as follows by Dunn's generalized valuation:

$v(\varphi) = T \Leftrightarrow t \in v(\varphi)$ and $f \notin v(\varphi)$: φ is true only;

$v(\varphi) = F \Leftrightarrow t \notin v(\varphi)$ and $f \in v(\varphi)$: φ is false only;

$v(\varphi) = B \Leftrightarrow t \in v(\varphi)$ and $f \in v(\varphi)$: φ is both true and false;

$v(\varphi) = N \Leftrightarrow t \notin v(\varphi)$ and $f \notin v(\varphi)$: φ is neither true nor false.

If a sentence has the truth value T, it is said to be *exactly true*; if it has one of the values T or B, it can be viewed as *at least true*, and analogously for falsehood. By defining entailment as a relation between sentences, one may rely on a basic understanding that valid inference always preserves truth as well as non-falsity—from a premise to the conclusion. Belnap implements this understanding in such a way that if the premise is *at least* true, so is the conclusion, and if the conclusion is *at least* false, so is the premise (cf. [4, p. 519]). We have thus the following definition:

Definition 2. $\varphi \vDash_{\mathcal{FDE}} \psi =_{df} \forall v : t \in v(\varphi) \Rightarrow t \in v(\psi)$.

Observation 3. *The entailment relation* $\vDash_{\mathcal{FDE}}$ *as defined by Definition 2, explicitly takes* $\{T, B\}$ *as the set of designated truth values. That is, Definition 2 can be reformulated as follows:* $\varphi \vDash_{\mathcal{FDE}} \psi =_{df} \forall v : v(\varphi) \in \{T, B\} \Rightarrow v(\psi) \in \{T, B\}$.

Remarkably, within Dunn's setting, the first half of Belnap's understanding of entailment implies the second, and *vice versa*. Indeed, it can be shown (see, e.g., Proposition 4 in [14]), that the following holds:

Lemma 4. $\varphi \vDash_{\mathcal{FDE}} \psi \Leftrightarrow \forall v : f \in v(\psi) \Rightarrow f \in v(\varphi)$.

Observation 5. *Lemma 4 can be taken as a definition of entailment relation, in which case Definition 2 becomes provable as a lemma. The set of designated truth values is then* $\{T, N\}$*, and definition of entailment can be reformulated as follows:*

$$\varphi \vDash_{\mathcal{FDE}} \psi =_{df} \forall v : v(\varphi) \in \{T, N\} \Rightarrow v(\psi) \in \{T, N\}.$$

It should also be observed, that the set of valid entailments under Definition 2 (and Lemma 4) is exactly the set of tautological entailments in Belnap's sense. In this way, Anderson and Belnap's approach to finding criteria "for picking out from among first-degree entailments ... those that are valid" [2, p. 9] is enshrined in a semantic framework of Dunn's generalized valuation and Belnap's generalized truth values.

As already said, the distinctive feature of this approach consists in explicating entailment as a relation *between truth-functional formulas*, i.e. between *single* formulas, which do not include expressions of entailment in any form. Such a relation can serve then as a general base and starting point for further expansions and generalizations, by considering, e.g., entailments between sets of formulas, entailments between entailments, etc. For a comprehensive overview of various extensions (in the same vocabulary) and expansions of the logic of first-degree entailment see a survey paper [25]. Remarkably, by all these expansions the core relation of first-degree entailment remains basic, providing thus a sound general framework for the whole conception of entailment.

It is noteworthy, that this core relation, as originally introduced by Belnap and Dunn, can itself be extended in different directions, thus giving rise to the *first-degree entailment* *fragments* of various non-classical logics, such as Kleene's strong three-valued logic or Priest's logic of paradox.

In this paper I will explain a specific proof-theoretic machinery of binary consequence systems, which can be seen as most suitable for a deductive formalization of the first-degree entailment relation. I will show how this relation can be adequately axiomatized by a diversity of binary consequence systems, which are all deductively equivalent, and which allow a number of different extensions of the initial relation. A family of binary consequence systems for these extensions will be presented in a systematic manner.

This is a survey article, which recapitulates and summarizes certain work of myself and others (see, in particular [32, 33, 35, 36]), and organizes the corresponding material from a genuinely first-degree entailment perspective. The proofs of lemmata and theorems are omitted and can be restored from the corresponding references if needed.

2 Logical frameworks and binary consequence systems

Once again, first-degree entailments in the strict sense are implicational expressions of the form $\varphi \to \psi$, where φ and ψ can be "truth functions of any degree but cannot contain any arrows" [3, p. 150]. Currently it is more common to employ "binary consequence expressions" of the form $\varphi \vdash \psi$ (to be read as "φ has ψ as a consequence" [12, p. 302]), where $\varphi, \psi \in \mathcal{CDN}$. The corresponding proof systems (called "binary consequence systems" by Dunn [13, p. 24], and "symmetric consequence systems" by Chrysafis Hartonas [20, p. 5]) manipulate binary consequences as formal objects. Such systems are of interest in their own right, as an important particular way of presenting logical structures.

Dunn and Hardegree in [16, p. 185] differentiate between four kinds of consequence relations:

(1) unary assertional systems, $\vdash \phi$;

(2) binary implicational systems, $\phi \vdash \psi$;

(3) asymmetric consequence systems, $\Gamma \vdash \phi$;

(4) symmetric consequence systems, $\Gamma \vdash \Delta$.

It is observed that (1) can be viewed as a special case of (3), and (3) is a special case of (4), whereas (2) is a special case of both (3) and (4). They also remark that "binary implicational systems are perhaps the presentation that most fits the idea of thinking of logics as ordered algebras" [16, p. 186].

In a similar vein, Lloyd Humberstone [22] elaborated on the idea of *logical frameworks* as specific structures for manipulating *sequents* of various kinds. A particular logical framework assigns to each language a class of sequents permissible within this framework, cf. [22, p. 103]. For example, the logical framework SET-FMLA, "takes a sequent ... to have the form $\Gamma \succ B$ where Γ is a finite (possibly empty) set of formulas ... and B is a formula" [22, p. 103]. (Here $\succ$ is a special symbol that "combines formulas into sequents", used by Humberstone as a "sequent separator". Because I deal with consequence expressions, I follow Dunn by employing the sign of a consequence relation $\vdash$ in this place.)

Humberstone, in particular, describes in [22, p. 108] the following logical frameworks, each of which determines the corresponding set of sequents consisting of all expressions $\Gamma \vdash \Delta$ (with Γ and Δ being any finite sets of formulas, maybe empty), subject to the following restrictions:

for SET-SET: no restrictions neither on Γ, nor on Δ;

for SET-FMLA: no restrictions on Γ, $|\Delta| = 1$;

for SET$_1$-FMLA: $|\Gamma| \geq 1$, $|\Delta| = 1$;

for SET-FMLA$_0$: no restrictions on Γ, $|\Delta| \leq 1$;

for FMLA-FMLA: $|\Gamma| = 1$, $|\Delta| = 1$;

for FMLA: $|\Gamma| = 0$, $|\Delta| = 1$.

He observes that there could be further variations of logical frameworks, such as SET$_1$-SET$_1$ or FMLA-SET, and that they all "are specializations of SET-SET in the sense that for any given language, the sequents of that language according to the given framework are all sequents according to SET-SET" [22, p. 108].

Humberstone's classification of logical frameworks includes prominently a separate category for the FMLA-FMLA sequents, considered to be "a suitable setting in which to concentrate on entailment as a binary relation between formulas" [22, p. 108]. In full agreement with the above-cited remark by Dunn and Hardegree, Humberstone [22, p. 246] explains how one can naturally design adequate algebraic semantics for the FMLA-FMLA sequents, based on the relation of pre-order $\leq$ (which can also be restricted to a partial order if needed). In terms of consequences, if one defines a homomorphism h from a given language to a set of propositions, then a binary consequence $\varphi \vdash \psi$ is said to *hold* on this h when $h(\varphi) \leq h(\psi)$, and it is said to be *valid* when it holds on every such homomorphism. It is worth noting that the underlying set of propositions is usually taken to form a *lattice*, which enables us to have conjunction and disjunction in our language. Thus, the FMLA-FMLA consequence systems may play an important role in determining the corresponding algebraic structures, cf. discussion on *relational logical algebra* in [9, Ch. 4].

Let me briefly summarize. First degree entailments in their original and most genuine sense fall into the category of *binary consequence expressions*. A binary consequence expression (or simply binary consequence) is an expression of the form $\varphi \vdash \psi$, where $\varphi, \psi \in \mathcal{CDN}$. A *binary consequence system* can be defined then as a proof system, which manipulates binary consequences as formal objects. A binary consequence is thus a particular case of a Gentzenian 'sequent' where both antecedent and succedent are restricted to singletons. In this respect, a binary consequence system is a "sequent style system" [29, p. 246] constructed in a FMLA-FMLA "logical framework".

A *logic* over the given language (in the FMLA-FMLA setting) is meant to be a set of binary consequences closed at least under the usual Tarskian conditions of *reflexivity* and *transitivity*:

(ref) $\varphi \vdash \varphi$

(tr) $\varphi \vdash \psi,\ \psi \vdash \chi \Rightarrow \varphi \vdash \chi$

as well as Łoś and Suszko's condition of *structurality* (substitution-invariance, for every uniform substitution function s on $\mathcal{CDN}$):

(si) $\varphi \vdash \psi \Rightarrow s(\varphi) \vdash s(\psi).$

Moreover, a logic is said to be *consistent* (or non-trivial) whenever $\vdash \neq \mathcal{CDN} \times \mathcal{CDN}$. A particular consequence system is consistent if the logic generated by this system (the set of all valid consequences derived in this system) is consistent. Clearly, the same logic can be captured by various consequence systems. If several binary consequence systems determine the same set of provable consequences, these systems are said to be *deductively equivalent*.

In the next section I will consider the deductive characterization of the logic of first-degree entailment by various (deductively equivalent) binary consequence systems (more detailed exposition of this material see in [32]).

3 Consequence systems for first-degree entailment and structural reasoning

The logic of first-degree entailment has been first axiomatized in [3, p. 158] by the following binary consequence system:

System $\mathbf{E}_{\mathrm{fde}}$

(ce_1) $\varphi \wedge \psi \vdash \varphi$

(ce_2) $\varphi \wedge \psi \vdash \psi$

(di_1) $\varphi \vdash \varphi \vee \psi$

(di_2) $\psi \vdash \varphi \vee \psi$

(dis_1) $\varphi \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee \chi$

(ni) $\varphi \vdash \sim\sim\varphi$

(ne) $\sim\sim\varphi \vdash \varphi$

(tr) $\varphi \vdash \psi,\ \psi \vdash \chi \,/\, \varphi \vdash \chi$

(ci) $\varphi \vdash \psi,\ \varphi \vdash \chi \,/\, \varphi \vdash \psi \wedge \chi$

(de) $\varphi \vdash \chi,\ \psi \vdash \chi \,/\, \varphi \vee \psi \vdash \chi$

(con) $\varphi \vdash \psi \,/\, \sim\psi \vdash \sim\varphi$

It is essentially the original formulation from [3, p. 158] (with $\vdash$ instead of $\rightarrow$), where this system is characterized as "a Hilbert-style formalism" with "seven axioms and four rules", conceived as "the first degree entailment fragment of the calculus **E**" (hence its name).

This formulation is quite elegant, presenting a complete characterization of each propositional connective by a group of exactly three principles, two of which being the "direct consequences" taken as axioms: (ce_1), (ce_2) for conjunction elimination; (di_1), (di_2) for disjunction introduction; (ni), (ne) for negation introduction and elimination, and another one formulated as a rule of inference, saying how to obtain some consequence with certain connective from other consequence(s): (ci) for conjunction introduction; (de) for disjunction elimination and (con) for negation contraposition. There is also an additional axiom of distributivity (dis_1) reflecting an interconnection between conjunction and disjunction, and one inference rule of transitivity (tr), which involves no propositional connectives and deals purely with the consequence relation as such. Thus, by using Gentzen's terminology, see [19, p. 191], (ci), (de) and (con) are "rules for logical symbols" ("Logische-Zeichen-Schlußfiguren"),[3] and (tr) is the only "structural inference rule" [ibid].

An inference (proof) in $\mathbf{E}_{\text{fde}}$ is a finite list of consequence expressions where every list item is either an axiom or results from preceding elements of the list by an inference rule application. It is well known that four De Morgan laws ($\sim\varphi \wedge \sim\psi \vdash \sim(\varphi \vee \psi)$, $\sim(\varphi \wedge \psi) \vdash \sim\varphi \vee \sim\psi$, $\sim\varphi \vee \sim\psi \vdash \sim(\varphi \wedge \psi)$, $\sim(\varphi \vee \psi) \vdash \sim\varphi \wedge \sim\psi$) are derivable in $\mathbf{E}_{\text{fde}}$. Moreover, $\mathbf{E}_{\text{fde}}$ + $\varphi \wedge \sim\varphi \vdash \psi$ amounts to a consequence system of classical logic, cf. [30, pp. 255-256].[4]

$\mathbf{E}_{\text{fde}}$ is a rather strong system that hardly allows interesting non-classical extensions in the same vocabulary. The only non-classical consistent extension of $\mathbf{E}_{\text{fde}}$ can be obtained by adding to it the axiom $\varphi \wedge \sim\varphi \vdash \psi \vee \sim\psi$ (see, e.g., [11, p. 157 and note 7] and [18, p. 53]), which is the characteristic principle for the first-degree entailment fragments of both "R-mingle" and "E-mingle", the Dunn–McCall systems obtained by extending systems **R** and **E** by the "mingle axiom" $\varphi \rightarrow (\varphi \rightarrow \varphi)$ and "restricted mingle axiom" $(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \psi))$ respectively.

[3] One call them just "logical rules".

[4] Clearly, $\psi \vdash \varphi \vee \sim\varphi$ becomes then derivable by (con).

To make the logic of first-degree entailment more “flexible”, and particularly more sensitive to further interesting extensions, it is possible to give up some of its initial inference rules, and first of all, contraposition. In this way one obtains another well-known formulation of this logic without the contraposition rule, but with De Morgan laws taken instead as axioms. One can find this formulation, e.g. in [14, p. 12]:

System $\mathbf{R}_{\text{fde}}$

(ce_1) $\varphi \wedge \psi \vdash \varphi$

(ce_2) $\varphi \wedge \psi \vdash \psi$

(di_1) $\varphi \vdash \varphi \vee \psi$

(di_2) $\psi \vdash \varphi \vee \psi$

(dis_1) $\varphi \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee \chi$

(ni) $\varphi \vdash {\sim}{\sim}\varphi$

(ne) ${\sim}{\sim}\varphi \vdash \varphi$

(dm_1) ${\sim}(\varphi \vee \psi) \vdash {\sim}\varphi \wedge {\sim}\psi$

(dm_2) ${\sim}\varphi \wedge {\sim}\psi \vdash {\sim}(\varphi \vee \psi)$

(dm_3) ${\sim}(\varphi \wedge \psi) \vdash {\sim}\varphi \vee {\sim}\psi$

(dm_4) ${\sim}\varphi \vee {\sim}\psi \vdash {\sim}(\varphi \wedge \psi)$

(tr) $\varphi \vdash \psi,\ \psi \vdash \chi \,/\, \varphi \vdash \chi$

(ci) $\varphi \vdash \psi,\ \varphi \vdash \chi \,/\, \varphi \vdash \psi \wedge \chi$

(de) $\varphi \vdash \chi,\ \psi \vdash \chi \,/\, \varphi \vee \psi \vdash \chi$

Dunn uses the label $\mathbf{R}_{\text{fde}}$ to highlight the fact that the first-degree entailment fragments of systems **R** and **E** are the same. He also shows that contraposition, although *not derivable* in $\mathbf{R}_{\text{fde}}$, is still *admissible*, see [14, Proposition 11]. Thus, $\mathbf{E}_{\text{fde}}$ and $\mathbf{R}_{\text{fde}}$ are deductively equivalent. Yet, the latter system is in a way weaker than the former, so that it has fewer derivable rules, and allows thus certain non-classical (and non-trivial) extensions, which are impossible with $\mathbf{E}_{\text{fde}}$.

Namely, as stated in [14, Theorem 12], $\mathbf{R}_{\text{fde}}$ + $\varphi \wedge {\sim}\varphi \vdash \psi$ gives us the first-degree entailment fragment of Kleene’s logic, and $\mathbf{R}_{\text{fde}}$ + $\psi \vdash \varphi \vee {\sim}\varphi$ results in the first-degree entailment fragment of Priest’s “logic of paradox”. Clearly, the contraposition rule is no longer admissible in these extensions of $\mathbf{R}_{\text{fde}}$.

Let $\varphi \vdash_{\text{Efde}} \psi$ mean that $\varphi \vdash \psi$ is provable in $\mathbf{E}_{\text{fde}}$, and analogously for $\mathbf{R}_{\text{fde}}$. We have then the following soundness and completeness result for both $\mathbf{E}_{\text{fde}}$ and $\mathbf{R}_{\text{fde}}$ (see, e.g., Theorem 7 in [14]):

Theorem 6. *For every* φ, ψ*:* $\varphi \vdash_{\text{Efde}} \psi \Leftrightarrow \varphi \models_{\mathcal{FDE}} \psi \Leftrightarrow \varphi \vdash_{\text{Rfde}} \psi$.

A direct comparison of $\mathbf{E}_{\text{fde}}$ and $\mathbf{R}_{\text{fde}}$ shows that deleting contraposition from the list of initial inference rules, and compensating it by a set of suitable axioms (specifically, De Morgan laws), allows for more non-classical extensions of the first-degree entailment logic. Remarkably, the analogous methodology can be employed to eliminate the remaining *two* non-structural inference rules of $\mathbf{R}_{\text{fde}}$, for conjunction introduction and disjunction elimination.

Namely, consider the following system of first-degree entailment with conjunction introduction as the only *logical* inference rule (together with the structural rule of transitivity):

System **FDE**(ci)

(ce_1) $\quad \varphi \wedge \psi \vdash \varphi$

(ce_2) $\quad \varphi \wedge \psi \vdash \psi$

(di_1) $\quad \varphi \vdash \varphi \vee \psi$

(dco) $\quad \varphi \vee \psi \vdash \psi \vee \varphi$

(did) $\quad \varphi \vee \varphi \vdash \varphi$

(das) $\quad \varphi \vee (\psi \vee \chi) \vdash (\varphi \vee \psi) \vee \chi$

(dis_2) $\quad \varphi \vee (\psi \wedge \chi) \vdash (\varphi \vee \psi) \wedge (\varphi \vee \chi)$

(dis_3) $\quad (\varphi \vee \psi) \wedge (\varphi \vee \chi) \vdash \varphi \vee (\psi \wedge \chi)$

(dni) $\quad \varphi \vee \psi \vdash {\sim}{\sim}\varphi \vee \psi$

(dne) $\quad {\sim}{\sim}\varphi \vee \psi \vdash \varphi \vee \psi$

(ddm_1) $\quad {\sim}(\varphi \vee \psi) \vee \chi \vdash ({\sim}\varphi \wedge {\sim}\psi) \vee \chi$

(ddm_2) $\quad ({\sim}\varphi \wedge {\sim}\psi) \vee \chi \vdash {\sim}(\varphi \vee \psi) \vee \chi$

(ddm_3) $\quad {\sim}(\varphi \wedge \psi) \vee \chi \vdash ({\sim}\varphi \vee {\sim}\psi) \vee \chi$

(ddm_4) $\quad ({\sim}\varphi \vee {\sim}\psi) \vee \chi \vdash {\sim}(\varphi \wedge \psi) \vee \chi$

(tr) $\quad \varphi \vdash \psi\,,\ \psi \vdash \chi\,/\,\varphi \vdash \chi$

(ci) $\quad \varphi \vdash \psi,\ \varphi \vdash \chi\,/\,\varphi \vdash \psi \wedge \chi$

FDE(ci) is obtained from $\mathbf{R}_{\text{fde}}$ by removing disjunction elimination (de) from the list of initial inference rules, and compensating this loss by a stock of additional axiom schemata, most crucially, "disjunctive versions" of the double negation and De Morgan laws.

The following lemma, which enables us to get rid of redundant disjunctions, is inspired by Proposition 3.2 from [17]:

Lemma 7. *For every schema* (dni), (dne), (ddm_1)–(ddm_4) *of the form* $\alpha \vee \gamma \vdash \beta \vee \gamma$ *the following consequences are derivable in* **FDE**(ci):

(a) $\alpha \vdash \beta$;

(b) $\alpha \wedge \gamma \vdash \beta \wedge \gamma$.

Also the following lemma is important for establishing the adequacy of **FDE**(ci) for the logic of first-degree entailment:

Lemma 8. *The rules of disjunction elimination* (de) *and contraposition* (con) *are admissible in* **FDE**(ci).

System **FDE**(ci) opens the way for more interesting extensions of the logic of first-degree entailment, in particular, it gives a base for axiomatizing the FMLA-FMLA formulation of "exactly true logic" introduced by Andreas Pietz[5] and Umberto Rivieccio in [27]. I will come back to this (and other) extensions of first-degree entailment in the next section.

Duality between the rules of conjunction introduction and disjunction elimination suggests a construction of another version of the logic of first-degree entailment with only one logical inference rule (accompanied by the structural rule of transitivity), but now for disjunction elimination.

System **FDE**(de)

(di_1) $\varphi \vdash \varphi \vee \psi$
(di_2) $\psi \vdash \varphi \vee \psi$
(ce_1) $\varphi \wedge \psi \vdash \varphi$
(cco) $\varphi \wedge \psi \vdash \psi \wedge \varphi$
(cid) $\varphi \vdash \varphi \wedge \varphi$
(cas) $(\varphi \wedge \psi) \wedge \chi \vdash \varphi \wedge (\psi \wedge \chi)$
(dis_4) $(\varphi \wedge \psi) \vee (\varphi \wedge \chi) \vdash \varphi \wedge (\psi \vee \chi)$
(dis_5) $\varphi \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$
(cne) $\sim\sim\varphi \wedge \psi \vdash \varphi \wedge \psi$
(cni) $\varphi \wedge \psi \vdash \sim\sim\varphi \wedge \psi$
(cdm_4) $(\sim\varphi \vee \sim\psi) \wedge \chi \vdash \sim(\varphi \wedge \psi) \wedge \chi$
(cdm_3) $\sim(\varphi \wedge \psi) \wedge \chi \vdash (\sim\varphi \vee \sim\psi) \wedge \chi$
(cdm_2) $(\sim\varphi \wedge \sim\psi) \wedge \chi \vdash \sim(\varphi \vee \psi) \wedge \chi$
(cdm_1) $\sim(\varphi \vee \psi) \wedge \chi \vdash (\sim\varphi \wedge \sim\psi) \wedge \chi$
(tr) $\varphi \vdash \psi\,,\, \psi \vdash \chi \,/\, \varphi \vdash \chi$
(de) $\varphi \vdash \chi\,,\, \psi \vdash \chi \,/\, \varphi \vee \psi \vdash \chi$

By dualizing Lemma 7 one easily obtains:

[5] After he changed his name—Andreas Kapsner.

Lemma 9. *For every schema* (cne), (cni), (cdm_4)–(cdm_1) *of the form* $\alpha \wedge \gamma \vdash \beta \wedge \gamma$ *the following consequences are derivable in* **FDE**(de):

(a) $\alpha \vdash \beta$;

(b) $\alpha \vee \gamma \vdash \beta \vee \gamma$.

Also the following can be proved analogously to Lemma 8:

Lemma 10. *The rules of conjunction introduction* (ci) *and contraposition* (con) *are admissible in* **FDE**(de).

Again, **FDE**(de), having less derivable rules than $\mathbf{R}_{\mathrm{fde}}$ enables further non-trivial and non-classical extensions, which will be considered in the next section in greater detail.

Anderson and Belnap, when characterizing their system $\mathbf{E}_{\mathrm{fde}}$ as a "Hilbert-style formalism", observe: "This formulation suffers, from a proof-theoretical point of view, in having lots of rules; Hilbert would have preferred just one" [3, p. 158]. Well, both **FDE**(ci) and **FDE**(de) rectify this shortcoming significantly by reducing the number of rules to two, but still, they fail to meet "Hilbert's ideal" of *rule singularity*.

Remarkably, both systems have the same structural rule of transitivity (tr) in common, and each of them additionally has its own logical rule absent in the other system. This removability of logical inference rules by replacing them with appropriate axioms for the corresponding propositional connectives suggests another formalization of the first-degree entailment logic with *the only* (structural) inference rule of transitivity, obtained by a straightforward combination of **FDE**(ci) and **FDE**(de). In [33, p 320] such a combined system **FDE**(-) has been proposed as a consequence system that presumably fulfills "Hilbert's dream". In this system disjunctive and conjunctive contexts are attached *separately* to the antecedents and succedents of the consequences for De Morgan laws and the laws of double negation. However, it turns out that this system might not be closed under the rules of disjunction elimination and conjunction introduction, and thus, its deductive equivalence with $\mathbf{E}_{\mathrm{fde}}$ and $\mathbf{R}_{\mathrm{fde}}$ is very much in question.[6]

Therefore consider another "really Hilbertian" consequence system $\mathbf{FDE}_{\mathrm{S}}$, which manipulates *combined* disjunctive-conjunctive contexts as appropriate:

System $\mathbf{FDE}_{\mathrm{S}}$

(di_1) $\quad \varphi \vdash \varphi \vee \psi$

(dco) $\quad \varphi \vee \psi \vdash \psi \vee \varphi$

(did) $\quad \varphi \vee \varphi \vdash \varphi$

$(das^{\vee})$ $\quad (\varphi \vee (\psi \vee \chi)) \vee \xi \vdash ((\varphi \vee \psi) \vee \chi) \vee \xi$

[6] I am grateful to Adam Přenosil for pointing out this fact to me, and for noting that combined contexts fix the problem.

(ce_1) $\varphi \wedge \psi \vdash \varphi$

(cco) $\varphi \wedge \psi \vdash \psi \wedge \varphi$

(cid) $\varphi \vdash \varphi \wedge \varphi$

$(cas^{\wedge})$ $((\varphi \wedge \psi) \wedge \chi) \wedge \xi \vdash (\varphi \wedge (\psi \wedge \chi)) \wedge \xi$

$(dis_2^{\vee\wedge})$ $((\varphi \vee (\psi \wedge \chi)) \vee \xi) \wedge \tau \vdash (((\varphi \vee \psi) \wedge (\varphi \vee \chi)) \vee \xi) \wedge \tau$

$(dis_3^{\vee\wedge})$ $(((\varphi \vee \psi) \wedge (\varphi \vee \chi)) \vee \xi) \wedge \tau \vdash ((\varphi \vee (\psi \wedge \chi)) \vee \xi) \wedge \tau$

$(dis_4^{\vee\wedge})$ $(((\varphi \wedge \psi) \vee (\varphi \wedge \chi)) \vee \xi) \wedge \tau \vdash ((\varphi \wedge (\psi \vee \chi)) \vee \xi) \wedge \tau$

$(dis_5^{\vee\wedge})$ $((\varphi \wedge (\psi \vee \chi)) \vee \xi) \wedge \tau \vdash (((\varphi \wedge \psi) \vee (\varphi \wedge \chi)) \vee \xi) \wedge \tau$

$(ni^{\vee\wedge})$ $(\varphi \vee \psi) \wedge \chi \vdash (\sim\sim\varphi \vee \psi) \wedge \chi$

$(ne^{\vee\wedge})$ $(\sim\sim\varphi \vee \psi) \wedge \chi \vdash (\varphi \vee \psi) \wedge \chi$

$(dm_1^{\vee\wedge})$ $(\sim(\varphi \vee \psi) \vee \chi) \wedge \xi \vdash ((\sim\varphi \wedge \sim\psi) \vee \chi) \wedge \xi$

$(dm_2^{\vee\wedge})$ $((\sim\varphi \wedge \sim\psi) \vee \chi) \wedge \xi \vdash (\sim(\varphi \vee \psi) \vee \chi) \wedge \xi$

$(dm_3^{\vee\wedge})$ $(\sim(\varphi \wedge \psi) \vee \chi) \wedge \xi \vdash ((\sim\varphi \vee \sim\psi) \vee \chi) \wedge \xi$

$(dm_4^{\vee\wedge})$ $((\sim\varphi \vee \sim\psi) \vee \chi) \wedge \xi \vdash (\sim(\varphi \wedge \psi) \vee \chi) \wedge \xi$

(tr) $\varphi \vdash \psi, \psi \vdash \chi / \varphi \vdash \chi$

This system might appear rather bulky, yet it is quite manageable, as the following lemma shows:

Lemma 11. *For axioms $(dis_2^{\vee\wedge})$–$(dm_4^{\vee\wedge})$ of the form $(\alpha \vee \chi) \wedge \xi \vdash (\beta \vee \chi) \wedge \xi$:*

(1) The respective consequences (dis_2)–(dm_4) of the form $\alpha \vdash \beta$ are derivable;

(2) The respective dual consequences $(dis_2^{\wedge\vee})$–$(dm_4^{\wedge\vee})$ of the form $(\alpha \wedge \chi) \vee \xi \vdash (\beta \wedge \chi) \vee \xi$ are derivable;

(3) The respective consequences $(dis_2^{\vee})$–$(dm_4^{\vee})$ of the form $\alpha \vee \chi \vdash \beta \vee \chi$, and $(dis_2^{\wedge})$–$(dm_4^{\wedge})$ of the form $\alpha \wedge \chi \vdash \beta \wedge \chi$ are derivable.

Moreover, standard formulations of associativity for disjunction (das) $\varphi \vee (\psi \vee \chi) \vdash (\varphi \vee \psi) \vee \chi$, and conjunction (cas) $(\varphi \wedge \psi) \wedge \chi \vdash \varphi \wedge (\psi \wedge \chi)$ are derivable as well.

The full admissibility of the inference rules of the first-degree entailment logic in $\mathbf{FDE}_{\mathrm{S}}$ can also be established.

Lemma 12. *Rules (ci), (de), and (con) are all admissible in* $\mathbf{FDE}_{\mathrm{S}}$.

We thus have the following result:

Lemma 13. *Systems* $\mathbf{E}_{\mathrm{fde}}$*,* $\mathbf{R}_{\mathrm{fde}}$*,* **FDE**(ci)*,* **FDE**(de)*, and* $\mathbf{FDE}_{\mathrm{S}}$ *are all deductively equivalent in the sense that each of these systems determines the same set of provable consequences.*

As a direct byproduct of Lemma 13 we get the soundness and completeness of all the systems formulated in this section with respect to the entailment relation determined by Definition 2. Let $\varphi \vdash_{\text{fde}} \psi$ means that consequence $\varphi \vdash \psi$ is provable in any of the systems $\mathbf{E}_{\text{fde}}$, $\mathbf{R}_{\text{fde}}$, $\mathbf{FDE}$(ci), $\mathbf{FDE}$(de), or $\mathbf{FDE}_{\text{S}}$. Then we have:

Theorem 14. *For every* φ, ψ*:* $\varphi \vdash_{\text{fde}} \psi \Leftrightarrow \varphi \models_{\mathcal{FDE}} \psi$.

Among all these systems $\mathbf{FDE}_{\text{S}}$ is of particular interest, being the only purely *structural system* in certain precise sense of the term. This sense essentially goes back to the Gentzenian division between structural rules and rules for logical symbols. As it is well known, Gentzen's sequent calculi comprise both kinds of rules "with small number (usually one) of primitive sequents" [23, p. 1297]. There is also another approach to the sequent calculi construction, which stems directly from the work of Paul Hertz (see, e.g., [21]), according to which "rules have purely structural character and all logical content is contained in primitive sequents" [23, p. 1309].

Schroeder-Heister defines *structural reasoning* as "reasoning in a sequent style system using structural rules only" [29, p. 246]. He stresses the importance of this kind of reasoning as a "subject in its own right", particularly "in the light of modern developments such as logic programming", and also observes that "calculi developed by Paul Hertz in the 1920s are structural systems in this independent sense" [29, p. 247].

In this section a natural way of evolving the logic of first-degree entailment towards structural reasoning has been outlined—from $\mathbf{E}_{\text{fde}}$ up to $\mathbf{FDE}_{\text{S}}$. Namely, it has been shown how we can forgo logical rules due to additional axiom schemata. Starting from $\mathbf{E}_{\text{fde}}$ with a special inference rule for each of the three propositional connectives of language $\mathcal{CDN}$, one can work the way through a system without the rule for negation ($\mathbf{R}_{\text{fde}}$), then to a system without the rule either for disjunction ($\mathbf{FDE}$(ci)) or for conjunction ($\mathbf{FDE}$(de)), and ending up with a system with no logical rules at all ($\mathbf{FDE}_{\text{S}}$). The latter is thus a purely structural sequent style system of the "Hertz-kind".

Moreover, derivations in all these systems are constructed not in tree form but are linearly ordered. An inference (proof) of a consequence is defined here as a finite *consecutive* list of (occurrences of) consequence expressions, each of which either is an axiom or comes by an inference rule from some consequence expressions preceding it in the list (cf. [24, p. 34]). $\mathbf{FDE}_{\text{S}}$ deserves thus attention as a system, which represents *linear structural reasoning*, and can serve as a "proof ground" for investigating the properties of this kind of reasoning.

The section concludes with an example of deriving in $\mathbf{FDE}_{\text{S}}$ consequence, which states the closure of conjunction comutativity under a disjunctive context (note how Lemma 11 is effectively used in this derivation).

$(cco^{\vee})$ $\quad (\varphi \wedge \psi) \vee \chi \vdash (\psi \wedge \varphi) \vee \chi$:

1.	$(\varphi \wedge \psi) \vee \chi \vdash \chi \vee (\varphi \wedge \psi)$	(dco)
2.	$\chi \vee (\varphi \wedge \psi) \vdash (\chi \vee \varphi) \wedge (\chi \vee \psi)$	(dis_2)
3.	$(\chi \vee \varphi) \wedge (\chi \vee \psi) \vdash (\chi \vee \psi) \wedge (\chi \vee \varphi)$	(cco)
4.	$(\chi \vee \psi) \wedge (\chi \vee \varphi) \vdash \chi \vee (\psi \wedge \varphi)$	(dis_3)
5.	$\chi \vee (\psi \wedge \varphi) \vdash (\psi \wedge \varphi) \vee \chi$	(dco)
6.	$(\varphi \wedge \psi) \vee \chi \vdash (\psi \wedge \varphi) \vee \chi$	1–5: (tr), four times

4 Extensions of first-degree entailment: a family of binary consequence systems

$\mathbf{E}_{\mathrm{fde}}$, $\mathbf{R}_{\mathrm{fde}}$, **FDE**(ci), **FDE**(de) and $\mathbf{FDE}_{\mathrm{S}}$ jointly present a rather illustrative example of deductively equivalent systems which nevertheless disagree in their derivable rules of inference. "Downgrading" a rule from a derivable to merely admissible within some system may be useful, opening the door for interesting new extensions of the system, which could be impossible otherwise. The less derivable rules a system has, the more subtle (non-trivial) extensions it allows. For example, whereas adding $\varphi \wedge {\sim}\varphi \vdash \psi$ to $\mathbf{E}_{\mathrm{fde}}$ collapses it into classical logic (due to the rule of contraposition), the same manipulation with $\mathbf{R}_{\mathrm{fde}}$ (where contraposition is not derivable, but merely admissible) produces the first-degree entailment fragment of Kleene's strong three-valued logic, where contraposition is no longer admissible.

Importantly, logic as a set of expressions is always closed under the inference rules of an (underlying) proof system, which explicates the properties of the corresponding consequence relation. Therefore, adding some expression to the given set may result in different outcomes, depending on the system taken to be basic, and thus, on the corresponding closure. It means that any accurate (syntactic) consideration of "logical extensibility" is essentially "system-dependent", so that when we speak of extensions of some "logic", we always have in mind (explicitly or implicitly) a particular *proof system* for this logic, subject to the extensions in question. The stock of derivable rules of a system sets the boundaries of its possible extensions, since such rules are preserved by all these extensions (by contrast with merely admissible rules, which need not remain intact in extended systems).

From this perspective the purely structural system for the logic of first-degree entailment $\mathbf{FDE}_{\mathrm{S}}$ is most promising, opening a way for a fine-tuning of a whole bundle of systems, which can be located between first-degree entailment and classical logic. Indeed, $\mathbf{FDE}_{\mathrm{S}}$ presents the "limiting case" among the deductively equivalent systems for the logic of first-degree entailment, permitting the largest number of its possible extensions.

Consider the following consequences, which are *not* derivable in any of the first-degree entailment systems deductively equivalent to $\mathbf{FDE}_{\mathrm{S}}$:

(ds) $\quad {\sim}\varphi \wedge (\varphi \vee \psi) \vdash \psi$
(dds) $\quad \varphi \vdash {\sim}\psi \vee (\psi \wedge \varphi)$
(efq) $\quad \varphi \wedge {\sim}\varphi \vdash \psi$
(veq) $\quad \varphi \vdash \psi \vee {\sim}\psi$
$(efq^{\vee})$ $\quad (\varphi \wedge {\sim}\varphi) \vee \psi \vdash \psi$
$(veq^{\wedge})$ $\quad \varphi \vdash (\psi \vee {\sim}\psi) \wedge \varphi$
(saf) $\quad \varphi \wedge {\sim}\varphi \vdash \psi \vee {\sim}\psi$
$(saf^{\vee\wedge})$ $\quad (\varphi \wedge {\sim}\varphi) \vee \chi \vdash (\psi \vee {\sim}\psi) \wedge \chi$
$(saf^{\wedge})$ $\quad (\varphi \wedge {\sim}\varphi) \wedge \chi \vdash (\psi \vee {\sim}\psi) \wedge \chi$
$(saf^{\vee})$ $\quad (\varphi \wedge {\sim}\varphi) \vee \chi \vdash (\psi \vee {\sim}\psi) \vee \chi$

Label (ds) stands for "disjunctive syllogism", whereas (dds) marks its dual version. Consequences (efq) and (veq) are the famous "ex falso quodlibet" and "verum ex quodlibet", and (saf) stands for "safety", see explanations in [15, p. 443]. Indexing these rules with $\wedge$ or $\vee$ (or both) signifies the conjunctive and disjunctive variations thereof.

A family of *structural consequence systems*, which are all extensions of $\mathbf{FDE}_{\mathrm{S}}$ (a structural FDE-family), can be defined as follows:

$\mathbf{SM}_{\mathrm{S}} = \mathbf{FDE}_{\mathrm{S}} + (saf)$; $\quad \mathbf{RM}_{\mathrm{S}} = \mathbf{FDE}_{\mathrm{S}} + (saf^{\vee\wedge})$;
$\mathbf{RM}^{\wedge}_{\mathrm{S}} = \mathbf{FDE}_{\mathrm{S}} + (saf^{\wedge})$; $\quad \mathbf{RM}^{\vee}_{\mathrm{S}} = \mathbf{FDE}_{\mathrm{S}} + (saf^{\vee})$;
$\mathbf{EFQ}_{\mathrm{S}} = \mathbf{FDE}_{\mathrm{S}} + (efq)$; $\quad \mathbf{VEQ}_{\mathrm{S}} = \mathbf{FDE}_{\mathrm{S}} + (veq)$;
$\mathbf{ETL}_{\mathrm{S}} = \mathbf{FDE}_{\mathrm{S}} + (ds)$; $\quad \mathbf{NFL}_{\mathrm{S}} = \mathbf{FDE}_{\mathrm{S}} + (dds)$;
$\mathbf{K3}_{\mathrm{S}} = \mathbf{FDE}_{\mathrm{S}} + (efq^{\vee})$; $\quad \mathbf{LP}_{\mathrm{S}} = \mathbf{FDE}_{\mathrm{S}} + (veq^{\wedge})$;
$\mathbf{RM}^{f}_{\mathrm{S}} = \mathbf{RM}_{\mathrm{S}} + (efq)$; $\quad \mathbf{RM}^{v}_{\mathrm{S}} = \mathbf{RM}_{\mathrm{S}} + (veq)$;
$\mathbf{K3}^{v}_{\mathrm{S}} = \mathbf{K3}_{\mathrm{S}} + (veq)$; $\quad \mathbf{LP}^{f}_{\mathrm{S}} = \mathbf{LP}_{\mathrm{S}} + (efq)$;
$\mathbf{SCL}_{\mathrm{S}} = \mathbf{RM}_{\mathrm{S}} + (veq), (efq)$; $\quad \mathbf{CL}_{\mathrm{S}} = \mathbf{FDE}_{\mathrm{S}} + (efq^{\vee}), (veq^{\wedge})$.

Let me shortly characterize systems from the structural FDE-family. This family consists of the purely structural systems for the first-degree entailment fragments of various logics, some of which are well known, some less known, and some are almost completely unknown.

In particular, $\mathbf{SM}_{\mathrm{S}}$, $\mathbf{RM}^{\wedge}_{\mathrm{S}}$, $\mathbf{RM}^{\vee}_{\mathrm{S}}$, and $\mathbf{RM}_{\mathrm{S}}$ are different mingle-logics obtained from $\mathbf{FDE}_{\mathrm{S}}$ by extending it with various versions of safety. Since *both* conjunction introduction and disjunction elimination are not derivable in $\mathbf{FDE}_{\mathrm{S}}$, extending it with simple safety

(saf) results in a very weak logic, which can be called a "submingle"-system $\mathbf{SM}_{\mathrm{S}}$. The distinctive feature of $\mathbf{SM}_{\mathrm{S}}$ is a non-derivability of the following consequences:

$$(\varphi \wedge {\sim}\varphi) \vee (\psi \wedge {\sim}\psi) \vdash (\chi \vee {\sim}\chi), \tag{1}$$

$$(\varphi \wedge {\sim}\varphi) \vdash (\psi \vee {\sim}\psi) \wedge (\chi \vee {\sim}\chi), \tag{2}$$

$$(\varphi \wedge {\sim}\varphi) \vee (\psi \wedge {\sim}\psi) \vdash (\chi \vee {\sim}\chi) \wedge (\xi \vee {\sim}\xi). \tag{3}$$

Now, (1) becomes derivable in $\mathbf{RM}^{\vee}_{\mathrm{S}}$, (2) in $\mathbf{RM}^{\wedge}_{\mathrm{S}}$, and $\mathbf{RM}_{\mathrm{S}}$ restores the first-degree entailment fragment of R-mingle in full generality.

$\mathbf{K3}_{\mathrm{S}}$ and $\mathbf{LP}_{\mathrm{S}}$ are purely structural first-degree entailment fragments of Kleene's strong three-valued logic and Priest's logic of paradox respectively. $\mathbf{VEQ}_{\mathrm{S}}$ and $\mathbf{ETL}_{\mathrm{S}}$ represent certain narrowings of the first of these logics, whereas $\mathbf{EFQ}_{\mathrm{S}}$ and $\mathbf{NFL}_{\mathrm{S}}$ are narrowings of the second. Remarkably, $\mathbf{ETL}_{\mathrm{S}}$ is a binary consequence system of Pietz and Rivieccio's exactly true logic, and $\mathbf{NFL}_{\mathrm{S}}$ is a binary consequence system of the "non-falsity logic" from [35].

Systems $\mathbf{RM}^{f}_{\mathrm{S}}$, $\mathbf{RM}^{v}_{\mathrm{S}}$, $\mathbf{K3}^{v}_{\mathrm{S}}$, $\mathbf{LP}^{f}_{\mathrm{S}}$ are the results of extending the corresponding systems either with (efq), or with (veq). $\mathbf{CL}_{\mathrm{S}}$ is a structural system for the binary first-degree entailment relation of classical logic. $\mathbf{SCL}_{\mathrm{S}}$ is a peculiar "sub-classical" logic, which is *almost classical*, with the only difference that it is not closed under conjunction introduction and disjunction elimination.

The relations between these systems (including $\mathbf{FDE}_{\mathrm{S}}$) constitute a lattice of structural FDE-family as presented in Figure 1. This lattice outlines a general framework for logics based on the first-degree entailment, with benchmark systems defined above, and (infinitely) many other systems which can be placed between (some of) them.

Now consider the following definition:

Definition 15. Let φ, ψ be any sentences of $\mathcal{CDN}$, and let φ^d be obtained from φ by interchanging between $\wedge$ and $\vee$, and replacing every atomic sentence with its negation (and likewise for ψ and ψ^d). Then $\psi^d \vdash \varphi^d$ is said to be *dual* to $\varphi \vdash \psi$. Logical system S is *self-dual* iff $\varphi \vdash_{\mathsf{s}} \psi \Leftrightarrow \psi^d \vdash_{\mathsf{s}} \varphi^d$; logical systems S_1 and S_2 are *mutually dual* iff $\varphi \vdash_{\mathsf{s}_1} \psi \Leftrightarrow \psi^d \vdash_{\mathsf{s}_2} \varphi^d$ (for any φ and ψ).

We have then the following result:

Lemma 16. *Systems* $\mathbf{FDE}_{\mathrm{S}}$*,* $\mathbf{SM}_{\mathrm{S}}$*,* $\mathbf{RM}_{\mathrm{S}}$*,* $\mathbf{SCL}_{\mathrm{S}}$ *and* $\mathbf{CL}_{\mathrm{S}}$ *are self-dual. The following pairs of systems are mutually dual:* $\mathbf{RM}^{\wedge}_{\mathrm{S}}$*–*$\mathbf{RM}^{\vee}_{\mathrm{S}}$*,* $\mathbf{EFQ}_{\mathrm{S}}$*–*$\mathbf{VEQ}_{\mathrm{S}}$*,* $\mathbf{ETL}_{\mathrm{S}}$*–*$\mathbf{NFL}_{\mathrm{S}}$*,* $\mathbf{RM}^{f}_{\mathrm{S}}$*–*$\mathbf{RM}^{v}_{\mathrm{S}}$*,* $\mathbf{K3}_{\mathrm{S}}$*–*$\mathbf{LP}_{\mathrm{S}}$*,* $\mathbf{K3}^{v}_{\mathrm{S}}$*–*$\mathbf{LP}^{f}_{\mathrm{S}}$*.*

Various extensions of the logical construction of Belnap and Dunn (called also "super-Belnap" logics) has received a considerable attention recently, see, e.g., [1, 28], where

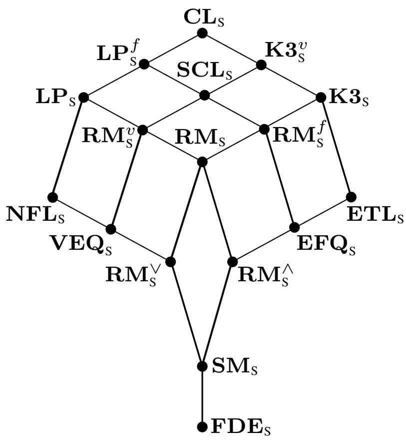

Figure 1: Lattice of structural FDE-family

this issue was approached primarily from an algebraic standpoint, concentrating on logical systems of the SET-FMLA type belonging mainly to the path evolving between **SM** and **K3**. In particular, it has been shown in [28] that there are infinitely many logics between Belnap-Dunn's and Kleene's logics. These results can be straightforwardly adjusted to the FMLA-FMLA framework, and also extend to the dual versions of the systems under consideration.

Namely, consider a series of axiom schemata:

$(efq_n) \quad (\varphi_1 \wedge {\sim}\varphi_1) \vee \ldots \vee (\varphi_n \wedge {\sim}\varphi_n) \vdash \psi,$

defined for any $n \geq 1$, and let system $\mathbf{EFQ}_{\mathrm{S}_n}$ be obtained by adding (efq_n) to $\mathbf{EFQ}_\mathrm{S}$ (clearly, $\mathbf{EFQ}_{\mathrm{S}_1}$ is just **EFQ**). It can be shown that (efq_{n+1}) is not derivable in $\mathbf{EFQ}_{\mathrm{S}_n}$, for any $n \geq 1$. Thus, there exists a denumerable chain of systems

$$\mathbf{EFQ}_\mathrm{S} < \mathbf{EFQ}_{\mathrm{S}_2} < \ldots < \mathbf{EFQ}_{\mathrm{S}_n} < \ldots \mathbf{EFQ}_{\mathrm{S}_\infty} < \mathbf{ETL}_\mathrm{S},$$

such that $\mathbf{EFQ}_{\mathrm{S}_n} < \mathbf{EFQ}_{\mathrm{S}_{n+1}}$ for any $n \geq 1$ (where $<$ is proper inclusion with respect to provable consequences). Analogously, one can define $\mathbf{ETL}_{\mathrm{S}_n}$ by adding (efq_n) to $\mathbf{ETL}_\mathrm{S}$, and obtain the corresponding infinite chain of systems between $\mathbf{ETL}_\mathrm{S}$ and $\mathbf{K3}_\mathrm{S}$. Likewise, there are infinitely many systems between $\mathbf{EFQ}_\mathrm{S}$ and $\mathbf{RM}^f_\mathrm{S}$.

Moreover, there are infinitely many systems between $\mathbf{VEQ}_\mathrm{S}$ and $\mathbf{NFL}_\mathrm{S}$, $\mathbf{VEQ}_\mathrm{S}$ and $\mathbf{RM}^v_\mathrm{S}$, as well as between $\mathbf{NFL}_\mathrm{S}$ and $\mathbf{LP}_\mathrm{S}$. These new chains are obtained by using additional rules

$(veq_n) \quad \varphi \vdash (\psi_1 \vee {\sim}\psi_1) \wedge \ldots \wedge (\psi_n \vee {\sim}\psi_n),$

and defining corresponding systems for every $n \geq 1$.

Note, that systems $\mathbf{SM}_{\mathrm{S}}$, $\mathbf{RM}_{\mathrm{S}}^{\wedge}$, $\mathbf{RM}_{\mathrm{S}}^{\vee}$, and $\mathbf{RM}_{\mathrm{S}}$ constitute a "diamond of mingle-logics", which is a sublattice of the lattice of structural FDE-family. This sublattice deserves special consideration, in particular, by analysis of an infinity of systems between $\mathbf{SM}_{\mathrm{S}}$ and $\mathbf{RM}_{\mathrm{S}}$ obtained by involving series of axioms like:

$(saf_n^{\vee})$ $(\varphi_1 \wedge {\sim}\varphi_1) \vee \ldots \vee (\varphi_n \wedge {\sim}\varphi_n) \vdash \psi \vee {\sim}\psi$, and

$(saf_n^{\wedge})$ $\varphi \wedge {\sim}\varphi \vdash (\psi_1 \vee {\sim}\psi_1) \wedge \ldots \wedge (\psi_n \vee {\sim}\psi_n)$.

These and other possible extensions of $\mathbf{FDE}_{\mathrm{S}}$ are worthy of special consideration.

References

[1] Hugo Albuquerque, Adam Přenosil, and Umberto Rivieccio *An algebraic view of super-Belnap logics*, Studia Logica 105, 2017, pp. 1051-1086.

[2] Alan Ross Anderson, and Nuel D. Belnap Jr. *Tautological entailments*, Philosophical Studies 13, 1962, pp. 9-24.

[3] — Entailment: The Logic of Relevance and Necessity vol. I, Princeton University Press, Princeton, NJ 1975.

[4] Alan Ross Anderson, Nuel D. Belnap Jr., and Jon Michael Dunn Entailment: The Logic of Relevance and Necessity vol. II, Princeton University Press, Princeton, NJ 1992.

[5] Nuel D. Belnap Jr. The Formalization of Entailment, University doctoral dissertation, Yale 1959.

[6] — *Tautological entailments (abstract)*, Journal of Symbolic Logic 24, 1959, p. 316.

[7] — *A useful four-valued logic*, in: Jon Michael Dunn, and George Epstein (eds.), Modern Uses of Multiple-Valued Logic, D. Reidel Publishing Company, Dordrecht & Boston MA 1977, pp. 5-37.

[8] — *How a computer should think*, in: Gilbert Ryle (ed.), Contemporary Aspects of Philosophy, Oriel Press 1977, pp. 30-55.

[9] Haskell B. Curry Foundations of Mathematical Logic, McGraw-Hill Book Company, New York, Etc. 1963.

[10] Jon Michael Dunn The Algebra of Intensional Logics, Doctoral Dissertation, University of Pittsburgh, Ann Arbor, 1966 (University Microfilms). (Published as: Jon Michael Dunn, The Algebra of Intensional Logics, with an introductory essay by Katalin Bimbó, Logic PhDs, vol. 2, College Publications, 2019.)

[11] — *Intuitive semantics for first-degree entailment and coupled trees*, Philosophical Studies 29, 1976, pp. 149-168.

[12] — *Positive modal logic*, Studia Logica 55, 1995, pp. 301-317.

[13] — *A comparative study of various model-theoretic treatments of negation: a history of formal negation*, in: Dov Gabbay, and Heinrich Wansing (eds.), What is Negation?, Kluwer Academic Publishers, Dordrecht 1999, pp. 23-51.

[14] — *Partiality and its dual*, Studia Logica 66, 2000, pp. 5-40.

[15] — *Contradictory information: too much of a good thing*, Journal of Philosophical Logic 39, 2010, pp. 425-452.

[16] Jon Michael Dunn, and Gary Hardegree Algebraic Methods in Philosophical Logic, Clarendon Press, Oxford, New York, Etc. 2001.

[17] Josep Maria Font *Belnap's four-valued logic and De Morgan lattices*, Logic Journal of the IGPL 5, 1997, pp. 413-440.

[18] Bas van Fraassen *Gentlemen's wagers: Relevant logic and probability*, Philosophical Studies 43, 1983, pp. 47-61.

[19] Gerhard Gentzen *Untersuchungen über das logische Schließen. I*, Mathematische Zeitschrift 39, 1935, pp. 176-210.

[20] Chrysafis Hartonas *Order-dual relational semantics for non-distributive propositional logics*, Logic Journal of the IGPL 25, 2017, pp. 145-182.

[21] Paul Hertz *Über Axiomensysteme für beliebige Satzsysteme*, Mathematische Annalen 101, 1929, pp. 457-514.

[22] Lloyd Humberstone The Connectives, MIT Press, Cambridge MA 2011.

[23] Andrzej Indrzejczak *A survey of nonstandard sequent calculi*, Studia Logica 102, 2014, pp. 1295-1322.

[24] Lloyd Humberstone Mathematical Logic, Wiley, New York 1967.

[25] Hitoshi Omori, and Heinrich Wansing *A 40 years of FDE: an introductory overview*, Studia Logica 105, 2017, pp. 1021-1049.

[26] — (eds.) New Essays on Belnap-Dunn Logic, Springer, Cham 2019.

[27] Andreas Pietz, and Umberto Rivieccio *Nothing but the Truth*, Journal of Philosophical Logic 42, 2013, pp. 125-135.

[28] Umberto Rivieccio *An infinity of super-Belnap logics*, Journal of Applied Non-Classical Logics 422, 2012, pp. 319-335.

[29] Peter Schroeder-Heister *Resolution and the origins of structural reasoning: early proof-theoretic ideas of Hertz and Gentzen*, Bulletin of Symbolic Logic 8, 2002, pp. 246-265.

[30] Yaroslav Shramko *A modal translation for dual-intuitionistic logic*, The Review of Symbolic Logic 9, 2016, pp. 251-265.

[31] — *Truth, falsehood, information and beyond: the American plan generalized*, in: Katalin Bimbó (ed.), J. Michael Dunn on Information Based Logics, Springer 2016, pp. 191-212.

[32] — *First-degree entailment and structural reasoning*, in: Hitoshi Omori, and Heinrich Wansing (ed.), New Essays on Belnap-Dunn Logic, Springer, Cham 2019, pp. 311-324.

[33] — *Dual-Belnap logic and anything but falsehood*, Journal of Applied Logics – IfCoLoG Journal of Logics and their Applications 6, 2019, pp. 413-433.

[34] Yaroslav Shramko, Jon Michael Dunn, and Tatsutoshi Takenaka *The trilaticce of constructive truth values*, Journal of Logic and Computation 911, 2001, pp. 761-788.

[35] Yaroslav Shramko, Dmitry Zaitsev, and Alexander Belikov *First-degree entailment and its relatives*, Studia Logica 105, 2017, pp. 1291-1317.

[36] — *The* FMLA-FMLA *axiomatizations of the exactly true and non-falsity logics and some of their cousins*, Journal of Philosophical Logic 48, 2019, pp. 787-808.

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Point-free Theories of Space and Time

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Abstract

The paper is in the field of Region Based Theory of Space and Time (RBTST). This is an extension of the Region Based Theory of Space (RBTS) in which we incorporate also time. RBTS is a kind of point-free theory of space based on the notion of *region*. Another name of RBTS is *mereotopology*, because it combines notions and methods of mereology and topology [71]. The origin of this theory goes back to some ideas of Whitehead, De Laguna and Tarski to build the theory of space without the use of the notion of point. More information on RBTS, mereotopology and their applications can be found in [76, 8, 42, 66]. The notion of *contact algebra* [26] presents an algebraic formulation of RBTS and in fact gives axiomatizations of the Boolean algebras of regular closed sets of various classes of topological spaces with an additional relation of *contact*. *Dynamic contact algebra* (DCA) is introduced by the present author in [77, 78, 79] and can be considered as an algebraic formulation of RBTST. It is a generalization of contact algebra studying regions changing in time and presents a formal explication of Whitehead's ideas of integrated point-free theory of space and time. DCA is an abstraction of a special *dynamic model of space*, called also *snapshot* or *cinematographic* model. In the present paper we introduce a simplified version of DCA with the aim to be used as a representative example of DCA and to develop for this example not only the snapshot models but also topological models and the expected topological duality theory, generalizing in a certain sense the well known Stone duality for Boolean algebras. Due to these

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models DCA can be called also *dynamic mereotopology.* Abstract topological models of DCAs present a new view on the nature of space and time and show what happens if we are abstracting from their metric properties.

Preface

The present work can be considered as a continuation of the essay 'Region-Based Theory of Space: Algebras of Regions, Representation Theory and Logics' ([76]). The essay contains a short history of the Region-Based Theory of Space (RBTS) and a survey of the corresponding literature (till 2006), an exposition of the mathematical apparatus of this approach based on contact algebras and a description of some propositional spatial logics related to RBTS. In this approach 'region-based' means that the notion of region, taken as an abstraction of material or geometric body, is considered as one of the base notions of the theory. The theory is also 'point-free' in a sense that the typical geometric notion of 'point' is not considered as a primitive (undefinable) notion of the theory and should be defined in a later stage of the theory. Later on we consider RBTS and 'point-free theory of space' as synonyms.

The motivation of the point-free approach to the theory of space was formulated for the first time by Alfred North Whitehead in 1915 in his lecture *Space, Time, and Relativity* (published as chapter VIII of [87]). In the same lecture Whitehead also claims that the same approach should also be applied to the theory of time, and, motivated by the relativity theory, that the theory of time should not be developed separately from the theory of space and they both should be developed in one integrated point-free theory of space and time. In this context 'point-free' means that neither space points, nor time points (instances of time, moments) are considered as primitive notions of the theory.

The present essay is devoted mainly to the *point-free theories of space and time* and so is the title. Point-free theories of space and time are also 'region-based' because they consider changing or moving regions. So, we consider also another equivalent name: Region-Based Theory of Space and Time - RBTST.

The text of the paper is structured as follows. *Section 1* is the *Introduction.* We start with some discussion about point-free theory of space and time and present with more details the discussions about the nature of space and time between Leibnitz and Newton, Leibnitz's relational view on space and time and Newton's *absolute space* and *absolute time.* We consider the Whitehead's viewpoint on this subject and his motivations why the theory of space and time should be point-free and region-based. We describe shortly Whitehead's contributions to this idea and some other sources and finally we present our concrete strategy of how to build an integrated point-free theory of space and time. In *Section 2* we summarize some facts of *con-*

tact algebras and *precontact algebras* taken from [25, 76, 30] to be used later on. In *Section 3* we introduce a concrete point-based model of *dynamic space* called *snapshot model* or *cinematographic model*. This model is used as a source of motivated axioms for a various versions of the abstract notion of dynamic contact algebra. *Section 4* is devoted to the abstract notion of one special version of dynamic contact algebra (DCA), considered as a representative example of DCA. The main result in this section is the representation theorem of DCA by means of snapshot models. In *Section 5* we introduce topological point-based models called *dynamic mereotopological spaces* (DMS) and develop the intended topological representation theory. *Section 6* is devoted to the expected topological duality theory for DCAs and DMSes, generalizing the famous Stone Duality Theorem for Boolean algebras. *Section 7* is for some conclusions, discussions and open problems. In a separate Appendix we present a very short survey of results on RBTS obtained after 2007 making in this way a more close connection with the present essay [76].

We consider [70], [35] and [58] as standard reference books correspondingly for Boolean algebras, topology and category theory.

1 Introduction

1.1 Point-based and point-free theories of space and time

In mathematics the theory of space is identified with *geometry* which includes various geometrical disciplines. Well-known example is the classical Euclidean geometry. Typical for all axiomatically presented geometries is that they follow the standard Euclidean approach to consider the notion of 'point' as one of the basic undefinable notions of the theory and similarly for the notions 'strait line' and 'plane'. Sometimes strait lines and planes are considered as certain sets of points satisfying some additional axioms, so, point in geometry is always a primitive notion. But neither points, nor strait lines and planes have a separate existence in reality, so the truths for these notions do not correspond to some observational truths for the real things. In a sense 'points', 'straight lines' and 'planes' are some kind of *suitable fictions* and it is not good to put fictions on the base of the so respectable mathematical theory as geometry, considered as a certain theory of reality. This issue gives rise to serious discussions, which we will comment on below.

So, what is a point-free theory of space? Contemporary example is the point-free topology [50]. Standardly topology is considered as an abstract theory of space formalizing the notion of continuity and is considered as a set of points with some distinguished subsets called open sets. Instead, point-free topology is based on lattice theory considering the members of the lattice representing open sets. In general by a point-free theory of space we mean an axiomatic theory of space in which the notion of point is not assumed as a primitive notion. For a given (point-based) geometry, for instance Euclidean geometry, its point-free reformulation means it to be reaxiomatized equivalently on a point-free basis of primitive notions. This means that points are nor disregarded at all but are given by certain definitions in the new axiomatization. Among the first authors who criticized the standard Euclidean point-based approach to the theory of space and appealing to a point-free bases for the theory I can mention Whitehead [87, 88, 89, 90, 91], De Laguna [53, 54, 55] and Tarski [73].

According to time we can say that there is no specific pure mathematical area like geometry, which is devoted exclusively to to the theory of time. Only some investigations on temporal logic (TL) (see, for instance, [5]) introduced the so called *time structures* devoted to a separate study of time. Time structures are systems in the form $(T, \prec)$, where T is a nonempty set whose elements are called *time points* or *moments of time* and $\prec$ is a binary relation between time points called *before-after relation*, reading: $i \prec j$ - i is before j, or equivalently j is after i (other relations between time points are also possible). Such structures are studied to be used as a semantics of TL. The before-after relation may satisfy various sets of some meaningful conditions which fact makes possible to have various different time structures and hence different TL systems. If, for instance, T is the set of real numbers and $\prec$ is the strong inequality $<$, then $(T, \prec)$ is called 'real time structure', and similarly for 'rational' or 'integer (discrete) time structure'. Thus, by definition all temporal structures of the above kind are point-based. But moments of time, like space points, also are some abstract fictions without a separate existence in reality. So the problem to avoid time points in TL also exists. And indeed there are TL systems with a more realistic semantics based on *time intervals* and some relations between them according to their possible positions to each other. However, the intuition of time intervals and their interrelations is based on their representation as ordered pairs of time points (x, y) such that $x \prec y$ and $x \neq y$, and x, y taken from some linearly ordered time structure (for instance real numbers). So, time intervals and their interrelations again are reduced to time points. There is also a point of view to consider interval structures as intuitively more clear and to extract from their structure the notion of time point and a kind of before-after relation. But time intervals are also 'suitable fictions', abstract tings, so the above criticism also holds.

Both time and space are central notions in physics, but physics takes its mathematical apparatus from mathematics (unless we can treat mathematical physics just as a part of mathematics). Newtonian physics adopts, for instance, Newtonian notions of *absolute space* and *absolute time* considered them independent from the material things, independent from each other and having a separate existence in reality (see for this view, for instance [34, 49]). In relativistic physics space and time are not independent and are considered as one spacetime system. In special relativity, this is the Minkowski spacetime in which points are called events and are identified with tuples of real numbers (x_1, x_2, x_3, x_4) where x_1, x_2, x_3 are meant as space coordinates of the event and x_4 is meant as its time coordinate. So in Minkowski spacetime time is the fourth coordinate, which makes the system to be four dimensional with 3 spatial dimensions and one time dimension. Minkowski spacetime differs from the 4-dimensional Euclidean space because it has a different metrics convenient for describing special relativity in which gravitation is not considered (see the paper [68] which discusses the relationship between Einstein's Special relativity [33] and Minkowki's notion of spacetime.)

An axiomatic presentation of Minkowskian spacetime geometry is given by A. A. Robb in [67]. Robb's system has only two primitive notions: 'instant' intuitively meant as a spacetime point and the 'before-after' relation between spacetime points interpreted intuitively as a *causal ordering* of things. Robb named his relation *after* and its converse *before* and presented for it an appealing illustration by means of the Euclidean conic model of 3-dimensional Minkowski spacetime, which motivated him to call this relation a *conic order*. Because *after* is a temporal relation and space features (as well as all other notions of the system) are definable by it, this fact motivates Robb to state that time is more fundamental than space and to call his system *geometry of time and space* putting time on the first place. Probably this shows in a certain sense that both time and space are based on a more deep concept of causality. Spacetime systems based on before-after relation interpreted as a causality relation are called *causality theories of spacetime* (see, for instance [93]).

A readable axiomatic treatment of Minkowski spacetime and some related spacetimes based on a more natural and classically oriented basis of primitive concepts is given by R. Goldblatt in [37]. Modal logics with a relational semantics based on some versions of Minkowski spacetime relation 'after' are also studied — see Goldblatt [36] and Shehtmann [69].

General relativity theory is a generalization of special relativity by assuming the effects of gravitation. An intensive research on axiomatic foundations of relativity theories is initiated by a Hungarian group of logicians organized by I. Nemeti and H. Andreka [2]. But, let us note again, both Newtonian and relativistic spacetime theories are not point-free and the problem of their point-free reformulation is still open (the situation in quantum physics is still unclear).

Spacetime systems in which space and time are considered together like in relativity theory are used in applied mathematics for describing certain systems of dynamically changing spatial objects. Such spacetime systems are combinations of some spatial structure (geometry) and some temporal structure (theory of time). For one such construction of concrete spacetime system see, for instance, [52]. It was based on the so called *snapshot construction* and it is natural to be named *snapshot spacetime*. As a rule such spatio-temporal systems are also point-based, so their point-free reaxiomatization is an open problem. Later on we will discuss such systems with more details and will consider them as a starting point for various versions of an integrated point-free theory of space and time.

1.2 Relational theory of space and time: Newton, Leibniz and Whitehead

The question of whether points of space and time have to be considered as real things, raises hot philosophical discussions and puts the more serious question whether space and time itself are also 'suitable fictions'. A typical example is the discussion between Leibnitz and Newton about the nature of space and time. Leibnitz' position is known now as the 'relational view of space and time': space and time are mathematical fictions and the tings in reality are connected by some spacetime relations and the mathematical theories of space and time just describe the properties of these relations. Space expresses the coexistence of things, while time expresses an order of successive things. Newton's position advocates the view of 'absolute space' and 'absolute time' discussed in the previous section (for more details about the discussion between Leibniz and Newton see, for instance, [34, 49]).

At the beginning of 20th Century probably the first who adopted in some form Leibnitz's relational view of space and time and formulated the problem of its correct mathematical reinterpretation as a point-free theory of space and time was Alfred North Whitehead.

Whitehead is well-known among logicians as a co-author with Bertrand Russell in their famous book Principia Mathematica, published in three volumes in 1910-1913 and dedicated to the foundation of mathematics [92]. It is said in the preface of volume III of the book that geometry is reserved for the final volume IV. But probably due to some disagreements between the authors about the nature of space (and probably of time), volume IV had not been written.

The best articulation of the original Whitehead's view about space and time is given in the following quote (pages 194,195 of [87]) of Whitehead's lecture *Space, Time, and Reality*:

> "...We may conceive of the points of space as self-subsistent entities which have the indefinable relation of being occupied by the ultimate stuff (matter, I will call it) which is there. Thus, to say that the sun is there (wherever it is) is to affirm the relation of occupation between the set of positive and negative

> electrons which we call the sun and a certain set of points, the points having an existence essentially independent of the sun. This is the absolute theory of space. The absolute theory is not popular just now, but it has very respectable authority on its side Newton, for one so treat it tenderly. The other theory is associated with Leibnitz.
>
> Our spare concepts are concepts of relations between things in space. Thus there is no such entity as a self-subsistent point. A point is merely the name for some peculiarity of the relations between the matter which is, in common language, said to be in space.
>
> It follows from the relativity theory that a point should be definable in terms of the relations between material things. So far as I am aware, this outcome of the theory has escaped the notice of mathematicians, who have invariably assumed the point as the ultimate starting ground of their reasoning. Many years ago I explained some types of ways in which we might achieve such a definition, and more recently have added some others. Similar explanations apply to time. Before the theories of space and time have been carried to a satisfactory conclusion on the relational basis, a long and careful scrutiny of the definitions of points of space and instants of time will have to be undertaken, and many ways of effecting these definitions will have to be tried and compared. This is an unwritten chapter of mathematics, in much the same state as was the theory of parallels in the eighteenth century."

It can be concluded from this quote that Whitehead accepted Leibnitz's relational theory of space and time in a more relaxed form: we have to build the theory of space staring from more realistic primitive notions avoiding points, lines and planes and introducing them by suitable definitions. From his other writings, for instance from his main philosophical book Process and Reality [91] (which we will discuss with more details after words) such more realistic notions are regions as abstractions of material bodies and some natural relations between them. In contemporary terminology the above quote is nothing but a program for building of a point-free theory of space, and also for building of an integrated point-free theory of space and time as it is considered in relativity theory. From the phrase

> "This is an unwritten chapter of mathematics, in much the same state as was the theory of parallels in the eighteenth century"

we may conclude that Whitehead considered this as a difficult and a serious problem. This problem has two forms, first, concerning only space, and second, concerning both space and time taken together. Since geometry as a theory of space exists as a branch of mathematics separately from the theory of time, this is the problem to build the point-free theory of space. And since the theory of time appeared mostly in mathematical physics as an integrated theory of space and time - this is just the related problem to build an integrated point-free theory of space and time.

1.3 Whitehead's contribution and other roots of point-free theories of space and time

In the lecture *The Anatomy of Some Scientific Ideas* (Chapter VII in the same book cited above [88]) Whitehead describes, among others, how such a 'point-free theory' should be built. First he considers as a base notion the notion of 'event' a feature existing in space and in time. Second, the theory should be based on the theory of 'whole and a part' (named by other authors mereology - see, for instance [71] and more recently [65] and [83]) and definitions of the 'points of time' and 'points of space' to be done by his 'principle of convergence', renamed in his later publications by 'the method of extensive abstraction'.

An attempt to present such a theory is given in the Whitehead's books [88] and [89]. This attempt was criticized from philosophical and from methodological points of view by De Laguna in the papers [53, 54, 55], where he presented his own approach for point-free theory of space based on mereology. De Laguna's system has the primitives *solid* as an abstraction of physical body and a ternary relation between solids named *can connect*. Intuitively the solids a, b and c are in the relation *can connect* if a can be moved so that to connect b and c. Here *to connect* means to touch or to overlap. De Laguna showed how to define points, lines and surfaces using a modification of Whitehead's method of extensive abstraction. We will not comment De Laguna's critical remarks, but it has to be mentioned that Whitehead considered them seriously and changed radically his system, published in Process and Reality (**P**&**R**) [91] (see page 440 of **P**&**R** [91] where Whitehead correctly gives credits to De Laguna's criticism and comments how to avoid the defects of his approach to the definition of point presented in [88] and [89]). Instead of De Laguna's notion of *solid* Whitehead uses the term *region* with the same intuitive meaning, and instead of the De Laguna's ternary relation *can connect* he used the simplified binary relation of *connection* (called in the recent literature *contact*). The main idea of Whitehead's new approach is described in Part IV of the book - 'The theory of extension' and the mathematical details are presented in Chapters II and III of **P**&**R**. The exposition is almost mathematical and consists of a series of enumerated definitions and assumptions without any attempt 'to reduce these enumerated characteristics to a logical minimum from which the remainder can be deduced by strict deduction' (p. 449). By means of the connection relation, Whitehead defines in Chap. II some other relations between regions: part-of, overlap, external connection, and tangential inclusion. Chapter II ends with the definition of a point (Def. 16). Chapter III contains all preliminary formal definitions and assumptions needed in the definitions of a straight line (Def. 6) and definition of a plane (Def. 8) as certain sets of regions using the method of extensive abstraction. Because the text is sketchy these two chapters of **P&R** have to be considered as an *extended program* containing all needed details in order to develop Whitehead's new theory of space in a strictly mathematical manner. Namely, this is what is called now the root of 'region-based theory of space' (RBTS), or equivalently - point-free theory of space. Another root is, of course, De Laguna's papers [53, 54, 55], but still De Laguna's system has no precise contemporary interpretation with adequate models and representation theory. As another root it has to be mentioned Tarski [73], who developed a point-free version of Euclidean geometry called 'Foundations of the geometry of solids'. It is based on mereology extended with the primitive notion of ball which is used in the definition of point. Also we owe to Tarski the reinterpretation of mereology (the mereological system of Lesniewski) to the notion of Boolean algebra (BA) (namely complete BA with deleted zero) and also the good topological model of complete BA as algebra of regular open (or regular closed) subsets of a topological space. In an algebra of regular closed sets solids (or regions) are just the regular closed sets and the relation of 'contact' has a very natural definition - having a common point. These facts can be considered as the roots of the first definitions of the notion of contact algebra (CA) as an extension of BA with the contact relation (for the history of CA see [76]). Now the version of CA from [26] is commonly considered as the simplest point-free formulation of RBTS with standard models the algebras of regular closed sets of topological spaces. This fact motivates some authors to use another name of RBTS - *mereotopology* - a combination of mereology with topology: the BA represents mereologycal component and the contact relation which has a topological nature represents the topological component of the system.

Let us mention that RBTS as a point-free approach to the theory of space can be considered now as a well established branch of mathematics with applications in computer science which is open for further research. For the results of RBTS till 2006 see our essay [76] as well as the survey papers [8, 66], and [42] which contains also information of applications of RBTS in computer science. Some possibly incomplete information on the further development of RBTS and some related topics after 2007 is given in the Appendix of this paper.

Let us return to the integrated point-free theory of space and time. As we have mentioned spacetime systems from mathematical physics are not point-free and the Whitehead's early program formulated in his lecture *Space, Time, and Relativity* can be considered as a kind of program or a wish to build such a theory. Whitehead's view on the nature of time developed in his books [87, 88, 90, 91] is mainly philosophical and changed over years. For instance in [87, 88] he uses a more common time terminology: instances of time, moments, but in [90, 91] he renamed his theory of time as 'epochal theory of time' (ETT) considering *epochs* as certain atomic instances of time. Probably the reason for this new terminology is that the Whitehead's notion of epoch is one of the central notions of his later theory of time. Whitehead did not propose how ETT can be formalized and integrated with the point-free theory of space. Unlike his quite detailed program for building point-free mathematical theory of space, presented in **P&R** Whitehead did not describe analogous program for his ETT. He introduced and analyzed many notions related to ETT but mainly in an informal way using his own quite complicated philosophical terminology which makes extremely difficult to obtain clear mathematical theory corresponding to ETT.

An attempt to build a theory incorporating both space and time was recently made in [31, 32], but the system is not point-free with respect to time: time points are presented directly in the system.

1.4 The first attempts in building of an integrated point-free theories of space and time and a possible strategy for such a task

Having in mind the situation about building an integrated point-free theory of space and time discussed at the end of the previous section, the present author decided to make the first steps in building such a theory (or examples of such theories). The results till now appeared in the series of papers started from 2010: [77, 78, 79], and (jointly with P. Dimitrov) in [13]. Because the notions of space and time are so rich, our aim in this project was to start with a simple system describing in a point free manner (some aspects of) both space and time and their mutual relationships, and then to refine the system step by step removing some defects and extending its expressive power. First we had to find a strategy how to build such systems and what requirements they should satisfy in order to treat them as point-free axiomatic systems of space and time.

We found that the following requirements will be useful.

1. **In order to follow Whitehead style the system should be region-based and should be based on mereology**. Regions will correspond to changing or moving objects and following Tarski the regions should form a Boolean algebra.

2. **The regions should be equipped with a number of basic spatio-temporal relations with well motivated meaning.** The relations are called basic because they have to be used in the definitions of some other meaningful relations. The meaning of the basic relations should be determined by an appropriate

set of axioms. What does this mean? - see the next two requirements:

3. **The system should have a meaningful standard adequate set-theoretical point based spacetime model describing the change of regions and the meaning of the spatio-temporal relations**. 'Meaningful' means that the model is in accordance with our point-based spatial and temporal intuition which we obtained during our basic education in mathematics and physics. 'Standard' means that we consider that this model give the intended point-based intuition of the basic relations.

4. **'Adequate' in 3. means that we can extract from the system in a canonical way a standard model, called the 'canonical model of the system', and to define an isomorphism mapping of the system into its canonical model**. Here 'to extract' means to define both space points and time points within the system and also all other ingredients needed to construct the model. 'To construct the model' means to use only the axioms of the system and standard set-theoretical constructions. So, the theory should have the form of ordinary axiomatical mathematical theory.

5. **The main problem in realization of 2 and 4 is how to find the needed axioms.** This is the most difficult part of the realization of the program. One way, which we follow, is to start with the standard model and to proof for it enough statements considered further as possible axioms. But which true sentences to accept as axioms? Practically this is the following informal task: make an initial hypothesis of the possible steps of the construction of the canonical model and look for the axioms which are needed to prove the correctness of the given step. If the required axioms are not in the list, see if they are true in the standard model and add them to the list. This is a long experimental mathematical procedure which is not always successful, and, as Whitehead commented in the quote from section 1.2, 'many attempts have to be done in order to obtain a satisfying result'.

If we succeed in the realization of the above five requirements then obviously the resulting system will be point-free, the standard models indeed will be models of the system and the isomorphism of the system into its canonical model will show that the choice of the axioms is successful and that the standard point-based model and the point-free axiomatic systems are in certain sense equivalent. The expressivity power of the system will depend on the choice of the basic spatio-temporal relations between regions, so further steps of improving the system is to consider larger and a richer system of basic relations.

As we have seen, the realization of such a strategy is to start with the standard point-based model of spacetime and to find a successful construction of space points, time points and other ingredients of the model. Whitehead does this by his method of 'extensive abstraction' which results to a complicated constructions.

In contemporary mathematics, for instance in the Stone representation theory of Boolean algebras [72] and the theory of proximity spaces [60, 74] there are more good methods for defining abstract points: ultrafilters, clans, clusters and others. The success of the realization of the above scheme depends also on what kind of concrete point-based model is chosen to start with. Because standard point-based models are concrete constructions involving space points and time points, we adopted a special construction called 'snapshot construction' and the resulting models - called 'snapshot spacetime models'. This is a very simple and intuitive construction which we mentioned in Section 1.1 [52]. Intuitively the snapshot construction is a formalization and generalization of the real method of describing an area of changing objects by making a video: for each moment of time the video camera makes a snapshot of the current spatial configurations of the objects and the series of the snapshots can be used to construct the point based spacetime model of change (see Remark 3.2 about the limitation of the analogy of the method of 'snapshot construction with making video).

The first paper [77] from the above mentioned series of papers was experimental - we just wanted to see if the above described strategy works. That is why we included only two spatio-temporal relations between changing objects which do not suppose that time flaws: $aC^{\forall}b$ - stable contact (a and b are always in a contact) and $aC^{\exists}b$ - unstable contact (a and b are sometimes in a contact). The paper [78] makes the next step assuming that time flaws and in the point based model the moments of time are equipped with 'before-after' relation. It contains two relations which do not depend on before-after relation: *space contact* $aC^{s}b$ - there is a moment of time in which a and b are in a space contact, *time contact* $aC^{t}b$ - there is a moment of time in which a and b exist simultaneously. The third relation, called *preceding* just uses the before-after relation: there is a moment s in which a exists and a later moment t, $s \prec t$, in which b exists. This is a quite rich system for space and time, but it was not able to describe *past* , *present* and *future*. This was possible in the system from [79] in which we added the notion of the so called time representative, a region existing only at a given moment of time, or epoch in Whitehead's terminology, which is using as a name of the corresponding epoch, for instance 'the epoch of Leonardo'. The paper [13] studies some new spacetime systems extending the system from [79] with new axioms and some propositional (quantifier-free) logics based on these systems. Other results in this direction are included in the papers [63] and [61, 62] which generalize [77] putting the system on pure relational base and without operations on regions.

In this paper, starting from Section 3, we will present with some details one not very complicated spacetime system just in order to show how the method works. The new thing is that we will supply the system not only with snapshot models, but also with topological models which will give more information on the nature of space points and time points.

2 Contact and precontact algebras

In this section we summarize some facts about contact and precontact algebras which are needed later on. We assume a familiarity of the reader with the basic theory of Boolean algebras, filters, ideals, ultrafilters and the Stone representation of Boolean algebra by ultrafilters. We consider only non-degenerate Boolean algebras, i.e. algebras with $0 \neq 1$.

2.1 Definitions of contact and precontact algebras

Definition 2.1. Contact algebra [26]. *Let $(B, 0, 1, \leq, +, ., *)$ be a Boolean algebra with complement denoted by $*$ and let C be a binary relation in B. C is called a* ***contact*** *relation in B if the following axioms are satisfied:*

(C1) If aCb then $a \neq 0$ and $b \neq 0$,

(C2) If aCb and $a \leq a'$ and $b \leq b'$ then $a'Cb'$,

(C3') If $aC(b+c)$ then aCb or aCc, (C3") If $(a+b)Cc$ then aCc or bCc,

(C4) If aCb then bCa,

(C5) If $a.b \neq 0$ then aCb.

We write $\overline{C}$ for the complement of C. If C is a contact relation in B, then the algebra $A = (B, C)$ is called a contact algebra.

If we do not assume axioms (C4) and (C5), then C is called a ***precontact*** *relation in B and the pair (B, C) is called a precontact algebra.*

If $A = (B, C)$ is a precontact (contact) algebra then we will write also $A = (B_A, C_A)$, where $B_A = (B, 0, 1, \leq, +, ., *)$ and $C_A = C$.

In this paper we will consider also Boolean algebras with several precontact and contact relations satisfying some interacting axioms.

Let us mention that if we assume (C4) only one of the axioms (C3') and (C3") is needed. Note also that (C5) is equivalent (on the base of the precontact axioms) to the following more simple axiom

(C5') If $a \neq 0$ then aCa.

From (C5') and (C1) it follows that $a \neq 0$ iff aCa.

In the present context we treat the Boolean part of the contact algebra as its *mereological component* and the contact relation - as its *mereotopological component.*

In our treating of mereology we consider the zero element 0 as a *non-existing region* and this can be used to define the ontological predicate of existence $E(a)$: 'a ontologically exists', in the following way:

$E(a)$ iff $a \neq 0$.

For simplicity, instead of 'ontologically exists' we will say simply 'exists' and from the context it will be clear that this is not the existential quantifier.

The definitions of mereological relations 'part-of' and 'overlap' are the following:

• a is part of b iff $a \leq b$, i.e. part-of is just the Boolean ordering,

• a overlaps b (in symbols $a\mathrm{O}b$) iff there exists a region $c \neq 0$ such that $c \leq a$ and $c \leq b$ iff $a.b \neq 0$.

Note that by the definition of overlap the axiom (C5) can be presented in this way: $a\mathrm{O}b$ implies aCb.

Remark 2.2. *It is easy to see that the relation O of overlap satisfies all axioms of contact relation and by axiom (C5) it can be considered as the smallest contact in B. Non-degenerate Boolean algebras have also another contact C_{max} definable by "$a \neq 0$ and $b \neq 0$". It follows by axiom (C1) that this is the largest contact in B.*

By means of the contact relation we may reproduce the definitions of some mereotopological relations considered by Whitehead:

• *external contact:* $aC^Eb \leftrightarrow_{def} aCb$ and $a.b = 0$, the common points of a and b are on their boundaries.

• *non-tangential inclusion* $a \ll b \leftrightarrow_{def} a\overline{C}b^*$, called also deep inclusion - a is included in b not touching the boundary of b.

• *tangential inclusion:* $a \leq^T b \leftrightarrow_{def} a \leq b$ and $a \not\ll b \leftrightarrow a \leq b$ and aCb^*, a is included in b and touches the boundary of b.

Intuitive examples: A cup on a table is in an external contact with the table. If a nail is driven into the table then it is tangentially included into the table. If the nail is deeply embedded into the table so that its head is not seen, then the nail is non-tangentially included in the table.

Contact relation has the following interesting property, stated in the next lemma.

Lemma 2.3. *(*[77], Lemma 1.1. (vi)) For any $a, b, p, q \in B$: if pCq and $a\overline{C}b$ then either $(p.a^*)C(q.a^*)$ or $(p.b^*)C(q.b^*)$.

Precontact algebras were considered under the name of proximity algebras in [30]. We will be interested later on contact and precontact algebras satisfying the following additional axiom:

(CE) If $a\overline{C}b$ then $(\exists c)(a\overline{C}c$ and $(c^*\overline{C}b)$.

This axiom is sometimes called *Efremovich axiom*, because it is used in the definition of *Efremovich proximity spaces* [60]. Let us note that the largest contact C_{max} satisfies the Efremowich axiom.

2.2 Examples of contact and precontact algebras

Topological example of contact algebra. The intended example of contact algebra is a topological one and can be defined in the following way. Let X be a topological space and Cl and Int be the operations of closure and interior of a subset of X. A set $a \subseteq X$ is called *regular closed* if $a = Cl(Int(a))$. The set $RC(X)$ of regular closed subsets of X is a Boolean algebra with respect to the following operations and constants: $0 = \emptyset$, $1 = X$, $a + b = a \cup b$, $a.b = Cl(Int(a \cap b))$, $a^* = Cl(X \setminus a) = Cl(-a)$. The algebra $RC(X)$ becomes a contact algebra with respect to the the following contact relation $C_X : aC_Xb$ iff $a \cap b \neq \emptyset$, i.e. if a and b have a common point. The contact algebra $RC(X)$ and any contact subalgebra of $RC(X)$ is considered as a standard topological contact algebra. In the next section we will see that each contact algebra is isomorphic to a standard contact algebra.

Remark 2.4. *Defining regions as regular closed sets is a good choice, because all known good geometrical regions in Euclidean geometry are regular closed sets of points: balls, cubes, pyramids, etc. Let us note, however, that some authors do not agree with this definition showing examples of regular closed sets in Euclidean geometry which can not be considered intuitively as regions. That is true, but the situation is quite similar to the formal $\epsilon - \delta$-definition of continuous function - there are counterintuitive examples but, nevertheless, the definition is accepted.*

Relational examples of precontact and contact algebras. Let X be a non-empty set, whose elements are considered as points and R be a reflexive and symmetric relation in X. Pairs (X, R) with reflexive and symmetric R are called by Galton *adjacency spaces* (see [30]).

One can construct a contact algebra from an adjacency space as follows: take a class B of subsets of X which form a Boolean algebra under the set-theoretical operations of union $a + b = a \cup b$, intersection $a.b = a \cap b$ and complement $a^* = X \setminus a$ and define contact C_R between two members of B as follows: aC_Rb iff there exist $x \in a$ and $y \in b$ such that xRy. It can easily be verified that all axioms of contact are satisfied.

Let us note that there are more general adjacency spaces in which neither reflexivity nor symmetry for the relation R are assumed (see [30]). We reserve the name 'adjacency space' for such more general spaces and for the special case where R is a reflexive and symmetric relation we will say 'adjacency spaces in the sense of Galton'. If we repeat the above construction then the axioms (C1), (C2), (C3') and (C3") will be true but in general the axioms (C4) and (C5) will not be satisfied and in this way we obtain examples of precontact algebras which are not contact algebras. The relational models of contact and precontact algebras are called also *discrete models.*

The following lemma will be of later use:

Lemma 2.5. Characterization of reflexivity, symmetry and transitivity. [30] *Let* (X, R) *be an adjacency space and* $(B(X), C_R)$ *be the precontact algebra over all subsets of* X*. Then the following conditions hold:*

(i) R *is a symmetric relation in* X *iff* $(B(X), C_R)$ *satisfies the axiom (C4) If* aC_Rb *then* bC_Ra*,*

(ii) R *is reflexive relation in* X *iff* $(B(X), C_R)$ *satisfies the axiom (C5) If* $a.b \neq \emptyset$ *then* aCb*,*

(iii) R *is a transitive relation in* X *iff* $(B(X), C_R)$ *satisfies the axiom*

(CE) If $a\overline{C}b$ *then* $(\exists c)(a\overline{C}c$ *and* $c^*\overline{C}b)$*.*

In the proof of the above lemma the following equivalent definition of the precontact relation aC_Rb will be helpful. For a subset $a \subseteq X$ define $\langle R\rangle a =_{def} \{x \in X : (\exists y \in a)(xRy)\}$. Then obviously we have: aC_Rb iff $a \cap \langle R\rangle b \neq \varnothing$. The operation $\langle R\rangle a$ comes from the relational semantics of modal logic and represents the operation of possibility (for more information for this connection see [4]). The following property of the operation $\langle R\rangle a$ can be proved: R is transitive relation on X iff for all $a \subseteq X$: $\langle R\rangle\langle R\rangle a \subseteq \langle R\rangle a$. Then by pure set-theoretical transformations one can show that the Efremovich axiom (CE) is equivalent to this property, which proves (iii).

2.3 Algebras with several precontact relations

In this section we will introduce Boolean algebras with two precontact relations satisfying two special interacting axioms which will be used in the definition of dynamic contact algebra. First, we will present their relational examples.

Let (W, R, S) be a relational system with two relations. We consider the following two first-order conditions for R and S:

$(R \circ S \subseteq S)$ If xRy and ySz, then xSz (The composition of R with S is included in S).

$(S \circ R \subseteq S)$ If xSy and yRz, then xSz (The composition of S with R is included in S).

The system (W, R, S) defines in an obvious way set-theoretical Boolean algebra with two precontact relations C_R and C_S.

Consider the following two conditions for the precontact relations C_R and C_S which are similar to the Efremowich axiom (CE):

(C_RC_S) If $a\overline{C}_Sb$, then there exists $c \subseteq W$ such that $a\overline{C}_Rc$ and $c^*\overline{C}_Sb$, and

(C_SC_R) If $a\overline{C}_Sb$, then there exists $c \subseteq W$ such that $a\overline{C}_Sc$ and $c^*\overline{C}_Rb$.

We call the conditions (C_RC_S) and (C_SC_R) **compositional axioms** for C_R and C_S.

Lemma 2.6. *(i) The condition (C_RC_S) is fulfilled between precontact relations C_R and C_S iff the condition $(R \circ S \subseteq S)$ is satisfied,*

(ii) The condition (C_SC_R) is fulfilled between precontacts relations C_R and C_S iff the condition $(S \circ R \subseteq S)$ is satisfied.

The proof is similar to the proof of Lemma 2.5 (iii). In the proof of (i) use the following equivalences: $(R \circ S \subseteq S)$ iff for all $a \subseteq X$ $\langle R\rangle\langle S\rangle a \subseteq \langle S\rangle a$ iff (C_RC_S) and similarly for (ii) by exchanging the places of R and S.

2.4 Discrete (relational) representation of contact and precontact algebras.

One way to obtain a representation theory of precontact algebras with relational representation of precontact is to consider ultrafilters as the set of abstract points of a given precontact algebra $A = (B, C)$ (as in the Stone representation theory of Boolean algebras) and to define the relation R in the set of ultrafilters $Ult(A)$ of A as follows. For $U, V \in Ult(A)$:

$URV \leftrightarrow_{def} (\forall a, b \in B)(a \in U \text{ and } b \in V \Rightarrow aCb)$.

For $a \in B$ define also the Stone embedding: $s(a) = \{U \in Ult(A) : a \in U\}$.

Definition 2.7. *The relational system $(Ult(A), R)$ with just defined R is called a* canonical adjacency space *over A and R is called the* canonical adjacency relation *on $Ult(A)$.*

Note that the definition of the canonical relation R is meaningful for arbitrary filters. In order to prove some facts for the canonical relation some constructions of filters and ideals will be needed and some technical lemmas have to be introduced.

First we remind the well known Separation Lemma for filters and ideals in Boolean algebra and the Extension Lemma for proper filters.

Lemma 2.8. *(i)* **Separation Lemma**. *If F is a filter and I is an ideal in a Boolean algebra such that $F \cap I = \varnothing$, then there exists an ultrafilter U such that $F \subseteq U$ and $U \cap I = \varnothing$.*

(ii) **Extension Lemma**. *Every proper filter can be extended into an ultrafilter.*

The sum of two filters: If F and G are filters, then $F \oplus G =_{def} \{a.b : a \in F, b \in B\}$ is the smallest filter containing both F and G. $0 \in F \oplus G$ iff there exists $a \in F$ and $a^* \in G$.

Lemma 2.9. Technical lema for the canonical relation. *Let $A = (B, C)$ be a precontact algebra, F and G be filters in A and FRG be the canonical relation between them corresponding to C. Define the following sets:*
$I_1^C(F) = \{b : (\exists a \in F)(a\overline{C}b)\}$, $I_2^C(G) = \{a : (\exists b \in G)(a\overline{C}b)\}$,
$F_1^C(F) = \{b : (\exists a \in F)(a\overline{C}b^*)\}$, $F_2^C(G) = \{a : (\exists b \in G)(a^*\overline{C}b)\}$.
Then the following equivalencies are true:
(i) FRG iff $I_1^C(F) \cap G = \varnothing$, and $I_1^C(F)$ is an ideal.
(ii) FRG iff $F \cap I_2^C(G) = \varnothing$, and $I_2^C(G)$ is an ideal.
(i') If G is an ultrafilter then FRG iff $F_1^C(F) \subseteq G$, and $F_1^C(F)$ is a filter.
(ii') If F is an ultrafilter, then FRG iff $F_2^C(G) \subseteq F$, and $F_2^C(G)$ is a filter.

Proof. The proof follows by a direct verification of the corresponding definitions. □

Lemma 2.10. [30] **R-extension Lemma.** *Let U_0 and V_0 be filters in a precontact algebra (B, C) and let U_0RV_0. Then there exist ultrafilters U and V such that $U_0 \subseteq U$, $V_0 \subseteq V$ and URV.*

Proof. By Lemma 2.9 U_0RV_0 iff $I_1^C(U_0) \cap V_0 = \varnothing$. Then by the Sepation Lemma for filters and ideals 2.8 there exists an ultrafilter V such that $V_0 \subseteq V$ and $I_1^C(U_0) \cap V = \varnothing$. From $I_1^C(U_0) \cap V = \varnothing$ again by Lemma 2.9 we obtain U_0RV. So we have extended U_0 into the ultrafilter U. Similarly repeating this procedure for V_0 we can extend it into an ultrafilter V. □

Lemma 2.11. [30] **Canonical Lemma 1.**
(i) aCb iff there exist ultrafilters U, V such that URV, $a \in U$ and $b \in V$.
(ii) aCb iff $s(a)C_Rs(b)$.

Proof. For (i) define first the filters generated by a and b: $[a) = \{c : a \leq c\}$ and $[b) = \{c : b \leq c\}$. Second, aCb implies $[a)R[b)$ and then apply the R-extension Lemma 2.10. Condition (ii) follows from (i). □

Lemma 2.12. [30] **Canonical Lemma 2.** *Let $A = (B, C)$ be a precontact algebra. Then:*
(i) R is a symmetric relation in $Ult(A)$ iff C satisfies the axiom (C4).
(ii) R is a reflexive relation in $Ult(A)$ iff C satisfies the axiom (C5).
(iii) R is transitive relation in $Ult(A)$ iff C satisfies the Efremovich axiom (CE) $a\overline{C}b \Rightarrow (\exists c)(a\overline{C}c$ and $c^*\overline{C}b$.

Proof. We will demonstrate only the proof of (iii).

Proof of ($\Longrightarrow$). Suppose that R is a transitive relation. We will prove (CE). Suppose $a\overline{C}b$ and in order to obtain a contradiction suppose that $(\exists c)(a\overline{C}c$ and $c^*\overline{C}b)$ is not true. We will show that there are ultrafilters U, V and W such that URV, VRW, but $U\overline{R}W$ which contradicts the assumption on transitivity of R.

Let $[a) =_{def} \{c : a \leq c\}$ and $[b) =_{def} \{b : b \leq c\}$ and define (see Lemma 2.9):$\Gamma = F_1^C([a)) \oplus F_2^C([b))$. Γ is a proper filter containing $F_1^C([a))$ and $F_2^C([b))$. If we assume that $0 \in \Gamma$, then there is a c such $c^* \in F_1^C([a))$ and $c \in F_2^C([b))$. This implies that $a\overline{C}c$ and $c^*\overline{C}b$ contrary to the assumption that there is no such c. So Γ is a propper filter and can be extended into an ultrafilter V such that $F_1^C([a)) \subseteq V$ and $F_2^C([b)) \subseteq V$. By Lemma 2.9) (i') and (ii') we obtain $[a)RV$ and $VR[b)$. By Lemma 2.10 extend $[a)$ and $[b)$ to ultrafilters U and W such that URV and VRW, $a \in U$ and $b \in W$. But by assumption we have $a\overline{C}b$ which shows that $U\overline{R}W$ - the desired contradiction.

Proof of ($\Longleftarrow$). Suppose that (CE) holds and for the sake of contradiction that R is not transitive. Then there exist ultrafilters U, V and W such that URV, VRW, but $U\overline{R}W$. So, there exist $a \in U$ and $b \in W$ such that $a\overline{C}b$. By (CE) there exists c such that $a\overline{C}c$ and $c^*\overline{C}b$. We have two cases for c:

Case 1: $c \in V$. But $a \in U$ and URV, so aCc - a contradiction with $a\overline{C}c$.

Case 2: $c \notin V$, so $c^* \in V$. But $b \in W$ and VRW imply c^*Cb - a contradiction with $c^*\overline{C}b$. □

The following lemma will be used later on. It is the canonical analog of Lemma 2.6 concerning algebras with several precontact relations.

Lemma 2.13. Canonical Lemma 3. *Let $A = (B, C_1, C_2)$ be a Boolean algebra with two precontact relations C_1 and C_2 and let R_1 and R_2 be their canonical relations in the canonical structure $(Ult(A), R_1, R_2)$. Then the following conditions are true:*

(i) A satisfies the condition
(C_1, C_2) $a\overline{C}_1b \Rightarrow (\exists c)(a\overline{C}_1c$ *and* $c^*\overline{C}_2b)$ *iff*
$(Ult(A), R_1, R_2)$ *satisfies the condition*
$(R_1 \circ R_2 \subseteq R_1)$ UR_1V *and* $VR_2W \Rightarrow UR_1W$.
(ii) A satisfies the condition
(C_2, C_1) $a\overline{C}_1b \Rightarrow (\exists c)(a\overline{C}_2c$ *and* $c^*\overline{C}_{21}b)$ *iff*
$(Ult(A), R_1, R_2)$ *satisfies the condition*
$(R_2 \circ R_1 \subseteq R_1)$ UR_2V *and* $VR_1W \Rightarrow UR_1W$.

Proof. The proof is similar to the proof of condition (iii) of 2.12. □

Theorem 2.14. Relational representation theorem for precontact and contact algebras [30]. *Let $A = (B, C)$ be a precontact algebra, $(Ult(A), R)$ be the canonical adjacency space of A and s be the stone embedding. Then:*

(i) s is an embedding of (B, C) into the precontact algebra over the canonical adjacency space $(Ult(A), R)$.

(ii) If (B, C) is a contact algebra then the precontact algebra over the canonical adjacency space over (B, C) is a contact algebra.

Proof. The proof follows from Lemma 2.11 and Lemma 2.12 and the fact that s is an isomorphic embedding of the Boolean algebra B into the algebra of all subsets of $Ult(A)$. □

The above representation theorem for the case of contact algebras is not the intended one because the contact is not of Whiteheadian type, namely sharing a common point. In the next section we will describe another representation of contact algebras using topology, which presents an Whiteheadian type contact between regions. As we see, the reason is that ultrafilters as abstract points are not enough to model the Whiteheadean contact and we need to introduce another kind of abstract points.

2.5 Topological representation of contact algebras. Clans.

First we will introduce another kind of abstract points in contact algebras called clans.

Definition 2.15. Definition of clan. [26] *Let $A = (B, C)$ be a contact algebra. A subset $\Gamma \subseteq B$ is called a **clan** in (B, C) if it satisfies the following conditions:*

(i) $1 \in \Gamma$ and $0 \notin \Gamma$,

(ii) It $a \in \Gamma$ and $a \leq b$ then $b \in \Gamma$,

(iii) If $a + b \in \Gamma$ then $a \in \Gamma$ or $b \in \Gamma$

(iv) If $a, b \in \Gamma$ then aCb.

*Γ is a **maximal clan** if it is a maximal set under the set inclusion. We denote by $Ult(\Gamma)$ the set of all ultrafilters contained in Γ and by $Clans(A)$ - the set of all clans of A.*

*Subsets of B satisfying (i), (ii) and (iii) are called **grills**. So clans are grills satisfying (iv).*

The above definition is an algebraic abstraction from an analogous notion in the proximity theory (see, for instance, [74], from where we adopt the name *clan*).

Let us note that ultrafilters are clans, but there are other clans and they can be obtained by the following construction.

Let $\sum$ be a nonempty set of ultrafilters of (B, C) such that if $U, V \in \sum$, then URV, where R is the canonical adjacency relation of C on the set of ultrafilters of (B, C). Such sets of ultrafilters are called *R-cliques*. An R-clique is maximal, if it is a maximal set under the set-inclusion. By the axiom of choice every R-clique is contained in a maximal R-clique. Let Γ be the union of all ultrafilters from $\sum$. Then it can be verified that Γ is a clan. Moreover, every clan can be obtained by this construction from an R-clique and there is an obvious correspondence between maximal cliques and maximal clans. All these facts about clans are contained in the following technical lemma:

Lemma 2.16. [26] ***Clan Lemma.*** *(i) Every ultrafilter is a clan.*

(ii) The complement of a clan is an ideal.

(iii) Every clan is contained in a maximal clan (by the Zorn Lemma),

(iv) Let $\sum$ be an R-clique and $\Gamma(\sum) = \bigcup_{\Gamma \in \sum} \Gamma$. Then $\Gamma(\sum)$ is a clan.

(v) If $U, V \in Ult(\Gamma)$ then URV, so $Ult(\Gamma)$ is an R-clique,

(vi) If Γ is a clan and $a \in \Gamma$ then there is an ultrafilter $U \in Ult(\Gamma)$ such that $a \in U$,

(vii) Let Γ be a clan and $\sum$ be the R-clique $Ult(\Gamma)$. Then $\Gamma = \Gamma(\sum)$, so every clan can be defined by an R-clique as in (iv),

(viii) If $\sum$ is a maximal R-clique then $\Gamma(\sum)$ is a maximal clan,

(ix) If Γ is a maximal clan then $Ult(\Gamma)$ is a maximal R-clique,

(x) For all ultrafilters U, V: URV iff there exists a (maximal) clan Γ such that $U, V \in Ult(\Gamma)$,

(xi) For all $a, b \in B$: aCb iff there exists a (maximal) clan Γ such that $a, b \in \Gamma$,

(xii) For all $a, b \in B$: $a \not\leq b$ iff there exists clan (ultrafilter) Γ such that $a \in \Gamma$ and $b \notin \Gamma$.

Proof. We invite the reader to prove the lemma by himself or to consult [26]. As an example we will give proofs only of some parts of the lemma in order to connect it with the discrete representation of contact algebras.

(vi) Let Γ be a clan and $a \in \Gamma$. Then obviously $[a) \subseteq \Gamma$ and consequently $[a) \cap \overline{\Gamma} = \varnothing$. But $[a)$ is a filter, $\overline{\Gamma}$ is an ideal (by (ii)) and by the Separation Theorem for filters and ideals there exists an ultrafilter U such that $[a) \subseteq U$ and $U \cap \overline{\Gamma} = \varnothing$. This implies that $a \in U$ and $U \subseteq \Gamma$.

(ix) ($\Rightarrow$). Let aCb. Then by Lemma 2.11 there exist ultrafilters U, V such that URV, $a \in U$ and $b \in V$. Since R is a reflexive and symmetric relation, then $\sum = \{U, V\}$ is a clique and by (iv) $\Gamma = U \cup V$ is a clan such that $a, b \in \Gamma$.

(ix) ($\Leftarrow$). This direction follows by the definition of clan. $\square$

Lemma 2.17. [26] *Let Γ be a clan in a contact algebra $A = (B, C)$. Then the following holds for any $a \in B$:*
$a^* \in \Gamma$ *iff* $(\forall b \in B)(a + b = 1 \Rightarrow b \in \Gamma)$.

Proof. By a direct verification. $\square$

The topological representation theory of contact algebras is based on the following construction taken from [26]. Let $A = (B, C)$ be a contact algebra and let $X = Clans(A)$ and for $a \in B$, define $g(a) =_{def} \{\Gamma \in Clans(B) : a \in \Gamma\}$. We introduce a topology in X taking the set $\mathbf{B} = \{g(a) : a \in B\}$ as the base of closed sets in X. The obtained topological space X is called the canonical topological space of (B, C).

Lemma 2.18. [26]
(i) $g(0) = \varnothing$, $g(1) = X$,
(ii) $g(a + b) = g(a) \cup g(b)$,
(iii) $a \leq b$ iff $g(a) \subseteq g(b)$.
(iv) $a = 1$ iff $g(a) = X$.
(v) $g(a^*) = Cl_X(X \setminus g(a)) = Cl_X - g(a)$
(vi) $g(a)$ is a regular closed subset of X.

Proof. (i) and (ii) follow directly from the definition of clan, (iii) follows from Lemma 2.11 (xii) and (iv) follows from (iii). (v) follows from the following sequence of equivalencies:

for any clan Γ: $\Gamma \in g(a^*)$ iff $a^* \in \Gamma$ iff (by Lemma 2.17) $(\forall b \in B)(a + b = 1 \Rightarrow b \in \Gamma)$ iff (by (ii) and (iv)) $(\forall b \in B)(g(a) \cup g(b) = X \Rightarrow \Gamma \in g(b))$ iff $(\forall b \in B)(X \setminus g(a) \subseteq g(b) \Rightarrow \Gamma \in g(b))$ iff $Cl_X(X \setminus g(a)) = Cl_X - g(a)$.

For (vi) By (v) $g((a^*)^*) = Cl_X - Cl_X - g(a) = Cl_X(Int_X(a))$. $\square$

Theorem 2.19. Topological representation theorem for contact algebras [26] (see also [76]). *(i) The mapping g is an embedding from (B, C) into the canonical contact algebra $RC(X)$ of (B, C).*

(ii) The canonical space of (B, C) is T0, compact and semiregular.

Note that a topological space is semiregular if it has a base of regular-closed sets.

Proof. We will give a proof only of (i). By Lemma 2.18 we see that g isomorphically embeds B into $RC(X)$ where $X = Clans(A)$ and the topology is determined by the closed basis $\{g(a) : a \in B\}$. It remains to show that g preserves contact:

aCb iff (by Lemma 2.16 (ix)) there exists a clan Γ such that $a \in \Gamma$ and $b \in \Gamma$ iff there exists a clan Γ such that $\Gamma \in g(a)$ and $\Gamma \in g(b)$ iff $g(a) \cap g(b) \neq \emptyset$, i.e. $g(a)$ and $g(b)$ have a common point. $\square$

Let us note that in the above representation theorem two kinds of abstract points have been used: ultrafilters and clans which are not ultrafilters (ultrafilters as clans are used in the Clan Lemma (xii)). Note that in the relational representation (Theorem 2.14) contact is chracterized by the adjacency relations between ultrafilters. It is possible that two regions are in a relational contact and not share an ultrafilter. By adding more points (namely clans) this situation is excluded because we can find a clan-like point in both regions. We may consider ultrafilter points as simple *atoms*. Since clans are unions of adjacent ultrafilters, this suggests to consider clans as *molecules* composed by atoms. It is interesting to know how these two kinds of points are distributed in the set $g(a)$ of points associated with a given region a. For instance, it can be proved that the set $BP(a) = g(a) \setminus Int(g(a)$ of boundary points of $g(a)$ do not contain any ultrafilter point. In some sense the above facts throw a new light on the ancient atomistic view of space.

Remark 2.20. *Let us note that the clans corresponding to the largest contact C_{max} (which can be named C_{max}-clans) are just the gills and that there is only one maximal grill - just the union of all ultrafilters. Analogously the clans and maximal clans corresponding to the smallest contact, the overlap relation O in a Boolean algebra (O-clans) are ultrafilters (see Example 3.1 in [26]).*

2.6 Factor contact algebras determined by sets of clans.

The following is a construction of a contact algebra from a given contact algebra A and given set of clans of A. The construction is taken from [77] and the reader is invited to consult the paper for the details.

Let Δ be an ideal in a Boolean algebra B. It is known from the theory of Boolean algebras that the relation $a \equiv_\Delta b$ iff $a.b^* + a^*.b \in \Delta$ is a congruence relation in B and the factor algebra $B/\equiv_\Delta$ under this congruence (called also factor algebra under Δ and denoted by B/Δ) is a Boolean algebra. Denote the congruence class determined by an element a of B by $|a|_\Delta$ (or simply by $|a|$). Boolean operations in B/Δ are defined as follows: $|a| + |b| = |a + b|$, $|a|.|b| = |a.b|$, $|a|^* = |a^*|$, $0 = |0|$, $1 = |1|$. Recall that Boolean ordering in B/Δ is defined by $|a| \leq |b|$ iff $a.b^* \in \Delta$ (see [70] for details).

Let A be a contact algebra and $\alpha \subseteq Clans(A)$, $\alpha \neq \varnothing$. Now we will define a construction of a contact algebra B_α corresponding to α. Define $I(\alpha) = \{a \in B : \alpha \cap g(a) = \varnothing\}$. It is easy to see that $I(\alpha)$ is a proper ideal in B, i.e. $1 \notin I(\alpha)$. The congruence defined by $I(\alpha)$ is denoted by $\equiv_\alpha$. So we have $a \equiv_\alpha b$ iff $a^*.b + a.b^* \in I(\alpha)$ iff $a^*.b \in I(\alpha)$ and $a.b^* \in I(\alpha)$. Now define B_α to be the Boolean algebra $B/I(\alpha)$. We define a contact relation C_α in B_α as follows: $|a|_\alpha C_\alpha |b|_\alpha$ iff $\alpha \cap g(a) \cap g(b) \neq \varnothing$, where $g(a) = \{\Gamma \in Clans(B) : a \in \Gamma\}$ (see the topological representation theorem of contact algebras).

Lemma 2.21. *(B_α, C_α) is a contact algebra.*

Let us note that in the Boolean algebra B_α the following conditions are true: $|a|_\alpha \neq |0|_\alpha$ iff $a \notin I(\alpha)$ iff there exists a clan $\Gamma \in \alpha$ such that $a \in \Gamma$.

2.7 Contact algebras satisfying the Efremovich axiom (CE). Clusters.

We will show in this section that in contact algebras satisfying the Efremovich axiom (CE) we can introduce a new kind of abstract points called clusters. Our definition is an algebraic abstraction of the analogous notion used in the compactification theory of proximity spaces (see for instance [60]). Clusters will be used later on to define time points in dynamic contact algebras.

Definition 2.22. Clusters. [26] *Let (B, C) be a contact algebra. A subset $\Gamma \subseteq B$ is called a **cluster** in (B, C) if it is a clan satisfying the following condition:*
(Cluster) If $a \notin \Gamma$ then there exists $b \in \Gamma$ such that $a\overline{C}b$.
The set of clusters of $A = (B, C)$ is denoted by Clusters(A).

Lemma 2.23. *Let $A = (B, C)$ be a contact algebra satisfying the Efremovich axiom (CE). Then:*
(i) Γ is a cluster in (B, C) iff Γ is a maximal clan in (B, C).
(ii) Every clan is contained in a unique cluster.

Proof. Let us note that the above lemma is a lattice-theoretic version of a result of Leader about clusters in proximity spaces mentioned in [74]. One can prove this lemma having in mind the following facts. First, it follows from Lemma 2.12 that if C is a contact relation satisfying the Efremovich axiom (CE), then the canonical relation for C is an equivalence relation. Second, the maximal R-cliques of an equivalence relation are the equivalence classes of R. And third, clusters in the presence of (CE) are unions of such R-equivalence classes (by 2.16). □

Lemma 2.24. *Let (B, C) be a contact algebra satisfying the Efremovich axiom (CE). Then for any $a, b \in B$: aCb iff there is a cluster Γ containing a and b.*

Proof. aCb iff (by Lemma 2.16) there exists a maximal clan Γ containing a and b. By Lemma 2.23 Γ is a cluster. □

Note that we can not prove a representation theorem for contact algebras satisfying the Efremovich axiom as subalgebras of regular closed sets using only clusters as abstract points, because we can not distinguish in general different regions by means of clusters. Ultrafilters can distinguish different regions, but in general they are not clusters.

The following lemma states how we can distinguish clusters.

Lemma 2.25. *Let $A = (B, C)$ be a contact algebra satisfying the Efremovich axiom and let Γ, Δ be clusters. Then the following conditions are equivalent:*
(i) $\Gamma \neq \Delta$*,*
(ii) there exist $a \in \Gamma$ *and* $b \in \Delta$ *such that* $a\overline{C}b$*,*
(iii) there exists $c \in B$ *such that* $c \notin \Gamma$ *and* $c^* \notin \Delta$*.*

Proof. $(i) \Rightarrow (ii)$ Suppose $\Gamma \neq \Delta$, then, since they are maximal clans, there exists $a \in \Delta$ and $a \notin \Gamma$. Consequently, there exists $b \in \Gamma$ such that $a\overline{C}b$, so (ii) is fulfilled.

$(ii) \Rightarrow (iii)$ Suppose that there exist $a \in \Gamma$ and $b \in \Delta$ such that $a\overline{C}b$. From $a\overline{C}b$ we obtain by the Efremovich axiom that there exists c such that $a\overline{C}c$ and $c^*\overline{C}b$. Conditions $a \in \Gamma$ and $a\overline{C}c$ imply $c \notin \Gamma$. Similarly $b \in \Delta$ and $c^*\overline{C}b$ imply $c^* \notin \Delta$.

$(iii) \Rightarrow (i)$ Suppose that there exists $c \in B$ such that $c \notin \Gamma$ and $c^* \notin \Delta$ and for the sake of contradiction that $\Gamma = \Delta$. Since $c + c^* = 1$ then ether $c \in \Gamma$ or $c^* \in \Delta$ - a contradiction. □

Remark 2.26. *We have mentioned in Remak 2.20 that C_{max}-clans are grills and that there is only one maximal C_{max}-clan just the union of all ultrafilters. Because C_{max} satisfies the Efremowich axiom, then there is only one C_{max}-cluster - the maximal grill.*

3 A dynamic model of space and time based on snapshot construction

In this section, following mainly [79, 13] we will give a specific point-based space-time structure called dynamic model of space and time (DMST) built by a special construction mentioned in Section 1 and called **snapshot construction** Because the notion of time structure is one of the base ingredients of the construction we start with this notion.

3.1 Time structures

Time structures of the forma $\underline{T} = (T, \prec)$ were introduced in Section 1.1 as relational systems used as a semantic basis of temporal logic. Let us remind that T is a non-empty set whose elements are called 'time points' (moments, Whitehead's epochs). The binary relation $\prec$ is called 'before-after' relation (or 'time order') with the standard intuitive meaning of $i \prec j$: the moment i is before the moment j, or equivalently, j is after i. We also suppose that T is supplied with the standard notion of equality denoted as usual by $=$. We do not presuppose in advance any fixed set of conditions for the relation $\prec$. One possible list of first-order conditions for $\prec$ which are typical for some systems of temporal logic, are the following. We describe them with their specific names and notations which will be used in this paper.

- **(RS)** *Right seriality* $(\forall m)(\exists n)(m \prec n)$,
- **(LS)** *Left seriality* $(\forall m)(\exists n)(n \prec m)$,
- **(Up Dir)** *Updirectedness* $(\forall i, j)(\exists k)(i \prec k$ and $j \prec k)$,
- **(Down Dir)** *Downdirectedness* $(\forall i, j)(\exists k)(k \prec i$ and $k \prec j)$,
- **(Circ)** *Circularity* $(\forall i, j)(i \prec j \to (\exists k)(j \prec k$ and $k \prec i))$
- **(Dens)** *Density* $i \prec j \to (\exists k)(i \prec k$ and $k \prec j)$,
- **(Ref)** *Reflexivity* $(\forall m)(m \prec m)$,
- **(Irr)** *Irreflexivity* $(\forall m)($ not $m \prec m)$,
- **(Lin)** *Linearity* $(\forall m, n)(m \prec n$ or $n \prec m)$,
- **(Tri)** *Trichotomy* $(\forall m, n)(m = n$ or $m \prec n$ or $n \prec m)$,
- **(Tr)** *Transitivity* $(\forall ijk)(i \prec j$ and $j \prec k \to i \prec k)$.

We call the set of formulas (RS), (LS), (Up Dir), (Down Dir), (Circ), (Dens), (Ref), (Irr), (Lin), (Tri), (Tr) *time conditions.* If the relation $\prec$ satisfies the condition (Irr) it will be called "strict". If $\prec$ satisfies (Ref) the reading of $i \prec j$ should be more precise: "i is equal or before j".

Note that the above listed conditions for time ordering are not independent. Taking some meaningful subsets of them we obtain various notions of time order. Of course this list is not absolute and is open for extensions but in this paper we will consider only these 11 conditions.

3.2 The snapshot construction and the dynamic model of space and time

The snapshot construction is a specific method of constructing a dynamic model of space. It is a formalization of the following intuitive idea. Suppose we are observing an area of changing regions, called 'dynamic regions' and we want to describe this area. In our everyday life such a description can be realized by a video camera making a video. In this way the *camera can be interpreted as a fixed observer*. The description is realized by making a snapshot of the observed area for each moment of the camera's time. Namely the series of these snapshots can be considered as a realization of the description of the area of changing or moving regions and each snapshot can be considered as a static spatial description of the area for the corresponding time moment. This procedure can be formalized and generalized as follows. First, we start with certain time structure $\underline{T} = (T, \prec)$, described in the previous section. The formalization of the action 'making snapshots' is the following. To each moment $i \in T$ we associate a contact algebra $A_i = (B_i, 0_i, 1_i, \leq_i, +_i, \cdot_i, *_i, C_i) = (B_i, C_i)$, called 'coordinate contact algebra'. We assume that the algebra (B_i, C_i) realizes the static description of the dynamic regions at the moment $i \in T$ and can be considered as the corresponding 'snapshot' of the area at the moment $i \in T$. In this way each dynamic region a is represented by a series $\langle a_i \rangle_{i \in T}$ such that for each $i \in T$, $a_i \in B_i$. The series $\langle a_i \rangle_{i \in T}$ is considered also as a life history of a. We identify a with the series $\langle a_i \rangle_{i \in T}$ and will write $a = \langle a_i \rangle_{i \in T}$. The set of all dynamic regions is denoted by $\mathbf{B}$. We consider $\mathbf{B}$ as a Boolean algebra with Boolean operations defined coordinate-wise. For instance:

$a + b = \langle a_i +_i b_i \rangle_{i \in T}$, $0 = \langle 0_i \rangle_{i \in T}$, $1 = \langle 1_i \rangle_{i \in T}$, etc.

Let us define the Cartesian product (direct product) $\mathbb{B}$ of the coordinate Boolean algebras B_i, $i \in T$, namely $\mathbb{B} = \prod_{i \in T} B_i$. Obviously $\mathbf{B}$ is a subalgebra of $\mathbb{B}$. Now we introduce the following important definition

Definition 3.1. *By a* ***dynamic model of space and time*** *(DMST) we understand the system* $\mathcal{M} =< (T, \prec), \{(B_i, C_i) : i \in T\}, \mathbf{B}, \mathbb{B}\rangle$. *We say that* $\mathcal{M}$ *is a* ***full model*** *if* $\mathbf{B} = \mathbb{B}$*, and that* $\mathbb{M}$ *is a* ***rich model*** *if* $\mathbf{B}$ *contains all regions* $a = \langle a_i \rangle_{i \in T}$ *such that for all* $i \in T$ *either* $a_i = 0_i$*, or* $a_i = 1_i$*. (obviously every full model is a rich model).*

Dynamic model of space and time will be sometimes called 'snapshot model' or 'cinematographic model'.

Let us note that DMST is a very expressive model with the main component the Boolean algebra $\mathbf{B}$ of dynamic regions which can be supplied with additional structure by various ways using the other components of the model. Before doing this let us make some observations and introduce some terminology.

Let $a = \langle a_i \rangle_{i \in T}$ and $b = \langle b_i \rangle_{i \in T}$ be two dynamic regions. Then $a \leq b$ (in the Boolean algebra $\mathbf{B}$ or in $\mathbb{B}$) iff $(\forall i \in T)(a_i \leq_i b_i)$. If $a_i \neq 0_i$ for some $i \in T$ we say that a exists at the moment i. It is possible for some dynamic region $a \neq 0$ to have many successive (with respect to $\prec$) moments of time in which it is alternatively existing and non-existing (for example viruses in biology). Also it is quite possible for two different regions a and b that there exists a moment of time i (possibly not only one) such that $a_i = b_i$. Example: before the World War II we have one Germany, after that for some time - two Germanies, West Germany and East Germany, now again one Germany, and what will be in the future we do not know. Note that in DMST coordinate contact algebras are presented as point-free spatial systems, but they can equivalently be presented by their point-based representative copies according to the representation theory of contact algebras. So, in DMST we do not have one space, but for each $i \in T$ a concrete local space X_i with his own set of points. Of course all such observations put some ontological questions about the meaning of 'existence', 'equality' and other abstract metaphysical concepts which we will not discuss in this paper.

Remark 3.2. *Let us note that the analogy of 'snapshot construction' with making a video have to be considered more carefully and not literally, because video is based on visual observation. Normally what we (or camera) see is considered as existing at the moment of observation. But this is true only for objects which are not far from the observer. For instance seeing a star on the sky does not mean that this star is existing at the moment of observation - it is quite possible that this star had ceased to exist a billion years before and this fact is based on the finite velocity of light. So, if we use a video (or some optic devices) for obtaining information for dynamically changing area of regions, for some of them which are far from the observer we need additional information for their status of existing and spatial configuration at the moment of observing. For instance, if I observe the Sun from which the light travels to the Earth several minutes I can conclude that it exists at the moment of observation, just because it is not possible for it to stop existing for such a short time. Having in mind the above, the phrase 'snapshot at the moment t of the area of dynamic regions' has to be considered just as attaching to t the contact algebra (B_t, C_t) considered as the real (actual) static description of spatial configurations of regions of the area at the moment t no matter how we can obtain this information. The analogy with video film is considered only as a way to illustrate the snapshot construction.*

3.3 Standard dynamic contact algebras

Let $\mathcal{M} =< (T, \prec), \{(B_i, C_i) : i \in T\}, \mathbf{B}, \mathbb{B}\rangle$ be a given DMST. As we mentioned in the previous section, the Boolean algebra $\mathbf{B}$ of dynamic regions can be supplied with some additional relational structure in different ways. In this section we will give the first step introducing three spatio-temporal relations in $\mathbf{B}$.

- **Space contact** $aC^s b$ iff $(\exists m \in T)(a_m C_m b_m)$.

Intuitively space contact between a and b means that there is a time point $i \in T$ in which a and b are in a contact C_i in the corresponding coordinate contact algebra (B_i, C_i).

• **Time contact** aC^tb iff $(\exists m \in T)(a_m \neq 0_m$ and $b_m \neq 0_m)$.

Intuitively time contact between a and b means that there exists a time point in which a and b exist simultaneously. Note that $a_m \neq 0_m$ and $b_m \neq 0_m$ means just that a and b exist at the time point m. This relation can be considered also as a kind of **simultaneity relation** or **contemporaneity relation** studied in Whitehead's works and special relativity.

• **Local precedence** or simply **Precedence** $a\mathcal{B}b$ iff $(\exists m, n \in T)(m \prec n$ and $a_m \neq 0_m$ and $b_n \neq 0_n)$.

Intuitively a is in a local precedence relation with b (in words *a precedes b*) means that there is a time point in which a exists which is before a time point in which b exists, which motivates the name of $\mathcal{B}$ as a (local) precedence relation. Note the following similarity between the relations C^t and $\mathcal{B}$: if in the definition of $\mathcal{B}$ we replace the relation $\prec$ with $=$, then we obtain just the definition of C^t.

Lemma 3.3. *Let $\mathcal{M} =< (T, \prec), \{(B_i, C_i) : i \in T\}, \mathbf{B}, \mathbb{B}\rangle$ be a a rich DMST. Then the relations C^s, C^t and $\mathcal{B}$ satisfy the following abstract conditions:*

(i) C^s is a contact relation,

(ii) C^t is a contact relation satisfying the following additional conditions:

$(C^s \subseteq C^t)$ $aC^sb \rightarrow aC^tb$.

(C^tE) $a\overline{C}^tb \rightarrow (\exists c \in \mathbf{B})(a\overline{C}^tc$ and $c^*\overline{C}^tb)$ - the Efremovich axiom for C^t.

(iii) $\mathcal{B}$ is a precontact relation satisfying the following additional conditions (see for these conditions Section 2.3):

$(C^t\mathcal{B})$ $a\overline{\mathcal{B}}b \Rightarrow (\exists c \in \mathbf{B})(a\overline{C}c$ and $c^*\overline{\mathcal{B}}b)$,

$(\mathcal{B}C^t)$ $a\overline{\mathcal{B}}b \Rightarrow (\exists c \in \mathbf{B})(a\overline{\mathcal{B}}c$ and $c^*\overline{C}b)$,

Proof. Let us note that the requirement that the model $\mathcal{M}$ is rich is needed only in the verifications of the conditions (C^tE), $(C^t\mathcal{B})$ and $(\mathcal{B}C^t)$ which required constructions of new regions. As an example we shall verify only the condition $(\mathcal{B}C^t)$. The proof for the other conditions is similar.

Suppose $a\overline{\mathcal{B}}b$ and define c coordinate-wise:

$$c_k = \begin{cases} 0_k, & \text{if } a_k \neq 0_k \\ 1_k, & \text{if } a_k = 0_k. \end{cases}$$

Since the model is rich then c certainly belongs to $\mathbf{B}$. The verification of the conclusion $a\overline{\mathcal{B}}c$ and $c^*\overline{C}^tb$ is straightforward.

□

Definition 3.4. ***Standard Dynamic Contact Algebra.*** *Let Let* $\mathcal{M} =< (T, \prec), \{(B_i, C_i) : i \in T\}, \mathbf{B}, \mathbb{B}\rangle$ *be a be a DMST and let us suppose that the algebra* $\mathbf{B}$ *of dynamic regions enriched with the relations* C^s*,* C^t *and* $\mathcal{B}$ *satisfies the conclusions of Lemma 3.3. Then the system* $(\mathbf{B}, C^s, C^t, \mathcal{B})$ *is called* ***standard dynamic contact algebra*** *(standard DCA) over DMST.*

Let us note that Lemma 3.3 ensures that standard DCAs exist. We call them 'standard', because they are concrete and will be considered as standard models of abstract DCA (to be introduced and study later on). Shortly speaking the definition of abstract DCA is to rephrase the present definition in an abstract way. Let us remind that the aim to start with concrete point-based model for spacetime is to use it as a source of motivated axioms.

3.4 A characterization of the abstract properties of time structures with some time axioms

We do not presuppose in the formal definition of DMST that the time structure $(T, \prec)$ satisfies some abstract properties of the precedence relation. In this section we shall see that all abstract properties of the precedence relation mentioned in Section 3.1 are in an exact correlation with some special conditions of time contact C^t and precedence relation $\mathcal{B}$ called **time axioms**. The correlation is given in the next table:

(RS) *Right seriality* $(\forall m)(\exists n)(m \prec n) \iff$

(rs) $a \neq 0 \to a\mathcal{B}1$,

(LS) *Left seriality* $(\forall m)(\exists n)(n \prec m) \iff$

(ls) $a \neq 0 \to 1\mathcal{B}a$,

(Up Dir) *Updirectedness* $(\forall i, j)(\exists k)(i \prec k$ and $j \prec k) \iff$

(up dir) $a \neq 0$ and $b \neq 0 \to a\mathcal{B}p$ or $b\mathcal{B}p^*$,

(Down Dir) *Downdirectedness* $(\forall i, j)(\exists k)(k \prec i$ and $k \prec j) \iff$

(down dir) $a \neq 0$ and $b \neq 0 \to p\mathcal{B}a$ or $p^*\mathcal{B}b$,

(Circ) $i \prec j \to (\exists k)(k \prec i$ and $j \prec k) \iff$

(cirk) $a\mathcal{B}b \to b\mathcal{B}p$ or $p^*\mathcal{B}a$

(Dens) *Density* $i \prec j \to (\exists k)(i \prec k \wedge k \prec j) \iff$

(dens) $a\mathcal{B}b \to a\mathcal{B}p$ or $p^*\mathcal{B}b$,

(Ref) *Reflexivity* $(\forall m)(m \prec m) \iff$

(ref) $aC^tb \to a\mathcal{B}b$,

(Irr) *Irreflexivity* $(\forall m)(\ m \not\prec m) \Longleftrightarrow$

(irr) $a\mathcal{B}b \to (\exists c, d)(aC^tc$ and bC^td and $c\overline{C}^td)$,

(Lin) *Linearity* $(\forall m, n)(m \prec n \vee n \prec m) \Longleftrightarrow$

(lin) $a \neq 0$ and $b \neq 0 \to a\mathcal{B}b$ or $b\mathcal{B}a$,

(Tri) *Trichotomy* $(\forall m, n)(m = n$ or $m \prec n$ or $n \prec m) \Longleftrightarrow$

(tri) $(a \neq 0$ and $b \neq 0 \to aC^tb$ or $(a\mathcal{B}b$ or $b\mathcal{B}a)$,

(Tr) *Transitivity* $i \prec j$ and $j \prec k \to i \prec k \Longleftrightarrow$

(tr) $a\overline{\mathcal{B}}b \to (\exists c)(a\overline{\mathcal{B}}c$ and $c^*\overline{\mathcal{B}}b)$.

Lemma 3.5. Correspondence Lemma 1. *Let* $\mathcal{M} = \langle\langle(T, \prec), \{(B_i, C_i) : i \in T\}, \mathbf{B}, \mathbb{B}\rangle$ *be a rich DMST and let* $\mathbf{B}$ *be enriched with the relations* C^t *and* $\mathcal{B}$*. Then all the correspondences in the above table are true in the following sense: the left site of a given equivalence is true in* $(T, \prec)$ *iff the right site is true in* $\mathbf{B}$*.*

Proof. We will show the proof for two cases: **(Irr)** and **(Circ)**.

Case 1: **(Irr)$\Longleftrightarrow$(irr)**.

(Irr)$\Longrightarrow$(irr). Suppose **Irr**. This condition is also equivalent to the following one: $m \prec n \to m \neq n$. To prove **(irr)** suppose $a\mathcal{B}b$. Then there exist i, j such that $a_i \neq 0_i$, $b_j \neq 0_j$ and $i \prec j$ which implies $i \neq j$. Define the regions c and d coordinate-wise as follows:

$$c_k = \begin{cases} 1_k, & \text{if } k = i \\ 0_k, & \text{if } k \neq i. \end{cases},\ d_k = \begin{cases} 1_k, & \text{if } k = j \\ 0_k, & \text{if } k \neq j. \end{cases}$$

From here we obtain $c_i = 1_i \neq 0_i$ and $d_j = 1_j \neq 0_j$. Since $a_i \neq 0_i$ we get aC^tc. Since $b_j \neq 0_j$ we get bC^td. In order to show that $c\overline{C}d$ suppose the contrary: cC^td. This implies that there is $k \in T$ such that $c_k \neq 0_k$ and $d_k \neq 0_k$. By the definitions of c and d we get that $c_k = 1_k$ (and hence $k = i$) and $d_k = 1_k$ (and hence k=j) and consequently - $i = j$ - a contradiction. Thus $c\overline{C}^td$ which has to be proved.

(irr)$\Longrightarrow$(Irr). Suppose **(irr)** and that **(Irr)** is not true. Then there exists i such that $i \prec i$. Define a coordinate-wise as follows:

$$a_k = \begin{cases} 1_k, & \text{if } k = i \\ 0_k, & \text{if } k \neq i. \end{cases}$$

From here we get that $a_i = 1_i \neq 0_i$ and since $i \prec i$ we obtain $a\mathcal{B}a$. By **(irr)** There are c and d such that aC^tc, aC^td and $c\overline{C}d$. From the definition of a we have that $a_k \neq 0_k$ only for $k = i$. From this and aC^tc we get that $c_i \neq 0_i$ and from aC^td that $d_i \neq 0_i$. Consequently cC^td a contradiction with $c\overline{C}d$, which ends the proof.

Case 2: (Circ)$\Longleftrightarrow$(circ).

(Circ)$\Longrightarrow$(circ). Suppose that **(Circ)** is true. To prove **(circ)** suppose $a\mathcal{B}b$. Then there are $i, j \in T$ such that $a_i \neq 0_i$, $b_j \neq 0_j$ and $i \prec j$. By **Circ** there is a $k \in T$ such that $j \prec k$ and $k \prec i$. Let p be arbitrary dynamic region. There are two cases: **Case a:** $p_k \neq 0_k$ which implies $p\mathcal{B}a$.

Case b: $p_k = 0_k$. Then $p^*_k = 1_k \neq 0_k$ which implies $b\mathcal{B}p^*$.

(circ)$\Longrightarrow$(Circ). Suppose **(circ)** holds. In order to prove **(Circ)** suppose $i \prec j$. Define a, b and p as follows:

$$a_m = \begin{cases} 1_m, & \text{if } m = i \\ 0_m, & \text{if } m \neq i. \end{cases}, \; b_n = \begin{cases} 1_n, & \text{if } n = j \\ 0_n, & \text{if } n \neq j. \end{cases}, \; p_k = \begin{cases} 1_k, & \text{if } k \prec i \\ 0_k, & \text{if } k \not\prec i. \end{cases}.$$

By the definitions of a and b we obtain that $a_i = 1_i \neq 0_i$ and $b_j = 1_j \neq 0_j$. Since $i \prec j$ we get $a\mathcal{B}b$. By **(Circ)** we obtain $b\mathcal{B}p$ or $p^*\mathcal{B}a$. Consider the two cases separately.

Case I: $b\mathcal{B}p$. This implies that there exist $m, k \in T$ such that $n \prec k$, $b_n \neq 0_m$ (hence $b_n = 1_n$ and $n = j$) and $p_k \neq 0_k$ (and hence $p_k = 1_k$ and $k \prec i$). From here we get $j \prec k$ and $k \prec i$ -just what have to be proved.

Case II: $p^*\mathcal{B}a$. This implies that there exist $k, m \in T$ such that $k \prec m$, $p^*_k \neq 0_k$ (and hence $p^*_k = 1_k$, $p_k = 0_k$ and $k \not\prec i$) and $a_m \neq 0_m$ (and hence $a_m = 1_m$ and $m = i$). From here we get $k \prec i$ which contradicts $k \not\prec i$. So this case is impossible and the previous case implied what is needed. □

Definition 3.6. *The formulas **(rs)**, **(ls)**, **(up dir)**, **(down dir)**, **(circ)**, **(dens)**, **(ref)**, **(irr)**, **(lin)**, **(tri)**, **(tr)**, included in the above table are called **'time axioms'** and will be considered as additional axioms for abstract DCAs.*

The above lemma is very important because it states that the abstract properties of the time structure of a given rich model of space are determined by the time axioms which contain only variables for dynamic regions and time points are not mention. This correlation suggests to consider (abstract) DCAs satisfying some of the time axioms.

3.5 Time representatives and NOW

In this section, following [79] we present another enrichment of the expressive power of standard DCA by new constructs called *time representatives*, *universal time representatives* and **NOW**. Since this material will not be used later on in this paper, the presentation is sketchy and without proofs.

First about the intuitions behind these notions. Consider the phrases: "the epoch of Leonardo", "the epoch of Renaissance", "the geological age of the dinosaurs", "the time of the First World War", etc. All these phrases indicate a concrete unit of time named by something which happened or existed at that time and not in some other moment (epoch) of time. These examples suggest to introduce in DMST a special set of dynamic regions called *time representatives*, which are regions existing at a unique time point. The formal definition is the following:

Definition 3.7. *A region c in a DMST is called a **time representative** if there exists a time point $i \in T$ such that $c_i \neq 0_i$ and for all $j \neq i$, $c_j = 0_j$. We say also that c is a representative of the time point i and indicate this by writing $c = c(i)$. In the case when $c_i = 1_i$, c is called **universal time representative**. We denote by TR the set of universal time representatives and by UTR the set of universal time representatives.*

Time representatives and universal time representatives always exist in rich models. Let $i \in T$, then the following region $c = c(i)$ is the universal time representative corresponding to the time point i:

$$c_k = \begin{cases} 1_k, & \text{if } k = i \\ 0_k, & \text{if } k \neq i. \end{cases}$$

.

If for a given $i \in T$ there exists a such that $a_i \neq 0_i$ and $a_i \neq 1_i$ then $c.a$ is time representative of i which is not universal time representative.

The existence of universal time representatives for each $i \in T$ suggests to consider enriched time structures $(T, \prec, \mathbf{now})$, where **now** is a fixed element of T corresponding to the present epoch. We denote by **NOW** the universal time representative of **now**. Let us note that the extension of the language of standard DCA with time representatives and **NOW** enriches considerably its expressive power and makes possible to consider Past, Present and Future. Examples:

- a exists now - $aC^t\mathbf{NOW}$,
- a will exist in the future - $\mathbf{NOW}\mathcal{B}a$,
- a will always exist in the future - $(\forall c \in TR)(\mathbf{NOW}\mathcal{B}c \to aC^tc)$,
- a was existing in the past - $a\mathcal{B}\mathbf{NOW}$,
- a is in a contact with b now - $a.\mathbf{NOW}C^sb$,
- a will be in a contact with b - $(\exists c \in UTR)(\mathbf{NOW}\mathcal{B}c$ and $a.cC^sb)$,
- a and b are always in a contact - $(\forall c \in UTR)(a.cC^sb)$.

4 Dynamic contact algebra (DCA)

We adopt in this paper the following definition of abstract dynamic contact algebra.

Definition 4.1. *The algebraic system $A = (B_A, C^s_A, C^t_A, \mathcal{B}_A)$ is called dynamic contact algebra (DCA) provided the following conditions are stisfied:*

(BA) $B_A = (B_A, \leq, 0, 1, +, ., *)$ is a nondegenerate Boolean algebra.

*(CC^s) C^s_A is a contact relation in B_A, called **space contact**,*

*(CC^t) C^t_A is a contact relation in B_A, called **time contact** and satisfying the following two axioms:*

$(C^s \subseteq C^t)$ $aC^s_A b \Rightarrow aC^t_A b$.

$(C^t E)$ $a\overline{C}^t_A b \Rightarrow (\exists c)(a\overline{C}^t_A c$ and $c^*\overline{C}^t_A)$, the Efremovich axiom for C^t_A.

*$(PreC\mathcal{B})$ $\mathcal{B}_A$ is a precontact relation in B_A, called **local precedence** and satisfying the following two axioms:*

$(C^t\mathcal{B})$ $a\overline{\mathcal{B}}_A B \Rightarrow (\exists c)(a\overline{C}^t_A c$ and $c^*\overline{\mathcal{B}}_A b)$.

$(\mathcal{B}C^t)$ $a\overline{\mathcal{B}}_A B \Rightarrow (\exists c)(a\overline{\mathcal{B}}_A c$ and $c^*\overline{C}^t_A b)$.

*We considerer also DCA satisfying additionally some of the time axioms **(rs)**, **(ls)**, **(up dir)**, **(down dir)**, **(circ)**, **(dens)**, **(ref)**, **(lin)**, **(tri)**, **(tr)** (see Definition 3.6).* (Note that here the axiom **(irr)** is excluded for reasons which will be explained later, see Remark 4.10).

Since DCAs are algebraic systems we adopt the standard algebraic notions of isomorphism between two DCAs A_1 and A_2 and isomorphic embedding of A_1 into A_2. If A_1 and A_2 are isomorphic we will denote this by $A_1 \cong A_2$.

Note that the name 'dynamic contact algebra' is used in the papers [77, 78, 79, 13] as an integral name for point-free theories of space and time with different definitions in different papers. This is just for economy of names. The definition used in [79] incorporates also time representatives but for the purposes of this paper we decided to adopt more simple definition which is based only on the relations C^s, C^t and $\mathcal{B}$. It is similar to the definition of DCA from [78], but the present definition is based on a more strong axioms, so it has a different theory. Note also that the just introduced DCA has models - these are the standard DCAs from Definition 3.4 and they will be considered as standard models of the present definition of DCA. Our first aim is to show that DCAs are representable by means of models.

Lemma 4.2. *DCA is a generalization of CA.*

Proof. Let $A = (B_A, C_A)$ be a contact algebra. Set $C^s_A = C_A$, $aC^t_A b$ iff $a \neq 0$ and $b \neq 0$ (the maximal contact of A) and $\mathcal{B}_A = C^t_A$. Then it is easy to see that A with thus defined relations is a DCA. □

Remark 4.3. *One note to the Lemma 4.2. If we interpret contact algebras as dynamic contact algebras as in Lemma 4.2 the obtained reinterpretation of contact algebra has topological models which are different from the standard topological models of contact algebras (see section 5.5). So the stated equivalence in the Lemma 4.2 is only about the corresponding algebraic structures.*

It is true if we consider CA with an additional contact - the definable maximal contact $(C_{max})_A$ with $a(C_{max})_A b \Leftrightarrow_{def} a \neq 0$ and $b \neq 0$. Such extended contact algebras have topological models which are different from the standard topological models of contact algebras (see section 5.5).

4.1 Facts about ultrafilters, clans and clusters in DCA

Let $A = (B_A, C^s_A, C^t_A, \mathcal{B}_A)$ be a DCA. We denote by $Ult(A)$ the set of ultrafilters of A and by R^s_A, R^t_A and $\prec_A$ we denote correspondingly the canonical relations of C^s_A, C^t_A and $\mathcal{B}_A$ (for the definition of canonical relation see Definition 2.7). Since C^s_A and C^t_A are contact relations, then R^s_A and R^t_A are reflexive and symmetric relations (Lemma 2.12). Since C^t_A satisfies the Efremovich axiom (C^tE), the relation R^t_A is transitive (Lemma 2.12), which implies the following statement:

The relation R^t_A is an equivalence relation. (1)

By the axioms $(C^t\mathcal{B})$ and $(\mathcal{B}C^t)$ the relation $\prec_A$ satisfies the following conditions (see Lemma 2.13) for arbitrary $U, V, W \in Ult(A)$:

$(R^t\circ \prec\subseteq\prec)$ $UR^t_A V$ and $V \prec W \Rightarrow U \prec W$, (2)

$(\prec \circ R^t \subseteq\prec)$ $U \prec V$ and $VR^t_A W \Rightarrow U \prec W$. (3)

Conditions (2) and (3) imply the following more general condition

$UR^t_A U_0$ and $U_0 \prec_A V_0$ and $V_0 R^t_A V \Rightarrow U \prec_A V$. (4)

The axiom $(C^s \subseteq C^t)$ implies that the relation R^s_A is included in the relation R^t_A, namely the following condition is satisfied for arbitrary $U, V \in Ult(A)$:

$UR^s V \Rightarrow UR^t_A V$. (5)

The clans determined by the contact C^s_A are called s-clans and their set is denoted by s-Clans(A). The clans determined by C^t_A are called t-clans and their set is denoted by t-Clans(A). By axiom $(C^s \subseteq C^t)$ every s-clan is a t-clan. Note that every ultrafilter is both an s-clan and a t-clan. So we have the inclusions:

Ult(A)⊆s-Clans(A)⊆t-clans(A).

If Γ is a t-clan we denote by $Ult(\Gamma)$ the set of ultrafilters included in Γ. (6)

By axiom (C^tE) maximal t-clans are clusters and by Lemma 2.24 they are unions of the equivalence classes of ultrafilters determined by the equivalence relation R^t_A. The set of clusters is denoted by Clust(A). Note that (see Lemma 2.23)

$$\text{Every t-clan (s-clan) is contained in a unique cluster.} \tag{7}$$

So there is a function γ_A:t-$Clans(A) \rightarrow Clusters(A)$ with the following properties;

(γ1) If $\Gamma \in$ t-$Clans(A)$, then $\gamma_A(\Gamma) \in Clust(A)$,

(γ2) If $\Gamma \in Clust(A)$, then $\gamma_A(\Gamma) = \Gamma$. (8)

Now we extend the relation $\prec$ to hold between t-clans (and hence between clusters) by the same definition used for ultrafilters: for $\Gamma, \Delta \in t - Clans(A)$

$$\Gamma \prec_A \Delta \Leftrightarrow_{def} (\forall a, b \in B_A)(a \in \Gamma \text{ and } b \in \Delta \Rightarrow a\mathcal{B}_A b). \tag{9}$$

Lemma 4.4. *The following conditions are equivalent for any* $\Gamma, \Delta \in$ *t-*$Clans(A)$*:*

(i) $\Gamma \prec_A \Delta$*,*

(ii) For all $U \in Ult(\Gamma)$ *and* $V \in Ult(\Delta)$*:* $U \prec_A V$*,*

(iii) There exist $U_0 \in ULT(\Gamma)$ *and* $V_0 \in Ult(\Delta)$ *such that:* $U_0 \prec_A V_0$*.*

Proof. (i)$\Rightarrow$(ii). Suppose (i) holds and to prove (ii) suppose $a \in U \in Ult(\Gamma)$ and $b \in V \in Ult(\Delta)$. Then $a \in \Gamma$ and $b \in \Delta$ and by (i) and (9) we get $a\mathcal{B}b$ which proves (ii).

(ii)$\Rightarrow$(iii) is obvious.

(iii)$\Rightarrow$(i). Suppose (iii): $U_0 \prec_A V_0$ for some $U_0 \in Ult(\Gamma)$ and $V_0 \in Ult(\Delta)$. In order to show (i) suppose $a \in \Gamma$ and $b \in \Delta$ and proceed to show that $a\mathcal{B}_A b$. Since $a \in \Gamma$, then there exist an ultrafilter U such that $a \in U \in Clans(\Gamma)$ and an ultrafilter V such that $b \in V \in Clans(\Delta)$ (see Lemma 2.16). Then $UR^t_A U_0$ and $V_0 R^t_A V$. Since $U_0 \prec_A V_0$, then by (4) we get $U \prec_A V$. But $a \in U$, $b \in V$ and $U \prec_A V$ imply $a\mathcal{B}_A b$. □

Lemma 4.5. *For all t-clans* Γ, Δ *if* $\Gamma \prec_A \Delta$*, then there exists a cluster* Γ' *and a cluster* Δ' *such that* $\Gamma \subseteq \Gamma'$ *and* $\Delta \subseteq \Delta'$ *and* $\Gamma' \prec_A \Delta'$*.*

Proof. The proof follows from the fact that every t-clan can be extended into unique cluster and the relation $\prec_A$ between extensions is preserved by the properties of this relation stated in Lemma 4.4.

□

The next three definitions will be used later on. For $a \in B_A$ set:

$$g_A(a) =_{def} \{\Gamma \in t\text{-}Clans(A) : a \in \Gamma\}, \tag{10}$$

$$g^s_A(a) =_{def} \{\Gamma \in s\text{-}Clans(A) : a \in \Gamma\} = g_A(a) \cap s\text{-}Clans(A), \tag{11}$$

$$g^{clust}_A(a) =_{def} \{\Gamma \in Clusters(A) : a \in \Gamma\} = g_A(a) \cap Clusters(A). \tag{12}$$

Lemma 4.6. *The following equivalencies are true for arbitrary* $a, b \in B_A$*:*

(i) $aC^t_A b$ *iff there exists a t-clan (cluster)* Γ *containing* a *and* b *iff* $g_A(a) \cap g_A(b) \neq \varnothing$ *(* $(g^{clust}_A(a) \cap g^{clust}_A(b) \neq \varnothing)$ *(see (10) and (12)).*

(ii) $aC^s_A b$ *iff there exists an s-clan* Γ *containing* a *and* b *iff* $g^s_A(a) \cap g^s_A(b) \neq \varnothing$ *(see (11)),*

(iii) $a\mathcal{B}_A b$ *iff there exist t-clans (clusters)* Γ, Δ *such that* $\Gamma \prec \Delta$*,* $a \in \Gamma$ *and* $b \in \Delta$ *iff there exist t-clans (clusters)* Γ, Δ *such that* $\Gamma \prec \Delta$ *and* $g_A(a) \neq \varnothing$*,* $g_A(b) \neq \varnothing$ $(g^{clust}_A(a) \neq \varnothing$*,* $g^{clust}_A(b) \neq \varnothing)$ *(see (10) and (12)).*

Proof. (i) and (ii) follow from Lemma 2.16 and definitions (10), (11) and (12). For (iii) suppose $a\mathcal{B}_A b$. Then by Lemma 2.11 there are ultrafilters U, V such that $U \prec_A V$. Then there are clusters Γ, Δ such that $U \subseteq \Gamma$ and $V \subseteq \Delta$, so $a \in \Gamma$ and $b \in \Delta$. By Lemma 4.4 we obtain that $\Gamma \prec_A \Delta$. The converse implication follows from the definition of $\prec$. □

The next lemma is a more detailed reformulation of Lemma 4.6 which will be used in Section 4.3.

Lemma 4.7. *(i)* $aC^s_A b$ *iff there exists a cluster* Γ *and an s-clan* Δ *containing* a *and* b *such that* $\Delta \subseteq \Gamma$*.*

(ii) $aC^t_A b$ *iff there exist a cluster* Γ *and s-clans* Δ, Θ *such that* $a \in \Delta$*,* $b \in \Theta$ *and* $\Delta, \Theta \subseteq \Gamma$*.*

(iii) $a\mathcal{B}_A b$ *iff there exist clusters* Γ, Δ*, such that* $\Gamma \prec \Delta$ *and there exist s-clans* $\Theta \subseteq \Gamma$ *and* $\Lambda \subseteq \Delta$*,* $a \in \Theta$ *and* $b \in \Lambda$*.*

(iv) $a \nleq b$ *iff* $a.b^* \neq 0$ *iff there exists a cluster* Γ *and an s-clan* $\Delta \subseteq \Gamma$ *such that* $a.b^* \in \Delta$*.*

Proof. The proof follows from Lemma 4.6 and the fact that every s-clan and t-clan is contained in a cluster. □

The system (s-Clans(A),t-Clans(a), Clusters(A), $\gamma_A, \prec_A$) is called the *clan structure* of A.

Since any contact algebra is a DCA (Lemma 4.2) it is interesting to know which are s-clans, t-clans and clusters of A. Obviously s-clans are just the clans of A with (respect to C), t-clans are just the grills of A (they are unions of ultrafilters). There is only one maximal grill in A - the union of all ultrafilters and this is the unique cluster in A (with respect to C^t_A). The relation $\prec$ is just the universal relation in the set of all grills.

4.2 Extracting the time structure of DCA

Let $A = (B_A, C^s_A, C^t_A, \mathcal{B}_A)$ be a DCA. The first step to represent A in some DMSP by the snapshot construction is to extract the time structure of A. This means to define the time points of A and the corresponding 'before-after' relation. From Lemma 4.6 we see that the relations C^t_A and $\mathcal{B}_A$ which have a temporal nature can be characterized by means of clusters. This suggests the time points of A to be identified with the clusters of A and the before-after relation to be identified with the relation $\prec$ defined by (9) and restricted to the set of clusters. So, we have the following

Definition 4.8. Canonical time structure. *The system*

$T_A = (Clusters(A), \prec_A)$ *where* $\prec_A$ *is restricted to Clusters(A) is considered as the canonical time structure of A.*

It is interesting to see if there is a correspondence between time properties of T_A and the corresponding time axioms like in Lemma 3.5. This is possible for all time conditions except **(Irr)** . First we will present ultrafilter characterization of time axioms by means of conditions on the set Ult(A) expressible by the canonical relations R^t_A and $\prec_A$, considered as a relation between ultrafilters (so these conditions will be for the structure $(Ult(A), \prec_A, R^t_A)$). The corresponding table is the following. Note that the names of ultrafilter conditions are the same for the names for the corresponding time conditions from Section 3.1. enclosed by curly brackets. U, V, W below are considered as variables ranging on ultrafilters.

$\langle$ **RS** $\rangle$ $(\forall U)(\exists V)(U \prec_A V) \Longleftrightarrow$

(rs) $a \neq 0 \rightarrow a\mathcal{B}1$,

$\langle$ **LS** $\rangle$ $(\forall U)(\exists V)(V \prec_A U) \Longleftrightarrow$

(ls) $a \neq 0 \rightarrow 1\mathcal{B}a$,

$\langle$ **Up Dir** $\rangle$ $(\forall U, V)(\exists W)(U \prec W$ and $V \prec W) \Longleftrightarrow$

(up dir) $a \neq 0 \wedge b \neq 0 \Rightarrow a\mathcal{B}p$ or $b\mathcal{B}p^*$,

$\langle$ **Down Dir** $\rangle$ $(\forall U, V)(\exists W)(W \prec U$ and $W \prec V) \Longleftrightarrow$

(down dir) $a \neq 0 \wedge b \neq 0 \Rightarrow p\mathcal{B}a$ or $p^*\mathcal{B}b$,

$\langle$ **Circ** $\rangle$ $U \prec_A V \rightarrow (\exists W)(W \prec_A U$ and $V \prec W) \Longleftrightarrow$

(cirk) $a\mathcal{B}b \Rightarrow b\mathcal{B}p$ or $p^*\mathcal{B}a$

$\langle$ **Dens** $\rangle$ $U \prec_A V \rightarrow (\exists W)(U \prec W$ and $W \prec V) \Longleftrightarrow$

(dens) $a\mathcal{B}b \Rightarrow a\mathcal{B}p$ or $p^*\mathcal{B}b$,

$\langle$ **Ref** $\rangle$ $(\forall U)(U \prec_A U) \Longleftrightarrow$

(ref) $aC^t b \Rightarrow a\mathcal{B}b$,

$\langle$ **Lin** $\rangle$ $(\forall U, V)(U \prec V$ or $V \prec U) \Longleftrightarrow$

(lin) $a \neq 0$ and $b \neq 0 \Rightarrow a\mathcal{B}b$ or $b\mathcal{B}a$,

$\langle$ **Tri** $\rangle$ $(\forall U, V)(UR^t_A V$ or $U \prec_A V$ or $V \prec_A U) \Longleftrightarrow$

(tri) $(a \neq 0$ and $b \neq 0 \Rightarrow aC^t_A b$ or $a\mathcal{B}_A b$ or $b\mathcal{B}_A a$,

$\langle$ **Tr** $\rangle$ $U \prec_A Vj$ and $V \prec_A W \Rightarrow U \prec_A W \Longleftrightarrow$

(tr) $a\overline{\mathcal{B}}b \Rightarrow (\exists c)(a\overline{\mathcal{B}}c$ and $c^*\overline{\mathcal{B}}b)$.

The table with clusters can be obtained from the above one replacing ultrafilter variables U, V, W with cluster variables Γ, Δ, Θ and R^t_A (which occurs only in the condition $\langle$ **Tr** $\rangle$) with equality $=$.

Lemma 4.9. Correspondence Lemma 2. *The following equivalencies are true for each raw of the above table:*

(i) The left-side condition is true in the structure $(Ult(A), \prec_A, R^t_A)$.

(ii) The left-side condition in its cluster interpretation is true in the canonical time structure $(Clusters(A), \prec_A)$.

(iii) The right-side condition is true in DCA A.

Proof. We illustrate the proof checking three examples. Let us start with the easiest case - **(ref)**. We will prove the following implications:

(i) $(\forall U \in Ult(A))(U \prec_A U) \Longrightarrow$ (ii) $(\forall \Gamma \in Clusters(A))(\Gamma \prec_A \Gamma) \Longrightarrow$ (iii) $(\forall a, b \in B_A)(aC^t_A b \Rightarrow a\mathcal{B}_A b) \Longrightarrow$ (i).

(i)$\Longrightarrow$(ii). Suppose (i) and to prove (ii) suppose that $\Gamma \in Clusters(A)$ and that an ultrafilter $U_0 \subseteq \Gamma$. By (i) $U_0 \prec_A U_0$ and by Lemma 4.4 we get that $\Gamma \prec_A \Gamma$.

(ii)$\Longrightarrow$(iii). Suppose (ii) and in order to show (iii) suppose $aC^t_A b$ and proceed to show $a\mathcal{B}_A b$. Condition $aC^t_A b$ implies that there is a cluster Γ containing a and b. By (ii) we have $\Gamma \prec_A \Gamma$. But $a \in \Gamma$ and $b \in \Gamma$ implies (by the definition of $\prec$) that $a\mathcal{B}_A b$.

(iii)$\Longrightarrow$(i). Suppose (iii) and in order to prove (i) suppose that $U \in Ult(B)$ and $a, b \in U$. Then $a.b \neq 0$ which implies $aC^t_A b$ (C^t_A is a contact relation) and hence by (iii) we get that $a\mathcal{B}_A b$. By the definition of the canonical relation $\prec_A$ for ultrafilters, this shows that $U \prec_A U$.

The next example is **(tri)**. We will prove the following implications:

(i) $UR^t_A V$ or $U \prec_A V$ or $V \prec_A U \Longrightarrow$ (ii) $\Gamma = \Delta$ or $\Gamma \prec_A \Delta$ or $\Delta \prec_A \Gamma \Longrightarrow$ (iii) $(aC^t c$ and $bC^t d$ and $c\overline{C}^t d) \Rightarrow (a\mathcal{B}b$ or $b\mathcal{B}a) \Rightarrow$ (i).

(i)$\Rightarrow$ (ii). Suppose (i) and let $\Gamma, \Delta \in Clusters(A)$. To show (ii) suppose that $\Gamma, \Delta \in Clusters(A)$. If $\Gamma = \Delta$, then (ii) is OK. Suppose $\Gamma \neq \Delta$. Then by Lemma 2.25 there exist $a \notin \Gamma$ and $b \notin \Delta$ such that $a\overline{C}^t_A b$. Consequently there are ultrafilters U, V such that $a \in U \in Ult(\Gamma)$ and $b \in V \in Ult(\Delta)$. Since $a\overline{C}^t_A b$, then $U\overline{C}^t_A V$. This implies by (i) that $U \prec_A V$ or $V \prec_A U$. Since $U \subseteq \Gamma$ and $V \subseteq \Delta$, then by Lemma 4.4 we get $\Gamma \prec_A \Delta$ or $\Delta \prec_A \Gamma$.

(ii)$\Rightarrow$ (iii). Suppose (ii) and in order to show (iii) suppose $a \neq 0$ and $b \neq 0$. Then there are $\Gamma, \Delta \in Clusters(A)$ such that $a \in \Gamma$ and $b \in \Delta$. By (ii) there are three cases:

Case I: $\Gamma = \Delta$. Then $aC^t_A b$.

Case II: $\Gamma \prec_A \Delta$. Then $a\mathcal{B}_A b$.

Case III: $\Delta \prec_A \Gamma =$. Then $b\mathcal{B}_A a$.

(iii)$\Rightarrow$ (i). Suppose (iii) and for the sake of contradiction assume that (i) is not true. Then there are ultrafilters U, V such that $U\overline{R}^t_A V$, $U\overline{\mathcal{B}}_A V$ and $V\overline{\mathcal{B}}_A U$. Then there are a_1, b_1 such that $a_1 \in U$, $b_1 \in V$ and $a_1\overline{C}^t_A b_1$, there are a_2, b_2 such that $a_2 \in U$, $b_2 \in V$ and $a_2\overline{\mathcal{B}}_A b_2$, and there are a_3, b_3 such that $a_3 \in U$, $b_3 \in V$ and $b_3\overline{\mathcal{B}}_A ba_3$. Let $a = a_1.a_2.a_3$ and $b = b_1.b_2.b_3$. Since U, V are ultrafilters then $a \in U$ and $b \in V$, so $a \neq 0$ and $b \neq 0$. It can be shown also that $a\overline{C}^t_A b$, $a\overline{\mathcal{B}}_A b$ and $b\overline{\mathcal{B}}_A a$ which contradicts (iii).

Let us consider as a last example **(tr)**. By Lemma 2.12 we already know that (i) $\Leftrightarrow$ (iii). It remains to show (i) $\Leftrightarrow$ (ii).

(i) $\Longrightarrow$ (ii). Suppose (i) and in order to prove (ii) suppose that $\Gamma \prec_A \Delta$ and $\Delta \prec_A \Theta$. Suppose for the contrary that $\Gamma \not\prec_A \Theta$. Then by Lemma 4.4 there are ultrafilters $U \in Ult(\Gamma)$ and $W \in Ult(\Theta)$ such that $U \not\prec_A W$. Then by (i) $U \not\prec_A V$ or $V \not\prec_A W$ for any $V \in Ult(B)_A$. Take some $V \in Ult(\Delta)$.

Case I: $U \not\prec_A V$. Then $U \in Ulta(\Gamma)$, $V \in Ult(\Delta)$ and $\Gamma \prec_A \Delta$ implies $U \prec_A V$ - a contradiction.

Case II: $V \not\prec_A W$. Then $V \in Ult(\Delta)$, $W \in Ult(\Theta)$ and $\Delta \prec_A \Theta$ implies $V \prec W$ - a contradiction.

(ii) $\Longrightarrow$ (i). Suppose (ii) and in order to show (i) suppose $U \prec_A V$ and $V \prec_A W$, $U, V, W \in Ult(B)_A$. Then there are clusters Γ, Δ, Θ such that $U \subseteq \Gamma$, $V \subseteq \Delta$ and $W \subseteq \Theta$. By Lemma 4.4 we get $\Gamma \prec_A \Delta$ and $\Delta \prec_A \Theta$. By (ii) this implies $\Gamma \prec_A \Theta$. But $U \subseteq \Gamma$ and $W \subseteq \Theta$ which implies $U \prec_A W$.

One remark for the proofs of the remaining cases of this lemma is to show first the implication (i)$\Longrightarrow$ (iii) which follow the style of the proof of Lemma 2.12 and Lemma 2.13. Then the proof of (i) $\Longrightarrow$ (ii) is more easy by application of Lemma 4.4. $\square$

Remark 4.10. *Let us explain why we excluded the axiom* ***(irr)*** *from the list of time axioms and the Correspondence Lemma. The reason is that we can not prove the equivalence* $\langle$ ***Irr*** $\rangle \Longleftrightarrow$***(irr)****. One can easily proof the implication* $\langle$ *Irr* $\rangle \Longrightarrow$ ***(irr)****, but we do not know if the converse has a proof (we believe not) or if there is a stronger first-order sentence like* ***(irr)*** *for which the equivalence holds. This equivalence is true in rich standard DCA and the reason is the possibility to define special regions due to richness. The language of the abstract version of DCA can not express a property similar to richness but in a DCA enriched with time representatives discussed in Section 3.5 the treatment of this case is possible because the language is more expressive (see [79]).*

Since any contact algebra A is a DCA which is the canonical time structure of A? The set T of time points is the singleton set $\{\Gamma\}$ where Γ is the maximal grill in A (the union of all ultrafilters) and $\prec$ is just the equality. So the time of A has only one moment and the clock of A is not ticking - the time is 'stopped' or **degenerated**. That is why contact algebras can be considered as *static* (no time is hidden in them) and the RBTS based on contact algebras - as a *static mereotopology*.

4.3 Extracting canonical coordinate contact algebras and the canonical standard DCA

Let $A = (B_A, C^s_A, C^t_A, \mathcal{B}_A)$ be a DCA and let $T_A = (Clusters(A), \prec_A)$ be the canonical time structure of A. The next step in the snapshot construction is for each $\Gamma \in Clusters(A)$ to define in a canonical way the coordinate contact algebra $A_\Gamma = (B_\Gamma, C_\Gamma)$.

Because Γ is a cluster, consider the set

$\widehat{\Gamma} = \{\Delta \in s\text{-Clans(A)}: \Delta \subseteq \Gamma\}$.

We will consider the construction of factor contact algebra determined by sets of clans described in Section 2.6. So we adopt the following definition.

Definition 4.11. Canonical coordinate contact algebra. *We define* $(B_{\widehat{\Gamma}}, C_{\widehat{\Gamma}})$ *denoted for simplicity by* $B_\Gamma = (B_\Gamma, C_\Gamma)$ *to be the contact algebra defined by the factor construction from Sections 2.6 applied to the contact algebra* (B_A, C^s_A) *and the set of s-clans* $\widehat{\Gamma}$*. The algebra* (B_Γ, C_Γ) *is called the* ***canonical coordinate contact algebra*** *corresponding to the time point* Γ.

Remind that the elements of B_Γ are now of the form $|a|_\Gamma$ defined by the congruence $\equiv_{\widehat{\Gamma}}$ (see Section 2.6) and $|a|_\Gamma C_\Gamma |b|_\Gamma$ iff $\widehat{\Gamma} \cap g(a) \cap g(b) \neq \varnothing$, where

$g(a) = \{\Gamma \in s\text{-Clans(A)}: a \in \Gamma\}$.

Definition 4.12. Canonical standard DCA. *Having the canonical time structure $T_B = (Clusters(A), \prec_A)$ and the set of canonical contact algebras $A_\Gamma = (B_\Gamma, C_\Gamma)$, $\Gamma \in Clusters(A)$ we define by the snapshot construction described in Sections 3.2 and 3.3 the full* ***canonical standard DCA*** $\mathbf{A}^{can} = (\mathbb{B}, C^s, C^t, \mathcal{B})$, *where* $\mathbb{B} = \prod_{\Gamma \in Clusters(A)} B_\Gamma$ *is the Cartesian product of the coordinate Boolean algebras.*

We define an embedding function h from A into $\mathbf{A}^{can}$ coordinatewise as follows: for $a \in B_A$ and for each $\Gamma \in Clusters(A)$, $h_\Gamma(a) = |a|_\Gamma$.

The next lemma is important because it shows that the time axioms are preserved by the construction of the full canonical standard DCA.

Lemma 4.13. *Let A be a DCA and $\mathbf{A}^{can}$ be the full canonical standard dynamic contact algebra associated to A. Then for each time axiom α from the list of* time axioms ***(rs)****,* ***(ls)****,* ***(up dir)****,* ***(down dir)****,* ***(circ)****,* ***(dens)****,* ***(ref)****,* ***(lin)****,* ***(tri)****,* ***(tr)*** *the following equivalence is true: α holds in A iff α holds in $\mathbf{A}^{can}$.*

Proof. By Lemma 4.9 α is true in A iff the corresponding condition $\widehat{\alpha}$ is true in the canonical time structure $T_A = (Clusters(A), \prec_A)$ iff (by Lemma 3.5) α is true in the full standard DCA $\mathbf{A}^{can}$. $\square$

Lemma 4.14. Embedding Lemma. *Let A be a DCA and h be the mapping defined in Definition 4.12. Then:*

(i) h preserves Boolean operations.

(ii) $aC^s_A b$ in A iff there exists $\Gamma \in Clusters(a)$ such that $|a|_\Gamma\ C_\Gamma |b|_\Gamma$ iff $h(a)C^s_{\mathbf{A}^{can}}h(b)$ in $\mathbf{A}^{can}$.

(iii) $aC^t_A b$ in A iff there exists $\Gamma \in Clusters(A)$ such that $|a|_\Gamma \neq |0|_\Gamma$ and $|b|_\Gamma \neq |0|_\Gamma$ iff $h(a)C^t_{\mathbf{A}^{can}}h(b)$ in $\mathbf{A}^{can}$.

(iv) $a\mathcal{B}_A b$ in A iff there exist $\Gamma, \Delta \in Clusters(A)$ such that $\Gamma \prec \Delta$ and $|a|_\Gamma \neq |0|_\Gamma$ and $|b|_\Delta \neq |0|_\Delta$ iff $h(a)\mathbf{B}(A)_{\mathbf{A}^{can}}h(b)$ in $\mathbf{A}^{can}$.

(v) $a \not\leq b$ in A iff there exist $\Gamma \in Clusters(A)$ such that $|a|_\Gamma \not\leq_\Gamma |b|_\Gamma$ iff $h(a) \not\leq h(b)$ in $\mathbf{A}^{can}$.

(vi) $a = b$ iff $h(a) = h(b)$, i.e. h is an embedding.

Proof. (i) The statement is obvious, because the elements of the coordinate algebras are equivalence classes determined by a congruence relations in A and that Boolean operations in $\mathbf{A}^{can}$ are defined coordinatewise.

(ii) $aC^s_A b$ in A iff (by Lemma 4.7)there exist a cluster Γ and s-clans Δ, Θ such that $a \in \Delta$, $b \in \Theta$ and $\Delta, \Theta \subseteq \Gamma$ iff (by the definition of $\widehat{\Gamma}$ and g, see (11), (12)) there exists $\Gamma \in Clusters(A)$ such that $\widehat{\Gamma} \cap g(a) \cap g(b) \neq \varnothing$ iff (by the factorization construction) there exist $\Gamma \in Clusters(A)$ such that $|a|_\Gamma C_\Gamma |b|_\Gamma$ iff $h(a)C^s_{\mathbf{A}^{can}}h(b)$ in $\mathbf{A}^{can}$.

(iii) $aC^t_A b$ in A iff (by Lemma 4.7) there exist clusters Γ, Δ, such that $\Gamma \prec \Delta$ and there exist s-clans $\Theta \subseteq \Gamma$ and $\Lambda \subseteq \Delta$, $a \in \Theta$ and $b \in \Lambda$ iff there exist $\Gamma \in Clusters(A)$ such that $\widehat{\Gamma} \cap g(a) \neq \varnothing$ and $\widehat{\Gamma} \cap g(b) \neq \varnothing$ iff (by the factorization construction) there exist $\Gamma \in Clusters(A)$ $|a|_\Gamma \neq |0|_\Gamma$ and $|b|_\Gamma \neq |0|_\Gamma$ iff $h(a)C^t_{\mathbf{A}^{can}}h(b)$ in $\mathbf{A}^{can}$.

(iv) $a\mathcal{B}_A b$ in A iff (by Lemma 4.7) there exist clusters Γ, Δ, such that $\Gamma \prec \Delta$ and there exist s-clans $\Theta \subseteq \Gamma$ and $\Lambda \subseteq \Delta$, $a \in \Theta$ and $b \in \Lambda$ iff there exist $\Gamma, \Delta \in Clusters(A)$ such that $\Gamma \prec_A \Delta$, $\widehat{\Gamma} \cap g(a) \neq \varnothing$ and $\widehat{\Delta} \cap g(b) \neq \varnothing$ iff (by the factorization construction) there exist clusters Γ, Δ, such that $\Gamma \prec \Delta$, $|a|_\Gamma \neq |0|_\Gamma$ and $|b|_\Delta \neq |0|_\Delta$ iff $h(a)\mathcal{B}_{\mathbf{A}^{can}}h(b)$ in $\mathbf{A}^{can}$.

(v) $a \not\leq b$ in A iff $a.b^* \neq 0$ iff there exists a cluster Γ and an s-clan $\Delta \subseteq \Gamma$ such that $a.b^* \in \Delta$ iff there exists $\Gamma \in Clans(A)$ such that $\widehat{\Gamma} \cap g(a.b^*) \neq \varnothing$ iff (by the factorization construction) $|a|_\Gamma \not\leq_\Gamma |b|_\Gamma$ iff $h(a) \not\leq h(b)$ in $\mathbf{A}^{can}$.

(vi) $a = b$ iff $h(a) = h(b)$ - by (v) and the fact that $a = b$ iff $a \leq b$ and $b \leq a$. $\square$

4.4 Representation Theorem for DCAs by means of snapshot models

Theorem 4.15. Representation Theorem for DCA by means of snapshot models. *Let A be a DCA. Then there exists a full standard DCA $\mathbb{B}$ and an isomorphic embedding h of A into $\mathbb{B}$. Moreover, A satisfies some of the time axioms iff the same axioms are satisfied in $\mathbb{B}$.*

Proof. The proof is a direct corollary of Lemma 4.14 and Lemma 4.13 by taking $\mathbb{B} = \mathbf{A}^{can}$. $\square$

This Theorem shows that the meaning of the (point-based) standard DCA built by the snapshot construction is coded by the axioms of the abstract DCA which is point-free. Note, however, that this representation theorem is of embedding type, like the representation theorem for Boolean algebras as algebras of sets: every Boolean algebra can be isomorphically embedded into the Boolean algebra of subsets of some universe. The theorem does not guarantee one-one correspondence between set models and algebras via some isomorphism. The same situation is with DCAs and standard (point-based) DCAs. But adding topology we may characterize more deeply point models and like in the Stone topological representation theorem for Boolean algebras to establish a one-one correspondence between algebras and topological models. That is why we introduce and develop in the next Section topological models for DCAs.

5 Topological models for dynamic contact algebras

In this section we introduce topological models for DCA and prove the expected topological representation theorem for DCA possibly extended with some time axioms. We develop the topological duality theory for the category of all DCAs and some related categories.

5.1 What kind of topological models for DCA we need?

What kind of topological models for DCA we need? We need topological spaces X such that their algebra $RC(X)$ of regular closed subsets to model the algebra of regions. Note that regions in this algebra are related by three different relations - space contact C^s, time contact C^t and precedence $\mathcal{B}$, the first two acting as contact relations and the third - as precontact relation. This means that the realization of the contact aC^sb should be a and b to have a common point and for aC^tb also a and b to have a common point and these common points should be of different kind - points characterized space contact - space points, and points characterized time contact - time points. So regions should contain at least two kinds of points - space and time points and aC^sb should hold if they share a space point, and aC^tb should hold if a and b share time point. According to the third relation $\mathcal{B}$, it should act as a precontact by means of some binary relation between time points. Also, in order to characterize C^t as a simultaneity relation we need a special subclass of 'bigger' time points to be interpretted as 'moments of time' and the other time points to be considered as parts of the bigger time points, such that simultaneous time points to form different disjoint classes. So space should have different classes of points similar to the clan structure of DCA. The topology in this space, as in the representation theory for contact algebras, should be generated by a subalgebra of the Boolean algebra of regular closed subsets of the space taken as a closed base for the topology. And finally, in order to prove topological representation theorem for DCA, we should be able to extract in a canonical way the same type of topological space from the structure of DCA. Obviously the abstract points of such a topology should be the different kinds of clans in DCA and their interrelations. So, this is the intuition which we will put in the definition of the special topological spaces introduced in Section 5.3 called Dynamic Mereotopological Spaces (DMS). Since DCA is a generalizations of contact algebra, we follow some terminology and ideas from the representation and duality theory for contact algebras given recently by Goldblatt and Grice in [38]. Since we will represent a given DCA A as a subalgebra of the regular closed subsets $RC(S)$ of certain DMS S, we need some 'lifting' conditions guaranteeing that A satisfies some abstract conditions (for instance the time axioms and some others) iff $RC(X)$ satisfies the same axioms. This will be subject of the next section.

5.2 Lifting conditions

Let $A_i = (B_{A_i}, C^s_{A_i}, C^t_{A_i}, \mathcal{B}_{A_i})$, $i = 1, 2$ be two algebras with a signature of DCA such that $C^s_{A_i}$ and $C^t_{A_i}$ be contact relations and $\mathcal{B}_{A_i}$ be a precontact relation. We assume also that A_1 is a subalgebra of A_2. This means that B_{A_1} is a Boolean subalgebra of B_{A_2} and that the relations from the list $C^s_{A_1}, C^t_{A_1}, \mathcal{B}_{A_1}$ are restrictions of the corresponding relations from the list $C^s_{A_2}, C^t_{A_2}, \mathcal{B}_{A_2}$ to B_{A_1}. We need some abstract 'lifting' conditions guarantying that A_1 satisfies the remaining axioms of DCA and possibly some time axioms from the list *time axioms* **(rs)**, **(ls)**, **(up dir)**, **(down dir)**, **(circ)**, **(dens)**, **(ref)**, **(lin)**, **(tri)**, **(tr)** iff A_2 satisfies the same axioms. The conditions are given in the next definition and are similar to analogical conditions considered in [76](pages 283-4) only for contact algebras. For convenience the elements from the set B_{A_i} are denoted correspondingly by $a_i, b_i, c_i, ...$ etc.

Definition 5.1. Lifting conditions. *Having in mind the above notations we say that the Boolean subalgebra A_1 is said to be a Boolean **dense subalgebra** of A_2 if*

(Dense) $(\forall a_2)(a_2 \neq 0 \Rightarrow (\exists a_1)(a_1 \neq 0$ *and* $a_1 \leq a_2)$,

*and to be a **co-dense subalgebra** of A_2 if*

(Co-dense) $(\forall a_2)(a_2 \neq 1 \Rightarrow (\exists a_1)(a_1 \neq 1$ *and* $a_2 \leq a_1)$.

It is easy to see that (Dense) is equivalent to (Co-dense).

Let C be any of the relations $C^s_{A_2}, C^t_{A_2}, \mathcal{B}_{A_2}$ and its restriction to B_{A_1} to be denoted also by C. We say that A_1 is a C-separable subalgebra of A_2 if the following condition is satisfied:

(C-separation) $(\forall a_2, b_2))(a_2 \overline{C} b_2 \Rightarrow (\exists a_1, b_1)(a_1 \overline{C} b_1$ *and* $a_2 \leq a_1$ *and* $b_2 \leq b_1)$.

*Conditions (Dense), (Co-dense) and (C-separable) for all C from the set $\{C^s_{A_2}, C^t_{A_2}, \mathcal{B}_{A_2}\}$ are called lifting conditions. If all lifting conditions are satisfied then A_1 is said to be a **stable subalgebra** of A_2.*

*If g is an isomorphic embedding of A_1 into A_2, then g is said to be a **dense (co-dense)** embedding provided that $g(A_1)$ is a dense (co-dense) subalgebra of A_2. We say that g is a C-separable embedding if $g(A_1)$ is a C-separable subalgebra of A_2. If all lifting conditions are satisfied, then g is called a **stable embedding** of A_1 into A_2.*

Lemma 5.2. Lifting Lemma. *Let $A_i = (B_{A_i}, C^s_{A_i}, C^t_{A_i}, \mathcal{B}_{A_i})$, $i = 1, 2$ be two algebras with a signature of DCA such that $C^s_{A_i}$ and $C^t_{A_i}$ be contact relations and $\mathcal{B}_{A_i}$ be a precontact relation and let A_1 be a stable subalgebra of A_2. Let **Ax** be any* *of the following list of axioms of DCA : $(C^s \subseteq C^t)$, (C^tE), $(C^t\mathcal{B})$, $(\mathcal{B}C^t)$, or any from the list of time axioms. Then* $\boldsymbol{Ax}$ *is true in* A_1 *iff* $\boldsymbol{Ax}$ *is true in* A_2.

Proof. Let us start with the case when **Ax** is the axiom $(C^s \subseteq C^t)$ $aC^sb \Rightarrow aC^tb$. Suppose first that $(C^s \subseteq C^t)$ is true in A_1 and for the sake of contradiction that it is not true in A_2. Then for some a_2, b_2 we have: $a_2C^sb_2$ and $a_2\overline{C}^tb_2$. Then by the condition (C^t-separation) we obtain: there exist a_1, b_1, such that $a_2 \leq a_1$, $b_2 \leq b_1$ and $a_1\overline{C}^tb_1$. From here and $a_2C^sb_2$ we get $a_1C^sb_1$ which by $a_1\overline{C}^tb_1$ shows that the axiom $(C^s \subseteq C^t)$ is not true in A_1 - a contradiction. Suppose now that the axiom is true in A_2. Since it is an universal formula, then it is trivially true in A_1.

Consider now that **Ax** is the axiom (C^tE) $a\overline{C}^tb \Rightarrow (\exists c)(a\overline{C}^tc$ and $c^*\overline{C}^tb)$. Suppose first that (C^tE) is true in A_1. In order to show that it is true in A_2 suppose $a_2\overline{C}^tb_2$. Then by the condition (C^t-separation) there exist a_1, b_1 such that $a_1\overline{C}^tb_1$, $a_2 \leq a_1$ and $b_2 \leq b_1$. By the assumption that (C^tE) is true in A_1, $a_1\overline{C}^tb_1$ implies that $(\exists c_1)(a_1\overline{C}^tc_1$ and $c_1^*\overline{C}^tb_1)$. From here we obtain $a_2\overline{C}^tc_1$ and $c_1^*\overline{C}^tb_2$. Obviously c_1 and c_1^* are in B_{A_2} which shows that (C^tE) is true in A_2.

Suppose now that (C^tE) is true in A_2 and in order to prove it in A_1 suppose $a_1\overline{C}^tb_1$. Since a_1, b_1 are also in B_{A_2}, then by the assumption there is c_2 such that $a_1\overline{C}^tc_2$ and $c_2^*\overline{C}^tb_1$. Then by the condition (C^t-separation) applied to $a_1\overline{C}^tc_2$ there exist a_1', c_1' such that $a_1 \leq a_1'$, $c_2 \leq c_2 \leq c_1'$ and $a_1'\overline{C}^tc_1'$. Analogously from $c_2^*\overline{C}^tb_1$ we infer that there exist c_1'', b_1' such that $b_1 \leq b_1'$, $c_2^* \leq c_1''$, $b_1 \leq b_1'$ and $c_1''\overline{C}^tb_1'$. Manipulating with inequalities and monotonicity conditions for C^t we finally obtain $a_1\overline{C}^tc_1'$ and $c_1'^*\overline{C}^tb_1$ which shows that (C^tE) holds in A_1.

In a similar way one can treat the case for the axioms $(C^t\mathcal{B})$ and $(\mathcal{B}C^t)$.

As an example we will treat one case for time axioms just to show that the tings go in a similar way. We consider the axiom **(lin)** $a \neq 0$ and $b \neq 0 \Rightarrow a\mathcal{B}b$ or $b\mathcal{B}a$. Suppose first that **(lin)** is true in A_1 and in order to show that it is true in A_2 suppose $a_2 \neq 0$ and $b_2 \neq 0$. Then by the condition (dence) there exists $a_1 \neq 0$ such that $a_1 \leq a_2$ and there exists $b_1 \neq 0$ such that $b_1 \leq b_2$. By the assumption $a_1 \neq 0$ and $b_1 \neq 0$ imply $a_1\mathcal{B}b_1$ or $b_1\mathcal{B}a_1$. By monotonicity conditions for $\mathcal{B}$ we get $a_2\mathcal{B}b_2$ or $b_2\mathcal{B}a_2$ which finishes the proof for this direction. For the converse direction suppose that **(lin)** is true in A_2. Since **(lin)** is an universal sentence it trivially holds in the subalgebra A_1. □

5.3 Dynamic Mereotopological Spaces (DMS)

Definition 5.3. Dynamic Mereotopological Space. *A system* $S = (X_S^t, X_S^s, T_S, \prec_S, M_S)$ *is called Dynamic Mereotopological Space (DMS, DM-space) if the next axioms are satisfied.*

The axioms of DMS:

• *(S1) X^t_S is a nonempty topological space, the elements of X^t_S are called* ***partial time points of*** *S.*

• *(S2) M_S is a subalgebra of the algebra $RC(X^t_S)$ of regular closed sets of X^t_S and M_S is a closed base of the topology of X^t_S.*

• *(S3) The sets X^t_S, X^s_S and T_S are non-empty sets satisfying the following inclusions:*

$X^s_S \subseteq X^t_S$, $T_S \subseteq X^t_S$.

The elements of X^s_S are called ***space points of*** *S, hence every space point is a partial time point. The elemnts of T_S are called* ***time points of*** *S.*

• *(S4) For $a \in RC(X^t_S)$: if $a \neq \varnothing$, then $a \cap X^s_S \neq \varnothing$ and*

• *(S5) $\prec_S$ is a binary relation in X^t_S called* ***before-after relation****. The subsystem $(T_S, \prec_S)$ is called the* ***time structure of*** *S.*

Definitions*: For $a, b \in RC(X^t_S)$ define:*

> $aC^t_S b$ *iff* $a \cap b \neq \varnothing$, ***time contact***,
>
> $aC^s_S b$ *iff* $a \cap b \cap X^s_S \neq \varnothing$, ***space contact***,
>
> $a\mathcal{B}_S b$ *iff there exist* $x, y \in X^t_S$ *such that* $x \prec_S y$, $x \in a$ *and* $y \in b$, ***precedence***,
>
> $RC(S) =_{def} (RC(X^t_S), C^t_S, C^s_S, \mathcal{B}_S)$, ***regular-sets algebra*** *of* S,
>
> *For* $x \in X^t_S$ *set* $\rho_S(x) =_{def} \{a \in M_S : x \in a\}$.
>
> $S^+ =_{def} (M_S, C^t_S, C^s_S, \mathcal{B}_S)$ *with the above defined relations restricted to* M_S.

It can easily be seen that C^s_S and C^t_S are contact relations in $RC(X^t_S)$ and that $\mathcal{B}$ is a precontact relation (for C^s_S use axiom (S4)).

• *(S6) The system S^+ is a DCA. S^+ is called the* ***canonical DCA of*** *S or the* ***dual of*** *S.*

• *(S7) For $x, y \in X^t_S$, $x \prec_S y$ iff $(\forall a, b \in M_S)(x \in a, y \in b \Rightarrow a\mathcal{B}_S b)$.*

• *(S8) If $x \in T_S$ then $\rho_S(x)$ is a cluster in S^+,*

We say that S is a $T0$ space if X^t_S is a $T0$ space.

Let $\widehat{Ax}$ be a subset of the time conditions from the list (RS), (LS), (Up Dir), (Down Dir), (Circ), (Dens), (Ref), (Lin), (Tri), (Tr). We say that S satisfies the axioms from the list $\widehat{Ax}$ if the time structure $(T_S, \prec_S)$ satisfies these conditions.

Intuitively DMS is abstracted from the clan-structure of DCA by introducing in it a topology.

Lemma 5.4. *Let $S = (X^t_S, X^s_S, T_S, \prec_S, M_S)$ be a DMS. Then:*

(i) If $x \in X^t_S$, then $\rho_S(x)$ is a t-clan in S^+.

(ii) If $x \in X^s_S$, then $\rho_S(x)$ is an s-clan in S^+.

(iii) If $x \in T_S$, then $\rho_S(x)$ is a cluster in S^+.

(iv) Let $\prec_{S^+}$ be the canonical relation of $\mathcal{B}$ between t-clans of S^+ (see (9) for the definition). Then Axiom (S7) of DMS is equivalent to the following statement: for all $x, y \in X^t_S$, $x \prec_S y$ iff $\rho_S(x) \prec_{S^+} \rho_S(y)$.

(v) S is T0 space iff $(\forall x, y \in X^t_S)(\rho_S(x) = \rho_S(y) \Rightarrow x = y)$. (or, equivalently, S is T0 iff ρ_S is an injective mapping from X^t_S into the t-clans of S^+).

Proof. For (i) and (ii) - by an easy verification of the corresponding definitions. For (iii) this is just the axiom (S8) for DMS. (iv) is trivial on the base of the definition of the relation $\prec_{M^+}$. (v) is easy if we take in consideration the definition $T0$ property, the definition of ρ_s and the fact that M_S is a closed base of the topology of X^t_S. □

Definition 5.5. *(1) A t-clan (s-clan, t-cluster) Γ of S^+ is called a* ***point t-clan*** *(s-clan, t-cluster) if there is a point $x \in X^t_S$ ($x \in X^s_S$, $x \in T_S$) such that $\Gamma = \rho_S(x)$.*

(2) S is a ***DM-compact*** *(dynamic mereoompact) space if every t-clan, s-clan and t-cluster of S^+ is respectively a point t-clan, s-clan and a t-cluster.*

The following Lemma is obvious.

Lemma 5.6. *Let S be a DMS. Then the following two conditions are equivalent:*

(i) S is DM-compact,

(ii) ρ_S is a surjective mapping from X^t_S onto the set of all t-clans of S^+. More over ρ_S maps X^s_S onto the set of all s-clans of S^+ and it maps T_S onto the set of all clusters of S^+.

Corollary 5.7. *Let S be a T0 and DM-compact DMS. Then ρ_S is a one-one mapping from X^t_S onto the set of all t-clans of S^+ which preserves the sets X^s_S and T_S.*

Proof. By Lemma 5.4 and Lemma 5.6 □

Remark 5.8. *The notions of DM-space and DM-compactness can be considered as dynamic versions of the notions of mereotopological space and mereocompactness introduced by Goldblatt and Grice in [38]. Their definitions are the following. A mereotopological space is a pair $S = (X_S, M_S)$ where X is a topological space and M_S is a subalgebra of the Boolean algebra $RC(X_S)$ of regular closed sets of X_S considered as closed base of the topology of X. Let S^+ be the contact algebra (M_S, C_S) where C_S is the standard topological contact between regular closed sets. S is mereocompact if every clan of the contact algebra S^+ is a point clan in the sense of*

Definition 5.5 (in fact the definition of mereocompactness in [38] is slightly different but equivalent to the given here). So, if $S = (X^t_S, X^s_S, T_S, \prec_S, M_S)$ is a DM-space then the pair (X^t, M_S) is a mereotopological space and if S is DM-compact then (X^t, M_S) is mereocompact. Mereotopological spaces have been introduced by Goldblatt and Grice in order to develop a topological duality theory for contact algebras. Similarly, we introduce the notion of DM-space to be used in the topological representation theory and duality theory of DCAs.

Because our exposition is quite similar to that of Goldblatt and Grice and in some sense is an adaptation of their method to the case of DCAs, we recommend the paper [38] to the reader of the present text. For convenience we even use similar and compatible notations with [38].

Lemma 5.9. *Let S be a DM-compact space. Then the topological space X^t_S is compact.*

Proof. According to Remark 5.8 DM-compactness of S implies that the pair (X^t_S, M_S) is a mereocompact space and then the statement follows from Theorem 4.2.(3) of [38]. We present below the proof illustrating our definition of DM-compactness.

In order to prove the compactness of X^t_S, it suffices to prove the following. Let $I \subseteq M_S$ be a nonempty set and let $A = \bigcap\{a \in M_S : a \in I\}$. If for every finite $I_0 \subseteq I$ the set $\bigcap\{a \in M_S : a \in I_0\} \neq \varnothing$, then $A \neq \varnothing$. The fact that $\bigcap\{a \in M_S : a \in I_0\} \neq \varnothing$ for every finite subset I_0 of I guarantees the existence of an ultrafilter U in the subset of all subsets of X^t_S such that $\{a \in M_S : a \in I\} \subseteq U$. Let $\Gamma = \{a \in M_S : a \in U\}$. Then it is easy to see that Γ is a t-clan. Then by DM-compactness there exists $x \in X^t_S$ such that $\Gamma = \rho_S(x)$. Hence for every $a \in I$ we have the following:

$a \in I \Longrightarrow a \in U \Longrightarrow a \in \Gamma \Longrightarrow a \in \rho_S(x) \Longrightarrow x \in a \Longrightarrow x \in A \Longrightarrow A \neq \varnothing$ □

Lemma 5.10. *Let $S = (X^t_S, X^s_S, T_S, \gamma_S, \prec_S, M_S)$ be a DM-compact DMS. Then the set X^s_S of space points of S with a subset topology is a $T0$ dense subset of X^t_S.*

Proof. Let Cl denote the closure operation of X^t_S. We have to show that $ClX^s_S = X^t_S$. Suppose that this is not true, i.e. there exists $x \in X^t_S$ such that $x \notin ClX^s_S$. Since M_S is a closed base of the topology of X^t_S then there exits $a \in M_S$ such that $X^s \subseteq a$ and $x \notin a$. Then $a \notin \rho_S(x)$, which is a t-clan in S^+. Then for all ultrafilters $U \subseteq \rho_S(x)$ we have that $a \notin U$, and let U be such one. But U is an s-clan, so by DM-compactness there is a point $y \in X^s_S$ such that $U = \rho_S(y)$. Because $U \subseteq \rho_S(x)$ we obtain $\rho_S(y) \subseteq \rho_S(x)$. From here we obtain that $a \notin \rho_S(y)$ and consequently $y \notin a$. But $y \in X^s \subseteq a$, so $y \in a$ - a contradiction. □

Lemma 5.11. ([12], page 271) Let X be a dense subspace of a topological space Y and let $RC(X)$ and $RC(Y)$ be the corresponding Boolean algebras of regular closed sets of X and Y. Let for $a \in RC(X)$, $h(a) = Cl_Y(a)$. Then $h : RC(X) \to RC(Y)$ is an isomorphism from $RC(X)$ onto $RC(Y)$. For $b \in RC(Y)$ converse mapping h^{-1} acts as follows: $h^{-1}(b) = b \cap X$.

Corollary 5.12. *The Boolean algebra $RC(X^s_S)$ of regular closed subsets of X^s_S is isomorphic to the Boolean algebra $RC(X^t_S)$.*

Proof. The lemma is a corollary of Lemma 5.10 and Lemma 5.11. □

In the next section we study some other consequences of DM-compactness.

5.4 Canonical filters in DM-compact spaces

We assume in this section that S is a DM-compact space. The aim of the section is to introduce a technical notion - *canonical filter*, generalizing a similar notion from [76]. By means of canonical filters and the assumption of DM-compactness of a given S we will establish that the algebra S^+ is a stable subalgebra of $RC(S)$ in the sense of Definition 5.1 which fact implies several important consequences.

Definition 5.13. *Let $A \in RC(X^t_S)$. Then the set $F_A =_{def} \{a \in M_S : A \subseteq a\}$ is called canonical filter of S^+.*

Lemma 5.14. *Let $A, B \in RC(X^t_S)$. Then:*

(i) F_A is a filter in S^+.

(ii) $\forall x \in X^t_S$: $x \in A$ iff $F_A \subseteq \rho_S(x)$.

(iii) $A \neq X^t_S$ iff there exists $a \in M_S$ such that $A \subseteq a$ and $a \neq X^t_S$.

Let $R^t, R^s, \prec$ be the canonical relations between filters corresponding to the relations $C^t_S, C^s_S, \mathcal{B}_S$ from the DCA algebra S^+.

(iv) The following conditions are equivalent:

(1.1) AC^t_SB. (1.2) $F_AR^tF_B$. (1.3) $A \cap B \cap T_S \neq \varnothing$.

(v) The following conditions are equivalent:

(2.1) AC^s_SB. (2.2) $F_AR^sF_B$.

(vi) The following conditions are equivalent

(3.1) $A\mathcal{B}_SB$. (3.2) $F_A \prec F_B$. (3.3) There exist $x \in A \cap T_S$ and $y \in B \cap T_S$ such that $x \prec_S y$.

Proof. (i) The proof is by a direct checking of the corresponding definitions.

(ii) The implication from left to right is by straightforward checking. For the converse direction we will reason by contraposition. Suppose $x \notin A$. Now we will apply the fact that M_S is a closed base of the topology of X. Because A is a regular closed set then A is a closed set and then there exists $a \in M_S$ such that $A \subseteq a$ and $x \notin a$. Then $a \in F_A$ and $a \notin \rho_S(x)$, so $F_A \not\subseteq \rho_S(x)$.

(iii) can be derived by direct application of (ii).

(iv) (1.1)$\Rightarrow$(1.2) Suppose $AC^t_S B$. Then there is a point $x \in X^t_S$ such that $x \in A$ and $x \in B$. By (ii) this implies

(1) $F_A \subseteq \rho_S(x)$ and

(2) $F_B \subseteq \rho_S(x)$.

In order to show $F_A \prec F_B$ suppose $a \in F_A$ and $b \in F_B$ and proceed to show $F_A R^t F_B$. Then by (1) and (2) we get $a \in \rho_S(x)$ and hence $x \in a$, and $b \in \rho_S(x)$ and hence $x \in b$, which shows $a \cap b \neq \varnothing$. So, $aC^t_{S^+} b$ which proves that $F_A R^t F_B$.

(1.2)$\Rightarrow$(1.3) Suppose $F_A R^t F^b$. By Lemma 2.10 there exist ultrafilters U, V such that $F_A \subseteq U$, $F_B \subseteq V$ and UR^tV. Let $\Gamma = U \cup V$. Obviously $F_A \subseteq \Gamma$ and $F_B \subseteq \Gamma$. By Lemma 2.16 Γ as a union of R^t-related ultrafilters is a t-clan in S^+ and then it can be extended into a cluster Δ. By DM-compactness there is $x \in T_S$ such that $\Delta = \rho_s(x)$. Hence $F_A \subseteq \rho_s(x)$ and $F_B \subseteq \rho_s(x)$. By (ii) $x \in A$ and $x \in B$ hence $A \cap B \cap T_S \neq \varnothing$.

(1.3)$\Rightarrow$(1.1) Suppose $A \cap B \cap T_S \neq \varnothing$. Then $A \cap B \neq \varnothing$, so $AC^t_S B$.

(v) the proof is similar to (iv)- it is used that if Γ is an s-clan in S^+ then by the DM-compactness there is point $x \in X^s_S$ such that $\Gamma = \rho_S(x)$.

(vi) (3.1)($\Rightarrow$) (3.2) Suppose $A\mathcal{B}_S B$. Then there exist $x \in A \cap X^t_S$ and $y \in B \cap X^t_S$ such that $x \prec_S y$. Then by (ii) we obtain $F_A \subseteq \rho_S(x)$, $F_B \subseteq \rho_S(y)$ and by Lemma 5.4 we have $\rho_S(x) \prec \rho_S(y)$ and $\rho_S(x)$ and $\rho_S(y)$ are t-clans. Then by the definition of $\prec$ in the set of t-clans we get $F_A \prec F_B$.

(3.2)($\Rightarrow$) (3.3) Suppose $F_A \prec F_B$. Then by Lemma 2.10 there are ultrafilters U, V such that $F_A \subseteq U$, $F_B \subseteq V$ and $U \prec V$. Ultrafilters are t-clans and we can extend them into clusters preserving the relation $\prec$, namely: there exist clusters Γ, Δ such that $U \subseteq \Gamma$, $V \subseteq \Delta$ and $\Gamma \prec \Delta$. By DM-compactness there are $x', y' \in T_S$ such that $\Gamma = \rho_S(x')$ and $\Delta = \rho_S(y')$, so $\rho_S(x') \prec \rho_S(y')$ and hence $x' \prec_S y'$. We can obtain also $F_A \subseteq \rho_S(x')$ and hence $x' \in A$, and $F_B \subseteq \rho_S(y')$ and hence $y' \in B$. All this says: $\exists x' \in A \cap T_S$, $\exists y' \in B \cap T_S$ such that $x' \prec_S y'$.

(3.3)($\Rightarrow$) (3.1). This implication is obvious because $T_S \subseteq X^t_S$. $\square$

Note that conditions (i), (ii) and (iii) of the above lemma does not depend on the assumption of DM-compactness.

Lemma 5.15. *The following conditions are true for S:*

(i) The algebra S^+ is a stable Boolean sub-algebra of $RC(S)$.

(ii) $RC(S)$ is a DCA.

Proof. (i) We first show that S^+ satisfies the lifting conditions (see Definition 5.1) and then (i) is a corollary of Lemma 5.2. First we verify the lifting condition (co-dense). Suppose $A \in RC(X_S^t)$ and $A \neq X_S^t$. Then by Lemma 5.14 (iii) there exists $a \neq M_S$ such that $a \neq X_S^t$ and $A \subseteq a$. We do not treat (dense) because it is equivalent to (co-dense).

To verify the condition (C-separation) for $C \in \{C_S^t, C_S^s, \mathcal{B}_S\}$ we proceed as follows. Looking at the conditions (iv), (v), (vi) of Lemma 5.14 we see that they have the following common form. Let R be the canonical relation between filters corresponding to the relation C. Then for any $A, B \in RC(X_S^t)$: ACB iff $F_A R F_B$. Taking the negation in both sides we obtain: $A\overline{C}B$ iff $F_A\overline{R}F_B$ iff there exists $a, b \in M_S$ such that $a \in F_A$, $b \in F_B$ and $a\overline{C}b$ iff there exists $a, b \in M_S$ such that $A \subseteq a$, $B \subseteq b$ and $a\overline{C}b$. Thus: $F_A\overline{R}F_B$ implies that for some $a, b \in M_S$, $A \subseteq a$, $B \subseteq b$ and $a\overline{C}b$ which is the (C-separation) condition. Note that just this implication needed DM-compactness in Lemma 5.14.

(ii) is a corollary of (i) and the fact that S^+ is a DCA, so by Lemma 5.2 the axioms ($C^s \subseteq C^t$), (C^tE), ($C^t\mathcal{B}$ and ($\mathcal{B}C^t$) are lifted from S^+ to $RC(S)$. □

Lemma 5.16. *Let* (φ) *be any of the time axioms:* ***(rs), (ls), (up dir), (down dir), (circ), (dens), (ref), (lin), (tri), (tr)****. Then the following conditions are equivalent:*

(i) (φ) *is true in the algebra* S^+*.*

(ii) (φ) *is true in the algebra* $RC(S)$*.*

Proof. The proof follows from Lemma 5.15 (i) and Lemma 5.2. □

Lemma 5.17. *Let* S *be DM-compact DMS,* $RC(S)$ *be its regular-sets algebra,* $(T_S, \prec_S)$ *be its time structure and let* $(T_{S^+}, \prec_{S^+})$ *be the canonical time structure of* S^+ *(see Definition 4.8). Let* (Φ) *be the time condition from the list (RS), (LS), (Up Dir), (Down Dir), (Circ), (Dens), (Ref), (Irr), (Lin), (Tr) (condition (Tri) is excluded). Then the following conditions are true:*

(i) (Φ) *is true in* $(T_S, \prec_S)$ *iff* (Φ) *is true in* $(T_{S^+}, \prec_{S^+})$*.*

(ii) If S *is* $T0$ *DMS, then: (Tri) is true in* $(T_S, \prec_S)$ *iff (Tri) is true in* $(T_{S^+}, \prec_{S^+})$*.*

Proof. (i) Let us remind that the members of T_{S^+} are clusters of S^+ , which we will denote by $\Gamma, \Delta, \Theta, \ldots.$ We will demonstrate the proof considering the case (Dense), the proofs for the other cases go in the same manner.

(Dense) $(\forall i, j)(i \prec j \Rightarrow (\exists k)(i \prec k \text{ and } k \prec j)$.

($\Rightarrow$) Suppose (Dense) is true in $(T_S, \prec_S)$ and let $\Gamma, \Delta \in T_{S^+}$ and $\Gamma \prec_{S^+} \Delta$. Then by DM-compactness there exist $x, y \in T_S$ such that $\Gamma = \rho_S(x)$, and $\Delta = \rho_S(y)$, so $\rho_S(x) \prec_{S^+} \rho_S(y)$. By Lemma 5.4 (iv) we obtain $x \prec_S y$ and by (Dence) there exists

$z \in T_S$ such that $x \prec_S z \prec_S y$. Again by Lemma 5.4 (iv) we obtain $\rho_S(x) \prec_{S^+} \rho_S(z) \prec_{S^+} \rho_S(y)$. Because $\rho_S(z)$ is a cluster in S_+ we put $\Theta = \rho_S(z)$ and obtain $\Gamma \prec_{S^+} \Theta \prec_{S^+} \Delta$ which shows that (Dense) is true in $(T_{S^+}, \prec_{S^+})$.

($\Leftarrow$) Suppose (Dense) is true in $(T_{S^+}, \prec_{S^+})$, $x, y \in T_S$ and $x \prec_S y$. Then $\rho_S(x) \prec_{S^+} \rho_{S^+}(y)$. By (Dence) there exists a cluster Θ (hence there exists $z \in T_S$ with $\rho_S(z) = \Theta$) such that $\rho_S(x) \prec_{S^+} \rho_S(z) \prec_{S^+} \rho_{S^+}(y)$. This implies $x \prec_S z \prec_S y$ which shows that (Dense) is true in $(T_S, \prec_S)$.

(ii) The case of (Tri) $(\forall i, j)(i = j$ or $i \prec j$ or $j \prec i$.

($\Rightarrow$) The proof of this implication is straightforward and requires neither DM-compactness nor $T0$ property.

($\Leftarrow$) Suppose (Tri) is true in $(T_{S^+}, \prec_{S^+})$ and let $x, y \in T_S$. Then $\rho_S(x), \rho_S(y)$ are clusters in S^+. Then by (Tri) we have $\rho_S(x) = \rho_S(y)$ or $\rho_S(x) \prec_{S^+} \rho_S(y)$ or $\rho_S(y) \prec_{S^+} \rho_S(x)$.

Case 1: $\rho_S(x) = \rho_S(y)$. Since $\rho_S(x)$ and $\rho_S(y)$ are also t-clans then by the assumption that S is a $T0$ space case 1 implies $x = y$ (by Lemma 5.4 (v)).

Case 2: $\rho_S(x) \prec_{S^+} \rho_S(y)$. By Lemma 5.4 (iv) this implies $x \prec_S y$.

Case 3: $\rho_S(y) \prec_{S^+} \rho_S(x)$. Again by Lemma 5.4 (iv) this implies $y \prec_S x$. Thus, (Tri) is fulfilled in the time structure $(T_S, \prec_S)$. □

Lemma 5.18. ***Topological definability.*** *Let $(T_S, \prec_S)$ be the time structure of S, (Φ) be the time condition from the list (RS), (LS), (Up Dir), (Down Dir), (Circ), (Dens), (Ref), (Lin), (Tri) (Tr) and (φ) be the corresponding time axiom from the list* ***(rs), (ls), (up dir), (down dir), (circ), (dens), (ref), (lin), (tri), (tr)****. Then the following conditions are equivalent (for the case of (Tri) we assume also that S is $T0$):*

(i) (Φ) is true in $(T_S, \prec_S)$

(ii) (φ) is true in $(RC)(S)$.

Proof. (Φ) is true in $(T_S, \prec_S)$ iff (by Lemma 5.17) (Φ) is true in the canonical time structure of S^+, $(T_{S^+}, \prec_{S^+})$ iff (by Lemma 4.9 (φ) is true in S^+ iff (by Lemma 5.16) (φ) is true in the algebra $RC(S)$. □

5.5 Canonical DMS for DCA and topological representation theorem for DCA

Let $A = (B_A, C^t_A, C^s_A, \mathcal{B}_A)$ be a DCA. We associate to DCA in a canonical way a DM-space denoted by A_+ and called the **canonical DMS of A** or the **dual DMS of** A as follows:

- $A_+ =_{def} (X^t_A, X^s_A, T_A, \prec_A, M_A)$, where:

- X^t_A =t-$Clans(A)$, X^s_A =s-$Clans(A)$ and $T_A = Clusters(A)$.
- $\prec_A$ is the before-after relation in the set X^t_A defined by (9). The structure $(T_A, \prec_A)$ - the time structure of A is now the time structure of A_+.

M_A is defined as follows and is used to introduce a topology in the set X^t_A considering it as a basis of the closed sets in the topology:

- For $a \in B_A$ let $g_A(a) = \{\Gamma \in$t-$Clans(A) : a \in \Gamma\}$ and put
- $M_A = \{g_A(a) : a \in B_A\}$.

By the topological representation theory of contact algebras (see Section 2.5) the set $\{g_A(a) : a \in B_A\}$ defines a topology in the set X^t_A and g_A is an isomorphic embedding of B_A into the algebra $RC(X^t_A)$ and M_A is a Boolean subalgebra of $RC(X^t_A)$ isomorphic to B_A.

We define the algebra $(A_+)^+$ - the dual of A_+ as follows.

- $(A_+)^+ =_{def} (M_A, C^t_{A_+}, C^s_{A_+}, \mathcal{B}_{A_+})$.

Having in mind the topological representation theory of contact algebras (see Section 2.5 and Lemma 4.6 it can be seen that g_A is also an isomorphism from A onto $(A_+)^+$, so $(A_+)^+ = g_A(B_A)$ which proves the following lemma.

Lemma 5.19. *A is isomorphic to $(A_+)^+$ and hence $(A_+)^+$ is a DCA.*

By definition we have $\rho_{A_+} =_{def} \{g_A(a) \in M_A : \Gamma \in g_A(a)\} = \{g_A(a) \in M_A : a \in \Gamma\}$.

Lemma 5.20. *(i) For any $\Gamma \in X^t_A$ $\rho_{A_+}(\Gamma)$ is a t-clan in $(A_+)^+$.*
(ii) For any $\Gamma \in X^s_A$ $\rho_{A_+}(\Gamma)$ is a s-clan in $(A_+)^+$.
(iii) For any $\Gamma \in T_A$ $\rho_{A_+}(\Gamma)$ is a cluster in $(A_+)^+$.

Proof. The proof is by a routine verification of the corresponding definitions and using the results about the clan structure of DCA developed in Section 4.1. As an example we will demonstrate the proof of (iii).

Let $\Gamma \in T_A$. Then Γ is a cluster in A, so Γ is a t-clan in A. By (i) $\rho_{A_+}(\Gamma)$ is a t-clan in $(A_+)^+$. We will show that $\rho_{A_+}(\Gamma)$ is a cluster in $(A_+)^+$. Suppose that for some $a \in B_A$, $g_A(a) \notin \rho_{A_+}(\Gamma)$. Then $\Gamma \notin g_A(a)$, so $a \notin \Gamma$. Then there exists $b \in B_A$ such that $b \in \Gamma$ and $a\overline{C}_A b$. Then $g_A(b) \in \rho_{A_+}(\Gamma)$ and $g_A(a) \cap g_A(b) = \varnothing$, so $g_A(a)\overline{C^t}_{A_+} g_A(b)$. Note that (iii) verifies the DMS axiom (S7) for A_+. □

Lemma 5.21. *Let Γ, Δ be t-clans in A. Then: $\Gamma \prec_A \Delta$ iff for all $g_A(a), g_A(b) \in M_A$: if $\Gamma \in g_A(a)$ and $\Delta \in g_A(b)$, then $g_A(a)\mathcal{B}_{(A_+)^+} g_A(b)$.*

Proof. Let Γ, Δ be t-clans in A. Having in mind the relevant definitions the implication from left to the right is obvious. For the converse implication suppose that

($\sharp$) For all $g_A(a), g_A(b) \in M_A$, the conditions $\Gamma \in g_A(a)$ and $\Delta \in g_A(b)$ imply $g_A(a)\mathcal{B}_{(A_+)^+}g_A(b)$

and proceed to show $\Gamma \prec_A \Delta$. By (9) this means that for some $a \in \Gamma$ and $b \in \Delta$ we have $a\mathcal{B}_A b$. To this end suppose $a \in \Gamma$ and $b \in \Delta$. Then $\Gamma \in g_A(a)$ and $\Delta \in g_A(b)$. By ($\sharp$) we get $g_A(a)\mathcal{B}_{(A_+)^+}g_A(b)$ which by the definition of $\mathcal{B}_{(A_+)^+}$ means that for some t-clans Γ', Δ' in A we have $\Gamma' \in g_A(a)$, $\Delta' \in g_A(b)$ and $\Gamma' \prec_A \Delta'$. This implies $a \in \Gamma'$ and $b \in \Delta'$ and by the definition of $\Gamma' \prec_A \Delta'$ (see (9)) that $a\mathcal{B}_A b$ - end of the proof. Note that this lemma verifies the DMS axiom (S7) for A_+. □

Lemma 5.22. *Let A be a DCA and $\Gamma \subseteq M_A$. Define $\widehat{\Gamma} =_{def} \{a \in B_A : g_A(a) \in \Gamma\}$. Then the following conditions are true:*

(i) If Γ is a t-clan in $(A_+)^+$, then $\widehat{\Gamma}$ is a t-clan in A and $\rho_{A_+}(\widehat{\Gamma}) = \Gamma$.

(ii) If Γ is an s-clan in $(A_+)^+$, then $\widehat{\Gamma}$ is an s-clan in A and $\rho_{A_+}(\widehat{\Gamma}) = \Gamma$.

(iii) If Γ is a cluster in $(A_+)^+$, then $\widehat{\Gamma}$ is a cluster in A and $\rho_{A_+}(\widehat{\Gamma}) = \Gamma$.

Proof. (i) Let Γ be a t-clan in $(A_+)^+$. The verification of grill properties of $\widehat{\Gamma}$ is easy. Let us prove the t-clan property. Suppose $a, b \in \widehat{\Gamma}$. Then $g_A(a), g_A(b) \in \Gamma$. Then $(g_A(a))C^t_{(A_+)^+}((g_A(b))$. By the definition of $C^t_{(A_+)^+}$ we have $(g_A(a)) \cap ((g_A(b)) \neq \varnothing$. So there exists $\Gamma \in t-Clans(A)$ such that $a, b \in \Gamma$, which implies $aC^t_A b$.

Let us show the equality $\rho_{A_+}(\widehat{\Gamma}) = \Gamma$. The following sequence of equivalencies proves this:

$g_A(a) \in \rho_{A_+}(\widehat{\Gamma})$ iff $\widehat{\Gamma} \in g_A(a)$ iff $a \in \{b \in B_A : g_A(b) \in \Gamma\}$ iff $g_A(a) \in \Gamma$.

(ii) Let Γ be an s-clan in $(A_+)^+$. We will show the s-clan property of $\widehat{\Gamma}$. Suppose that $a, b \in \widehat{\Gamma}$. Then $(g_A(a))C^s_{(A_+)^+}((g_A(b))$. By the definition of $C^s_{(A_+)^+}$, there exists $\Gamma \in s-Clans(A)$ such that $\Gamma \in g_A(a) \cap g_A(b)$. This implies $a, b \in \Gamma$ and consequently $aC^s b$. The proof of $\rho_{A_+}(\widehat{\Gamma}) = \Gamma$ is as in (i).

(iii) Let Γ a cluster in $(A_+)^+$. So by it is a t-clan in $(A_+)^+$ and by (i) $\widehat{\Gamma}$ is a t-clan in A. We will show that $\widehat{\Gamma}$ is a cluster in A. Suppose $a \not{i}n\widehat{\Gamma}$. Then $g_A(a) \notin \Gamma$, hence there exists $g_A(b) \in M_A$ such that $g_A(b) \in \Gamma$ and $g_A(a)\overline{C^t}_{(A_+)^+}g_A(b)$. This implies $b \in \widehat{\Gamma}$ and $g_A(a) \cap g_A(b) = \varnothing$ which gives $a\overline{C^t}_A b$. This shows the cluster property of $\widehat{\Gamma}$. The proof of $\rho_{A_+}(\widehat{\Gamma}) = \Gamma$ is as in (i). □

Lemma 5.23. *Let A be a DCA, $A_+ = (X^t_A, X^s_A, T_A, \prec_A, M_A)$ be its dual space, $\alpha \in RC(X^t_A)$ and $\alpha \cap X^t_{A_+} \neq \varnothing$. Then $\alpha \cap X^s_{A_+} \neq \varnothing$.*

Proof. Suppose $\alpha \in RC(X^t_A)$ and $\alpha \cap X^t_A \neq \varnothing$. (remind that $X^t_A = t - Clans(A)$ and $X^s_A = s - Clans(A)$). Then there exists $\Gamma \in X^t_A$ such that $\Gamma \in \alpha$. Let F_α be the canonical filter of α (see Section 5.14). Then by Lemma 5.14 (ii) we have $F_\alpha \subseteq \rho_{A_+}(\Gamma)$. By Lemma 5.20 $\rho_{A_+}(\Gamma)$ is a t-clan in $(A_+)^+$. Then there exists an ultrafilter U in $(A_+)^+$ such that $F_\alpha \subseteq U \subseteq \rho_{A_+}(\Gamma)$. U is both a t-clan and an s-clan in $(A_+)^+$. By Lemma 5.22 (ii) there exists an s-clan $\widehat{U}$ such that $U = \rho_{A_+}(\widehat{U})$. So, we have $F_\alpha \subseteq \rho_{A_+}(\widehat{U})$ and $\widehat{U} \in X^s_A$. Again by Lemma 5.14 (ii) we get $\widehat{U} \in \alpha$ and consequently $\alpha \cap X^s_A \neq \varnothing$. □

Note that we have used Lemma 5.14 which presupposes DM-compactness. But as it was mentioned after the proof of this lemma condition (ii) which we used does not depend on DM-compactness. Note also that the above lemma verifies the DMS axiom (S4) for A_+.

Lemma 5.24. *A_+ is a DMS.*

Proof. The proof follows from Lemma 5.20, Lemma 5.23, Lemma 5.21 and Lemma 5.19 which establish the DMS axioms (S4), (S6), (S7) and (S8) for A_+. The other axioms are obviously true. □

The following theorem is important.

Theorem 5.25. *A_+ is T0 and DM-compact DMS.*

Proof. By Lemma 5.4(v) A_+ has $T0$ property iff for every two members Γ, Δ of X^t_A(=t-$Clans(A)$) the following holds: if $\rho_{A_+}(\Gamma) = \rho_{A_+}(\Delta)$, then $\Gamma = \Delta$. Suppose $\rho_{A_+}(\Gamma) = \rho_{A_+}(\Delta)$ and for the sake of contradiction that $\Gamma \neq \Delta$, so $\Gamma \not\subseteq \Delta$ or $\Delta \not\subseteq \Gamma$. Considering the first case this means that there exists a such that $a \in \Gamma$ and $a \notin \Delta$.Then by Lemma 5.20 $g_A(a) \in \rho_{A_+}(\Gamma)$ and $g_A(a) \notin \rho_{A_+}(\Delta)$ which shows that $\rho_{A_+}(\Gamma) \neq \rho_{A_+}(\Delta)$ - a contradiction. In a similar way the second case also implies a contradiction.

For DM-compactness we have to show the following three things:
(i) Every t-clan Γ of $(A_+)^+$ is a point t-clan,
(ii) Every s-clan of $(A_+)^+$ is a point s-clan,
(iii) Every cluster of $(A_+)^+$ is a point cluster.

Proof of (i). Let Γ be a t-clan of $(A_+)^+$. To show that Γ is a point t-clan we have to find $\Delta \in X^t_A$ (= t-Clans(A)) such that $\Gamma = \rho_{A_+}(\Delta)$. Let $\Delta = \widehat{\Gamma} = \{a \in B_A : g_A(a) \in \Gamma\}$. By Lemma 5.22 (i) $\widehat{\Gamma}$ is a t-clan in A and hence it is in X^t_A. More over we have $\rho_{A_+}(\widehat{\Gamma}) = \Gamma$.

The proofs of (ii) and (iii) are similar by using Lemma 5.22 (ii) and (iii). □

Theorem 5.26. Topological representation theorem for DCA. *Let A be a DCA. Then the following conditions for A are true:*

(i) $(A_+)^+$ is a stable subalgebra of the algebra $RC(A_+)$.

(ii) The algebra $RC(A_+)$ is a DCA.

(iii) The function g_A is a stable isomorphic embedding of A into $RC(A_+)$.

(iv) If $\boldsymbol{Ax}$ *is a time axiom, then* $\boldsymbol{Ax}$ *is true in A iff* $\boldsymbol{Ax}$ *is true in $RC(A_+)$.*

Proof. (i) By Theorem 5.25 A_+ is a DM-compact DMS and hence by Lemma 5.15 (i) $(A_+)^+$ is a stable Boolean subalgebra of $RC(A_+)$.

(ii) follows from (i) and Lemma 5.15 (ii).

(iii) By Lemma 5.19 g_A is an isomorphism from A onto $(A_+)^+$ and hence by (i) g_A is a stable isomorphic embedding of A into $RC(A_+)$.

(iv) follows from Lemma 5.25 and Lemma 5.16. □

5.6 Contact algebra as a special case of dynamic contact algebra

Let $A = (B_A, C_A)$ be a contact algebra. By Lemma 4.2 A can be considered as a DCA algebra on the base of the following definable relations: $a, b \in B_A$:

(1) $aC^t_A b \Leftrightarrow_{def} aC^{max}_A b \Leftrightarrow a \neq 0$ and $b \neq 0$.

(2) $a\mathcal{B}_A b \Leftrightarrow_{def} aC^t_A b$.

(3) $aC^s_A b \Leftrightarrow_{def} aC_A b$.

Let $A = (B_A, C^s_A, C^t_A, \mathcal{B}_A)$ be a DCA which satisfies the above conditions. Then it is obviously equivalent to the contact algebra (B_A, C_A). Condition (3) is just giving another name of C^s_A, and conditions (1) and (2) can be relaxed correspondingly to the following:

($1'$) If $a \neq 0$ and $b \neq 0$, then $aC^t_A b$,

($2'$) If $a \neq 0$ and $b \neq 0$, then $a\mathcal{B}_A b$.

Obviously ($1'$) implies (1) and ($2'$) implies (2). Hence if a DCA satisfies ($1'$) and ($2'$), t5hen it is equivalent to the contact algebra (B_A, C^s). Condition (1) then makes t-clans to coincide with grills, and in this case to have only one cluster, denote it by t_0 (the only time point of A which is just the union of all ultrafilters in A). Condition (2) implies that $\mathcal{B}_A = C^t_A$ which makes the relation $\prec_A$ to be the universal relation between grills and especially for t_0 to have $t_0 \prec_A t_0$. This suggests the following formal definition.

Definition 5.27. *We say that A is a* ***trivial DCA*** *if it satisfies the conditions* ($1'$) *and* ($2'$).

Thus for the dual space A_+ of a trivial DCA we have that $T_A = \{t_0\}$ is a singleton set and that t_0 is the only time point of A. This suggests to consider this as a characteristic property of a DMS corresponding in some sense to a trivial DCA and to adopt the following formal definition.

Definition 5.28. *We say that S is a trivial DMS if the set $T_S = \{t_0\}$ is a singleton with a single time point t_0 and $t_0 \prec_S t_0$*

Lemma 5.29. *Let S be a $T0$ and DM-compact space. Then the following two conditions are equivalent:*

(i) S is trivial DMS.

(ii) The dual algebra S^+ is a trivial DCA.

Proof. . (i)⇒(ii). Suppose that S is trivial DMS. First we will show that the DCA algebra S^+ has at most one cluster. Note that it has clusters. Let Γ, Δ be two clusters. By DM-compactness there is $x \in T_S$ such that $\rho_S(x) = \Gamma$ and $y \in T_S$ such that $\rho_S(y) = \Delta$. But T_S is a singleton, so $x = y$ which implies $\Gamma = \rho_S(x) = \rho_S(y) = \Delta$. So we have only one cluster, say Γ_0.

In order to show (ii) it is sufficient that the following is true for arbitrary regular closed sets $\alpha, \beta \in RC(X^t_S$:

If $\alpha \neq \varnothing$ and $\beta \neq \varnothing$, then $\alpha C^t_S \beta$ and then $\alpha \mathcal{B}_S \beta$.

Suppose $\alpha \neq \varnothing$ and $\beta \neq \varnothing$, then there exist $x \in \alpha$ and $y \in \beta$. Now we will apply the properties of canonical filters (see Lemma 5.14 from Section 5.4). Conditions $x \in \alpha$ and $y \in \beta$ imply $F_\alpha \subseteq \rho_S(x)$ and $F_\beta \subseteq \rho_S(y)$. $\rho_S(x)$ and $\rho_S(y)$ are t-clans in S^+ and can be extended into clusters. But there is only one cluster $\Gamma_0 = \rho_S(z)$ for some $z \in T_S$. Hence $F_\alpha \subseteq \rho_S(z)$ and $F_\beta \subseteq \rho_S(z)$. Then by the properties of canonical filters we get $z \in \alpha$ and $z \in \beta$, so $\alpha \cap \beta \neq \varnothing$ which shows $\alpha C^t_S \beta$. Because z is the only element of T_S we have $z \prec_S z$ which also shows that $\alpha \mathcal{B}_S \beta$.

(ii)⇒(i) Let S^+ be a trivial DCA. We mentioned that the condition (1) makes t-clans to coincide with grills. Because there exists only one maximal grill - the union of all ultrafilters, then there exists only one cluster, say Γ_0. By DM-compactness there exists $x \in T_S$ such that $\rho_S(x) = \Gamma_0$. We will show that T_S is a singleton. Suppose that $y \in T_S$. By axiom S8 of DMS $\rho_S(y)$ is a cluster an because we have only one cluster Γ_0 we have $\rho_S(y) = \Gamma_0$. So $\rho_S(x) = \rho_S(y)$. Because S is a $T0$ space this equality implies $x = y$. □

Theorem 5.30. *New topological representation theorem for contact algebras.* *Let $A = (B_A, C_A)$ be a contact algebra. Consider it as a trivial DCA. Then the following conditions are true.*

(i) The regular set-algebra $RC(A_+)$ is a trivial DCA.

(ii) The function g_A is a stable isomorphic embedding of A into $RC(A_+)$.

Proof. The Theorem is a consequence of of Theorem 5.26 - Topological representation theorem for DCA. Condition (iii) of the theorem says that the function g_A is a stable isomorphic embedding of A into $RC(A_+)$. This proves our condition (ii). Let us note that it is easy to see that the lifting Lemma 5.2 is true for the formulas $(1')$ and $(2')$. This implies that the conditions $(1')$ and $(2')$ are true in $RC(A_+)$, so $RC(A_+)$ is a trivial DCA and this proves our condition (i). □

6 Topological duality theory for DCA

In this section we extend the topological representation of DCAs to a topological duality theory of DCAs in terms of DMSes. We assume basic knowledge of category theory: categories, morphisms, functors and natural isomorphisms (see, for instance, Chapter I from [58]). Since DCA is a generalization of contact algebra, and DMS is a generalization of mereotopological space, the developed duality theory in this section will generalize the duality theory for contact algebras and mereotopological spaces presented by Goldblatt and Griece in [38] and some proofs below will be the same as in [38]. Other topological dualities for contact and precontact algebras are presented in [24] and it is possible to generalize them for DCAs, but in this paper we follow the scheme of [38] for two purposes: first, because the corresponding notion of DMS fits quite well to the topological representation theory for DCS-s, and second, because the proofs in this case are more short.

6.1 The categories DCA and DMS

Definition 6.1. *The category* ***DCA*** *consists of the class of all DCAs supplied with the following morphisms, called DCA-morphisms.*

Let $A_i = (B_{A_i}, C^s_{A_i}, C^t_{A_i}, \mathcal{B}_{A_i})$, $i = 1, 2$ *be two DCAs. Then* $f : A_1 \longrightarrow A_2$ *is a DCA-morphism if it is a mapping* $f : B_{A_1} \longrightarrow B_{A_2}$ *which satisfies the following conditions:*

(f 1) f *is a Boolean homomorphism from* B_{A_1} *into* B_{A_2}.

For all $a, b \in B_{A_1}$:

(f 2) if $f(a)C^s_{A_2}f(b)$, *then* $aC^s_{A_1}b$,

(f 3) if $f(a)C^t_{A_2}f(b)$, *then* $aC^t_{A_1}b$,

(f 4) if $f(a)\mathcal{B}^s_{A_2}f(b)$, *then* $a\mathcal{B}^s_{A_1}b$.

A_1 *is the domain of* f *and* A_2 *the codomain of* f.

We define $f_+ =_{def} f^{-1}$ *acting on t-clans of* A_2 *as follows: for* $\Gamma \in$ *t-Clans*(A_2), $f^{-1}(\Gamma) =_{def} \{a \in B_{A_1} : f(a) \in \Gamma\}$.

A DCA-morphism $f : A_1 \longrightarrow A_2$ *is a DCA-isomorphism (in the sense of category theory) if there is a DCA-morphism* $g : A_2 \longrightarrow A_1$ *such that the compositions* $f \circ g$

and $g \circ f$ are the identity morphism of their domains. It is a well known fact that this definition is equivalent to the standard algebraic definition of isomorphism in universal algebra.

Definition 6.2. *The category **DMS** consists of the class of all DMSes equipped with suitable morphisms called DMS morphism. The definition is as follows. Let $S_i = (X^t_{S_i}, X^s_{S_i}, T_{S_i}, \prec_{S_i}, M_{S_i})$, $i = 1, 2$ be two DMSes. A DMS-morphism is a mapping*

θ: $X^t_{S_1} \longrightarrow X^t_{S_2}$ such that:

(θ 1) if $x \in X^s_{S_1}$, then $\theta(x) \in X^s_{S_2}$,

(θ 2) If $x \prec_{S_1} y$, then $\theta(x) \prec_{S_2} \theta(y)$.

Let $a \subseteq X^t_{S_2}$ and $\theta^{-1}(a) =_{def} \{x \in X^t_{S_1} : \theta(x) \in a\}$. We define $\theta^+ =_{def} \theta^{-1}$.

The next two requirements for θ are the following:

(θ 3) If $a \in M_{S_2}$ then $\theta^{-1}(a) \in M_{S_1}$ and

(θ 4) the map $\theta^{-1} : M_{S_2} \longrightarrow M_{S_1}$ is a Boolean algebra homomorphism from (M_2) into (M_1).

*Note that in M_S the join operation is a set theoretical union of regular closed sets. Since meets in Boolean algebra is definable by the join and the complement *, for the condition (θ 4) it is sufficient to assume that θ^{-1} preserves complement.*

A DMS-morphism $\theta : S_1 \longrightarrow S_2$ is a DMS-isomorphism if there exists a converse DMS-morphism $\eta : S_2 \longrightarrow S_1$ such that the compositions $\theta \circ \eta$ and $\eta \circ \theta$ are identity morphisms in the corresponding domains.

The following lemma states an equivalent definition of DMS-isomorphism. Similar statement for mereotopological isomorphism is Theorem 2.2 from [38].

Lemma 6.3. *Let S, S' be DM-spaces and $\theta : S \mapsto S'$ be a DMS-morphism from S into S'. Let $a \subseteq X^t_S$ and define $\theta[a] = \{\theta(x) : x \in a\}$. Then the following two conditions are equivalent:*

(i) θ is a DMS-isomorphism from S onto S'.

(ii) θ is a DMS-morphism which is is a bijection from X^t_S onto $X^t_{S'}$ satisfying the following conditions:

(1) If $\theta(x) \in X^s_{S'}$, then $x \in X^s_S$.

(2) If $\theta(x) \prec_{S'} \theta(y)$, then $x \prec_S y$.

(3) If $a \in M_S$, then $\theta[a] \in M_{S'}$.

Proof. (i)⇒(ii) Suppose that θ is a DMS isomorphism from S onto S'. Then obviously θ is a bijection with converse η such that θ is a DMS-morphisms from S onto S' and η is a DMS-morphism from S' onto S such that the composition $\theta \circ \eta$ is the identity in S' and $\eta \circ \theta$ is the identity in S. To show (1) let $\theta(x) \in X^s_{S'}$. Then

$x = \eta(\theta(x)) \in X^s_S$, because η is a DMS-morphism from S' onto S. In a similar way we show (2). To show (3) let $a \in M_S$. Then $\eta^{-1}(a) \in M_{S'}$, because η is a DMS-morphism from S' onto S. This means that for any $x \in X^t_{S'}$ and $a \in M_S$ the following holds: $x \in \eta^{-1}(a)$ iff $\eta(x) \in a$ iff (by the definition of $\theta[a]$) $\theta(\eta(x)) \in \theta[a]$ iff (because $\theta(\eta(x)) = x$) $x \in \theta[a]$. This shows that $\theta[a] = \eta^{-1}(a)$, which shows that $\theta[a] \in M_{S'}$.

(i)$\Leftarrow$(ii) Suppose that θ is a DMS-morphism from S into S' and that (ii) is true. Conditions (1), (2) and (3) imply that η satisfy conditions (θ1), (θ2) and (θ3) for DMS-morphism. Since θ is a DMS morphism, it follows that the map $a \mapsto \theta^{-1}(a)$ is a Boolean homomorphism from $M_{S'}$ to M_S. Because θ is a bijection, it follows that for its converse η, the map $a \mapsto \eta^{-1}(a)$ is a Boolean homomorphism from M_S to $M_{S'}$, which shows that the condition (θ4) is also fulfilled. So η is a DMS morphism from S' to S. Because θ and η are converses to each other, their compositions are the identity mappings in the corresponding domains. So, θ is a DMS-isomorphism from S onto S'. □

Let $f : A_1 \longrightarrow A_2$ and $g : A_2 \longrightarrow A_3$ be two DCA-morphisms. The composition $h = f \circ g$ is a mapping $h : B_{A_1} \longrightarrow B_{A_3}$ acting as follows; for $a \in B_{A_1}$: $h(a) = g(f(a))$. In a similar way we define composition for DMS morphisms.

The following lemma has an easy proof.

Lemma 6.4. *(i) The composition of two DCA-morphisms is a DCA-mor phism. The identity mapping* 1_A *on each DCA A is a DCA-morphism. Hence* ***DCA*** *is indeed a category.*

(ii) The composition of two DMS-morphisms is a DMS-morphism. The identity mapping 1_S *on each DMS S is a DMS-morphism. Hence* ***DMS*** *is indeed a category.*

It follows from Lemma 6.4 that **DCA** and **DMS** are indeed categories.

We denote by **DMS*** the full subcategory of **DMS** of all T0 and DM-compact DMSes.

We introduce two contravariant functors

Φ: **DCA**→**DMS**, and Ψ: **DMS**→**DCA** as follows:

(I) For a given DCA A we put $\Phi(A) = A_+$ and for a DCA-morphism $f : A \longrightarrow A'$ we put $\Phi(f) = f_+$ and prove that f_+ is a DMS-morphism from $(A')_+$ into A.

(II) For a given DMS S we put $\Psi(S) = S^+$ and for a DMS-morphism $\theta : S \longrightarrow S'$ we put $\Psi(\theta) = \theta^+$ and prove that θ^+ : is a DMS morphism from $(S')^+$ into S.

(III) We show that for each DCA A the mapping $g_A(a) = \{\Gamma \in t\text{-}Clans(A) : a \in \Gamma\}$, $a \in B_A$ is a natural isomorphism (in the sense of category theory (see [58] Chapter I, 4.)) from A to $\Psi(\Phi(A)) = (A_+)^+$.

(IV) We show that for each T0 and DM-compact DMS S the mapping $\rho_S(x) = \{a \in M_S : x \in a\}$, $x \in X_S^t$, is a natural isomorphism from S to $\Phi(\Psi(S) = (S^+)_+$.

All this shows that the category **DCA** is dually equivalent to the category **DMS*** of T0 an DM-compact DMS. The realization of (I)-(IV) is given in the next subsection.

6.2 Facts for DCA-morphisms and DMS-morphisms

Lemma 6.5. *Every DMS-morphism is a continuous mapping.*

Proof. Let $\theta : S \longrightarrow S'$ be a DMS-morpism. Since θ^{-1} maps $M_{S'}$ (which is the closed basis of the topology of S') into M_S, then θ is continuous. □

Lemma 6.6. *Let $f : A \longrightarrow A'$ be a DCA-morphism. Then:*

(i) If Γ is a t-clan in A' then $f^{-1}(\Gamma) =_{def} \{a \in B_A : f(a) \in \Gamma\}$ is a t-clan in A.

(ii) If Γ is an s-clan in A' then $f^{-1}(\Gamma) =_{def} \{a \in B_A : f(a) \in \Gamma\}$ is an s-clan in A.

Proof. The proof consists of a routine check of the corresponding definitions of t-clan and s-clan. □

Lemma 6.7. *(i) Let A, A' be two DCAs and $f : A \longrightarrow A'$ be a DCA-morphism. Then f_+ is a DMS-morphism from $(A')_+$ to A_+.*

(ii)The mapping $g_A(a) = \{\Gamma \in t\text{-}Clans(A) : a \in \Gamma\}$, $a \in B_A$ is a natural DCA-isomorphism of A onto $\Psi(\Phi(A)) = (A_+)^+$.

Proof. (i) Remind that $(A')_+ = (t\text{-}Clans(A'), s\text{-}Clans(A'), Clusters(A'), \prec_{A'}, M_{A'})$. If $\Gamma \in t\text{-}Clans(A')$, then by Lemma 6.6 $f^{-1}(\Gamma)$ is a t-clan of A and similarly for the case when Γ is an s-clan. This shows that the condition ($\theta 1$) for DMS-morphisms is fulfilled. For the condition ($\theta 2$) let $\Gamma \prec_{A'} \Delta$, $\Gamma, \Delta \in t\text{-}Clans(A')$. We have to show that $f^{-1}(\Gamma) \prec_A f^{-1}(\Delta)$. By the definition of $\prec_A$ for clans (see (9)) this means the following. Let $a \in f^{-1}(\Gamma)$, $b \in f^{-1}(\Delta)$. Then $f(a) \in \Gamma$ and $f(b) \in \Delta$. But $\Gamma \prec_{A'} \Delta$, so $f(a)\mathcal{B}_{A'}f(b)$, which by (f 4) implies $a\mathcal{B}_A b$. This shows that $\Gamma \prec_A \Delta$.

The next step is to verify the condition ($\theta 3$) of DMS-morphisms, namely that $(f_+)^+$ maps the members of $M_{A'}$ into the members of M_A. Note that the members of M_A are of the form $g_A(a)$ for $a \in B_A$ and that $g_A(a) = \{\Gamma \in t{-}Clans(A) : a \in \Gamma\}$ and similarly for the members of $M_{A'}$. In order to verify ($\theta 3$) we will show that for any $a \in B_A$ the following equality holds which indeed shows that $(f_+)^+$ maps M_A into $M_{A'}$:

$$(f_+)^+(g_A(a)) = g_{A'}(f(a)) \tag{13}$$

To show (13) note that $(f_+)^+(g_A(a))$ is a subset of t-Clans(A'). So let $\Gamma \in$ *t-Clans*(A'). Then the following sequence of equivalences proves (13):

$\Gamma \in (f_+)^+(g_A(a))$ iff $\Gamma \in (f^{-1})^{-1}(g_A(a))$ iff $f^{-1}(\Gamma) \in g_A(a)$ iff $a \in f^{-1}(\Gamma)$ iff $f(a) \in \Gamma$ iff $\Gamma \in g_{A'}(f(a))$.

Now we verify the condition ($\theta 4$) of DMS-morphisms: $(f_+)^+$ preserves the Boolean complement. We show this by applying (13) and the facts that f and $g_{A'}$ acts as Boolean homomorphisms:

$(f^+)_+((g_A(a))^*)$=$(f^+)_+(g_A(a^*))$=$g_{A'}f(a^*)$=$(f^+)_+(g_A(a^*))$=
$((f^+)_+(g_A(a)))^*$.

(ii) The statement that g_A is a natural isomorphism in the sense of category theory means the following: first, that g_A is indeed an isomorphism from A onto A_+ (this is the Theorem 5.19) and second, that for any DCA-morphism $f : A \longrightarrow A'$, the following equality should be true: $g_{A'} \circ f = (f_+)^+ \circ g_A$. By the definition of the composition $\circ$ for DCA-morphisms this equality is equivalent to the following: for any $a \in B_A$ the following holds:

$g_{A'}(f(a)) = (f_+)^+(g_A(a))$, which is just (13).

In the language of category theory this means that the following diagram commutes for every DCA morphism f:

$$\begin{tikzcd} A \arrow[r, "g_A"] \arrow[d, "f"'] & (A_+)^+ \arrow[d, "(f_+)^+"] \\ A' \arrow[r, "g_{A'}"] & (A'_+)^+ \end{tikzcd}$$

□

Lemma 6.8. *Let S, S' be two DMS-s and $\theta : S \longrightarrow S'$ be a DMS-morphism from S to S'. Then θ^+ is a DCA-morphism from $(S')^+$ to S^+.*

Proof. We have to verify that $\theta^+ = \theta^{-1}$ satisfies the conditions (f1)-(f4) for DCA-morphism. Condition (f1) is fulfilled by the condition ($\theta 4$) for DMS-morphisms. For condition (f2) suppose that for some $a, b \in M_{S'}$, $\theta^{-1}(a)C^t_S\theta^{-1}(b)$ and proceed to show $aC^t_{S'}b$. This implies that there exists $x \in X^t_S$ such that $x \in \theta^{-1}(a)$ and $x \in \theta^{-1}(b)$. From here we obtain $\theta(x) \in a$, $\theta(x) \in b$ and $\theta(x) \in X^t_{S'}$ (by condition ($\theta 1$) for DMS morphism) which yields $aC^t_{S'}b$. In a similar way one can verify condition (f3).

For (f4) suppose $\theta^{-1}(a)\mathcal{B}_S\theta^{-1}(b)$ and proceed to show that $a\mathcal{B}_{S'}b$. Then there exist $x, y \in X^t_S$ such that $x \prec_S y$, $x \in \theta^{-1}(a)$, $y \in \theta^{-1}(b)$. This implies $\theta(x) \in a$, $\theta(y) \in b$, and by ($\theta 1$) and ($\theta 2$) that $\theta(x), \theta(y) \in X^t_{S'}$ and $\theta(x) \prec_{S'} \theta(y)$. This implies $a\mathcal{B}_{S'}b$. □

Before the formulation of the next statement let us see what is $(S^+)_+$ for a DMS S. S^+ is the dual of S which is the DCA algebra $(M_S, C^t_S, C^s_S, \mathcal{B}_S)$ (see Definition 5.3). Then $(S^+)_+$ is the dual space of the algebra S^+ which is $(S^+)_+ = (X^t_{S^+}, X^s_{S^+}, T_{S^+}, \prec_{S^+}, M_{S^+})$, where $X^t_{S^+}$ is the set of t-clans of S^+, $X^s_{S^+}$ is the set of s-clans of S^+, T_{S^+} is the set of clusters of S^+, $\prec_{S^+}$ is the relation defined by (9) between t-clans, and M_{S^+} is the set $\{g_{S^+}(a) : a \in M_S\}$, where $g_{S^+}(a) =_{def} \{\Gamma \in t-clans(S^+) : a \in \Gamma\}$ (see Section 5.5).

Lemma 6.9. *(i) Let S be a DMS. Then ρ_S is a DMS-morphism from S to $(S^+)_+$.*

(ii) Let S be a DM-compact DMS and let for $a \subseteq X^t_S$, $\rho_S[a] =_{def} \{\rho_S(x) : x \in a\}$. Then for $a \in M_S$: $\rho_S[a] = g_{S^+}(a)$ (for the function g_A for a DCA A see Section 5.5).

(iii) If S is $T0$ and DM-compact, then ρ_S is a DMS-isomorphism from S onto $(S^+)_+$.

(iv) If S is a $T0$ and DM-compact DMS, then ρ_S is a natural isomorphism from S to $\Phi(\Psi(S)) = (S^+)_+$.

Proof. (i) We have to verify whether ρ_S satisfies the conditions (θ1)-(θ4) for DMS-morphisms. By Lemma 5.4 $\rho_S(x)$ is a t-clan in S^+ for $x \in X^t_S$ and an s-clan in S^+ for $x \in X^s_S$. This verifies the conditions (θ1) and (θ2) for DMS-morphisms. Condition (θ2) is guaranteed by axiom (7) for DMS and Lemma 5.4 (iv). For condition (θ3) we have to show that $(\rho_S)^{-1}$ transforms the members from M_{S^+} into the members from M_S (recall that the members of M_{S^+} are of the form $g_{S^+}(a)$, $a \in M_S$, see the text before the lemma). This can be seen from the following equality

$$(\rho_S)^{-1}(g_{S^+}(a)) = a \tag{14}$$

Indeed, for $x \in X^t_S$ we have:

$x \in (\rho_S)^{-1}(g_{S^+}(a))$ iff $\rho_S(x) \in g_{S^+}(a)$ iff $a \in \rho_S(x)$ iff $x \in a$.

For condition (θ4) we have to show that $(\rho_S)^{-1}$ preserves Boolean complement. The following sequence of equalities proves this: $(\rho_S)^{-1}(g_{S^+}(a^*)) = a^* = ((\rho_S)^{-1}(g_{S^+}(a)))^*$, which is true on the base of (14).

(ii) Suppose $a \in M_S$ and let us show first $\rho_S[a] \subseteq g_{S^+}(a)$:

$\rho_S(x) \in \rho_S[a] \Rightarrow x \in a \Rightarrow a \in \rho_S(x) \Rightarrow \rho_S(x) \in g_{S^+}(a)$ (because $\rho_S(x)$ is a t-clan in the DCA algebra S^+). For the converse inclusion, let Γ be a t-clan in S^+. The by DM-compactness there exists $x \in X^t_S$ such that $\Gamma = \rho_S(x)$. Then for $a \in M_S$:

$\Gamma \in g_{S^+}(a) \Rightarrow a \in \Gamma \Rightarrow a \in \rho_S(x)$ and $x \in a \Rightarrow \rho_S(x) \in \rho_S[a] \Rightarrow \Gamma \in \rho_S[a]$.

(iii) Let S be $T0$ and DM-compact. Then by Lemma 5.7Then ρ_S is a one-one mapping from X^t_S onto the set of all t-clans of S^+, which are the points of $(S^+)_+$. By (i) ρ_S is a DMS-morphism from S to $(S^+)_+$. So in order to show that ρ_S is a DMS-isomorphism from S onto $(S^+)_+$ we have to see if ρ_S satisfies the conditions (1), (2) and (3) of Lemma 6.3 (ii).

For condition (1) suppose $\rho_S(x) \in X^s_{S^+}$. Then $\rho_S(x)$ is a t-clan in M_S. By DM-compactness there exists $y \in X^s_S$ such that $\rho_S(x) = \rho_S(y)$. By $T0$ condition this implies $x = y$, so $x \in X^s_S$.

For condition (2) suppose $\rho_S(x) \prec_{S^+} \rho_S(y)$. Then by Lemma 5.4 and axiom (S7) for DMS we obtain $x \prec_S y$.

For condition (3) suppose $a \in M_S$ and proceed to show that $\theta[a] \in M_{(S^+)_+}$. By (ii) $\theta[a] = g_{S^+}(a)$ and since $g_{S^+}(a) \in M_{(S^+)_+}$ we get $\theta[a] \in M_{(S^+)_+}$.

Thus the conditions (1), (2) and (3) are fulfilled which proves that ρ_S is a DMS-isomorphism from S onto $(S^+)_+$.

(iv) Let S be a T0 and DM-compact DMS. In order ρ_S to be a natural isomorphism from S to $(S^+)_+$ it has to satisfy the following two conditions: first, ρ_S have to be a DMS-isomorphism - this is guaranteed by (iii), and second, for every DMS morphism $\theta : S \Rightarrow S'$: the following equality should be true: $\theta \circ \rho_{S'} = \rho_S \circ (\theta^+)_+$. This equality is equivalent to the following condition: for $x \in X^t_S$

$$(\theta^+)_+(\rho_S(x) = \rho_{S'}(\theta(x)) \tag{15}$$

In the language of category theory this means that the following diagram commutes for every DMS morphism θ:

$$\begin{tikzcd}
S \arrow[r, "\rho_S"] \arrow[d, "\theta"'] & (S^+)_+ \arrow[d, "(\theta^+)_+"] \\
S' \arrow[r, "\rho_{S'}"] & (S'^+)_+
\end{tikzcd}$$

The following sequence of equivalencies proves (15). For $a \in M_{S'}$:

$a \in (\theta^+)_+(\rho_S(x))$ iff $a \in (\theta^+)^{-1}(\rho_S(x))$ iff $\Theta^+(a) \in \rho_S(x)$ iff $x \in \theta^+(a)$ iff $x \in \theta^{-1}(a)$ $\theta(x) \in a$ iff $a \in \rho_{S'}(\theta(x))$. $\square$

As applications of the developed theory we establish some isomorphism correspondences between the objects of the two categories. The isomorphism between two objects will be denoted by the symbol $\cong$.

Lemma 6.10. *Let A, A' be two DCAs. Then the following conditions are equivalent:*

(i) $A \cong A'$,

(ii) $A_+ \cong (A')_+$,

(iii) $(A_+)^+ \cong ((A')_+)^+$

Proof. **(i)⇔(iii)**.By Lemma 5.19 we have $A \cong (A_+)^+$ and $A' \cong (A'_+)^+$. This makes obvious the equivalence (i)⇔(iii).

(i)⇒(ii) Suppose $A \cong A'$, then there exists a on-one mapping f from A onto A' with a converse mapping h such that $f : A \mapsto A'$ is a DCA morphism from A onto

A' and $h : A' \mapsto A$ is a DCA- morphism from A' onto A such that the composition $f \circ h$ is the identity mapping in A' and the composition $h \circ f$ is the identity mapping in A. Then by Lemma 6.7 f_+ is a DMS-morphism from A'_+ onto A_+ and h_+ is a DMS-morphism from A_+ onto A'_+.

We shall show the following:

(1) The composition $f_+ \circ h_+$ is the identity in A'_+, and

(2) The composition $h_+ \circ f_+$ is the identity in A_+.

Then, by the definition of DMS- isomorphism this will imply that both f_+ and h_+ are DMS-isomorphisms in the corresponding directions.

Note that the members of A_+ are the t-clans of A and similarly for A'_+.

To show (1) let Γ be a point of the space A'_+, i.e. Γ is a t-clan in A'. We shall show that $(f_+ \circ h_+)(\Gamma) = \Gamma$ which will prove (1). This is seen from the following sequence of equivalencies where a is an arbitrary element of $B_{A'}$:

$a \in (f_+ \circ h_+)(\Gamma\ )$ iff $a \in (f_+(h_+(\Gamma))$ iff $a \in f^{-1}(h_+(\Gamma))$ iff $f(a) \in h_+(\Gamma)$ iff $f(a) \in h^{-1}(\Gamma)$ iff $h(f(a)) \in \Gamma$ iff $a \in \Gamma$.

Here we use that $h(f(a)) = a$ for $a \in B_{A'}$ because h is the converse of the one-one mapping f from B_A onto $B_{A'}$.

In a similar way we show (2).

(ii)⇒(iii) The proof is similar to the above one. Suppose $A_+ \cong (A')_+$, then there exists a one-one mapping θ and its converse η such that θ is a DMS-morphism from A_+ onto $(A')_+$ and η is a DMS-morphism from $(A')_+$ onto A_+. Then by Lemma 6.8 θ^+ is a DCA-morphism from $(A'_+)^+$ into $(A_+)^+$ and η^+ is a DCA-morphism from $(A'_+)^+$ into $(A_+)^+$. We shall show that both θ^+ and η^+ are DCA-isomorphisms in the corresponding directions by showing that their compositions are identities in the corresponding domains. Let us note that the domain of θ^+ is the members of the algebra $(A'_+)^+$ which are of the form $g_{A'}(a)$, $a \in_{B_{A'}}$, and similarly for the members of $(A_+)^+$. Namely we will show the following two things:

(3) $(\theta^+ \circ \eta^+)(g_{A'}(a)) = g_{A'}(a)$ for any $a \in B_{A'}$,

(4) $(\eta^+ \circ \theta^+)(g_{A'}(a)) = g_{A'}(a)$ for any $a \in B_A$,

To show (3) note that $g_{A'}(a) = \{\Gamma \in t - clans(A') : a \in \Gamma$. So let $\Gamma \in t - clans(A')$. Then the following sequence of equivalents proves (3):

$\Gamma \in (\theta^+ \circ \eta^+)(g_{A'}(a))$ iff $\Gamma \in (\theta^+(eta^+(g_{A'}(a))))$ iff $\Gamma \in (\theta^{-1}(eta^+(g_{A'}(a))))$ iff $\theta(\Gamma) \in (eta^+(g_{A'}(a)))$ iff $\theta(\Gamma) \in (eta^{-1}(g_{A'}(a)))$ iff $\eta(\theta(\Gamma)) \in g_{A'}(a)$ iff $\Gamma \in g_{A'}(a)$.

We have just used that $\eta(\theta(\Gamma)) = \Gamma$, because η is the converse of the one-one mapping θ from $X^t_{A_+} = t - Calans(A)$ onto $X^t_{(A')_+} = t - clans(A')$. The proof of (4) is similar. □

Lemma 6.11. *Let S, S' be two DMSes. Then the following conditions are equivalent:*

(i) $S \cong S'$,

(ii) $S^+ \cong (S')^+$,
(iii) $(S^+)_+ \cong ((S')^+)_+$.

Proof. The proof is analogous to the proof of Lemma 6.10 □

As a corollary from Lemma 6.10 and Lemma 6.11 we obtain the following addition to the topological representation theorem for DCAs.

Corollary 6.12. *There exists a bijective correspondence between the class of all, up to DCA-isomorphism DCAs, and the class of all, up to DMS-isomorphism DMSes; namely, for every DCA-algebra A the corrseponding DMS of A is A_+ - the canonical DM-space of A; and for every DMS S the corresponding DCA of S is S^+ – The canonical DC-algebra of S.*

6.3 Topological duality theorem for DCAs

In this section we prove the third important theorem of this paper.

Theorem 6.13. Topological duality theorem for DCAs. *The category **DCA** of all dynamic contact algebras is dually equivalent to the category **DMS*** of all $T0$ and DM-compact DMSes.*

Proof. The proof follows from Lemma 6.7, Lemma 6.8 and Lemma 6.9. □

The above theorem has several consequences to some important subcategories of **DCA** and **DMS**. The first example is the following. Let Ax be a subset of the set of temporal axioms (rs), (ls), (up dir), (down dir), (circ), (dens), (ref), (lin), (tri), (tr). Consider the class of all DCAs satisfying the axioms from Ax. It is easy to see that this class forms a full subcategory of the category of all DCAs under the DCA-morphism. Denote this subcategory by **DCA(Ax)**. Let $\widehat{Ax}$ be the subset of the corresponding to Ax time condition from the list (RS), (LS), (Up Dir), (Down Dir), (Circ), (Dens), (Ref), (Lin), (Tri), (Tr). Consider the class of all $T0$ and DM-compact DMSes which satisfy the axioms $\widehat{Ax}$. It is easy to see that this class is a full subcategory of the category **DMS*** of all T0 and DM-compact dynamic mereotopological spaces. Denote this subcategory by $\mathbf{DMS}(\widehat{Ax})^*$

Theorem 6.14. *The category **DCA(Ax)** of all dynamic contact algebras satisfying Ax is dually equivalent to the category* $\boldsymbol{DMS}(\widehat{Ax})^*$ *of all $T0$ and DM-compact DMSes satisfying $\widehat{Ax}$.*

Proof. Let S be a $T0$ and DM-compact DMS. It follows by Lemma 5.18 that S satisfies $\widehat{Ax}$ iff S^+ satisfies Ax. Now the theorem is a corollary of Theorem 6.13. □

Another subcategory of **DCA** is the class of all trivial DCAs with the same morphisms. Denote it by $\mathbf{DCA}_{trivial}$. The corresponding subcategory of $\mathbf{DMS}^*$ with the same morphisms is the class of all trivial $T0$ and DM-compact DMSes. Denote it by $\mathbf{DMS^*}_{trivial}$. The following theorem is also an obvious consequence of Theorem 6.13

Theorem 6.15. *The category $\boldsymbol{DCA}_{trivial}$ is dually isomorphic to the category $\boldsymbol{DMS^*}_{trivial}$.*

Remark 6.16. *Note that the category of contact algebras can be identified in an obvious way with the category of trivial DCAs by enriching contact algebras with some definable relations. Having in mind Lemma 5.10, Lemma 5.11 and Corollary 5.12 it can be shown that the category of mereocompact and $T0$ mereotopological spaces from [38] can also be identified with the category of $T0$ and DM-compact trivial DMS. This implies that the duality theorem for contact algebras from [38] can be derived from Theorem 6.15.*

7 Concluding remarks

Overview. The aim of this paper is to present with some details a version of point-free theory of space and time based on a special representative example of a dynamic contact algebra (DCA). The axioms of the algebra are true sentences from a concrete point-based model, the snapshot model, developed in Section 3. Theorem 4.15 - the Representation theorem for DCA by snapshot models snows that the chosen axioms are enough to code the intuition based on snapshot construction which can be considered as the cinematographic model of spacetime. In Section 4 we introduced topological models of DCAs giving them another intuition based on topology. These models are based on the notion of Dynamic mereotopological space (DMS). Let us note that the abstract definition of DCA can be considered as a 'dynamic generalization' of contact algebra, which in a sense is a certain point-free theory of space called also a mereotopology. In this relation contact algebras can be considered as a 'static mereotopology' while dynamic contact algebras can be considered as a 'dynamic mereotopology'. Let us note that topological models of contact algebras, which are considered as the standard models of this notion, contain one type of points, which are just the 'space points' while dynamic mereotopological spaces contain several kinds of points: partial time points, time points and space points, which in turn are also partial time points. Time points realize the time contact, while space points realize the space contact. The fact that each space point is a partial time point says that *space* in this model is reduced to *time*, a feature quite similar to the Robb's axiomatic treating of Minkowskian spacetime geometry in which space is reduced to time (see [67] and the discussion in Section 1.1). Another common feature of both snapshot and topological models is that the properties of the underline time structure corresponds to the validity of time axioms which are point-free conditions for dynamic regions formulated by the relations of time contact C^t and precedence relation $\mathcal{B}$. Because regions are observable things, then recognizing which time axioms they satisfy we may conclude which abstract properties satisfies the corresponding time structure.

Discussions and some open problems. Time contact relation aC^tb, and precedence relation $a\mathcal{B}b$ between two dynamic regions a and b in snapshot models are defined by the predicate 'existence' defined in Boolean algebras as follows: $E(a)$ iff $a \neq 0$. One may ask if this predicate is a good one. It has the following disadvantage - there are too many existing regions and only one non-existing - the zero region. For instance, we can not see the zero region, but we can see on the sky a non-existing star - see Remark 3.2. What we see is different from 0 but this does not mean that it is existing at the moment of observation. So, the adopted in this paper definition for 'existence' is approximate one and we need a more exact definition corresponding to what we intuitively mean by 'actual existence'. This is a problem discussed in our papers [81, 82] in which we introduce an axiomatic definition and corresponding models of predicate 'actual existence' (denoted by $AE(a)$) and a corresponding relation between regions called 'actual contact'. The predicate $E(a)$ satisfies the axioms of $AE(a)$ and is the simplest one, but AE is more general - it is possible for some region a to have $a \neq 0$ but not $AE(a)$ like 'non-existing stars' discussed in Remark 3.2. One of our future plans is to reconstruct the theory of the present paper on the base of the more realistic predicates of actual existence and actual contact.

Another subject of discussion is the relation aC^tb called 'time contact' which is a kind of simultaneity relation. Special relativity theory (SR) also studies a kind of simultaneity relation and states that it is not absolute and is relative to the observer. Is it possible to relate these two notions? In general these two relations are different because in our system this is a relation between regions and in SR it is between events, which are not regions but space-time points. Nevertheless we will try to find some correspondence. By event in SR one normally assume a space point, taken from our ordinary space, with attached time-point (a date), taken from a clock attached to the space point with the assumption that all attached clocks work synchronously (the possibility to have synchronized clocks in all points of our space is explained by Einstein in [33] by a special synchronization procedure). So, events are pairs (A, t), where A is a space point and t is a real number interpreted as a date. According to Einstein's natural definition, two events (A_1, t_1), (A_2, t_2)

are simultaneous if $t_1 = t_2$ which shows that simultaneity is an equivalence relation. Note that Einstein did not give formal definition of 'event', but in the terminology of Minkowski spacetime, which is a formal explication of SR spacetime, events are just spacetime points and two spacetime points are simultaneous if they have equal time coordinates. In our system we do not introduce the notion of event but in the abstract DCA an (approximate) analog of event can be identified with a pair (U, Γ) where U is an ultrafilter and Γ is a cluster containing U - U is a space point and Γ is a time point (see Section 5.5). Let (U_i, Γ_i), i=1,2 be two events in DCA. Then, according to the simultaneity relation between events it can be easily seen that (U_1, Γ_1) is simultaneous with (U_2, Γ_2) iff $U_1 R^t U_2$ which is just the canonical relation between ultrafilters corresponding to the contact relation C^t. Note that R^t is also an equivalence relation as the simultaneity relation in SR is. So an analog of SR simultaneity relation in our theory is the relation R^t considered between 'events' in the sense of DCA.

An analog of our before-after relation $\prec$ between events in SR is $(A_1, t_1) \prec (A_2, t_2)$ iff $t_1 < t_2$. This relation, like simultaneity, is not absolute and is relative to the observer. Note also that it is different from the Robb's causal relation 'before' taken as the unique basic relation between events in the axiomatic presentation of Minkowski geometry [67]). The natural analog of the above relation between DCAs 'events' is $(U_1, \Gamma_1) \prec (U_2, \Gamma_2) \Leftrightarrow_{def} \Gamma_1 \prec \Gamma_2$. But we have $\Gamma_1 \prec \Gamma_2$ iff $U_1 \prec U_2$ which shows that the relation coincides with the canonical relation $\prec$ between ultrafilters corresponding to the precontact relation $\mathcal{B}$. This shows that the canonical relation $\prec$ between ultrafilters which is used to characterize $\mathcal{B}$ is not an analog of the Robb's causal relation (let us denote it by $\prec_{Robb}$) which has a special definition in Minkowski spacetime by means of its metric. An analog of this definition in Einstein's SR is the following: $(A_1, t_1) \prec_{Robb} (A_2, t_2)$ iff $|A_1 A_2| \leq |t_1 - t_2|$ and $t_1 < t_2$. This relation is stronger than the relation $\prec$. It will be nice to have an abstract version of DCA containing stronger than $\mathcal{B}$ precontact relation corresponding to causality. We put this problem to the list of our future plans.

Comparing the presented in this paper theory with SR we see that there is another feature which differs the corresponding theories: RS considers many observers and can prove that some relations between events like simultaneity are relative to corresponding observer, while a given DCA A is based on only one observer, denote it by $O(A)$ (this observer can be identified with an abstract person describing the standard dynamic model of space which is isomorphic to A). So, because we have only one observer in our formalism, we can not give formal proofs whether the basic relations between regions are relative or not to the observer. Hence, building a theory like DCA incorporating many observers is the next open problem.

One possibility for a theory with many observers describing one and the same reality is to consider a family of DCAs with some relations between them. Let A and A' be two DCAs from such a set. Examples of possible relations between them are, for instance, the following:

(1) The observers $O(A)$ and $O(A')$ are at rest to each other, they have synchronous clocks, and have some possibilities to communicate. For instance, if we have two observers with equal cameras who are at rest to each other and are filming the same reality with their cameras. The communication is when one of them can point out to the other some of the observed objects.

(2) The observers $O(A)$ and $O(A')$ are not at rest to each other but have synchronous clocks and some possibilities to communicate. A situation similar to the above but one of the observers is moving with respect to the other.

Is it possible to find a meaningful abstract characterizations of such relations by using some morphism like relations between the algebras A and A'? An example of a set of DCAs with some morphisms between them is the category **DCA** considered as a small category (the class od DCAs is a set). Then a natural question is what are saying the DCA-morphisms between the algebras considered as algebras produced by observers describing one and the same reality. For instance, what is the meaning of the condition on DCA-morphism $f : A \longrightarrow A'$:

($\sharp$) If $f(a)C^t_{A'}f(b)$, then $aC^t_A b$

If we interpret f as a way for the observer $O(A)$ to point out some regions to the observer $O(A')$, then ($\sharp$) says that if $O(A')$ sees that the pointed regions are in a time contact, then the same has been seen by A. Similar interpretation have the other conditions on DCA-morphisms concerned C^s and $\mathcal{B}$. This means that $O(A')$ is seen the reality in the same way as $O(A)$ from which we may conclude that observers are at rest to each other. So an open problem is to study small categories of DCAs with different kinds of meaningful morphisms between them.

Let us finish this section by formulating one more open problem. The axiomatization of Minkowski geometry presented by Robb [67] is point-based: the primitive concepts are points and the binary relation ‘before’ on points satisfying some axioms. The problem is to present a point-free characterization of Minkowskian geometry similar to DCA eventually with more spatio-temporal primitive relations between regions and probably by axiomatizing some special regions in this geometry, for instance, Minkowski’s light cones. An analogous result for Euclidean geometry is the Tarski result in [73], where he presented an abstract axiomatization of Euclidean balls. Euclidean balls are the regions in Euclidean geometry from which it is possible to extract the Euclidean metrics. In Minkowskian geometry light cones coded in some way Minkowskian metrics. Similar proposal for a point-free characterization of affine geometry was proposed by Whitehead in [91] by an abstract characterization of the set of convex regions (called by Whitehead 'ovals').

References

[1] Marco Aiello, Ian Pratt-Hartmann, and Johan van Benthem (eds.), *Handbook of Spatial Logics*, Springer, 2007.

[2] Hajnal Andreka, Judit Madarasz, and Istvan Nemeti Logic of spacetime and relativity theory, Chapter 11 In: *Handbook of Spatial Logics*, Marko Aiello, Ian Pratt, and Johan van Benthem (Eds.), Springer, 2007, pp. 607–712.

[3] Philipe Balbiani, and Tatyana Ivanova Representation theorems for extended contact algebras based on equivalence relations, 2019, submitted.

[4] Philippe Balbiani, Tinko Tinchev and Dimiter Vakarelov Modal Logics for Region-based Theory of Space, *Fundamenta Informaticae*, Special Issue: *Topics in Logic, Philosophy and Foundation of Mathematics and Computer Science in Recognition of Professor Andrzej Grzegorczyk 81*, (1-3), 2007, pp. 29–82.

[5] Johan van Benthem *The Logic of Time:* A Model-Theoretic Investigation into the Varieties of Temporal Ontology and Temporal Discourse, Second Edition, Synthese Library : v. 156, Springer 1983.

[6] Guram Bezhanishvili Stone duality and Gleason covers through de Vries duality, *Topology and its Applications 157*, 2010, pp. 1064–1080.

[7] Guram Bezhanishvili, Nick Bezhanishvili, S. Sourabh, and Yde Venema Irreducible equivalence relations, Gleason spaces, and de Vries duality, *Applied Categorical Structures 25*, 2017, pp. 381–401.

[8] Brandon Bennett, and Ivo Düntsch Axioms, Algebras and Topology. In: *Handbook of Spatial Logics*, M. Aiello, I. Pratt, and J. van Benthem (Eds.), Springer, 2007, pp. 99–160.

[9] Sergio Celani Precontact relations and quasi-modal operators in Boolean algebras, *Actas del XIII Congreso Dr. Antonio A. R. Monteiro*, 2016, pp. 63–79.

[10] — Subordination Tarski algebras, *Journal of Applied Non-Classical Logics 29*, 2019, pp. 288–306.

[11] Sabine Copelberg, Iivo Düntsch, and Michael Winter Remarks on contact relations on Boolean algebras, *Algebra Universalis 68*, 2012, pp 353–366.

[12] W. Comfort, and S. Negrepoints *Chain Conditions in Topology,* Cambridge University Press, 1982.

[13] Plamen Dimitrov, and Dimiter Vakarelov Dynamic contact algebras and quantifier-free logics for space and time *Siberian Electronic Mathematical Reports 15*, 2018, pp. 1103–1144.

[14] Georgi Dimov A generalization of De Vries' Duality Theorem, *Applied Categorical Structures 17*, 2009, pp. 501–516.

[15] Georgi Dimov Some generalizations of the Fedorchuk Duality Theorem – I, *Topology and its Applications 156*, 2009, pp. 728–746.

[16] — A de Vries-type duality theorem for the category of locally compact spaces and continuous maps - I, *Acta Mathematica Hungarica 129(4)*, 2010, pp. 314-349.

[17] — A de Vries-type duality theorem for the category of locally compact spaces and continuous maps - II, *Acta Mathematica Hungarica 130(1)*, 2011, pp. 50-77.

[18] — A Whiteheadian-type description of Euclidean spaces, spheres, tori and Tychonoff cubes, *Rendiconti dell'Istituto di Matematica dell'Università di Trieste 44*, 2012, pp. 45–74.

[19] Georgi Dimov and Elza Ivanova Yet another duality theorem for locally compact spaces, *Houston Journal of Mathematics*, 42 (2), 2016, 675–700.

[20] — Two extensions of the Stone Duality to the category of zero-dimensional Hausdorf spaces, arXiv:1901.04537v2, pp. 1–33, May 2020.

[21] Georgi Dimov, Elza Ivanova-Dimova, and Ivo Duentsch On dimension and weight of a local contact algebra, *Filomat 32(15)*, 2018, pp. 5481–5500.

[22] Georgi Dimov, Elza Ivanova-Dimova and Walter Tholen Extensions of dualities and a new approach to the de Vries duality, arXiv:1906.06177, pp. 1–45, June 2019.

[23] — Extensions of dualities and a new approach to the Fedorchuk duality, *Topology and its Applications*, In Press, 2020, 26 pp., https://doi.org/10.1016/j.topol.2020.107207.

[24] Georgi Dimov, Elza Ivanova-Dimova, and Dimiter Vakarelov A generalization of the Stone Duality Theorem, *Topology and its Applications 221*, 2017, pp. 237–261.

[25] Georgi Dimov, and Dimiter Vakarelov Topological representation of precontact algebras, In: *ReLMiCS'2005, St Catharines, Canada, February 22-26, 2005, Proceedings*, Wendy MacCaul, Michael Winter, and Ivo Düntsch (Eds.), LNCS 3929, Springer, 2006, pp. 1–16.

[26] — Contact algebras and region-based theory of space: A proximity approach, I and II, *Fundamenta Informaticae 74, 2-3*, 2006 pp. 209–249, pp. 251-282,

[27] — Topological representation of precontact algebras and a connected version of the Stone Duality Theorem I, *Topology and its Applications 227*, 2017, pp. 64–101.

[28] — Topological representation of precontact algebras and a connected version of the Stone Duality Theorem II, *Serdica Mathematical Journal 44*, 2018, pp. 31–80.

[29] Ivo Duntsch, and Sanjian Li On the homogeneous countable Boolean contact algebra, *Logic and Logical Philosophy 22*, 2013, pp. 213–251

[30] Ivo Düntsch and Dimiter Vakarelov Region-based theory of discrete spaces: *A proximity approach.* In: Nadif, M., Napoli, A., SanJuan, E., and Sigayret, A. EDS, *Proceedings of Fourth International Conference Journées de l'informatique Messine*, 123-129, Metz, France, 2003. Journal version in: *Annals of Mathematics and Artificial Intelligence 49*, 2007, pp.: 5–14.

[31] Ivo Düntsch and Michael Winter Moving Spaces, In: Stephane Demri and Christian S. Jensen Eds. *Proceedings of the 15th International Symposium on Temporal Representation and Reasoning (TIME 2008)*, IEEE Computer Society, 2008, pp. 59–63.

[32] — Timed Contact Algebras, *Proceedings of the 16-th International Symposium on Temporal Representation and Reasoning (TIME 2009)*, IEEE Computer Society, 2009,

pp. 133–138.

[33] Albert Einstein On the electrodynamics of moving bodies, published as Zur Elektrodynamik bewegter Körper, *Annalen der Physik 17:891*, 1905, English translation in: http://www.fourmilab.ch/ 24.

[34] Basil Evangelidis Space and Time as Relations: The Theoretical Approach of Leibniz, *Philosophies 3* 2018, pp. 1–15.

[35] Ryszard Engelking *General Topology*, PWN, 1977.

[36] Robert Goldblatt Diodorean modality in Minkowski spacetime, *Studia Logica 39*, 198), pp. 219–236.

[37] — *Orthogonality and Spacetime Geometry* Springer-Verlag, 1987.

[38] Robert Goldblatt, and M. Grice Mereocompactness and duality for mereotopological spaces, in Katalin Bimbo (Ed), *J. Michael Dunn on Information Based Logics:* Outstanding Contributions to Logic, vol. 8. Springer International Publishing, 2016, 313–330.

[39] Andrzej Grzegorczyk Axiomatizability of geometry without points, *Synthese 12*, 1960, pp. 109–127.

[40] Rafal Gruszczyński, and Andrzej Pietruszczak A Study in Grzegorczyk Point-Free Topology Part I: Separation and Grzegorczyk Structures, *Studia Logica 106*, 2018, pp. 1197–1238, Part II: Spaces of Points, *Studia Logica 107*, 2019, pp. 809-843.

[41] — Full Development of Tarski's Geometry of Solids, *The Bulletin of Symbolic Logic 14(4)*, 2008, pp. 481–540.

[42] Torsten Hahmann, and Michael Gröuninger Region-based Theories of Space: Mereotopology and Beyond, in *Qualitative Spatio-Temporal Representation and Reasoning:* Trends and Future Directions, Shyamanta Hazarika ed, IGI Publishing, 2012, pp. 1-62.

[43] Torsten Hahman, Michael Winter, and Michael Gröuninger Stonian p-Ortholattices: A new approach to the mereotopology RT0 *Artificial Intelligence 173*, 2009, pp. 1424–1440.

[44] Geoffrey Hellman and Steward Shapiro *Varieties of Continua: From Regions to Points and Back*, Oxford University Press, 2018, ix + 208 pp.

[45] Tatyana Ivanova Logics for extended distributive contact lattices, *Journal of Applied Non-Classical Logics 28*, 2018, pp. 140–162.

[46] — Extended contact algebras and internal connectedness, *Studia Logica 108*, 2020, pp. 239-254.

[47] Tatyana Ivanova and Dimiter Vakarelov Distributive mereotopology: Extended distributive contact lattices, *Annals of Mathematics and Artificial Intelligence 77(1-2)*, 2016, pp. 3-41.

[48] Elza Ivanova-Dimova Some Isomorphism Theorems for MVD-algebras, *International Electronic Journal of Pure and Applied Mathematics*, 9 (2), 2015, 67–87.

[49] Max Jammer *Concepts of Simultaneity From Antiquity to Einstein and Beyond,* The Johns Hopkins University Press, Baltimore, 2006.

[50] Peter Jonstone The point of pointless topology, *Buletin of The American Mathematical Society (New Series) 8(1)*, 1983, pp. 41–53.

[51] Roman Konchakov, Agi Kurucz, Frank Wolter, and Michael Zakharyaschev Spatial logic + temporal logic=?? In Marco Aiello, Johan Van Benthem, and Ian Pratt-Hartmann (Eds), *Handbook of Spatial Logics*, Chapter 9, Springer, 2007, pp. 497–564.

[52] Roman Kontchakov, Ian Pratt-Hartmann, Frank Wolter, Michael Zakharyaschev Topology, connectedness, and modal logic, in R. Goldblatt and Carlos Areces (eds.), *Advances in Modal Logic,* Volume 7, College Publications, 2008, pp. 151–176.

[53] Teodore de Laguna The nature of space - I. *The Journal of Philosophy 19*, 1922, pp. 393–407.

[54] — The nature of space - II , *The Journal of Philosophy 19*, 1922, pp. 421–440.

[55] — Point, line and surface as sets of solids, *The Journal of Philosophy 19*, 1922, pp. 449–461.

[56] Tamar Lando Topology and measure in logics for region-based theories of space, *Annals of Pure and Applied Logic 169*, 2018, pp. 277–311.

[57] Tamar Lando, and Dana Scott A Calculus of regions respecting both measure and topology, *Journal of Philosophical Logic 48*, 2019, pp. 825–850.

[58] Saunders Mac Lane *Categories for the Working Mathematician*, 2nd edn, Springer-Verlag, 1998.

[59] Carsten Lutz and Frank Wolter Modal logics for topological relations, *Logical Methods in Computer Science 2(2-5)*, 2006, pp. 1-41.

[60] Somashekhar A. Naimpally, and Brian D. Warrack *Proximity Spaces*, Cambridge University Press, Cambridge, 1970.

[61] Vladislav Nenchev Logics for stable and unstable mereological relations, *Central European Journal of Mathematics 9(6)*, 2011, pp. 1354–1379.

[62] — Dynamic relational mereotopology. logics for stable and unstable relations, *Logic and Logical Philosophy 22*, 2013, pp. 295–325.

[63] Vvladislav Nenchev, and Dimiter Vakarelov An axiomatization of dynamic ontology of stable and unstable mereological relations, *Proceedings of 7th Panhellenic Logic Symposium*, Patras, Greece, 2009, pp. 137–141.

[64] Yavor Nenov and Dimiter Vakarelov Modal logics for mereotopological relations in Robert Goldblatt, and Carlos Areces (eds.), *Advances in Modal Logic 7*, College Publications, 2008, pp. 249–272.

[65] Andrzej Pietruszczak *Metamereology*, The Nicolaus Copernicus University Scientific Publishing House, Torun 2018 328 pages, PDF (open access).

[66] Ian Pratt-Hartmann First-order region-based theories of space, In: *Handbook of Spatial Logics* , Marco Aiello, Ian Pratt-Hartmann, and Johan van Benthem (Eds.), Springer, 2007, pp. 13–97.

[67] Alfred A. Robb *A Theory of Time and Space*, Cambrdge University Press, 1914, Revised edition, *Geometry of Time and Space*, 1936.

[68] Walter Scott Minkowski, Mathematicians, and the Mathematical Theory of Relativity

In: Goenner, Hubert; et al. (eds.). *The Expanding Worlds of General Relativity.* Boston: BirkhÃďuser, (1999), pp. 45ÂŰ86. ISBN 978-0-8176-4060-6.

[69] Valentin Shehtman Modal logics of domains on the real plane, *Studia Logica 42*, 1983, pp. 63–80.

[70] Roman Sikorski *Boolean Algebras*, Springer-Verlag, Berlin, 1964.

[71] Peter Simons *Parts: A Study in Ontology*, Oxford, Clarendon Press, 1987.

[72] Marshal Harvey Stone The theory of representations for Boolean algebras, *Transactions of the Americal Mathematical Society 40*, 1937, 37–111.

[73] Alfred Tarski Foundations of the geometry of solids, Joseph Henry Woodger, editor, *Logic, Semantics, Metamathematics*, pp. 24-29. Clarendon Press. Oxford, 1956. Translation of the summary of address given by Tarski to the First Polish Mathematical Congress, Lwów, 1927.

[74] Wolfgang J. Thron Proximity structures and grills, *Mathematische Annalen 206*, 1973, pp. 35–62.

[75] Tinko Tinchev, and Dimiter Vakarelov Logics of space with connectedness predicates: complete axiomatizations, *Advances in Modal logic 8*, College Publications, 2010, pp. 419–437.

[76] Dimiter Vakarelov Region-based theory of space: algebras of regions, representation theory and logics, In: Dov Gabbay, Sergey goncharov, and Michael Zakharyaschev (Eds.) *Mathematical Problems from Applied Logics.* New Logics for the XXIst Century. II. Springer, 2007, pp. 267-348.

[77] — Dynamic mereotopology I: A point-free theory of changing regions.: stable and unstable mereotopological relations, *Fundamenta Informaticae 100(1-4)*, 2010, pp. 159-180.

[78] — Dynamic mereotopology II: axiomatizing some Whiteheadian type spacetime logics, In:. Thomas Bolander, Torben Braüner, Silvio Ghilardi, and Lawrence Moss Eds. *Advances in Modal Logic 9*, College Publications, 2012, pp. 538-558.

[79] — Dynamic mereotopology III. Whiteheadean type of integrated point-free theories of space and time, Part I, *Algebra and Logic 53(3)*, 2014, pp. 191-205, Part II, *Algebra and Logic 55(1)*, 2016, pp. 9-197, Part III, *Algebra and Logic 55(3)*, 2016, pp. 181-197.

[80] — A Mereotopology based on sequent algebras, *Journal of Applied Non-Classical Logics 27(3-4)*, 2017, Published online: 22 Jan 2018.

[81] — Actual existence predicate in mereology and mereotopology, In: Lech Polkowski, Yiyu Yao, Piotr Artiermjew, Davide Ciucci, Dun Liu, Dominik Slezak, and Beata Zielosko (Eds.) *Rough Sets 2*, IJCRS 2017, pp. 138-157. Springer,

[82] — Mereotopologies with predicates of actual existence and actual contact, *Fundamenta Informaticae 156(3-4)*, 2017, pp. 413-432,

[83] Achille Varzi Mereology, *The Stanford Encyclopedia of Philosophy (Spring 2019 Edition), Edward N. Zalta (ed.),*
URL = <https://plato.stanford.edu/archives/spr2019/entries/mereology/>.

[84] H. de Vries *Compact spaces and Compactifications*, Van Gorcum, 1962.

[85] Michael Winter, Torsten Hahmann, and Michael Gruninger On the algebra of regular sets: properties of representable Stonian p-ortholattices, *Annals of Mathematics and Artificial Intelligence 65(1)*, 2012, pp. 25-60.

[86] — On the skeleton of Stonian p-ortholattices, in *Conference on Relational Methods in Computer Science (RelMiCS-11)*, LNCS 5827, Springer, 2009, pp. 351-365.

[87] Alfred North Whitehead The Organization of Thought, *Science 44*, 1916, pp. 409–419.

[88] — *An Enquiry Concerning the Principles of Natural Knowledge*, Cambridge Univ. Press, Cambridge, 1919.

[89] — *The Concept of Nature,* Cambridge Univ. Press, Cambridge, 1920.

[90] —*Science and the Modern World, Lowel Lectures*, Macmillan, New York, 1925.

[91] —*Process and Reality*, New York, MacMillan, New York, 1929.

[92] Alfred North Whitehead, and Bertrand Russell *Principia Mathematica*, Vol. I 1910, Vol. II 1912, Vol. III 1913, Cambridge University Press, Cambridge.

[93] John A. Winnie The Causal Theory of Space-Time, in John Earman, Clark Glymour, and John Stachel (eds.): *Foundations of Space-Time Theories,* University of Minnesota Press, 1977, pp. 134-205.

Appendix: Short review of papers on RBTS

In this appendix we present a short, probably incomplete review of papers on RBTS appeared after 1977 and not discussed in [76]. The papers are classified in several groups.

(I) Mereology and RBTS First I want to mention here some papers devoted to a detailed analysis of results obtained by Polish logicians in the field of mereology and RBTS. The book Metamereology [65] extends some results on mereology, the paper [40] is devoted to a detailed analysis of Grzegorczyk point-free theory of space [39] and the paper [41] - to a full analysis of Tarski geometry of solids ([73]). The paper [83] is also a good survey on recent results of Mereology and its connection to mereotopology. The book Varieties of Continua: From Regions to Points and Back [44] discusses the history of the idea of Continua and the point-based and region-based approaches to its foundation.

(II) Further results on contact and precontact algebras. The papers [29, 11] contain some technical results on contact algebras. The paper [21] transfers the notion of dimension from topology to the corresponding notion of some classes of contact algebras and the paper [75] extends contact algebras with connectedness predicates and studies the corresponding quantifier-free logics. The paper [18] characterizes contact algebras on Euclidean spaces. The papers [27, 28] presented topological representation theorem for precontact algebras and new representation theorems for some classes of contact algebras. Some Isomorphism Theorems for

MVD-algebras are studied in [48].

(III) Duality theory of contact and precontact algebras and some related systems. There are many papers generalizing De Vries duality theorem [84] mainly with applications to topology: [6], [7], [14], [15], [16], [17], [9] - for Boolean algebras with quasi-modal operators which are equivalent to precontact algebras, [10] - for subordination Tarski algebras with application to De Vries duality. Extensions of dualities and a new approach to the de Vries duality and Fedorchuk duality are studied in [22] and [23]. For some extensions of the Stone Duality to the category of zero-dimensional Hausdorf spaces see [20]. A new duality theorem for locally compact spaces is published in [19]. A paper about duality theory for contact and precontact algebras is [24] which includes also some generalizations of the Stone Duality Theorem. Another duality theorem for contact algebras based on mereotopological spaced is presented in [38].

(IV) Generalizations of contact algebras. The paper [64] contains a generalization of contact algebra based only on the standard mereological relations part-of, overlap and underlap plus standard mereotopological relations of contact, dual contact and non-tangential inclusion and studies also a modal logic based on these relations. The paper [47] studies generalizations of contact algebras based on distributive lattices with three basic mereotopological relations of contact, dual contact and non-tangential inclusion taken as primitive relations. Representation theorems for extended contact algebras based on equivalence relations is in the paper [3]. Generalization of contact algebra based on non-distributive lattices is presented in [43, 85, 86].

Another generalization of contact algebra is the notion of sequent algebra which presents Tarski and Scott consequence relations as mereotopological relations - see [80] and [46]. In standard models with regular closed subsets of a topological space Tarski consequence relation $a_1, \ldots, a_n \vdash b$ is defined as $a_1 \cap, \ldots, \cap a_n \subseteq b$, which makes possible to define n-ary contact by $C_n(a_1, \ldots, a_n) \Leftrightarrow_{def} a_1, \ldots, a_n \not\vdash 0$ and ordinary contact as $aCb \Leftrightarrow_{def} a, b \not\vdash 0$. Generalizations of contact algebras with predicates of actual existence and actual contact are subject of [81, 82]. In standard contact algebras the predicate of existence is defined as follows: $E(a) \Leftrightarrow_{def} a \neq 0$. This is a quite weak predicate, because the only non-existing region is 0. The generalization is to relax this definition as follows: take a fixed grill Γ (see Definition 2.15) and define $E(a) \Leftrightarrow_{def} a \in \Gamma$. Another line of generalizations is to consider Boolean algebras with contact relation and measure - see [56] and [57].

(V) Modal and Quantifier-free logics based on contact and precontact algebras. Modal logics based on mereological and mereotopological relations arising from contact algebras or topology are presented in [59] and [64]. Papers on quantifier-free logics in the style of [4] related to contact algebras and their extensions and generalizations are [75] for logics with connectedness predicates, [52] - studying them form computational point of view, [47],[45], [46] - for logics based on extended contact algebras. Quantifier-free logics related to contact algebras with measure are [56] and [57].

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Logic's Naturalistic Character

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"I challenge anyone here to show me a serious piece of argumentation in natural language that has been successfully evaluated as to its validity with the help of formal logic. I regard this as one of the greatest scandals of human existence."
Yehoshua Bar-Hillel[1]

Abstract

For most of its long history, concepts of logical interest were defined over, and instantiated by, constructions of natural language. This is one of the things I have in mind when I speak of logic's naturalistic character. For the most part, the tripartite distinction between a proposition S's *having* a proposition S' as a consequence, someone *spotting* S' as a consequence of S', and someone *drawing* that consequence S' from that proposition S, had a recognizable, if unannounced, presence in logical theory. A *full-service* logic of the consequence relation makes theoretical provision for each of the three ways in which it manifests itself. Given that agents are needed for spotting and drawing, decisions must be taken as to the best way of bringing cognitive agency into logical theory. With spotting and drawing, epistemology becomes ineradicably linked to a full-service logic of consequence. Until approximately 170 years ago, logic's agents were people, that is, beings like us, natural objects of the natural world. This is another of the things I have in mind in speaking of logic's naturalistic character. The spotting and drawing domains started to change when Pascal's axioms were adopted as rules of probabilistic inference and human agents were replaced by the mathematical fiction of ideally rational ones. Leibniz had a similar idea for all exact thought. In due course, deductive logic would also take the mathematicizing turn, thereby alienating human beings from the dynamics of spotting and drawing consequences. In the 1970s there arose a pushback that has yet to abate. It opened the road for the *restoration* of humans as they actually are in real life to the logics that are said to regulate their thinking.

[1] "Formal logic and natural languages (A Symposium)", J. F. Staal, editor, Foundations of Language, 5 (1969) 903–926.

In the late 1960s some epistemologists took the naturalizing turn in epistemology, in which the philosophy of knowledge established working relations with the sciences of cognition. In the late 1980s, some logicians proposed to do the same for logic. By the early 2000s the proposal started bearing fruit. I think it has a promising empirical future. This is a further manifestation of logic's naturalistic character.

1 Naturalistic logic

Until the latter part of the 19th century, the properties of interest to logicians were defined over and attributable to natural language constructions. Logic was framed as theory of natural-language argument and reasoning. It was a humanities subject, typically lodged in the curricula of Philosophy Faculties in institutions of advanced education. With scant exceptions (Leibniz, Pascal), for almost its whole history mathematics neither sought nor achieved footfall in logic.[2] Logic focused mainly on deductive matters, notably the three ways in which the deductive consequence relation manifests itself — *consequence-having* (entailment), *consequence spotting* (entailment-recognition) and *consequence-drawing* (inference). Consequence-having is a two place alethic relation defined over truth-evaluable sentences in what we could call *logical space*. Consequence-spotting is a three-place alethic-epistemic relation, whose third relatum is an epistemic agent's spotting-devices in *psychological space*. Consequence-drawing requires an inferer and, if his consequence-recognition devices are different from his inference-drawing devices, it would be a four-place relation instantiated in the agent's *inferential space*.[3] As we see, psychological and epistemological considerations are harboured in the last two of this ordered triple, which means that a fully developed logic of the consequence relation will draw upon

[2]Leibniz for deductive logic, and Pascal for probabilistic inductive logic.

[3]When a spotting occurs, the spotter forms a true belief in the form "S' is a consequence of $S_1, \ldots, S_n$." When a drawing occurs, the drawer forms the compound belief that S' is true in virtue of the fact that the S_i from which it follows are also true.

circumspect alliances with the empirical sciences of cognition and epistemology.[4,5] A further part of what I mean by logic's naturalistic character is that the best treatment of the epistemology to which spotting and drawing are tied is an epistemology naturalized in the manner of Goldman [63], Quine [117], Gabbay and Woods [58], Woods [148, 149] and Magnani [97].

It was recognized from logic's outset that, while human beings are good at reasoning, they also make errors. A corresponding interest has been in practicable measures for error-avoidance, and for the detection and correction of it after the fact. It was recognized that, like all natural creatures, the human animal has his limitations. Even when performing at his humanly possible best, there are certain idealized heights to which the human reasoner cannot and need not rise. An accompanying assumption is that logic has no business in laying down norms for rational performance which exceed the capacities of cognitively competent performers on the ground, that is to say, under the conditions of real life. The reason why is that the normal limits imposed on beings like us are not incapacitations. We can sum it up this way: Until the parting of the ways c. 1850, logic in all its iterations trended to agent-centred, resource-bound, goal-directed, interactive, time-and-action theories of human reasoning and argument in favourable psychological and epistemic circumstances. A main purpose of the sections to follow is to quash the idle notion

[4]See here Alvin I. Goldman, "A causal theory of knowing", *Journal of Philosophy,* 64 (1967), 357-372; W. V. Quine, "Epistemology naturalized" in his *Ontological Relativity and Other Essays*, pages 69-90, New York: Columbia University Press, 1969; Dov M. Gabbay and John Woods, *Agenda Relevance: A Study in Formal Pragmatics*, volume one of their *A Practical Logic of Cognitive Systems*, Amsterdam: North-Holland, 2003; and John Woods, "Logic naturalized", in Juan Redmond, Olga Pombo Martins and Ángel Nepomucheno Fernández, editors, *Epistemology, Knowledge and the Impact of Interaction*, pages 403-432, Cham, Switzerland: Springer, 2016. For a more expansive discussion of the role of a causal response epistemology in a naturalized logic of inference, readers could consult my *Errors of Reasoning: Naturalizing the Logic of Inference*, volume 45 of Studies in Logic, London: College Publications, 2013; reprinted with corrections in 2014; Lorenzo Magnani, *The Abductive Structure of Scientific Creativity: An Essay in the Ecology of Cognition*, Cham: Springer, 2017, "Naturalizing logic and errors of reasoning vindicated: Logic reapproaches cognitive science," *Journal of Applied Logic*, 13 (2015), 13-36, and "The urgent need of a naturalized logic", in G. Dodig-Crmkovic, M. J. Schroeder, editors, *Contemporary Natural Philosophy and Philosophies*, a special guest-edited number of *Philosophies*, 34 (2018), p. 44. In particular, we should trust the data that these sciences aim to account for and hold their theoretical ways of doing so to greater scrutiny. The social sciences sometimes embed philosophical mistakes.

[5]In this essay, I confine myself to logic in the Western tradition and intend no slight in omitting them here to rich traditions elsewhere. See, for example Jonardon Ganeri, editor, *Indian Logic: A Reader*, Milton Park, Oxon: Routledge, 2001; Salua Chatti, *Arabic Logic from al-Fārārabi to Averroes: A Study of the Early Arabic Categorical, Modal, and Hypothetical Syllogistic*, Basel: Birkhäuser, 2019; and Yiu-ming Fung, editor, *Dao Companion to Chinese Philosophy of Logic*, Cham: Springer, 2020.

that, in the absence of the mathematical tools which became available to deductive logicians mainly in the last half of the 19th century, logic was a subpar discipline that hadn't grown up yet. I also want to lay the ground for showing that, in taking the mathematical turn, modern logicians had largely changed the channel.

2 In the beginning

It is customary to locate the Western arrival of systematic logic in the six monographs of Aristotle's *Organon*. From its very first appearance logic provided the canonical regulatory framework for deductive science, for whose rigorous examination Aristotle originated metalogic. The key concept of this approach is a form of argument called the *syllogism*.[6] The concepts of argument and proof are goal-oriented activities of an agent. Arguments can have different objectives, but proofs always aim for truth. One way to spot a consequence S' from some S_i is to see that it follows from them. Another way is to provide a conditional proof that it does. Sometimes we are also able to know that S' is true. One way of doing it is by having a direct proof of it from S_i we know to be true. A logic lacking the concept of goal-oriented agency cannot be a full-service logic for the consequence relation.

Aristotle (384–322 BC) considers three basic kinds of syllogism: Direct syllogisms; indirect syllogisms; and hypothetical syllogisms. He also considers related kinds of proof rules: syllogistic rules (both direct and indirect), and common rules such as modus ponens and ekthesis. A proof is a direct syllogism if its conclusion arises from its premisses by direct syllogistic rules only. Direct syllogisms are conceptually prior to the others. A direct syllogism is a valid argument fulfilling further conditions. One is that its premisses and conclusions be *categorical*, that is, statements of the form "All S is P", "No S is P", "Some S is P" and "Some S is non-P". "S" and "P" are schematic letters that serve as place-holders for general terms. Syllogisms are sequences of exactly two distinct and non-redundant premisses and a single conclusion. Propositions containing terms not contained in a syllogism's conclusion — "terms from the outside" — are ineligible to serve in its premisses.

[6] Aristotle defines direct syllogisms in *On Sophistical Refutations* at 165a 1-3: "A *sulligismos* rests on certain propositions such that they involve necessarily the assertion of something, other than what has been stated, through what has been stated." The definition recurs in several other places in the *Organon*. Scholars are not of one mind about how secure these defining conditions are. This is not the place to litigate the matter further. Suffice it to note that an important difference is one between the principles of syllogistic reasoning and certain of those that regulate metalogical chains of reasoning whose various links are themselves syllogisms. Crucial to the success of chain-reasoning are rules that have no application to syllogistic reasoning, modus ponens being one of them. Syllogistic rules and only they are eligible for use in the crafting of syllogisms. But, as we will see, they do not suffice for running the proofs of Aristotle's metalogic.

The requirement that premisses bear some relevance to their conclusions is met by a rule for the distribution of terms. Each of the two distinct terms of a syllogism's conclusion must have exactly one occurrence in just one of the premisses. Because premisses and conclusions are required to be both internally and jointly consistent, syllogisms are hyperconsistent. Taken collectively, we have it from these conditions that the Aristotle's logic was nonmonotonic, relevantist, hyperconsistent, and a fair approximation of the intuitionist notion of deductively derived conclusions.[7] Aristotle also acknowledges proofs *per impossibile*. These are not, however, direct proofs and don't fall foul of the hyperconsistency requirement.[8] In what follows, I will mean by "syllogisms" direct syllogisms unless otherwise indicated.

It is easy to see that it is not possible for every (or even for few) proposition to follow *syllogistically* from any set of inconsistent premisses. In other words, the classical theorem that a contradictory sense that deductively entails every sentence fails to hold in the syllogistic.[9] If it did, this would violate the condition that the terms of the conclusions of direct syllogisms must have a solitary occurrence in one or other, but not both, of the premisses. All paraconsistent logics block the classical theorem. There are logics — some of them relevantist ones — that also block the theorem without being paraconsistentist. A logic is paraconsistent only if, in addition to blocking the theorem, it implies at least one inconsistent sentence. The goal of the paraconsistentist is to keep things from getting worse. The fundamental question is this: By what means is this containment to be achieved? Do we reconstruct the consequence-having relation, or do we observe the lived realities of consequence-*drawing* on the ground? I will come back to this, too.

The founder of logic had the nose of a modern logician. By this I mean that he had a nose for reductionism. In *On Interpretation*, Aristotle ventured, without proving it, the bold claim that anything stateable in Greek could be stated without relevant loss in the language of categorical propositions.[10] Although the categorical reduction claim is certainly false, we can see why Aristotle could have been drawn to it. In the matter of problem-solving, Aristotle was one of those theorists who framed for the big and solved for the small, and did so in a way that also took care of the big. This was done by *reducing* the large to the small. Three objectives lie at the centre of syllogistic logic. One was to provide a way of establishing that one's

[7]See here John Woods, *Aristotle's Earlier Logic*, 2nd edition, Studies in Logic volume 53, London: College Publications, 2014; p. 146. For the connection to intuitionism, see D. J. Shoesmith and T. J. Smiley, *Multiple-Conclusion Logic*, Cambridge: Cambridge University Press, 1978; p. 4.

[8]*Prior Analytics* 63b 31-64a 16.

[9]The theorem is commonly but inaccurately known as *ex falso quodlibet*. Its accurate name is *ex contradictione quodlibet* — in English "from a contradiction whatever [you like]". I will come back to this.

[10]*On Interpretation*, 17a 13, 18a 19ff., 18a 24.

opponent has made an inconsistent defence of some given thesis that he supports, and to show this without begging the question against him. The required procedures are laid out in *On Sophistical Refutations* and Book VIII of *Topics*. In solving the problem for syllogistic refutations, Aristotle solved it in a more general form. It can be summed up by the admonition not to argue against an opponent unless you have reason to believe that your premisses are propositions which your opponent would accede to. That way, if your refutation succeeds, you won't have begged the question against him.

In *Prior Analytics* Aristotle's objective is a metatheoretic one. Aristotle wants to construct a kind of decision-procedure for validity. The project is root and branch a venture in epistemology. Its goal is to say something instructive about the mechanics of coming to know something to be the case; in this instance, coming to know that the argument one is considering is valid. Aristotle's procedures would expose the validity of any syllogism to any competent speaker of Greek, and would do so in a step-by-step quasi-mechanical fashion with a practicable timeliness that made it user-friendly for the legendary "man in the street". [11] Such were the means of making the validity of an inapparent syllogism self-evident to anyone interested in knowing it. Aristotle's proof of the practicably effective recognizability of the validity of syllogisms almost succeeded, and was later shown by John Corcoran to be repairable in a nonconservative extension of a natural deduction system.[12] Were Aristotle's categorical reducibility thesis true, solving the recognizability problem for the validity of syllogisms would have solved it for all valid arguments. It emerges from *Prior Analytics* that there are just fourteen schemata whose instantiations are syllogisms in the direct sense. One could record the schemata on a tablet and wear it on a string around one's neck. Anyone doing so would have a practicable decision procedure for the validity of direct syllogisms. One would also have the same for the property of being a direct syllogism.

It might strike one as strange that validity is a primitive concept in Aristotle's logic. An argument is valid if and only if its conclusions follow of necessity from its premisses. Yet the logic contains no *theory* of the validity property or, relatedly, of the premiss-conclusion necessitation relation (*anagkaion*).[13] It bears repeating that

[11] Details can be found on pages 207 and 208 of *Aristotle's Earlier Logic*.

[12] John Corcoran, "Completeness of ancient logic", *Journal of Symbolic Logic*, 37 (1972), 696-702, and "Aristotle's natural deduction system", in John Corcoran, editor, *Ancient Logic and its Modern Interpretation*, pages 85-132, Dordrecht: Reidel, 1974. The Corcoran extension is (strongly) sound and (strongly) complete.

[13] Aristotle makes several fragmentary attempts to modalize the syllogistic, none of which quite made the grade of what he had accomplished for nonmodal syllogisms. See, for example, Storrs McCall, *Aristotle's Modal Syllogisms*, Amsterdam: North-Holland, 1963 and Adriane Rini, *Aristotle's Modal Proofs: Prior Analytics A8-22 in Predicate Logic*, Dordrecht: Springer, 2011.

Aristotle takes it for granted that the concept following of necessity from would be in the working vocabulary of any competent speaker of Greek. Aristotle's project was not to analyze validity or consequence-having, but rather to make inapparent validities and entailments apparent, that is, to make them spottable upon presentation. Though *predicated* on consequence-having, the metalogic of *Pr. An.* is deeply invested in the epistemology of consequence-*spotting*.

Let's move now to *Posterior Analytics*. Its principal task was to fashion a metatheory for the axiomatization of the mature sciences, and to do so in a way that proves that their demonstrative inferences in inference-chains from axioms are both truth-preserving and *knowledge-producing*.[14] Implicit in Aristotle's demonstrative logic is the full distinction between consequence-having, consequence-spotting, and consequence-drawing. Let me say again that although Aristotle's logic advances no theory of consequence-*having*, consequence-*spotting* is catered for in *Prior Analytics*, and consequence-*drawing* is handled in *Posterior Analytics*. The demonstrative rules of *Post. An.* teach an important lesson about axiom systems. In some quarters it is put about that the theorems of an axiom system lie entirely in their deductive closures, an overstatement to say the least. Consider the axiom that 1 is a natural number. Clearly "1 is a natural number or Nice is nice in November" is a consequence had by the axiom, hence sits in its deductive closure. But it is not a theorem of arithmetic. It tolerates "terms from the outside". Consequence-having is truth-preserving, but it is not subject-matter preserving, and not theorem-generating either.[15] When we demonstrate S' from the S_i, we *draw* it from them in a way that is a truth-preserving, content-preserving, theorem-generating, and knowledge-generating. Since syllogisms ban terms from the outside, the use of syllogistic rules in chains of demonstrative reasoning enables them to be subject-matter preserving. This is, as we see, a striking insight into the manipulation of epistemically fruitful deductive consequences. The logic of demonstration is a venture in the epistemology of axiomatization, the first systematic work in what Tarski calls "the methodology of deductive sciences". Here too, although predicated on consequence-having, it is entirely immersed in the mechanics of spotting, drawing and knowing. It is a relevant logic through and through.

A further feature of *Posterior Analytics* is its attempt to explain how fallible beings like us could come to grasp the certainty of the first principles of the demonstrative sciences, given that they themselves are not susceptible of independent demon-

[14] John Corcoran, "Aristotle's demonstrative logic", *History and Philosophy of Logic*, 30 (2009), 1-20.

[15] Nor, contrary to some philosophers of science, is consequence law-preserving. See, for example, Mark Lange, *Laws and Lawmakers: Science, Metaphysics and the Laws of Nature*, New York: Oxford University Press, 2009; p. 16.

stration.[16] There is no space here to detail Aristotle's courageously candid answer, but here is a short sketch. Let statement F be a candidate for first principleship in a demonstrative science D. Then the community of D experts repeatedly subject F to the sort of refutation arguments described in *On Sophistical Refutations*. If a refutation succeeds, F falls out of the race. But if F holds its ground against all expert attempts to refute it, and no other candidates are left standing, then Aristotle's contention is that the mind is causally induced to grasp with certainty that F is a first principle. The question, however, is whether upon further consideration a thithertofore unexamined expert refutation might now come forth and succeed. Then what? Do we outright scorn the challenger? Or do we listen to what he has to say? This is precisely the situation contemplated in the *Metaphysics* concerning the most certain of all first principles, the Law of Non-Contradiction:

> "It is impossible that the same thing belong and not belong to the same thing at the same time and in the same respect." (1005b 19-20):

However, rather remarkably, Aristotle immediately adds that "[w]e must presuppose, in the face of dialectical objections, any further qualifications which might be added." To understand this extraordinary capitulation, it is essential that we understand what Aristotle means by "dialectical". The notion of dialectic plays a twofold role in Aristotle's account of first principles. In one sense, the grasping of first principles requires the dialectics of attack-and-defend arguments. In its second sense, dialectic refers to beliefs endorsed by the wise, or in this case, the experts. What Aristotle's concession leaves room for is that future experts might have new refutations that succeed against this particular formulation of LNC. We may conclude from this (as I think we should) that Aristotle is a *fallibilist* about first principles and, by his own lights, a foundational *inductivist*.[17] That is to say that he saw the foundations of the deductive sciences as inductive.

Aristotle's syllogistic is a *term logic* in which there are five logical expressions, the subject-term modifiers "all", "no" and "some", the predicate-term complement particle "non-", and the copula "is". The nonlogical terms of the logic's vocabulary are schematic letters, "S", "P", "M", which serve as placeholders for general

[16]Frege had a similar concern about how the concept of number is grasped. He also shared Aristotle's conception of axioms or primitive truths. It is an old-fashioned conception, put into permanent retirement by the contradiction that Russell spotted in Frege's axiom V.

[17] For details see my "What did Frege take Russell to have proved?" *Synthese*, July 22, 2019. DOI 10.1007/s11229-019-02324-4. Aristotle does say that it is impossible to believe a contradictory sentence. If "contradictory sentence" implicitly carries the "in-the-same-respect" clause, Aristotle might well be right. But there is no point in claiming that the negation of LNC is a contradictory hence unbelievable sentence if one's interlocuter already has taken the Law to have failed. Aristotle is well aware of this in Book Gamma of the *Metaphysics*.

terms of Greek, subject to the requirement that they be applicable to one thing (at a time) and, when applied, they ascribe just one thing to it. Schematic letters are not variables. Variables appeared in mathematics only in the 16th century, and variable-binding had to wait another three centuries before Frege and Peirce, independently, provided the means for it. Deprived of variables, there are no quantifiers in Aristotle's logic. The expressions "all", "no" and "some" are general-term modifiers, functioning in the manner of adverbs. Aristotle's "is" is the "is" of predication, not the "is" of identity.[18] There is no conditional sign in Aristotle's categorical syllogistic and no metalogical term for entailment. The use of schematic letters invites the suggestion that an argument is a syllogism just in case it has the structure of a syllogistic schema. Even so, Aristotle has no doctrine of logical form in our sense, partly because he lacked a validity-preserving rule of substitutivity.[19]

It is hardly surprising that Aristotle's logical contemporaries and close descendants would chafe against the restriction of syllogisms to the categoricality of their component statements. Although Aristotle recognizes the modus ponens rule as an admissible "common principle" of metalogical reasoning, it is impossible to give it syllogistic formulation. In other words, the valid argument

i. If p then q

ii. p

iii. Therefore, q

is inexpressible as a syllogism. So, too, are the likes of

a. $p \wedge q$

b. Therefore p

and

c. p

[18] Although Aristotle is said to have wondered about that. See below section 3.

[19] The rule was a long time in coming. In his 1921 doctoral dissertation, Post pointed out its absence in *Principia Mathematica*. Post proved the rule and showed that when added to *PM*'s eight primitive propositions a complete and decidable theory of the propositional calculus can be got. See Emil L. Post, "Introduction to a general theory of elementary propositions", in Jean van Heijenoort, editor, *From Frege to Gödel: A Resource Book in Mathematical Logic*, 1879-1931, pages 265-283, Cambridge, MA: Harvard University Press, 1967. We shouldn't overlook Frege's substitution rule in §48 of the *Grundesetze*, volume 1. His rule is equivalent to an existence condition known as the Comprehension Principle for Concepts. This is problematic for Frege, who thought that existence claims were synthetic, hence not properly part of pure logic. I'll come back to Frege in section 5.

d. $p \vee q$.

The very fact that Aristotle tarried with the categorical reduction thesis shows an openness to the importance of having at hand the maximal field of the consequence relation and the maximal extension of the validity-property. That way, a theory's expressive capability matches the instantiation-scope of the items in question. Clearly, Aristotle fell short of this goal.

In the second greatest achievement of ancient logic, Megarian and Stoic logicians would examine concepts which Aristotle had made some use of without theoretical analysis, or had been wholly overlooked by him. In the first grouping we find the concepts of consequence-having or validity. In the second, material "implication" makes its first theoretical appearance. Only with the Stoics, does the ancient world make full-service stabs at the logic of consequence.[20] The Later Stoic logicians produced the first successful propositional logic. We also owe the material conditional to Philo of Megara (late 4th-early 3rd cent. BC). However, the Stoics lacked rules for simplification and disjunction-introduction. They also lacked "$p \to p$". Chrysippus (c. 280-207 BC) is credited by some commentators with having had the notion of truth-function. This is open to question. The concept of function announces itself only in the 17th century, in correspondence between Leibniz (1646-1716) and Bernoulli (1654-1705). It was a dominant and vexed subject of mathematical investigation from Euler (1707-1783) to Hilbert (1862-1943). The notion would stir in the mid-nineteenth century in the writings of De Morgan (1806-1871) and Boole (1815-1864) and make a prominent début in Frege's *Begriffsschrift* in 1879. There is nothing in the Stoic writings that captures Frege's notion of function or his provisions for the abstract objects *das Wahre* and *das Falsche*.[21] The Stoics also had

[20]Ancient sources include, Diogenes Laertius, *Lives of Eminent Philosophers.* R. D. Hicks, editor and translator, in two volumes, London: Loeb Classical Library 1925 and Diodorus Cronus, in *Die Megarikes: Kommentiere Sammlung der Testimonien*, Klaus Döreng, editor, pages 28-45 and 124-139, Amsterdam: Gruener, 1972. More recent is the golden oldie of Benson Mates, *Stoic Logic*, Berkeley and Los Angeles: University of California Press, 1953. More recent still are Julia Annas and Jonathan Barnes, editors, *Sextus Empiricus: Outlines of Scepticism*, 2nd edition, New York: Cambridge University Press, 2000; Susanne Bobzien, "Stoic logic", in Keimpe Algra, Jonathan Barnes, Jaap Mansfeld and Malcolm Schofield, editors, *The Cambridge History of Hellenistic Philosophy*, pages 92-157, Cambridge: Cambridge University Press, 1999; Walter Cavini, "Chrysippus on speaking truly and the Liar", in Klaus Döring and Theodor Ebert, editors, *Dialektikes und Stoiker: Zer Logic der Stoiker und ihrer Vorläufer*, Stuttgart: Franz Steiner, 1993; Susanne Bobzien, "Chrysippus' modal logic and its relation to Philo and Diodorius, in Döring and Ebert (1993); see also Michael Frede, *Die stoische Logik*, Göttingen: Vandenhoek & Rupert, 1974 and A. A. Long, "Language and thought in Stoicism", in his *Problems in Stoicism*, pages 75-113, London: Duckworth, 1971.

[21]Details of the evolution of the concept of function in 19th century mathematics and its impact on 19th century logic are reviewed in John Woods and Alirio Rosales, "Mathematics in Frege's

the conditionalization rule for logical implication. In addition to material implication, they had strict implication, and were aware of the so-called paradoxes thereof. Although its makers couldn't have known it, the Stoic syllogistic can be lodged in a nonmonotonic extension of a sound and complete Gentzen-style natural deduction system. Not every valid argument is a Stoic syllogism, but all are said to be subject to reductions that make them so. Although clearly different, the deductive systems of Aristotle and the Stoic needn't be thought of a rivals. For the most part they differ in the matters they cover. William and Martha Kneale are right to say in chapter III "The Megarians and the Stoics" of *The Development of Logic*, that the two systems can be seen as complementary. Still, it would fall to mediaeval logicians to try to unify the two approaches.

Aristotle's word for what we think of as his logic is *analytics*.[22] The Stoic's word for what we consider their logics is *logike*. The word "logic", made its first appearance with Alexander of Aphrodisias (2nd-3rd cent. A. D.).[23] Of particular note for what concerns us here is that *logikē* is usually taken as broader than analytics and logic. It also encompasses epistemology and the philosophy of natural language. On that view, although Aristotle's analytics are *agent-centred, resource-bound, time-and-action* logics for human reasoners, and have clear epistemological implications for inference, *logike* makes the connection to the allied modes of enquiry more expressly. The logic of human inference, even of the truth-preserving kind, would be a partnership between what is more usually thought of as logical theory, together with the theory of (human) *knowledge* and philosophy of (human) *language*. Speaking for myself, I find this view somewhat overstated. Not all of what matters for Aristotle's logic is encompassed in the *Organon* but, even within it, *On Interpretation* is an essay in the philosophy of language and, for all its metaphysical trappings, *Categories* is an essay on the ambiguity of the "is" of predication; *Rhetoric* is the home of implicit arguments (enthymemes), and *Metaphysics* the cite of a thorough

day", to appear.

[22]There is no scholarly consensus about what motivated Aristotle's choice of the word that would name his invention. Since he insisted on the utter originality of the concept of syllogism (*Soph. Ref.* 34, 183b 34-36, 184b 2-8), it stands to reason that any question about his chosen word's tie to the later notion of analyticity is moot. This, I think, is right. But another possibility is that the tie between Aristotle's word and the modern notion of the analytic to be found in the *Posterior Analytics'* notion of the axiomatization of a science (or scientific theory) which, in all essentials, was the device used by Frege to analyze the concept of number in ways that reveal arithmetic's analyticity. Details on this striking similarity can be found in "What did Frege take Russell to have proved?" (2019).

[23]Alexander of Aphrodias, *On Aristotle's Prior Analytics* 1-1-7, Jonathan Barnes, Susanne Bobzien, Kevin Flannery and Katerina Ierodiakonou, editors and translators, London: Duckworth, 1991.

discussion of contradiction.

We have seen how Stoic logicians found the categorical syllogistic too narrow to provide for truth-preserving argument in general. We are about to see the same complaint made against the Stoic alternative, especially as regards the implication relation. However, it would be wrong to leave the impression that the lifespan of term logic was nearing its end. Term logic would hold its ground well into the twentieth century. It was the dominant logic at Oxford in the 1920s, both when and after John Cook Wilson (1849-1915) held the Wykeham Chair of Logic.[24]

Nowhere in the ancient writings on logic is the tripartite division of the deductive consequence relation expressly drawn. Even so, its response to those divisions is discernible in its analytical provisions. Aristotle had no analysis of consequence-having, but offered robust analytical provision for spotting and drawing. The Stoics shifted the consequence relation from the confinements of categorical languages to languages more in tune with the realities of actual speech. These propositional settings catered for consequences in all three of its dimensions. Full-coverage was maintained throughout the mediaeval period and, as we shall see, made notable contributions to what had been a central methodological question for logic since Aristotle advanced the overhopeful doctrine of the reducibility of anything stateable at all to complete stateability in a language of categorical propositions.[25] Beyond knowing the doctrine to be false, the Stoics were able to show that the concepts which drew Aristotle's attention are subject to analytical treatments in languages other than categorical ones. As we move to the Middle Ages, this preoccupation with the appropriate language(s) for logic not only remains in place, but prompts the question of how strictly a language for logic must resemble the theorist's mother tongue.

3 The middle ages

In the 13th and 14th centuries, logicians sought theoretical accommodations of the consequence relation (*consequentiae*) that would work for a suitably unified language for the two logics of old.[26] This marked a significant juncture in the development

[24] Term logics still retain a place in the present day, and have followers. See, in particular, Fred Sommers, *The Logic of Natural Language*, Oxford: Clarendon Press, 1987; and George Englebretsen, *The New Syllogistic*, New York: Lang, 1987. I should mention that Frege's very different second order functional calculus also have a term-logic component in which all closed expressions and well-formed formulas are denoting terms.

[25] It should be noted however that Aristotle never recurred to this striking claim. So it is probably misleading to call it a doctrine.

[26] For a relevantist approach to consequence see Peter Abelard (1099-1142), *Dialectica*, L. M. de Rijk, editor, 2nd edition Assen: van Gorcum, 1950. Something closer to what we call classical logic

of logic. It showed that logic is able to benefit from well-designed mergers and acquisitions.[27] Some of the work on consequence was interwoven with developments in supposition theory, that is, what we now call theories of reference and truth, and in itself a clear adumbration of the semantic conception of consequence.[28] In some cases, suppositionism confined its focus to the workings of natural language, in which there are early suggestions of a recursive treatment of truth-conditions. This can be seen as an extension and enrichment of the Stoic's inclusion of the philosophy of Stoic logic in the logic of syllogisms. As the Stoics had done to Aristotle's, so too the suppositionists would do to both. They would re-express syllogisms in a new notation, a theme we'll get back to after a momentary return to earlier days. Let me say again that the ancients could not have had the tools to form the modern notion of quantifier. However, in the work of one of the Stoa's peripatetic successors of Aristotle, Theophrastus (c. 370-c. 288 B. C.), there is some indication of a struggle to give expression to something rather like it. Consider the sentence

"If [something is] A, [it is] B."

On a charitable reading, we might see that sentence is trying to capture something along the lines of

"If some given thing is A, then that very thing is B."

I mention this here, not to abandon the claim that quantifiers are the creations of Frege (1848-1925) and Peirce (1839-1914), but rather to concede that our forbears may well have felt the need of them. Accordingly, to simplify the exposition just below, I will take the liberty of placing quantifiers at the disposal of the mediaevals. Logicians of the period not only sought to provide rules for the effective recognizability of validity and invalidity, they also sought rules that would provide an account

is John Buridan (c. 1300- after 1360), *Tractatus de Consequentics* (14th C), in Herbert Hubien, *Iohannis Buridan tractatus de consequentus: Édition critique*, volume XVI of *Philosophes médiévaux*, Louvain: Université de Louvain, 1971. A good overview of consequence is provided by Catarina Dutilh Novaes, "Logic in the 14th century after Ockham", in Dov M. Gabbay and John Woods, editors, *Mediaeval and Renaissance Logic*, volume 2 of Gabbay and Woods, editors, *Handbook of the History of Logic*, section 3, "Consequences", pages 467-484, Amsterdam: North-Holland, 2008.

[27]Of course, Corcoran's handling of *Pr. An.* was both an acquisition and a merger. But the mediaevals did it first. For an interesting attempt to formalize supposition theory in first-order logic, see Graham Priest and Stephen Read, "The formalization of Ockham's theory of supposition", *Mind* 86 (1977), 109-113. I owe this reference to an anonymous reader, for which my thanks.

[28]Suppositionists of note include William of Sherwood (1200/10-1266/71), William of Ockham (c. 1285-1349) and John Buridan (c. 1300-after 1360). An excellent survey of this period is Terence Parsons, "The development of supposition theory in the later 12th through 14th centuries", in Gabbay and Woods (2008).

of what is to be valid or invalid. The rules in question are those regulating the quality and distribution of terms. The doctrine is flawed but, as Lawrence Powers has rightly observed, "it is an almost perfect answer to a problem that puzzled Aristotle."[29] Aristotle had observed some non-contingent parallel between the logic of syllogisms and the logic of identity, a parallelism which he was unable to explain. There is no space here to expound the suppositionists distribution doctrine and how it solves Aristotle's puzzle. Suffice it to examine how the mediaevals brought identity into the formulation of syllogisms. In what can be seen as early recognition of the concept of class, suppositionists had the distinction between "man" and "Socrates". They also recognized the distinction between kinds and their instantiations. Socrates, for example, is a thing of the human kind, just as π is a thing of the number kind and Nôtre Dame is a thing of the cathedral kind. Consider now the old classic "All humans are mortal". One way of schematizing it is Aristotle's way. Another is the suppositionists' way;

> "For every h there is some m to which h bears the identity relation."

In this rendering, the italicized lower-case letters function somewhat as the variables of multi-sorted quantifiers do, never mind that they aren't really variables and "every" and "some" aren't yet quantifiers. "Every" and "some" are arbitrary-term modifiers.[30] If we allowed the suppositionists real variables adreal quantifiers, we could say that "Every A is B" can be symbolized as "$\forall a \exists b(a = b)$". Indeed, as Powers has it, "every categorical statement is a quantified identity or non-identity." (p. 192) Powers overstates the case. He concedes that "the Mediaevals did not symbolize. But their analysis of the truth conditions of the various statements suggests the above symbolization ..." ; and these too: "$\forall a \forall b(a \neq b)$", "$\exists a \exists b(a = b)$" and "$\exists a \forall b(a \neq b)$" for "No A is B", "Some A is B" and "Some A is non-B" respectively. The main point of this brief visit with mediaeval logic is to emphasize logic's enduring attention to the load-bearing work done by a theory's *notation*.

This brings us to an important point. Powers has "re-imagined" or "reconceptualized" the suppositionists' syllogisms, using tools not then in their possession. All the same, the reconstructualizations are not implausible, and there is no need to resort to anachronism in ascribing it to the mediaevals. These artifacts were devices

[29] Lawrence H. Powers, *Non-contradiction*, with a Foreword by Hans V. Hansen, volume 39 of Studies in Logic, London: College Publications, 2012; p. 191. By "the logic of identity", Powers ascribes to Aristotle some implicit and non-extant theory.

[30] Consider the simplified example of a three-membered universe whose individuals instantiate one or more of the kinds A and B. Then "Every A is B" can be laid out as follows: $[(A_1 = B_1) \vee (A_1 = B_2) \vee (A_1 = B_3) \vee \ldots] \wedge [(A_2 = B_1) \vee (A_2 = B_2) \vee (A_2 = B_3) \vee \ldots] \wedge [(A_3 = B_1) \vee (A_3 = B_2) \vee (A_3 = B_3) \vee \ldots]$.

to strengthen and clarify the expressive powers of the home languages and their capacity for rigour. They were never conceived of as showing natural language's intrinsic unsuitability for logic. Rather, it raises the issue of the role of quantifiers and the identity sign in logic, one element of which had already caught Aristotle's attention. Let's call this the *quantifier/identity issue*. As we soon shall see, it is an issue that resurfaces from time to time, sometimes with real impact. It only remains to say that the mediaevals undoubtedly made abundant use of technical terms and neologisms. It is also true that they sometimes trended toward the distinction between the surface grammar of a natural-language construction and its purported depth grammar. But it would not be until Leibniz that the idea of a wholly artificial language for logic would be bruited.

4 Early modernity

Apart from some closing remarks on Leibniz, I'll be mainly concerned in this section with the contributions of Francis Bacon (1561-1626) and Antoine Arnauld (1612-1694). Since logic is our focus here, I'll mention a theme that is common to both men. Each in his own way is a critic of Aristotle's logic, but neither harbours destructive intent. Logic was now trending towards cumulative improvements on what had gone before. I'll not take the time to chronicle the polico-religious travails under which these developments were worked out. Suffice it to say that Bacon and Arnauld were very considerably men of parts.

Bacon was a rebel. He began his struggles with tradition as early as 1603, especially with classical antiquity and renaissance humanism. What he found wanting in these traditions is a comprehensive metascience of all the sciences.[31] Bacon conceded that Aristotle's *Posterior Analytics* had exposed the conditions under which the truths of all the mature sciences would lie open to knowledge in the demonstrative closures of their respective axioms, but he doubted that Aristotle's metascientific apparatus would serve the broader needs of the natural philosophy of Bacon's day. In matters of science, Bacon stresses the importance of trying out new ideas by putting them to experimental test.[32] He regarded as subpar Aristotle's preference for accounting for phenomena by finding their causes. In fact, it was Aristotle who had originated the idea of abductive reasoning in *Prior Analytics* II. 25. It will

[31] I draw here mainly on *The Advancement of Learning* (1605) and *Novum Organum* (1620). Bacon wrote prodigiously, and some of his shrewdest insights into traditional logic are to be found elsewhere. The whole lot can be found in the *Oxford Francis Bacon*, Graham Rees and Lisa Jardine, and Brian Vickers, general editors, Oxford: Oxford University Press, 1996-2011.

[32] In fairness, a dialectical version of submitting new ideas to trial was, as we saw, worked out by Aristotle in Book A of *Posterior Analytics*.

interest abduction scholars to know that Bacon lies closer to Peirce's own approach to it, but Aristotle's lies closer to the inference-to-the-best-explanation approach.[33]

Bacon's logic is thoroughly mentalistic. He considered it a part of rational psychology, whose remit is to expose the laws of thought that underlie all sound judgement and facilitate the detection of fallacy. It repays us to note that the fallacies project was a foundational element in Aristotle's logic, and remained a focus for logicians until logicians turned away to newer things — notably, the glorious turbulence of 19th century mathematics. Bacon organizes fallacies into three different classes. In the first are the sophistical fallacies, which closely resemble in identity and treatment Aristotle's sophistical refutations and paralogisms. In the second category, we find Bacon's fallacies of interpretation, in which errors arise from the misuse of common and general notions. The third grouping is that of "Idols" or false appearances, and it is they for which he reserves his largest effort, not least because they are

> "the deepest fallacies of the human mind: For they do not deceive in particulars, as the others do, by clouding and snaring the judgement; but by a corrupt and ill-ordered predisposition of mind, which as it were perverts and infects all the anticipations of the intellect."[34]

Bacon identifies four sorts of Idol. By Idols of the Tribe, he means that it lies in the very nature of a human being to make inapparent errors of the senses. By Idols of the Cave, he means inapparent errors to which we have been encultured and, by Idols of the Market Place, he means our various propensities to miscommunicate. In the final category lie the Idols of the Theatre, which are prejudices instilled by dogmatic philosophy or by faulty demonstration. Bacon remonstrates with us, bidding us to abjure and renounce our Idol ways, and adds that "the understanding [must be] thoroughly freed and cleansed." (*op. cit.*,69).

Bacon on the fallacies makes common cause with most who've been moved to write about them for the better part of the past two millenia. His adeptness in identifying the blemishes caused by the fallacies greatly exceeds his capacity for dampening down their pre-commission frequency. In this regard, Bacon and virtually all the others miss the vital connection wrought by Aristotle between *Prior Analytics*

[33]For the first, see my "Cognitive economics and the logic of abduction", *Review of Symbolic Logic*, 5 (2012), 148-161, and for the second, see Gilbert H. Harman, "The inference to the best explanation", *Philosophical Review*, 74 (1965), 88-95. For the difference between inference to the best explanation and abduction, see also my "Abduction and inference to the best explanation", in J. Anthony Blair, editor, *Studies in Critical Thinking*, volume 8 of Windsor Studies in Argumentation, pages 405-430, Windsor, ON: WSIA, 2019.

[34]*Advancement of Learning*, in volume IV of *OFB*, 431.

and *On Sophistical Refutations.* What Aristotle nearly brought off was, as we saw, his decision-procedure for the validity of any argument in syllogistic form). No one of intellectual honesty could think that any logician of this period holds a candle to the deductive logician they were trying to improve upon in other ways. This is especially true of Bacon, whose greatness as a revolutionary thinker rests upon his insights into the importance of permitting the sciences of nature grow lest they fall into the stiflement of dogma. In this respect, Bacon's greatest achievement in logic is in the logic of induction, a move of such importance to have led L. J. Cohen to fashion a name for it — *Baconian induction.*[35] I mention this here to give notice of something I'll say a bit later about the Port Royal Logique, to which I now turn.[36]

The threefold manifestations of deductive consequence receive no theoretical advancement in Bacon's logical writings. The reason why is not for efforts that failed, but rather because Bacon had switched the channel to the logic of induction. The Port-Royal Logic casts a wider net. It encompasses, and does so in ways reflected in its subtitle: *l'Art de penser*, the art (not science) of human thinking. By these lights, people who reasons sensibly are artists — masters of the practical — and not theorists. Although it doesn't originate here, the distinction between theoretical and practical reasoning is front and centre, and with an unmistakable preference for the latter as it plays out in the general reaches of human life. The *Logique* doesn't by any means scant scientific thinking. It harbours a notable advance in logical theory which not only throws itself into rivalrous, but unvoiced, opposition to Bacon, but also cuts across the grain of its respect for the practicalities of real-life human reasoning. This, of course, was the birth of the probability calculus.

The *Port-Royal Logic* was the most-used textbook in the whole period from Aristotle to the last flicker of the mid 19th century. The 1818 English edition was the textbook in use at Cambridge and Oxford for a generation. This is a fact of major importance if we are properly to come to terms with the state that logic was in when, at mid-century, mathematics began its efforts, which range from mergers and acquisitions on the cumulative side to hostile takeovers on the destructive side. At the heart of it all lay the question of just what, if anything, logic is that marks the divide from mathematics.

The *Port-Royal Logic* is lodged in the philosophical slipstream of Descartes

[35]L. Jonathan Cohen, *The Probable and the Provable*, Oxford: Clarendon Press, 1977, and "Bayesianism vs. Baconianism in the evaluation of medical diagnosis", *British Journal for the Philosophy of Science*, 33 (1980), 45-62.

[36]Antoine Arnauld and Pierre Nicole, *Logic or the Art of Thinking*, Jill Vance Buroker, translator, Cambridge: Cambridge University Press, 1996; first published under the title *La Logique ou l'art de penser* in 1662. Nicole (1625-1695) is the junior author.

(1596-1650), whose theory of knowledge is absorbed virtually unchanged.[37] In matters of logic, it is four-square with Aristotle's logic of categorical propositions and mediaeval term logic, onto which the cartesian theory of judgements is grafted (although not easily). In plainer words, the graft didn't take. There is neither space nor need for further details here. It suffices to say that the difficulty of grafting new theories onto old is itself as old as the hills (as the common expression has it). It is the standing problem for the cumulative improvement of received opinion, of making an old theory better without wrecking it.

As with Bacon, the fallacies loom large in the Port-Royal Logic under the name of *sophisms*. They are plenteous in number, with a grand total of 27. Details can be found in my File of Fallacies entry "Antoine Arnauld (1612-1694)".[38] Once again, we see the rates of identification greatly outpacing the measures of avoidance. All the same, there are points of real interest in the Royalist treatment of fallacies, made so in part, by its contrast with what I'll soon say about the *Logic's* treatment of probabilistic reasoning. First, since the reasonings of everyday life do not and need not aspire to standards of scientific rigour, people cannot be faulted for their failure to fulfill them. Secondly, when ordinary reasoning goes wrong, it will typically be for reasons different from those that afflict scientific reasoning. Thirdly, whereas scientific reasoning in the older tradition is the orderly demonstrative presentation of what is already known, the reasoning of ordinary life is more of an attempt to discover truth, which is the very thing that Bacon saw missing in Aristotle (and Descartes, too). There is a further respect in which Bacon and Arnauld seem to agree. Although Bacon thought that all branches of science lie open to a covering metascience, he did not think that it could be axiomatized in a way that would have won Descartes' approval. Arnaud, despite his attachment to Descartes' epistemology, has reservations about whether the particularities of human reasoning can adequately be grasped by an overarching scientific theory: from which, again, we see the point of the subtitle of the Logic — the *art* of thinking. Certainly he would have agreed that the art of thinking cannot be captured axiomatically.

I turn now to a widely held inclination of Royalist scholars to see the probability sections of the *Logic* as anonymously Pascal's own. If this is so, it matters in a way that matters to this day. It marks one-half of the rise of what is known as Bayesian probability, the theory that clashes significantly with Baconian probability.[39] It is the first time that mathematics made constructive inroad to any sector of

[37]Arnauld, however, is best known for the fourth set of Objections to Descartes' *Meditations* (1641).

[38]*Argumentation*, 14 (2000), 31-43.

[39]In the course of his correspondence with Pierre de Fermat (1607-1665), Blaise Pascal (1623-1662) recognized that states of uncertainty can be quantified using probability and expectations. In

logic. It marks the early beginning of a trend in logic towards the mathematical. It also matters in another way. In present-day inductive logic, the Bayesian influence is dominant, but its inference rules are implementable only by "rationally ideal" reasoners.

I said earlier that prior to the end of the first half of the 19th century mathematics made no footfall in logic, with the exception of work by Pascal and Leibniz. This is quite clear in the case of Pascal; real numbers are working elements of probability theory. With Leibniz it would be more strictly true to say that he introduced into logic a concept-notation system that bore some striking similarity to the one used by Frege, a *Begriffsschrift* built largely from mathematical concepts. The point on which they converge is the manifest unsuitability of natural languages for the heavy demands of exact thought. Leibniz had introduced the idea of a *calculus ratiocinator*, which anticipates the idea of leak-proof algorithmic proof-making. Also attributed to him was a *characteristica universalis*, which carries the idea of a fully expressive language for thought, a mode of representation that laid bare the internal conceptual interlinkages of its propositions. The expression *characteristica universals* is not to be found in Leibniz's writings but similar expressions are used there.[40] In the opening pages of *Begriffsschrift*, Frege says that, although he is following Leibniz's example, he has two reservations about it. One is that the Leibniz set-up is too ambitious. The other is that, while it held water on the *calculus rationator* side, it was wanting on the "*lingua universalis*" side. If one's language doesn't fully expose the internal conceptual make-up of its thoughts,[41] the good to be wrought from air-tight algorithmic proof rules can only be compromised. In particular, Leibniz's

the early 1760s, Thomas Bayes (1702-1761) proposed that learning can be represented probabilistically using the theorem that now bears his name. "These ideas serve as the basis for all Bayesian thought", according to James M. Joyce, "The development of subjective Bayesianism", in Dov M. Gabbay, Stephan Hartmann and John Woods, editors, *Inductive Logic*, volume 10 of Gabbay and Woods, editors, *Handbook of the History of Logic*, pages 415-475, Amsterdam: North-Holland, 2011. P. 415.

[40] Leibniz speaks variously of "lingua generalis", "lingua universalis", "lingua rationalis" and "lingua philosophica", and used "characterica" to name his general theory of signs. (Volker Peckhaus, "Schröder's logic" in Dov M. Gabbay and John Woods, editors, *The Rise of Modern Logic: From Leibniz to Frege*, volume 3 of Gabbay and Woods, editors, *Handbook of the History of Logic*, pages 557-609, Amsterdam: North-Holland, 2004. P. 599, n 57) Peckhaus adds that "Frege obviously took the term 'lingua characterisca' from Adolf Friedrick Adolf Trendelenburg who uses the expression 'lingua characterica universalis' ..."

[41] Everyone recognizes the peculiarities of Frege's ideography and favours the more "user-friendly" Shröder-Peirce notation or variations of Peano's. For Frege, however, user-friendliness was not the issue. What mattered utterly was the capacity of his notation-system to give full and conceptually fine-grained expression not only to each logical fact, but to the totality of them each in the relation to the others in the overall structure of logical reality.

language has a subject-predicate grammar, and Frege thought that any language thus structured was disabled for exact thought.

It is now time to move to the 19th century and closer to the trending breach. As we make our way into the century's first half, logic retains its naturalistic character. It remains an agent-centred, resource-bound, interactive time-and-action theory of human inference and argument in natural language settings. Indeed, with Bacon, it took on an expressly psychologistic character, and with the publication of Mill's *A System of Logic: A System of Logic, Ratiocinative and Inductive: Being a Connected View of the Principles of Evidence and the Methods of Scientific Investigation*, it took on an aggressively naturalistic form, with efforts to interpret normative matters as matters as they *normally* occur under normal conditions.[42] Mill's *Logic* displaced the Port-Royal *Logique* as the textbook of choice in Britain and elsewhere for most of the half-century to come. Also significant is De Morgan's *Essay on Probabilities* (1838).[43] It is to De Morgan (1806-1871), by the way, that we owe the principle of mathematical induction. It is unmistakable that *before* mid-century logic was trending in two opposite directions: towards the more empirically natural and concurrently towards the mathematical.

5 The nineteenth century

The extraordinary thing about traditional logic is that it lasted, more or less intact, for the better part of two millenia. With the exceptions of plane geometry and arithmetic, surely no other scientific theory even approximates to such venerability. In 1800 Kant (1724-1804) was prompted to observe, "Logic, by the way, has not gained much content since Aristotle's times and indeed it cannot, due to its nature. But it may well gain in exactness, definiteness and distinctness."[44] We may mark the

[42]This anticipates the NN convergence thesis of *Errors of Reasoning*, according to which, in matters of premiss-conclusion reasoning, the default position is that reasoning is normatively secure to the extent to which it approximates to reasoning as it normally happens. (p. 53-56) Mill (1806-1873) got into rather silly and undeserved trouble for making a similar move in ethics, when he claimed that the best reason for thinking that something is desirable is that it is desired by all. The best edition of Mill's *Logic* are volumes VII and VIII of the *Collected Works of John Stuart Mill*, edited by J. M. Robson, with an Introduction by R. F. McRae, Toronto: University of Toronto Press, 1973 and 1974 respectively. For Mill on the fallacies, readers could consult my "John Stuart Mill (1806-1873)" for the File of Fallacies in *Argumentation*, 13 (1999), 317-334.

[43]Augustus De Morgan, *An Essay on Probabilities, and Their Applications to Contingencies and Insurance Companies*, London: Longmans, Brown, Green and Longmans, 1838.

[44]Immanuel Kant, *Logic*, translated with an introduction by Robert S. Hartman and Wolfgang Schwartz, New York: Dover, 1974; p. 23. Emphases are Kant's. First published, in German, in 1800.

"official" break from that centuries-long tradition with Boole's attempt to reduce the syllogistic to algebra.[45] Were Kant to have had a greater familiarity with the richness of the logical Middle Ages, he could not have said that logic hadn't changed significantly since Aristotle's time. Even so, given the nature of the changes that lay just ahead, his remark had a certain prescience. It is perhaps not difficult to see why Boole (1815-1904) and the other algebraicists would have looked to mathematics for assistance in supplying the exactness, definiteness and distinctness that Kant had hoped for. However, Boole's intentions were more structurally ambitious, rather in the way that the suppostionists' intentions had been centuries earlier. The algebraicists would reduce syllogistic logic to something quite close to the algebra which later would bear Boole's name.[46] This was, in fully expressed form, the launch of mathematicism in logical theory. We should therefore note the extent to which, just thirty years later, Frege would engineer a volte-face, by switching the Boolean provisions for logic in mathematics to his own provisions for mathematics in logic. Frege's answer to mathematicism was logicism, the reduction of mathematics to logic. It would be a logic that had to await Frege's creation of it — a second-order functional calculus. Whatever their differences with Boole, Frege (and Dedekind too) were four-square with Kant on the necessity for exactness, definiteness and distinctness. Kant criticized their absence in logic. Frege found them missing in mathematics. One might think that he harboured doubts about, say, the role of geometry in 19th century analysis, as Dedekind certainly did. But he directed his fire elsewhere. He trained his guns on *school-boy arithmetic*.[47]

We come now to a crucial departure from the logical norm, the abandonment of natural language constructions as the bearers and relata of the provisions of formalized logics of deduction. This alone was a significant step away from logic's prior respect for the natural. In 1876, Richard Dedekind (1831-1916) wrote to his friend Lipschitz that, in contexts such as his theory of the irrationals,

> "[a]ll technical expressions [of a mathematical system are to be] replaced by arbitrary newly invented words; the edifice [= structure] must, if

[45]George Boole, *The Mathematical Analysis of Logic: Being an Essay Towards a Calculus of Deductive Reasoning*, Cambridge: Macmillan, Barclay & Macmillan; London: George Bell, 1847. Reprinted with an introduction by John Stater, Bristol: Thoemmes, 1998; "The calculus of logic", *Cambridge and Dublin Mathematical Journal*, 3 (1848), 183-198; and *An Investigation of the Laws of Thought on which are Founded the Mathematical Theories of Logic and Probabilities*, London: Walton and Maberly, 1854; New York: Dover, 1958.

[46]Theodore Hailperin, "Boole's algebra isn't Boolean", *Mathematics Magazine*, 54 (1984), 172-184.

[47]So did Dedekind in *Was sind und was sollen die Zahlen*? Brunswick: Vieweg, 1888. Translated as "The nature and meaning of numbers" in *Essays on the Theory of Numbers*, W. W. Beman, editor, pages 3-115, New York: Dover, 1963.

rightly constructed, not collapse."[48]

As already remarked, thirteen years later Frege writes approvingly of Leibniz's "perhaps overrated" *calculus philosophicus* or *ratiocinator*. Speaking of his own two-dimensional ideography, Frege marks its strangeness saying that

> "[t]hese deviations from what is traditional find their justification in the fact that logic hitherto always followed ordinary language and grammar too closely. In particular, I believe that the replacement of the concepts *subject* and *predicate* by *argument* and *function*, respectively, will stand the test of time."[49]

Nine years after that, Charles Peirce (1839-1914) picked up this theme. In his Cambridge conference lectures of 1898, he proposed that

> "[i]t is true that propositions must be expressed somehow; and for this reason formal logic, in order to disentangle itself completely from linguistic, or psychical, considerations, invents an artificial language of its own, of perfectly regular formation, and declines to consider any proposition under any form of statement than in that artificial language."[50]

Peirce goes on to say:

> "As for the business of translating from ordinary speech into precise forms, ... that is a matter of *applied logic* if you will. (p. 145; emphasis mine)

A page earlier, Peirce had said that his proposal

> "... is that logic, in the strict sense of that term, has nothing to do with how you think .... " (p. 143)[51]

[48]Richard Dedekind, "Briefe an Lipschitz (1-2)", in Dedekind, p. 7, *Gesammelte Mathematische Werke*, volumes 1-3, R. Friske, E. Noethen, and Ö. Ore, editors, Braunschweig: Vieweg, 1930-32. Reprinted in New York by Chelsea Publications in 1969. The wording is clumsy. It is better to read it as saying that if the structure is sound, it cannot collapse upon the arbitrary replacement of technical terms with nonsensical ones.

[49]Gottlob Frege, *Begriffsschrift, a formula language, modeled upon that of arithmetic, for pure thought*, in van Heijenoort; 7. Emphases are Frege's. First published, in German in 1879.

[50]C. S. Peirce, *Reasoning and the Logic of Things: The Cambridge Conference Lectures of 1898*, Kenneth Laine Ketner, editor, with an introduction by Ketner and Hilary Putnam, Cambridge, MA: Harvard University Press, 1992; pp. 144-145.

[51]This is misleading. "The mathematician *practices* deduction (2. 532; 4. 239; 4. 124; 4. 42), reasons deductively, whereas logic studies deductive reasonings and arguments. According to

The mathematical turn in logic is a revolution in at least three ways. It deposed nonmathematical natural language as the language of record for logical theory; it restocked logic's operating system with mathematical tools; and it subjected logic to a hostile take-over by algebra. In Frege's case, all prior ties would be cast aside and an entirely new thing was built from the ground up, and given the name of logic. I say again that the move to abstract artificial languages marks a critical rupture of a centuries-old tradition, and it is preceded by no demonstration of the incapacities of home languages to take the measure of its own successes and failures in the making and assessment of inference and argument.

In time, the following picture emerged. Artificialists sided with the new mathematics by insisting on new notations to carry the load of original concepts. But technical notation is one thing and wholesale language-abandonment another. Technical terms are usually purpose-built neologisms to be added to the home lexicon. Neo-languages that displace natural ones aren't that at all. Their lexicons are replete with *neosemanticisms*, in which common words — "language", "sentence", "predicate", [52] "interpretation", "semantics"[53] — are appropriated and given meanings they've never had before. These are the meanings wrought by outright theoretical stipulation. Special treatment is reserved for the natural language predicates "true" and "false" which, at home, are *predicates* of truth-evaluable sentences. In artificial settings, they are displaced by the undefined *objects* the True (*das Wahre*) and the False (*das Falsche*), which are values of functions, carrying none of the meanings of "true" or "false".[54]

a dictum of his father, Peirce characterizes mathematics as 'the science which draws necessary conclusions' (3. 588; 4. 229); logic, by contrast, is the science of drawing necessary conclusions."' (Claudine Engel-Turcelin, "Peirce's semiotic version of the semantic tradition in formal logic", in Neil Cooper and Pascal Engel, editors, *New Inquiries into Meaning and Truth*, pages 187-213, New York: St. Martin's Press, 1991.) Mathematics, for Peirce, has no intrinsic subject-matter. It is the practice of reasoning necessarily about all manner of things. The practice is as widely spread as the consequence relation is instantiated in human thought and speech. Logic, in turn, is the metatheory of the consequence relation. Moreover, formal logic "is nothing but mathematics applied to logic." (4. 263) We may draw from this the conclusion that drawing necessary conclusions about the science of drawing necessary conclusions evinces something in the way of a bootstrapping challenge.

[52]Albeit that Frege's predicates are functors.

[53]John P. Burgess, "Tarski's tort", in Burgess, *Mathematics, Models and Modality: Selected Philosophical Essays*, pages 149-168, New York: Cambridge University Press, 2008; paperback in 2011.

[54]There is no doubt that Frege aspired to establish a close connection between the property of being true and the relation of denoting the object *das Wahre*. In his attempt to explain how judgements that denote *das Wahre* express a fact of logic, Frege tried to preserve the commonplace that any sentence expressing a logical fact expresses a truth of logic, hence is itself a true proposition. Michael Hallett provides a sobering assessment of this effort in "Frege and Hilbert", in Michael Potter and Tom Ricketts, editors, *The Cambridge Companion to Frege*, ages 413-464, Cambridge:

It can rightly be said that logically artificial languages denature logic itself. To some — Frege, for example — this could only be good news. For others — Quine, for example — the alienating costs of denaturing might be somewhat offset by finding ways of re-importing artificially attained insights into the home language. In this way, logic's theorems might be matched to advantage to counterpart sentences at home.[55] We have it then that artificial "languages" stand in *formal representability relations* to significant fragments of the home language. They are relations under which artificial "sentences" of an artificial "language" stand in a one-to-one correspondence with selected sentences of the home language, thanks to which certain properties of the former are *backwards reflected* upon the latter — the property of logical truth, for example. It is a relationship that is easily misunderstood. For present purposes, it is enough to say that "logically true" in artificial settings does not mean what it means in home languages. Let $\mathcal{L}$ be a first-order artificial "language" and $\mathcal{N}$ any natural language of one's choice. It is widely accepted that, in $\mathcal{N}$, the sentential predicate "true" gathers at least some of its meaning from Convention T, according to which, for example, "Snow is white" is true just in case snow is white. The English predicate "true" does have an occurrence in the metatheory of $\mathcal{L}$. An artificial "sentence" Φ is true in an interpretation $\mathbf{I}$ iff every countably infinite sequence of objects in $\mathbf{I}$'s domain satisfies Φ. Any $\mathbf{I}$ in which Φ is true is said to be a model for Φ.[56] This is not the place to overwork this point. It is enough to note that the map that takes sentences of $\mathcal{N}$ to "sentences" of $\mathcal{L}$ is said to disclose the formers' *logical form* in $\mathcal{L}$. This leaves the suggestion that a sentence S of $\mathcal{N}$ is a *formal* logical truth just in case its image in $\mathcal{L}$ is a logical truth of $\mathcal{L}$. In fact, however it is no such thing. The sentence "Any red shirt is coloured" is logically true in N but not formally so in $\mathcal{L}$ (and not logically true there either). Similarly, "The shirt is red" logically implies "The shirt is coloured" notwithstanding that its logical form in $\mathcal{L}$ fails. Formal validity will reflect validity backwards into $\mathcal{N}$, but formal invalidity won't. Similarly, formal inconsistency reflects backwards in $\mathcal{N}$, but formal consistency doesn't. It bears on this that there is no empirically backed reason to suppose that S's own logical form in L is one and the same with its logical form *in* $\mathcal{N}$. This gives rise to a central question about the utility of formalized logics (as we may call them now) as assessment manuals for logical reasoning in our mother

Cambridge University Press, 2010; see especially section I.4 "Explicit definition and referential fixity".

[55]W. V. Quine, *Methods of Logic*, New York: Holt, 1950.

[56] As anyone reading these pages will know, an "interpretation" of an $\mathcal{L}$ is a set-theoretic structure defining abstract relations over abstract items of constructions from $\mathcal{L}$ and abstract constructions from $\mathcal{L}$'s domain of discourse, which is an infinitely large set of arbitrarily selected abstract items called "individuals". The details matter, but we needn't dwell on them here. We should also note that mass terms such as "snow" have no counterparts in $\mathcal{L}$.

tongues.[57]

As anyone will know who has taught the "translation" rules that map selected sentences of English to counterpart "sentences" in $\mathcal{L}$, there is little in the way of student-resistance, despite the tort of calling these mappings to $\mathcal{L}$ "translations" of sentences of English. Good teachers will point out that the English inputs for mapping to the atomic "sentences" of $\mathcal{L}$ must themselves contain no subordinate sentence and must also be logically independent of one another, not just formally so, but also in the semantic sense. Such helpful admonitions aren't routinely accompanied by express instruction. No one is told whether "She ran to the store" falls out of the input box by virtue of the occurrence within of "She ran". No one is told how to recognize sentential entailments and inconsistencies in the home language. There is a reason for this. These matters are known implicitly. We are all proficient logicians as a matter of course. The moral to draw is that we have to be good natural logicians to implement the translation manual for first-order logic. So whatever the merits of the latter, it cannot be to it that we owe our practical command of entailment, inconsistency and sentence-simplicity.

This might be an appropriate place to issue a caveat about the artificiers' naïve underestimation of the powers of natural speech. There isn't time to litigate the matter properly, but it will only take a minute to record my sympathy with the following view:

> "Projects in artificial intelligence developed large systems based on complex versions of logic, yet these systems are fragile and limited in comparison to the robust and immensely expressive languages. Formal logics are too inflexible to be the foundation for language; instead, logic and ontology are abstractions from language. This reversal turns many theories about language upside down, and it has profound implications for the design of automated systems for reasoning and language understanding."[58]

Let me now touch on a fuss within the "symbolic logic" community about the *core*

[57] We might note, by the way, that for all its distrust of home languages for their ambiguities, logicians of the "formal" persuasion make liberal use of a word that is at least eight-wise ambiguous. See here Catarina Dutilh Novaes, "The different ways in which logic is (said to be) formal", *History and Philosophy of Logic*, 32 (2011), 303-332. I have found ten varieties mentioned in her piece: formality as pertaining to forms; as schematic form; as variability; as indifference to particulars; as abstracted from content; as topic-neutral; as abstracted from meaning; as computability; as pertaining to regulative rules; as pertaining to constitutive rules.

[58] John F. Sowa, "The role of logic and ontology in language and reasoning", in Roberto Poli & Johanna Seibt, editors, *Theory and Applications of Ontology: Philosophical Perspectives*, pages 231-263, Berlin: Springer, 2014.

concept of logic and about its relationship, if any, with mathematics. In one of the testier exchanges, W. S. Jevons (1835-1882) jousted with John Venn (1834-1923) in embracing Boole's insistence that logic is equational and that, being so, it draws its strength from the cornerstone of mathematics — the relation of *equality*. On the other side of this debate were Peirce, Ernest Schröder (1841-1902), Hugh MacColl (1837-1909) and Frege, all of whom favoured the *implication relation* as the central concept of logic.[59] In the main, the British favoured equationalism, but its stoutest critic was MacColl, the early founder of modern modal logic. He was as aware as anyone else of the logical presence of implication since the Stoics. MacColl also resisted interpreting symbols as denoting classes and plumped for treating them as *statements*.[60] In time, Louis Couturat (1868-1914), in France, came to MacColl's defence, as did Bertrand Russell (1872-1970) in England. As we see, however, the identity/implication fight crisscrosses the divide between logic and mathematics.

The identity/implication issue is one half of the quantifier/identity issue, the issue of whether subject-predicate term logic can be rewritten as quantifier-identity symbolic logic. An affirmative answer is presaged in Frege's determination to supplant subject-predicate structures with argument-function structures.

An even more intense fight was for the soul of logic itself. Thomas Spencer Baynes (1823-1887) resisted the Boolean takeover, arguing that the

> "notion of extending the sphere of mathematics so as to include logic, is as theoretically absurd as its realization is practically impossible. To identify logic with mathematics is to make the whole equal to its part."[61]

Oddly enough, Jevons and Venn shared this worry, never mind their adoption of the Boolean notation for reasons of expository convenience.

Perhaps the strongest critic of the mathematical takeover was the aforementioned Wykeham Professor of Logic John Cook Wilson. Here is the flavour of his objections:

> "[S]ymbolic logic as such consists of a solution of particular problems, which are on the same plane as the solution of geometrical or algebraic problems, though concerned with the abstract forms of subject and predicate, as specially scientific as these mathematical processes — no more logic than they are, and related to logic precisely as they are. Incidentally there is a little elementary logic involved, but the real and serious

[59]Actually, Frege was focused on logical truth. But a logical truth can be characterized as a proposition logically implied by every proposition.

[60]We saw earlier adumbrations of this tension in our mediaeval section above.

[61]T. S. Baynes, *An Essay on the New Analytic of Logical Forms*, Edinburgh: Sutherland and Knox, 1850; p. 152.

> problems of logic proper do not appear, nor is the symbolic logician able to touch them. In comparison with the serious business of logic proper, the occupations of the symbolic logician are merely trivial."[62]

It would be wrong to think that Cook Wilson had no grasp of the mathematics of his time or to deem him unacquainted with the details of his Boolean rivals. Although his view persisted at Oxford until well into the 20th century, he had a steadfast dissident in the mathematician we all know as Lewis Carroll.[63]

A final example, which I'll touch on only slightly, is the debate between the two Frenchmen Henri Poincaré and Couturat (and also later Russell, the Englishman). I have only the space to say that Poincaré (1854-1912) took what we now call logicism to be an unnecessary drag on the autonomy of mathematics.[64] He writes,

> "However it be, Logistik must be refashioned, and it is not known how much of it can be saved. It is unnecessary to add that it is Cantorism and Logistik alone that are in that question. The true mathematics, the mathematics that is of some use, may continue to develop according to its own principles, taking no heed of the tempests that rage without, and step by step it will pursue it wonted conquests, which are decisive and have never to be abandoned."[65]

Before closing this section, it remains to say something about logicism. Frege was fighting a two-front war, one against logicians and the other against mathematicians. The knock against logicians such as Mill[66] and Benno Erdman (1851-1921)[67]

[62] John Cook Wilson, *Statement and Inference, with Other Philosophical Papers*, two volumes, Oxford: Clarendon Press, 1926; p. 637; posthumously published. Cook Wilson died in 1915. For modern reservations about the mathematical takeover, see for example, Hartley Slater, *Logic is not Mathematics*, volume 35 of Studies in Logic, London: College Publications, 2011.

[63] This the *nom de plume* of the Oxford mathematician Charles Lutwidge Dodgson (1838-1898), whose "What the Tortoise said to Achilles" (*Mind*, 1895) laid out the case for the *premissory* ineligibility of inference rules in logic.

[64] In these closing paragraphs of the present section, I have drawn on Amirouche Moktefi, "The social shaping of modern logic", in *Natural Argument: A Tribute to John Woods*, Dov M. Gabbay, Lorenzo Magnani, Ahti-Veikko Pietarinen and Woosuk Park, editors, volume 40 in the Tribute series, pages 503-520, London: College Publications, 2019.

[65] Henri Poincaré, *Science and Method*, London: Thomas Nelson, 1914; p. 189. Note that the word "logistik" is not a natural word of English, unlike "logistics" for example. Its referent is any system of mathematical logic contrived for the purpose of advancing the goals of what would be called "logicism" by the likes of Fraenkel and Carnap. For a penetrating discussion of Poincaré's attitude towards logicism, readers could consult Warren Goldfarb, "Poincaré against the logicists", in W. Aspray and P. Kitcher, editors, *Minnesota Studies in the Philosophy of Science*, 11 (1988), 61-81.

[66] *Grundlagen* §§27-27, 29-31 and 46.

[67] Gottlob Frege, *Grundgesetze der Arithmetik: Begriffschriftlich abgeleitet*, Jena: Herman Pohle,

was that they had polluted their respective logics with psychologistic and empirical considerations. He spoke hopefully of a renaissance in logic without troubling to cite the logic to which the subject might aspire to be reborn as. The knock against Frege's fellow mathematicians was their proclivity to rely on gappy proofs, that is to say, proofs in which some lines are assumed rather than expressly flagged and justified; some so-called axioms are actually provable; and many definitions are wanting in rigour. Logic now sought to collapse itself into a wholly re-engineered rescue-theory for mathematics. The founder of logicism had changed the subject even before he had produced the new subject that the name of logic would now name — a second-order functional calculus with axioms for sets.

Notwithstanding that the term "logicism" made no appearance in English until the close of the 1920s, several years after Frege had been thought to be the founder of its nominatum, and Russell, too, from 1903 and after, we have it now that logicism is

> "the approach to the philosophy of mathematics pioneered by Frege and Russell. According to logicism the truths of mathematics are logical truths, deducible by logical laws from basic logical axioms."[68]

Every 19th century mathematician was aware of Kant's epistemology of mathematics. Although there were predecessor distinctions, two played a key role for Kant. One is the distinction between knowledge *a priori* and *a posteriori*. The other is the divide between analytic truth and synthetic truth. A distinctive feature of Kant's treatment of mathematics is his subscription to the view that, although all the known theorems of arithmetic are known *a priori*, none is analytic. His question in the first *Critique* (1781) was to determine how *a priori* knowledge of synthetic truths is possible. We should note that Kant's conception of the analytic is information-containment notion, on which a statement is analytic just in case there is no information carried by its predicate that isn't contained in its subject.

1893 and 1903. Edited and translated into English as *Basic Laws of Arithmetic*, also edited by Philip A. Ebert and Marcus Rossberg, with Crispin Wright, Oxford: Oxford University Press, 2013; Foreword to the Introduction. Frege doesn't include the phenomenologist Edmund Husserl (1859-1938) in these denunciations, whose *On the Concept of Number* appeared in 1887, and was reworked as *Philosophie der Arithmetik* in 1891. His massive *Logische Untersuchungen* (1900-1913) partly overlapped the *Grundgesteze* (1893-1903). *Logische Untersuchungen*, was translated by J. N. Findlay for *Logical Investigations*, New York: Humanities Press, 1979. Husserl doesn't get much play in English-speaking philosophy of logic and mathematics. More's the pity, since he would be seen to have a substantial influence on keeping logic a naturalized science.

[68]Simon Blackburn, *Oxford Dictionary of Philosophy*, 2nd edition, Oxford: Oxford University Press, 2005; p. 215. A corrective is administered by Woods and Rosales (forthcoming), from *LogicismÕs Ashes: The Rise of Mathematical Philosophy.*

Since they contain no sentences in subject-predicate form, it is not clear how the migration of this notion to the 19th axioms for numbers could be achieved. Although the question is important, I shan't tarry with it here. In a way, it doesn't matter. While Frege adopted Kant's language, he actually collapsed the difference between analyticity and apriority. In *Grundlagen* (1884), Frege argues that to determine whether a proposition is analytic or synthetic, it is necessary that we examine its proof and determine whether it flows from "primitive truths", that is, from principles that neither need nor are susceptible of independent proof. If it turns out the primitive truths are wholly general laws which also validate the proof's definitions, then the proposition is analytic.[69] In almost the same words Frege says that, under these same conditions, the proposition is *a priori*.[70] The earlier idea that analyticity is a logical property and apriority an epistemological one now vanishes. Moreover, since a primitive truth is a law of logic just in case it is universally applicable irrespective of discipline, it cannot itself be considered either analytic or *a priori*. So conceived of, a law of logic cannot be proved, hence cannot fulfill the conditions on analyticity or apriority, one of which is that it be proved from logical laws. Unless this is simply a hiccough on Frege's part, we see that proofs from logical laws aren't analyticity-preserving. Again, it hardly matters. The same is true of Aristotle's demonstrative logic. Demonstration from first principles is truth-preserving and knowledge-producing; but it is not self-evidence preserving, something that is caught by the commonplace distinction between axioms and theorems. What Frege wants is some means of showing that any *theorem* proved from the laws is analytic and *a priori*. The laws, we may say, can take care of themselves. Their infallibility is intrinsic. They are self-evidently true. (But how so?)

With the birth of mathematical logic, logic is now mathematics, as Boole had earlier proposed. The attempt to prove every theorem of arithmetic in logic is widely regarded as the whole objective of logicism. Everyone knows that Frege's attempt failed because one of his axioms for sets — Basic Law V — implies the contradiction that Russell spotted in 1902. It is scarcely recognized at all that Frege's project fails for more fundamental reasons. For Frege to succeed, all the axioms of *Grundgesetze* must be content-free mathematical analyticities, and its proof rules must likewise be universally valid. Dedekind accepts and meets the second requirement. I doubt that he held fast to the first. Frege accepted both requirements, indeed insisted on them. Whatever our assessment of how he handled the second one, it is clear that he failed the first one. The first one cannot be met by any axiom system rich enough

[69]Frege: *Die Grundlagen der Arithmetik: Eine logische-mathematische Unter suchungen den Begriff der Zahl*, Breslau: Wilhelm Koebner, 1884. Translated by J. L. Austin as *Foundations of Arithmetic*, Oxford: Blackwell, 1950.

[70]*Ibid*, p. 4.

for arithmetic. Had Dedekind held fast with the first one, he too would have been met with this comeuppance. Several of the most fundamental laws, including all the laws for sets, cannot be re-expressed as content-indifferent universal validities.[71] The joining of quantification theory to set theory[72] was a marriage of convenience. It was a marriage doomed to fail. In due course, set theory would migrate to where it had all along belonged, and would become a flourishing branch of mathematics. Even so, quantification theory *itself* remains a theory built from thoroughly mathematical materials — variables, quantifiers, mathematical induction, recursive definitions, set-theoretic models, countably infinite series, one-to-one correspondences, and so on.

It remains only to show how the quantifier/identity issue now fares in the programme of showing the truths of arithmetic to be logical truths, made so by their deducibility by logical laws from basic logical axioms. The programme, writes Blackburn,

> "started with Frege's brilliant demonstration that the elementary truths of counting . . . can be formalized using only quantifiers and identity." (op. cit. p. 1, 215)[73]

Thus x is a natural number if and only if x falls under the concept of *not being self-identical*, or otherwise is any element of that concept's predecessor-series. It suffices to note that quantifiers are needed for the definition of the predecessor relation.

In one of its meanings, a formal treatment of a subject-matter defines properties of interest over schemata of natural-language constructions. In the Barbara-schema

[71]This is not a slip on Frege's part. It was he who emphasized the necessary presence of contentful terms in his logic — e.g. the concept of set-membership. On the other hand, if analyticity is to be what Dedekind took it to be, namely, universal validity independently of subject-matter, it can't be said that Frege has shown the theorems of his logic to be analytic. Nor can relief be found in returning to the older sense of analyticity, according to which a sentence is analytic just in case it is true by the meaning of its logical terms. As mentioned in an earlier footnote existence-asserting sentences don't fulfill this condition.

[72]For Boole, Russell, Dedekind and others, sets were classes. Frege eschewed sets for courses of values of concepts: "I have replaced the expression 'set' which is frequently used by mathematicians, with the expression customary in logic: 'Concepts"'. It is a false contrast. Although concepts play a role in the Laws of Thought tradition in 19th century logic, concepts play an equally embedded role in 19th century "conceptual" mathematics in the manner of Riemann. See Frege, "On formal theories of arithmetic" in Brian McGuiness, editor, *Collected Papers on Mathematics, Logic and Philosophy*, pages 112-121, Oxford: Oxford University Press, 1984; p. 112. Originally published in 1885, in German.

[73]Blackburn's characterization of the logicist programme omits mention of logical definitions of key concepts, for which Frege demanded both eliminability and conservatism. By the first requirement that defining term is intersubstitutable *salve veritate* with the to-be-defined term. By the second, definitions cannot be simply made-up.

"⟨All A are B, All B are C, All A are C⟩", A, B and C are place-holders for general terms of Greek. Any instantiation of the schema got by the uniform replacement of the placement by general terms of Greek is a syllogism. Whatever else we might think of them, the place holders of *Barbara* are not open to quantifier-binding.[74] Formalization by representation is a wholly artificial notation incapable of carrying sentential meaning is another thing entirely. It lies at two removes from what formal logic was subject to a hostile takeover and which Boole would play the mathematicism card. The year 1879 marks the second remove. Formal logic would now be transacted by measures set out in the *Begriffschrift*. By that stage, Frege's logic stood to traditional logic as Riemann's geometry stood to Euclid's, namely as radically different in kind: with the new geometry and the new logic, being geometry and logic in name only. By the 1920s the remove was finalized when Frege's second-order provisions went down-market to the first-order.[75] Then and now logic is the preserve of stipulated artificialities. The thirst for such measures over-lie the tripartite character of consequence-manifestation.[76] The effect overall is a further layer of difficulty. While the threefold distinctions remain, the system's artificialities occlude our understanding of their relata.

6 The 20th century

When we move from the old languages to the new ones, we move from the natural to the artificial and lose sight of the subject-matter at hand. Perhaps the point on which the two most differ pivots on the fact that in myriad ways artifice outreaches the potentialities of the natural. One can make artifacts do what nature would have no part of. A marked illustration of this difference lies in the ease with which the central concepts of logic proliferate. The dominant reality of modern logic is pluralism. Given the conditions under which it so easily comes about, it is hardly surprising that ambiguities on the scale evinced in pluralism are very hard to discern in nature; that

[74]"For all A, B, C, ⟨all A are B, all B are C, all A are C⟩" is ill-formed in English (and Greek).

[75] Like the name "quantifier", the term "first-order logic" was coined by Peirce. For well over a hundred years, first-order logics have been the logic of choice in analytically minded departments of philosophy. First-order logic is the strongest system that satisfies compactness, completeness and the downward Löwenheim-Skolem theorem. However, its theorems are not computationally enumerable. (Alonzo Church "The Richard paradox", *The American Mathematical Monthly*, 41 (1934), 356-361.) First-order logic also lacks a full command of mathematical induction. Stewart Shapiro makes a strong case for second-order logic in non-foundationalist approaches to the philosophy of mathematics. *Foundations Without Foundationalism: A Case for Second-Order Logic*, New York: Oxford University Press, 1991.

[76]In broad strokes, model theory more or less does for consequence-having theory more or less does for spotting more or less. Neither is much good for drawing, notwithstanding their efforts.

is, in the natural languages within which these concepts first rose. Pluralism, even at its most moderately venturesome, is further evidence of the modern denaturing of logic.[77] This, when juxtaposed with the absence of formalized representation proofs that pass on the fruits of artifice to the home languages. It is alienation on a scale that defies purchase in any epistemologically realistic provisions for the spottings and drawings of human life.

So, then, it can be said that 20th century logic stands to logic as 19th century mathematics stands to mathematics. In each case, a teeming prosperity of technical virtuosity and conceptual stipulation occasioned rising tides of tumult and uncertainty.[78] In the Preface to the first edition of *Methods of Logic*, Quine (1908-2000) penned the dismissive quip that logic is an old discipline, and since 1879 has been a great one. At mid-century mathematical logic had both trimmed down, set some boundaries, and yet expanded in other ways. Logic would largely shed its second-order origins and "simplify" itself to first-order. Finitary methods in proof theory had also taken root, ensuing from Hilbert (1862-1943) and developed by Gentzen (1909-1945) and others. Gentzen would give us natural deduction logic (natural in name only) and, thanks to the efforts of Gödel (1906-1978) and Tarski (1901-1983), model-theory would flower, and important limitation theorems would be proved.[79] Tarski would set the standard for the formal semantics of natural languages,[80] and C. I. Lewis (1883-1964) would axiomatize modal propositional logics in his systems of strict implication S1 to S5. Completeness and soundness results would be proved or disproved in variations of propositional and quantificational logics, increasing the motivation to recognize the system-relativities of pluralistic logics, including its most nihilistic expression in Carnap (1891-1970).[81] Many-valued logics had taken firm root.[82] Intuitionist logics would respond to the call of constructive mathematics.[83]

[77]JC Beall and Greg Restall, *Logical Pluralism*, Oxford: Clarendon Press, 2006. If Beall's and Restall's pluralism can be considered moderate, Carnap's can only be called rather mad. Rudolf Carnap, *The Logical Syntax of Language*, London: Routledge & Kegan Paul, 1937.

[78]The lovely phrase "teeming prosperity" is Quine's, which I've plucked from another context.

[79]Alfred Tarski, "Contributions to the theory of models I", *Indagationes Mathematicae*, 16 (1954), 572-581.

[80]C. I. Lewis and C. H. Langford, A Survey of Symbolic Logic, New York: Dover, 1959; originally published in 1932.

[81]Rudolf Carnap, *Philosophy and Logical Syntax*, London: Routledge & Kegan Paul 1935.

[82]Jan Łukasiewicz, "Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalklüs", *Comptes rendus des séance de la Société des Sciences et des Lettres de Varsovie Cl. III*, 23 (1930). Translated in Storrs McCall editor, *Polish Logic 1920-1939*, pages 40-65, Oxford: Clarendon Press, 1967.

[83] Kurt Gödel, "Eine Interpretation des intuitionistischen Aussagenkalküls", *Ergebnisse eines mathematisches Kolloquiums* 4 (1933) 39-40; translated in John Dawson, Solomon Feferman et al., editors, *Collected Works of Kurt Gödel, vol. 1, Publications 1929-1936*, pages 300-303, Oxford:

In 1950, with the publication of Carnap's *The Logical Foundations of Probability*,[84] the probability calculus would make mature landfall in inductive logic.

In the half-century to follow, modal logic would take on quantificational form, and would prove completeness and soundness results.[85] Modal logics would extend their reach to doxastic, epistemic, time and tense, and deontic frameworks. In the latter 1950s, Saul Kripke (1940-) provided the first really powerful semantics for the alethic modalities;[86] and concurrently relevant logic would start to stir in New Haven and Pittsburgh, soon taking root in the fertile soil of Oceania. In short order, paraconsistent systems would be up and running, and dialethic logic would soon appear.[87] Dialogue and interrogative logics would reappear in formalized theories and some of them would adapt themselves to game theory.[88] More generally, formal dialogue logics would be in full fettle.[89] Meanwhile, from Turing onwards, computer science would start the quest for a machine that's worth talking to, and would make inroads in philosophy of mind and enter into partnerships with psychology and the sciences of cognition. Nonmonotonic, defeasibility and default logics and their allied variations had hit their stride by the 1970s and 1980s.[90] Dynamic and justificationist logics were also finding their form.[91] Model-theoretic semantics in the manner of Tarski had engineered a dominating influence on philosophical theories of natural-language truth and meaning.[92] Running through virtually all these

Oxford University Press, 1986.

[84]Chicago: University of Chicago Press, 1950.

[85]E. L. Post, op. cit, and Kurt Gödel, "Die Vollstandigkeit der Axoms des logischen Funktionkalküls", *Mondatshefte for Mathematik und Physik*, 37 (1930), 349-360; Translated in van Heijenoort (1967), pages 582-591.

[86]Saul A. Kripke, "A completeness theorem in modal logic", *Journal of Symbolic Logic*, 24 (1959), 1-14.

[87]See Dov M. Gabbay and John Woods, editors, *Logic and the Modalities in the Twentieth Century*, vol. 7 of Gabbay and Woods, editors, *Handbook of the History of Logic*, Amsterdam: North-Holland, 2006. The usual spelling is "dialetheic", an imagined combination of the Greek *di* and *alethia*. In combining them in this way, neither Greek nor English requires the terminal 'e', no more than they require of "*aletheia*".

[88]Jaakko Hintikka, *The Principles of Mathematics Revisited*, Cambridge: Cambridge University Press, 1996.

[89]E. M. Barth and Erik C. W. Krabbe, *From Axiom to Dialogue: A Philosophical Study of Logic and Argumentation*, Berlin and New York: de Gruyter, 1982.

[90]E.g. Raymond Reiter, "On closed world bases", *Journal of Logic and Data Bases*, (1978), 55-76, and "A Logic for default reasoning", *Artificial Intelligence*, 12 (1980), 81-132.

[91]E.g., Johan van Benthem, "Dynamic logic for belief revision", *Journal of Applied Non-classical Logics*, 14 (2004), 1-26. See also his *Logic in Games*, Cambridge, MA: MIT Press, 2014.

[92]Alfred Tarski, "The concept of truth in formalized languages", in his *Logic, Semantics, Metamathematics: Papers from 1923-1938*, J. H. Woodger, translator, Oxford: Oxford University Press; second edition, John Corcoran, editor, pages 152-278, Indianapolis: Hackett, 1983.

variations in logic's fortunes, the methods of advancement are thoroughly mathematical. Inferential semantics in a form given it by Dag Prawitz (1936-) and later by Michael Dummett (1925-2011) had made its way to Pittsburgh and was brought to interesting form by Robert Brandom (1950-) and others.[93] Plausibility logics were developed by Nicholas Rescher (1928-) and others,[94] and the logic of fiction took formal shape in 1969 and 1974.[95] Meanwhile, logic was being put to work in various parts of psychology and other parts of cognitive science.[96] In physics, logics were purpose-built to accommodate the Boolean failures of quantum mechanics, and Hilary Putnam would put it about that logic might be an empirical science.[97] The idea that logic might be empirical would be taken up later, notably and more expansively by E. M. Barth (1928-2015) in the Netherlands and Maurice Finocchiaro (1942-) in the United States.[98]

In a 1970 paper, Gilbert Harman nailed two deep errors in the logical theories of the day.[99] The one is that the conditions on consequence-having (or entailment) can be reissued as rules of truth-preserving consequence-drawing (or inference). The other is that the calculation-rules of the probability calculus are those whose satisfaction ensure the inductive strength of reasoning and argument. Taking the last case first, consider a police investigation of a capital crime. At this stage, the evidence is strongly against suspect X but not sufficient to proceed to trial. It is now

[93] E. g., Dag Prawitz, *Natural Deduction: A Proof-Theoretical Study*, Stockholm: Almqvist Wiksell, 1965, and "Remarks on some approaches to the concept of logical consequence", *Synthese*, 62 (1985), 153-171; and Robert Brandom, *Making it Explicit*, Cambridge, MA: Harvard University Press, 1994.

[94] Nicholas Rescher, *Plausible Reasoning: An Introduction to the Theory and Practice of Plausible Inference*, Assen: Van Gorcum, 1976.

[95] John Woods, "Fictionality and the logic of relations", *Southern Journal of Philosophy* 7 (1969), 51-63, and *The Logic of Fiction: A Sounding of Deviant Logic*; 2nd edition with a Foreword by Nicholas Griffin, volume 23 of Studies in Logic, London: College Publications, 2009. First published by Mouton in 1974.

[96] P. N. Johnson-Laird, *Mental Models: Towards a Cognitive Science of Language, Inference and Consciousness*, Cambridge, MA: Harvard University Press, 1983. See also Daniel Kahneman, Paul Slovic and Amos Tversky, *Judgement Under Uncertainty: Heuristics and Biases*, Cambridge: Cambridge University Press, 1982. See also Francesco Berto, Anthia Solaki and Sonja Smets, "The logic of fast and slow thinking", *Erkenntnis*, DOI 10.1007/s10670-019-00128-z.

[97] Hilary Putnam, "The logic of quantum mechanics", in his *Mathematics, Matter and Method: Philosophical Papers, vol. 1*, pages 174-197, Cambridge: Cambridge University Press, 1975.

[98] E. M. Barth, "A new field: Empirical logic", *Synthese*, 63 (1985), 375-388, and Maurice Finocchiaro, "Methodological problems in empirical logic", *Communication and Cognition*, 22 (1989), 313-335.

[99] Gilbert Harman, "Induction: A discussion of the relevance of the theory of knowledge to the theory of induction", in Marshall Swain, editor, *Induction, Acceptance and Rational Belief*, pages 83-99, Dordrecht: Reidel, 1970.

the next morning, and overnight twenty new items of information have been logged. To upgrade the log in accordance with the update requirements of the Bayesian probability calculus would require ≈ one million calculations. Had thirty new items of evidence arrived, ≈ one billion calculations would have been required. It is at this juncture that we run headlong into a conflict about the *normative* conditions on good reasoning. One option is that the probability rules are descriptively accurate for the reasoning of ideally rational agents, and while descriptively inaccurate for us, are nevertheless normatively binding on us. Accordingly, the police in this case are handling the evidence correctly precisely to the extent that their update-calculations approximate to the provisions of Bayes' theorem. Whatever plausibility such a move might initially have had, it runs into two difficulties. One is that no one to date has successfully defined an approximation relation on real-life reasoning behaviour and the reasoning ascribed to the ideal reasoner. A second and related difficulty is that, to date no one has been able to defend against the objection that idealized reasoner-mongering is motivated solely by the boost it gives to the behaviour-distorting authority of mathematical equations.

In the first of the two Harman cases, we see the folly of a rule which prescribes that if one has $p \to q$ is in one's belief-box, then the arrival of information carrying p, either mandates or licenses that q be believed. In fact, other options present themselves under these same conditions. One option would be to infer q. Another would be to retain $p \to q$ and reject p. Yet another would be to retain p and reject q and thereby reject $p \to q$. A further purported condition on rational belief-update is that a rational agent is one who (which?) closes its belief under consequence. Since any belief has a minimum of ω deductive consequences, there is no finite degree to which the most perfectly possible human reasoner can approximate to this ideal. Difficulties such as these persuade Harman that the standard logics of deduction and the dominant logics of induction are catastrophically unfit to regulate deductive and inductive reasoning in the real-world circumstances of human life. At the heart of it all is this question;

- *When a theory and empirical data conflict, what is the rational thing to do? Save the theory? Or save the data?*[100]

The fact is that there is no general sure-fire, one-size-fits-all answer to this question. But in the present case, there is compelling reason to save the data. If we opted

[100]It would be wrong to leave the suggestion that Bayesianism is the only game in town for inductive logicians. For important but inequivalent alternatives, see among others, the ranking theory of Wolfgang Spohn in *The Laws of Belief: Ranking Theory and its Philosophical Applications*, New York: Oxford University Press, 2012 and the severe-test approach of Deborah G. Mayo, *Statistical Inference as Severe Testing: How to Get Beyond the Statistics Wars*, New York: Cambridge University Press, 2018. Both these approaches are heavily mathematicised.

otherwise, we'd saddle the logic of natural-language reasoning with a massive dose of big-box scepticism, according to which we humans are, in matters of reasoning and decision, colossal dim-wits.[101] None of this accords with the known facts of survival, evolutionary prosperity and, from time to time, the construction of the English common law and the cathedrals of mediaeval France. In 1970, first-order mathematical logic called the shots for deductive reasoning in $\mathcal{N}$ and the applied mathematics of probability would rule the waves of inductive reasoning. On Harman's telling, there is a common fault-line. It is (in my words) the failure to give adequate heed to the distinction between consequence-having and consequence-drawing, never mind whether the consequence relation in question be deductively structured or inductively so. Mathematical logicians haven't been unmindful of these failures, and have sought their repair in formal systems of considerably greater complexity than one finds in first-order environments. This is the way of what I call "heavy equipment technologies", in which the missing components of first-order systems — agents, goals, resources, time, action, social conventions are supplied in idealized form.[102]

If I say so myself, these are wonderful pieces of intellectual architecture, admirable for their mathematical beauty which, until I have cause to know better, is my sole interest in them. But on the applicational side, I have two things to say against them. One is the various ways in which they advance and sanction empirical falsehoods. The other is the failure convincingly to ground the presumption that the empirical falsehoods are redeemed by their purported normative authority.[103]

[101] Or to be fair, *somewhat* rational but not so hot reasoners.

[102] For example, in his dynamic epistemic logic, van Benthem calls into play categorical grammars, relational algebras, cognitive programming languages for information transfer, modal logic, the dynamic logic of programs, whereby insights are achieved (or purported) for process invariances and definability, dynamic inference and computational complexity logics. In the heavy equipment technologies for attack-and-defend networks (ADN), developed by Howard Barringer, Dov Gabbay and the present author, here too we find many moving parts — from unconscious neural nets to adjustments for various kinds of conscious reasoning. The ADN paradigm picks up along the way a number of other technical ideas currently in mathematical play — some of them pertaining to equational algebraic analyses of connection strength, where stability can be achieved by way of Brouwer's fixed point theorem. And so on. See Johan van Benthem, *Logical Dynamics of Information and Interaction*, New York: Cambridge University Press, 2011. See Howard Barringer, Dov M. Gabbay and John Woods, "Temporal argumentation networks", *Argument and Computation*, 2-3 (2012), 143-202, and "Modal argumentation networks", *op. cit.*, 203-227. There are many more efforts of note to bring theories of reasoning within recognizable reach of real-life reasoning — public announcement logics, knowledge-representation theories, to name just two. But none evades the observation that when we junk up theories of human performance with heavy-equipment mathematical virtuosities, we get theorems alright, but they lack recognizable presence in the lived realities of human cognitive life.

[103] I draw some of these remarks from my "The fragility of argument", in Fabio Paglieri, Laura Bonelli and Silvia Felletti, editors, *The Property of Argument: Cognitive Approaches to Argumen-*

To this I might add: "Might we have your formal representation theorems, please?"

Upon a little reflection, the idea of proving and inferring as rule-governed activities is difficult to square with empirically discernible facts of human cognitive behaviour. Let's pause with this a brief while. All going well, spotting and drawing facilitate the acquisition of knowledge and belief, and, in so doing, lay a foundation for decision and action. By their very natures, spotting and drawing are goal-involved. It would be unrealistic, however, to suppose that every goal-involved response to a consequence is goal-*directed*. Sometimes the human agent will spot a consequence or draw an inference without consciously aligning it with any particular goal. Sometimes the agent will align with a goal unconsciously. On some occasions the spotting or drawing will be wholly unaligned. Of course, as every logician knows, every truth-evaluable sentence or set of sentences has at least a countable infinity of consequences.[104] This has a considerable bearing on spotting and drawing. If there were a grand rule for such things, it could only be this:

> *Rule*: In the general case, accurate but unfettered spotting and drawing would be a waste of time. Do not resort to them without adequate cause.
>
> *Corollary I*: Being told that S' would be a *valid* and *sound* inference to draw gives no advice about whether to draw it. Being told that S' would be an *invalid* inference to draw would tell you not to draw it only if you aspire to validity.
>
> *Corollary II*: The proof rules of a *standardly formulated* logicistic system are scarcely worth the paper they are written on, except when goal- or interest- related.

Although entirely on track, our Rule scants its own more widely-spread and causally puissant counterpart provision, in which beings like us spot and draw inferences automatically and involuntarily, when causally induced to in the course of processing information at hand. Take Peano arithmetic as an example. Our Rule suggests that the Peano proof rules would be useless if either the theorem-prover lacked the goal-alignment knack or his causal-alignment mechanisms failed him. If so, we might find him drawing "1 is a natural number or Nice is nice in November" as a theorem of arithmetic. To vary the example, consider a close approximation to *Principia Mathematica*'s propositional logic. It has four axioms and three proof rules — substitutivity, modus ponens and conjunction-introduction. The system is

tation and Persuasion, volume 59 in Studies in Logic, pages 99-128, London: College Publications, 2016.

[104]Thanks in part to the consequences repeatedly obtained by double negation, conjunction- and disjunction-introduction.

known to be consistent, sound and complete. This tells us something. To excavate the whole demonstrative content of the *PM* axioms, no proof rule need be invoked other than these meagre three and no inference drawn save as licensed by them. To be sure, for every p and q in the $\mathcal{L}$ of *PM*, there is a modus ponens structure. Every q is *implied* in *PM* but is not in the output class of its proof rules. This tells us that the PM proof rules have been chosen with a particular goal in mind and that the goal is based on a certain belief. The belief is that the whole *theorematic* content of the *PM* propositional axioms can be got by repeated application of those three rules to (outputs of) its axioms. Of course, if the rules actually encompassed that restriction, they would be admissible but not valid. But not even this can save the day in the absence of the means to determine when a consequence of a theorem is itself a theorem, not just an implication. This is precisely what the proof rules don't provide. Enter now the no-terms-from-the-outside condition. Enter now relevance.

Call the spotting of a consequence *opportune* when it is relevant to an agent's interests, even interests undeclared and implicitly held. Call the drawing of a consequence *prudent* when the drawn belief is true and relevant to the drawer's interest. As we now see, both opportunity and prudence are subject to relevance constraints. We would expect it to be so in in relevant logics.[105] We might not expect this to be so in classical or intuitionist logics. But it is so there. This tells us that conditions on the entailment relation radically underdetermine opportune spotting and prudent drawing. The first truly *original* idea in proof theory was Aristotle's determination to exclude terms from the outside from syllogistic proofs and axiomatic demonstrations. That he felt able to impose the constraint without the need of lodging it in an analysis of the consequence relation itself is a solid indication that it doesn't belong there. Any proof theory worthy of the name will advance decently sized theories of the opportune and the prudent. None that is currently on offer fulfills this condition. It is not an easy condition to meet. And it takes but a moment's reflection to see that terms from the outside aren't the only source of irrelevance. It takes a moment after that to recognize that, by and large, beings like us don't have much of a clear idea of the myriad forms in which irrelevance can intrude or distract. The reason for this, I surmise, is that the proper management of irrelevance is subject to largely automatic and unconscious *filtration mechanism*.[106] Should this be so, we

[105]A small sample: A. D. Anderson and N. D. Belnap, Jr., *Entailment: The Logic of Relevance and Necessity*, volume 1, Princeton: Princeton University Press, 1975; Richard Routley, Val Plumwood, Robert K. Meyer and Ross Brady, *Relevant Lgoics and Their Rivals*, Atascadero: Ridgeview, 1984; and Stephen Read, *Relevant Logic,* Oxford: Blackwell, 1988. Among the classical rules declared invalid, disjunctive syllogism ranks high. In some cases, the deduction theorem is false if *ex falso* is true. Other relevantists find against the conjunction rules, and others against the transposition rule: These, as we now see, are slim pickings.

[106]Dov Gabbay and I have written about the management of irrelevance in *Agenda Relevance*

could easily see the fix that proof theory finds itself in.

The present point casts a long shadow, and yet hangs an obscuring cloud over relevant and paraconsistent logic. The misconception at the heart of each is the false belief that because *ex falso* licenses the inference of everything whatever, it must be denied a place among logic's theorems. However, as we now see, the grief over *ex falso* is not properly motivated. It is not a rule of inference, and to be so it need not be taken to mischaracterize entailment. We have seen that paraconsistent logics in their several variations have two things in common. One is that they block *ex falso* the classical theorem that says that any contradictory sentence has every sentence whatever as a deductive consequence.[107] The other is that, paraconsistent systems tolerate at least one instance of inconsistency. The general idea is to keep matters from getting worse by reconceptualizing the relation of consequence-having. What these logics overlook is the matter under current discussion. Not everything logically implied by a sentence is inferrable from it. It helps to keep firmly in mind that beings like us are knowledge-seeking processors of unceasing information flow. Like all natural objects capable of change, there are limits. With action-oriented beings, information is processed in knowledge-seeking ways, subject to the natural limits on cognitive resources — limits, not failures. Because at any given t there is more information at hand than the most adept of us can use to advantage, information-processing is therefore structured and shaped by the processor's cognitive interests at t and the time and other resources available to him for their satisfaction. Since human cognition is agenda-based, it only stands to reason that information irrelevant to its advancement not make its way into his working epistemic capital. These structural features bear directly on the mechanics of consequence-spotting. Again, every truth-evaluable sentence of English at a minimum has a denumerable infinity of deductive consequences, massively many of them of arbitrary length. Most of this plenitude exceeds the spotting capacity of the human individual, never mind the particularities of his cognitive agenda at t. Even when a consequence is spottable in principle, for the most part it won't be spotted. It wouldn't advance his agenda to spot it. (This was the message of "1 is a natural number or Nice is nice in November". It is a spottable consequence of its first disjunct, but it does not advance the spotter's number-theoretic agenda.) So I say again that for human cognition to

(2003). For a causal treatment of inference, readers could consult chapter 3 of *Errors of Reasoning* (2013).

[107]Coinage of the name is unknown. As we saw, the mediaevals had an accurate name for it: *ex contradictione quodlibet*. It was proved by Alexander Nekham of the School of Cologne in the year 1200. It also follows directly from the standard definition of logical consequence. The proof independently recurs in C. I. Lewis & C. H. Langford, *Symbolic Logic*, New York: Appleton Century-Croft, 1932; reprinted in New York by Dover, 1959: $\langle (p. \sim p), p, p \vee q, \sim p, q \rangle$ by simplification, addition, simplification again, and disjunctive syllogism in this order (p. 252).

work at all, spotting must be subject to powerful cut-down mechanisms, all the more so when the agenda is to acquire new knowledge. Most of what a knowledge-seeker spots at t will play not role in belief he's induced to have at t. Further cut-down mechanisms are required for drawing. The moral to draw is that Harman's case against the inferential legitimacy of modus ponens is an understatement. What is more, had the purveyors of these logics attended to what actually happens when information harbours an inconsistency, they would have seen that the limitations of the kind proposed for consequence-having are already and largely automatically at work in the domain of consequence-drawing. Human inference is paraconsistent, root and branch, and is so independently of the provisions of consequence-having or entailment. I say paraconsistent rather than relevant because, on strong empirical evidence, human beings store memories and background information in inconsistent quantities. Upon retrieval, however, the rising subsets tend to be consistent. But consider now a theory T whose inconsistency is not known, and suppose, as I do, that *ex falso* is true. Then we have the following interesting situation. To every theorem of T there corresponds a sentence that T implies and which contradicts it. Wouldn't this show that even a modestly inconsistent theory is a massive dialethia? We have already seen that the unwelcome negations fail to qualify as T's theorems notwithstanding their being implied there. We have also had occasion to reflect on the "same respect" clause of Aristotle's definition of the Law of Contradiction at *Metaphysics* 1005b 19-20. This gives us room, if we wanted to take it, to rescue T from dialethic saturation. Let S be a true sentence that is also a theorem of T. Its negation "not-S" will be a false sentence that T also implies. "Not-S" is not a theorem, and neither is "S and not-S". "Not-S" does not qualify as a theoretical disclosure of T.

What we have here is a nice example of not paying attention to the tripartite character of consequence-manifestation. Consider again the case of Frege. Prior to the revelation of the paradox, this was a consequence which Frege failed to spot. Does anyone really think that, on that very account, there was nothing his students in Jena could have learned about sets ($\approx$ value-ranges of concepts)? Does anyone really think that every statement whatever was a theorem of his set theory? Post-paradox Frege gave up on logicism entirely by 1906. But the reason for doing so was not that everything is now a theorem of his system. He gave up because he thought that reference of the concept value-ranges of concepts could not be consistently fixed with final assurance.

7 Restoring logic to its natural home

We come back now to the utility of formalization question. It brings us to a very substantial answer in the negative, which arose in the latter 1970s from the conviction of logic teachers that, beyond highly simplified fragments of linguistic practice, the fruits of formalizing natural language inference and argument ranged from bitter to poisonous. John Burgess was moved to say, somewhat noncommittally perhaps, that on the traditional "view of the subject, the phrase 'formal logic' is pleonasm and 'informal logic' oxymoron."[108] It was concomitantly believed the going formalisms of the day succeeded in the main for types of reasoning that hardly happened in the conditions of human cognitive life, mindful of the plain fact that most of a human being's sound reasoning is deductively invalid. In the informal logic research communities, probably still dogs the tail of ampliative reasoning. There is less inclination to disavow the probability calculus than the first-order predicate calculus. The informal logic movement has spawned an enormous literature — another of those teeming posperities of mainly rival conjecture — and there is no possibility here of widescreen coverage. I'll confine myself, to some important representative works from the beginning until now.[109] It is convenient and largely right to date modern logic's return to the natural with the publication in 1970 of Charles Hamblin's *Fallacies*.[110] It is not only a more developed work than Harman's paper on inference that same year, but it has had a larger impact. Hamblin, a formal logician and pioneering computer scientist, rebuked his fellow logicians for having given up on logic's fallacy project, and called for vigorous action to restore it to health.

There is in the informal logic community little express push to return logic to its naturalistic home, save for its general reluctance to call upon the notation and

[108] John P. Burgess, *Philosophical Logic*, Princeton: Princeton University Press, 2009; p. 2.

[109] J. A. Blair and R. H. Johnson, editors, "Recent developments of informal logic", in *Informal Logic: The First International Symposium*, pages 3-28, Inverness, CA: Edgepress, R. H. Johnson and J. A. Blair, editors, *New Essays in Informal Logic*, Windsor, ON: *Informal Logic*, 1994; James B. Freeman, *Acceptable Premises: An Epistemic Approach to an Informal Logic Problem*, New York: Cambridge University Press, 2005. David Hitchcock, and Bart Verheij, editors, *Arguing on the Toulmin Model*, Dordrecht: Springer, 2006. J. Anthony Blair and Ralph H. Johnson, editors, *Conductive Argument: An Overlooked Type of Defeasible Reasoning*, volume 33 of Studies in Logic, London: College Publications, 2011; J. Anthony Blair, *Groundwork in the Theory of Argumentation Selected Papers of J. Anthony Blair*, Dordrecht: Springer, 2012; David Hitchcock, *On Reasoning and Argument: Essays in Informal Logic and On Critical Thinking*, Cham, Switzerland: Springer, 2018. J. Anthony Blair, editor, *Studies in Critical Thinking*, volume 8 of Windsor Studies in Argumentation, Windsor, ON: WSIA, 2019. The organisational and chief residential research centre for informal logic is the Centre for Research in Reasoning, Argumentation and Rhetoric, in Ontario's University of Windsor, and the journal of record is *Informal Logic*.

[110] London: Methuen.

methods of mathematical knowledge. Although informal logicians invest heavily in the pragmatic dimension of language, there is little in the way of organized projects to unify informal logic with naturalized epistemology and the empirical sciences of cognition.[111] Partly this is because the informal logic "movement" has never had a manifesto.[112] It would also appear that their epistemological instincts run to the justified true belief model of knowledge and Bayesian theories of justification.

Engaging the practical is not the sole preserve of informal logicians. Many of them, derived their interest from the pragmatic dimension of language — language-in-use, so to speak, which is all it was ever good for. It wasn't typical of these logicians to enrich their understanding of how languages work by harkening to what can be learned from empirical linguistics. This, I think, has been a regrettable omission.[113] We should also note that the pragmatic dimension of language is not the sole route to the practical. Something deeper and richer can be got from *cognitizing* our enquiries into inference and argument. This is done by aligning one's logical interests with materially relevant disclosures of cognitive science and naturalized epistemology. Two figures of importance for these alliances are Dov Gabbay and Lorenzo Magnani.[114]

[111]As already mentioned, there are exceptions. One is Else Barth (*op. cit.*), and another is Maurice Finocchiaro (*op. cit.*). See also E. M. Barth and E. C. W. Krabbe, *From Axiom to Dialogue*, Berlin: de Gruyter, 1982, and Maurice Finocchiaro, Meta-argumentation: *An Approach to Logic and Argumentation Theory*, volume 42 of Studies in Logic, London: College Publications, 2013.

[112]Unlike the manifestos that periodically dot the pragma-dialectic landscape, as the School of Amsterdam weaves its steady and well-received way. The organizational site and chief residential research centre for the pragma-dialectical framework for critical conversations is the Group on Discourse Analysis, Argumentation and Rhetoric at the University of Amsterdam. The journal of record is Argumentation. Although deriving some initial inspiration from Barth and other logicians, pragma-dialecticians aren't logicians and show only limited interest in empirical considerations. We can safely say, then, that they aren't in the uptake draught of naturalistic renewal. The pragma-dialectical model has had an enormous influence on theories of argument in most of its disciplinary precincts. Of particular importance is F. H. van Eemeren and Rob Grootendorst, *A Systematic Theory of Argumentation: The Pragma-Dialectical Approach*, Cambridge: Cambridge University Press, 2004. See also van Eemeren, Peter Houtlosser and Francesca Snoeck Henkemans, *Argumentative Indicators in Discourse*, Dordrecht: Springer, 2007. On the quasi-empirical side, see van Eemeren, Bart Garsen and Bert Mueffels, *Fallacies and Judgements of Reasonableness: Empirical Research Concerning Pragma-Dialectical*, Dordrecht: Springer, 2009. I say "quasi-" in the belief that such investigations are motivated to some degree by a confirmation bias.

[113]An important example is Gregory N. Carlson and Francis Jeffry Pelletier, editors, *The Generic Book*, Chicago: University of Chicago Press, 1995. A deep lesson to draw from this book is the striking infrequency with which speakers of natural languages frame their generalizations as universally quantified conditionals

[114]Dov M. Gabbay, Ralph H. Johnson, Hans Jürgen Ohlbach and John Woods, editors, *Handbook of the Logic of Argument and Inference: The Turn Towards the Practical*, Amsterdam: North-

There are important exceptions to informal logic's quasi-indifference to naturalizing characteristics in its immediate "pre-history" of the emergence of informal logic. In 1953 Stephen Toulmin published a primer on the philosophy of science in which he took to task standard theories of probability for overwhelming the logic of ampliative reasoning with surplus-to-need mathematicizations of how such reasoning is regulated in real life.[115] In 1958, he followed up with *The Uses of Argument*, in whose chapter II he deepens the case against the over-mathematicization of probability. In chapter IV he takes against the ideal modals approach to logic, and in chapter V he insists on a working partnership between logic and epistemology.[116] Each book caused something of a *scandale*, and I remember that when, on being introduced to a large and excited audience in Ann Arbor, Toulmin was said to be "philosophy's most refuted living practitioner." If naturalized logicians ever sought the succor of a manifesto, they could do no better than selecting these two books as starters. I should also mention a point that I'll return to a bit later. Toulmin rejects mathematics as a profitable model for logic, and urges the English common law as much the superior alternative.

Also of considerable weight was the response to Hamblin's challenge to logicians to restore the fallacies project to its proper place in logic. What Hamblin himself had in mind were formalized versions of mediaeval dialogue logics, especially those of game-theoretic formulation, were much the coming thing in some precincts of mathematical logic, and still are. But a more direct response to Hamblin was what came to be known as the Woods-Walton Approach.[117] In the jointly authored papers from 1972 to 1982, Woods and Walton brought a general theoretical framework

Holland, 2002; Dov M. Gabbay and John Woods, *Agenda Relevance: A Study in Formal Pragmatics*, volume 1 of their A Practical Logic of Cognitive Systems, Amsterdam: North-Holland, 2003; Dov M. Gabbay and John Woods, *The Reach of Abduction: Insight and Trial*, volume 2 of their A Practical Logic of Cognitive Systems, Amsterdam: North-Holland, 2005; Lorenzo Magnani, *Abductive Cognition: The Epistemological and Eco-Cognitive Dimensions of Hypothetical Reasoning*, volume 3 of Cognitive Systems Monographs, Berlin: Springer, 2009; Lorenzo Magnani, *The Abductive Structure of Scientific Creativity: An essay on the Ecology of Cognition*, volume 37 of Studies in Applied Logic, Epistemology and Rational Ethics, Cham, Switzerland: Springer, 2017. See also Woosuk Park, *Abduction in Context: The Conjectural Dynamics of Scientific Reasoning*, volume 32 of Studies in Applied Philosophy, Epistemology and Rational Ethics, Cham, Switzerland: Springer, 2017.

[115]Stephen E. Toulmin, *Philosophy of Science*, London: Hutchins, 1953.

[116]Stephen E. Toulmin, *The Uses of Argument*, Cambridge: Cambridge University Press, 1958. Toulmin's first book has left little trace on the informal logics that followed it. Uses is mentioned by nearly everyone, studied by many fewer, has left little in the way of a structural impact on informal logic.

[117]John Woods and Douglas Walton, *Fallacies: Selected Papers 1972-1982*, 2nd edition, with a Foreword by Dale Jacquette, volume 7 of Studies in Logic, London: College Publications, 2007. First published in Dordrecht by Foris Publications, 1989.

to bear on the sprawling motley that has attracted the name of fallacy. Instead they worked on fallacies one-by-one, bringing to bear considerations, many of which were theoretical, to which they thought the target of an enquiry would be most responsive. If there were a common theme to their approach, it was that the safest home for a given fallacy was likely to be found in an application of some or other already established nonclassical logic — e.g. Kripke's intuitionistic logic for modelling circular reasoning or Tyler Burge's theory of aggregates for the modelling of composition and division. In some quarters of informal logic, the Woods-Walton Approach was resisted for its over-formality, but though I say it myself, the impact of the WWA was substantial, and has long outlasted the time at which is the mid-1980s its originators would move on in separate directions, with Walton assuming more of a pragma-dialectical orientation,[118] and Woods eschewing it entirely.

Aside from the WWA, there has been a good deal written about fallacies by informal logicians and argumentation theorists of all stripes, some of it extremely insightful. And probing.[119] Before centuries end, it was clear that logic had ceased to be the sole preserve of the theory of argument, what with developments in psychology and some of the allied social sciences, and interesting developments in departments of speech communication. For the most part, this work was patterned on the ideal reasoner model, from whence normative authority was mistakenly thought to be derived. There were, however, important exceptions on the empirical side, notably in fields such as conversational analysis.[120]

There is little in the works surveyed here — whether in informal logic, fallacy theory or argumentation theory — that could be considered to have harboured a naturalizing motivation for logic. So I will now turn to a pair of developments, mainly by computer scientists, in which the pulse of naturalism is more easily discerned. The one development ensues from pioneering work in nonmonotonic, defeasible and default logics, and allied ones such as autoepistemic logic. These logics cover what, in *Errors of Reasoning*, I call "third way reasoning", which is neither deductively structured nor responsive to the regulatory controls of the standard or classical

[118]Douglas Walton (1942-2020) published more books on the fallacies — individually analyzed or in small clusters — than can be easily counted. Over time they've come to rely heavily on Walton's methods for computer-modelling argument schemes. See, for example, Walton, C. Reed and F. Macagno, *Argumentation Schemes*, Cambridge: Cambridge University Press, 2008.

[119]See especially, Hans V. Hansen and Robert C. Pinto, editors, *Fallacies: Classical and Contemporary Readings*, University Park: Pennsylvania State University, 1995. See also Trudy Govier, The Philosophy of Argument, Newport News, VA: Vale Press, 2006, and Christopher W. Tindale, *Fallacies and Argument Appraisal*, Cambridge: Cambridge University Press, 2007.

[120]See for example, Sally Jackson, *Message Effects Research: Principles of Design and Analysis,* New York: Guilford, 1992, and Sally Jackson and Scott Jacobs, "The structure of conversational argument: Pragmatic Bases for the Enthymeme", *Quarterly Journal of Speech*, 66 (1980), 251-265.

logics of induction. Since space is our mistress here, I refer the interested reader to chapter 7 of that work, and also to chapter 8 which revisits the central concept of logically following from. Most of these computationally inspired logics are in some significant degree laid out formally, and there is liberal use made of mathematical methods. The reason why, for the most part, is that computer science was invented by mathematicians, many of whom were considered to have gone rogue.[121] The common impulse was not to make new mathematics, but rather to make computers that are worth our while to talk to.

For that to happen, computers had to figure out how to interact with one another, a feat accomplished by Carl Hewitt, inventor of the Actor model of concurrent computation. This marks a significant development. In human life, communicating with one another is robustly conversational. And he is wise to the fact that when humans converse with one another they bear a marked tendency to give conversational voice to differences of opinion. Man has been said to be the rational animal. It can equally be said that he is the arguing animal. So if, as a condition on building computers that are up to present and foreseeable demands on them, it is necessary to enable them to converse with us, they will have to learn to argue. At the present and closely foreseeable stage of computational evolution, Hewitt thinks it unlikely that computers will be simulacra of beings like us. But he does think that we're on track for computers and humans to communicate with one another *conversationally.* For this to happen, software engineers have to know enough about how we ourselves think, reason, converse and argue to enable them to build machines that are capable of conversations with *us.*[122] There is also the growingly vexed question of computational security. No one thinks that computer science could be foundationally secure in the absence of a suitably engineered mathematical logic. My own view is that first-order platforms simply aren't up to that job. Right or wrong, whatever the correct methods for the foundational security of computer science, that alone doesn't begin to answer the question of how human beings bring into efficacious play their cognitive resources in managing their interactions with one another, and with their selective and collective engagements with their environments.

The present volume is focused to research trends in logic. For most of this paper, I've been recounting, as much as I've had time for, logic's research-trend history. There is much talk these days about where research in logic is likely to take us. I have three suggestions in mind.

[121]Ray Reiter to John Woods in conversation in London, c. 1996.

[122]See, for example, "Formalizing common sense reasoning for scalable inconsistency-robust information using Direct Logic$^{\text{TM}}$ reasoning and the Actor model", in Carl Hewitt and John Woods, editors, *Inconsistency Robustness*, volume 52 of Studies in Logic, pages 3-103, London: College Publications, 2015.

- Mathematical logic will try to find ways to stabilize the foundational security of computer science. Of pressing and present concern, it must also provide for its *cybersecurity*.

- Naturalized logic will provide software engineers with what they need to know about how we make our cognitive ways — chiefly our argumentative ways — in the day-to-day conditions of human life, sufficiently so as to enable them to build a machine that's worth talking to and also worth being talked to in return.

To grasp the nettle of this point, it is necessary to understand that what we tell the engineers about our own naturalistic ways does not remotely guarantee or require that computers worth talking to be modelled on us. It is enough that the engineers know enough about how we operate to engineer a machine that's capable of *conversing* with us.

- More generally, in the absence of meaningful investments in productive cross-disciplinary research alliances, the movement of logic to a more naturalized form will sputter and gutter.

Not only must we cite works from partner disciplines, we must read them with all the critical care that we expend on reading ourselves. The best way to acquire standing in a partner discipline is to take an advanced degree in it. One of my friends has advanced degrees in philosophy, linguistics and computer science. Another is similarly accoutred, with credentials in philosophy, psychology and computer science. Perhaps this is a bit of a reach for most of us of naturalistic bent, but it should get us working on re-positioning logic in the curricula of our best — or should I say most aspiring — universities. Of particular importance is that cross-disciplinary traffic be subject to tight visa requirements, since the difficulties which naturalists seek to escape from can sometimes be ensconced in a partner discipline. Proper circumspection should red-flag neo-classical economics, for example, but issue a proceed-with-caution for behavioural economics, for example.

I close with semi-approving remarks on a trend currently underway to seek productive analogies for human reasoning or how matters are handled in law. I regard as his most important insight into the logic of human thinking, Toulmin's proposal to study the ways and means of thinking in the English common law. There are, to be sure, other ways of finding logical fodder in law. An especially brisk line of trade is to computerize schematic structures for argument-making and argument-appraisal.[123] There is much of interest in the readiness of argument-schemes to

[123]See Henry Prakken, *Logical Tools for Modelling Legal Argument*, Dordrecht: Kluwer, 1997;

admit of computer-rendering. This is nothing to make light of, but I am bound to think that much of this interaction misses Toulmin's point. If we take a common law criminal trial as our example, it stands out that nothing that occurs there takes the form of face-to-face argument. Counsel are not permitted to argue with witnesses, nor they with counsel. Counsel can give (brief) reasons for raising an objection, but he or she may not argue about it. Closing arguments are *monologues* that sum up counsel's theory of the evidence. When judges instruct the jury, they aren't arguing with them. They are laying down the law. The only occasion for face-to-face argument is *in camera* in the jury room. Toulmin's interest is much less argumentative than epistemological.[124] That is where the true value of the common law rests for the naturalization of logic, not least in the doctrine of unwritten law or *lex non scripta*. It is, in my respectful submission, a direction in which naturalized logic should trend.[125]

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Praaken And Giovanni Sartor, "A dialectical model of assessing conflicting arguments in legal reasoning", *Artificial Intelligence and Law*, 4 (1996), 331-368; Bart Verheij, "Dialectical argumentation with argumentation schemes: an approach to legal logic", *Artificial Logic and Law* 11 (2003), 1-19; and Douglas Walton and T. E. Gordon "Critical questions in computational models of legal argument", *Argumentation in Artificial Intelligence and Law*, IAAIL Workshop Series, in P. E. Dunne and T. Bench-Capon, editors, pages 103-111, Nijmegan: Wolf Legal Publishers, 2005; and Walton and Sartor, "Teledogical justifications of argumentation schemes", *Argumentation*, 27 (2013), 111-142.

[124] I should also mention Jaakko Hintikka's announcement that abductive inference lay at the centre of epistemology's not-yet solved problems. Much of the attention of naturalized logicians is focused on repairing this omission. See again Woosuk Park [102], Gabbay and Woods [59], Magnani [94, 97], and Woods [147, 148, 151].

[125] See Larry Laudan, *Truth, Error and Criminal Law: An Essay in Legal Epistemology*, New York: Cambridge University Press, 2006; Laudan, *The Law's Flaws: Rethinking Trials and Errors*, volume 3 of Law and Society, London: College Publications, 2016; and John Woods, *Is Legal Reasoning Irrational? An Introduction to the Epistemology of Law*, 2nd edition, revised and expanded, volume 2 of Law and Society, London: College Publications, 2018. First published in 2015. See also my "Evidence, probativity and knowledge: A troubled trio", in Hans V. Hansen, editor, *Proceedings of the 12th Meeting of the Ontario Society for the Study of Argument*, Windsor, ON: WSIA, to appear in 2020.

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References

[1] Peter Abelard (1099-1142), *Dialectica*, L. M. de Rijk, editor, 2nd edition Assen: van Gorcum, 1950.

[2] Alexander of Aphrodisias, *On Aristotle's Prior Analytics* 1-1-7, Jonathan Barnes, Susanne Bobzien, Kevin Flannery and Katerina Ierodiakonou, editors and translators, London: Duckworth, 1991.

[3] Alan Ross Anderson and Nuel D. Belnap, Jr., *Entailment: The Logic of Relevance and Necessity*, volume 1, Princeton: Princeton University Press, 1975.

[4] Julia Annas and Jonathan Barnes, editors, *Sextus Empiricus: Outlines of Scepticism*, 2nd edition, New York: Cambridge University Press, 2000.

[5] Antoine Arnauld and Pierre Nicole, *Logic or the Art of Thinking*, Jill Vance Buroker, translator, Cambridge: Cambridge University Press, 1996; first published under the title *La Logique ou l'art de penser* in 1662.

[6] Antoine Arnauld, the fourth set of Objections to Descartes' *Meditations* (1641). *Argumentation*, 14 (2000), 31-43.

[7] Francis Bacon, *Advancement of Learning*, in volume IV of OFB, 431.

[8] *Oxford Francis Bacon*, Graham Rees and Lisa Jardine, and Brian Vickers, general editors, Oxford: Oxford University Press, 1996-2011.

[9] Yehoshua Bar-Hillel, "Formal logic and natural languages (A Symposium)", J. F. Staal, editor, *Foundations of Language*, 5 (1969) 903-926.

[10] Else M. Barth and Erik C. W. Krabbe, *From Axiom to Dialogue: A Philosophical Study of Logic and Argumentation*, Berlin and New York: de Gruyter, 1982.

[11] Else M. Barth, "A new field: Empirical logic", *Synthese*, 63 (1985), 375-388.

[12] Thomas Spencer Baynes, *An Essay on the New Analytic of Logical Forms*, Edinburgh: Sutherland and Knox, 1850.

[13] JC Beall and Greg Restall, *Logical Pluralism*, Oxford: Clarendon Press, 2006.

[14] Francesco Berto, Anthia Solaki and Sonja Smets, "The logic of fast and slow thinking", *Erkenntnis*, DOI 10.1007/s10670-019-00128-z.

[15] Simon Blackburn, *Oxford Dictionary of Philosophy*, 2nd edition, Oxford: Oxford University Press, 2005.

[16] J. Anthony Blair, editor, *Studies in Critical Thinking*, volume 8 of Windsor Studies in Argumentation, Windsor, ON: WSIA, 2019.

[17] J. Anthony Blair and Ralph H. Johnson, editors, "Recent developments of informal logic", in *Informal Logic: The First International Symposium*, pages 3-28, Inverness, CA: Edgepress, 1977.

[18] —*Conductive Argument: An Overlooked Type of Defeasible Reasoning*, volume 33 of Studies in Logic, London: College Publications, 2011.

[19] J. Anthony Blair, *Groundwork in the Theory of Argumentation Selected Papers of J. Anthony Blair*, Dordrecht: Springer, 2012.

[20] Susanne Bobzien, "Chrysippus' modal logic and its relation to Philo and Diodorius", in Döring and Ebert (1993).

[21] —"Stoic logic", in Keimpe Algra, Jonathan Barnes, Jaap Mansfeld and Malcolm Schofield, editors, *The Cambridge History of Hellenistic Philosophy*, pages 92-157, Cambridge: Cambridge University Press, 1999.

[22] George Boole, *The Mathematical Analysis of Logic: Being an Essay Towards a Calculus of Deductive Reasoning*, Cambridge: Macmillan, Barclay & Macmillan; London: George Bell, 1847. Reprinted with an introduction by John Slater, Bristol: Thoemmes, 1999.

[23] —*An Investigation of the Laws of Thought on which are Founded the Mathematical Theories of Logic and Probabilities*, London: Walton and Maberly, 1854; New York: Dover, 1958.

[24] Robert Brandom, *Making it Explicit*, Cambridge, MA: Harvard University Press, 1994.

[25] John P. Burgess, *Philosophical Logic*, Princeton: Princeton University Press, 2009.

[26] —"Tarski's tort", in Burgess, *Mathematics, Models and Modality: Selected Philosophical Essays*, pages 149-168, New York: Cambridge University Press, 2008; paperback in 2011.

[27] John Buridan, *Tractatus de Consequentiis*, in Herbert Hubien, *Iohannis Buridan tractatus de consequentiis: édition critique*, volume XVI of *Philosophes médiévaux*, Louvain: Université de Louvain, 1971.

[28] Gregory N. Carlson and Francis Jeffry Pelletier, editors, *The Generic Book*, Chicago: University of Chicago Press, 1995.

[29] Rudolf Carnap, *Philosophy and Logical Syntax*, London: Routledge & Kegan Paul 1935.

[30] —*The Logical Syntax of Language*, London: Routledge & Kegan Paul, 1937.

[31] —*The Logical Foundations of Probability*, Chicago: University of Chicago Press, 1950.

[32] Walter Cavini, "Chrysippus on speaking truly and the Liar", in Klaus Döring and Theodor Ebert, editors, *Dialektikes und Stoiker: Zer Logic der Stoiker und ihrer Vorläufer*, Stuttgart: Franz Steiner, 1993.

[33] Salua Chatti, *Arabic Logic from al-Fārārabi to Averroes: A Study of the Early Arabaic Categorical, Modal, and Hypothetical Syllogistic*, Basel: Birkhäuser, 2019.

[34] Alonzo Church, "The Richard paradox", *American Mathematical Monthly*, 41 (1934), 356-361.

[35] Laurence Jonathan Cohen, *The Probable and the Provable*, Oxford: Clarendon Press, 1977, and "Bayesianism vs. Baconianism in the evaluation of medical diagnosis", *British Journal for the Philosophy of Science*, 33 (1980), 45-62.

[36] John Cook Wilson, *Statement and Inference, with Other Philosophical Papers*, two volumes, Oxford: Clarendon Press, 1926. Posthumously published.

[37] John Corcoran, "Completeness of ancient logic", *Journal of Symbolic Logic*, 37 (1972), 696-702.

[38] —"Aristotle's natural deduction system", in John Corcoran, editor, *Ancient Logic and its Modern Interpretation*, pages 85-132, Dordrecht: Reidel, 1974.

[39] —"Aristotle's demonstrative logic", *History and Philosophy of Logic*, 30 (2009), 1-20.

[40] Diodorus Cronus, in *Die Megarikes: Kommentiere Sammlung der Testimonien*, Klaus Döreng, editor, pages 28-45 and 124-139, Amsterdam: Gruener, 1972.

[41] Richard Dedekind, *Was sind und was sollen die Zahlen?* Braunschweig: Vienweg, 1884. Reprinted as "The nature and meaning of numbers", in his *Essays on the Theory of Numbers*, W. W. Benan, translator and editor, New York: Dover; p. 35. Original English publication by the Open Court Press of Chicago in 1901.

[42] —"Briefe an Lipschitz (1-2)", in Dedekind, p. 7, *Gesammelte Mathematische Werke*, volumes 1-3, R. Friske, E. Noethen, and Ö. Ore, editors, Braunschweig: Vieweg, 1930-32. Reprinted in New York by Chelsea Publications in 1969.

[43] Augustus De Morgan, *An Essay on Probabilities, and Their Applications to Contingencies and Insurance Companies*, London: Longmans, Brown, Green and Longmans, 1838.

[44] Catarina Dutilh Novaes, "Logic in the 14th century after Ockham", in Dov M. Gabbay and John Woods, editors, *Mediaeval and Renaissance Logic*, volume 2 of Gabbay and Woods, editors, *Handbook of the History of Logic*, section 3, "Consequences", pages 467-484, Amsterdam: North-Holland, 2008.

[45] —"The different ways in which logic is (said to be) formal", *History and Philosophy of Logic*, 32 (2011), 303-332.

[46] Charles Lutwidge Dodgson, "What the Tortoise said to Achilles," *Mind*, 4 (1895) 278-280.

[47] Claudine Engel-Turcelin, "Peirce's semiotic version of the semantic tradition in formal logic", in Neil Cooper and Pascal Engel, editors, *New Inquiries into Meaning and Truth*, pages 187-213, New York: St. Martin's Press, 1991.

[48] George Englebretsen, *The New Syllogistic*, New York: Lang, 1987.

[49] Maurice Finocchiaro, "Methodological problems in empirical logic", *Communication and Cognition*, 22 (1989), 313-335.

[50] Michael Frede, *Die stoische Logik*, Göttingen: Vandenhoek & Rupert, 1974.

[51] James B. Freeman, *Acceptable Premises: An Epistemic Approach to an Informal Logic Problem*, New York: Cambridge University Press, 2005.

[52] Gottlob Frege: *Die Grundlagen der Arithmetik: Fine logische-mathematische Untersuchungen den Begriff der Zahl*, Breslau: Wilhelm Koebner, 1884. Translated by J. L. Austin as Foundations of Arithmetic, Oxford: Blackwell, 1950.

[53] —*Begriffsschrift, a formula language, modeled upon that of arithmetic, for pure thought,* in Jean van Heijenoort, editor, *From Frege to Gödel: A Source Book in Mathematical Knowledge*, 1879-1931, pages 5-82, Cambridge, MA: Harvard University Press, 1967. First published, in German, in 1879.

[54] —"On formal theories of arithmetic", in Brian McGuiness, editor, *Collected Papers on Mathematics, Logic and Philosophy*, translated from the German, pages 112-121, Oxford: Oxford University Press, 1984.

[55] —*Grundgesetze der Arithmetik: Begriffschriftlich abgeleitet*, Jena: Herman Pohle, 1893 and 1903. Edited and translated into English as *Basic Laws of Arithmetic*, also edited by Philip A. Ebert and Marcus Rossberg, with Crispin Wright, Oxford: Oxford University Press, 2013.

[56] Yiu-ming Fung, editor, *Dao Companion to Chinese Philosophy of Logic*, Cham: Springer, 2020.

[57] Dov M. Gabbay, Ralph H. Johnson, Hans Jürgen Ohlbach and John Woods, editors, *Handbook of the Logic of Argument and Inference: The Turn Towards the Practical*, Amsterdam: North-Holland, 2002.

[58] Dov M. Gabbay and John Woods, *Agenda Relevance: A Study in Formal Pragmatics*, volume 1 of their A Practical Logic of Cognitive Systems, Amsterdam: North-Holland, 2003.

[59] —*The Reach of Abduction: Insight and Trial*, volume 2 of their A Practical Logic of Cognitive Systems, Amsterdam: North-Holland, 2005.

[60] Dov M. Gabbay and John Woods, editors, *Logic and the Modalities in the Twentieth Century*, vol. 7 of Gabbay and Woods, editors, Handbook of the History of Logic, Amsterdam: North-Holland, 2006.

[61] Jonardon Ganeri, editor, *Indian Logic: A Reader*, Milton Park, Oxon: Routledge, 2001.

[62] Kurt Gödel, "Eine Interpretation des intuitionistischen Aussagenkalküls", *Ergebnisse eines mathematisches Kolloquiums* 4 (1933) 39-40; translated in John Dawson, Solomon Feferman et al., editors, Collected Works of Kurt Gödel, vol. 1, *Publications* 1929-1936, pages 300-303, Oxford: Oxford University Press, 1986.

[63] Alvin I. Goldman, "A causal theory of knowing", *Journal of Philosophy*, 64 (1967), 357-372.

[64] Warren Goldfarb, "Poincaré against the logicists", in W. Aspray and P. Kitcher, editors, *Minnesota Studies in the Philosphy of Science*, volume 11, pages 61-81, Minneapolis: University of Minnesota Press, 1988.

[65] Trudy Govier, *The Philosophy of Argument*, Newport News, VA: Vale Press, 2006.

[66] Michael Hallett, "Frege and Hilbert", in Michael Potter and Tom Ricketts, editors, *The Cambridge Companion to Frege*, ages 413-464, Cambridge: Cambridge University Press, 2010.

[67] Charles Leonard Hamblin, *Fallacies*, London: Methuen, 1970.

[68] Hans V. Hansen and Robert C. Pinto, editors, *Fallacies: Classical and Contemporary Readings*, University Park: Pennsylvania State University, 1995.

[69] Gilbert Harman, "The inference to the best explanation", *Philosophical Review*, 74 (1965), 88-95.

[70] —"Induction: A discussion of the relevance of the theory of knowledge to the theory of induction", in Marshall Swain, editor, *Induction, Acceptance and Rational Belief*, pages

83-99, Dordrecht: Reidel, 1970.

[71] —*Change in View: Principles of Reasoning*, Cambridge, MA: MIT Press, 1986.

[72] Carl Hewitt, "Formalizing common sense reasoning for scalable inconsistency-robust information using Direct Logic$^{\text{TM}}$ reasoning and the Actor model", in Carl Hewitt and John Woods, editors, *Inconsistency Robustness*, volume 52 of Studies in Logic, pages 3-103, London: College Publications, 2015.

[73] Jaakko Hintikka, *The Principles of Mathematics Revisited*, Cambridge: Cambridge University Press, 1996.

[74] David Hitchcock, and Bart Verheij, editors, *Arguing on the Toulmin Model*, Dordrecht: Springer, 2006.

[75] David Hitchcock, *On Reasoning and Argument: Essays in Informal Logic and On Critical Thinking*, Cham, Switzerland: Springer, 2018.

[76] Edmund Husserl, *On the Concept of Number*, 1887, reworked as *Philosophie der Arithmetik*, 1891. *Logische Untersuchungen* (1900-1913), translated by J. N. Findlay, as *Logical Investigations*, New York: Humanities Press, 1979.

[77] Sally Jackson, *Message Effects Research: Principles of Design and Analysis*, New York: Guilford, 1992.

[78] Sally Jackson and Scott Jacobs, "The structure of conversational argument: Pragmatic Bases for the Enthymeme", *Quarterly Journal of Speech*, 66 (1980), 251-265.

[79] Ralph H. Johnson and J. Anthony Blair, editors, *New Essays in Informal Logic*, Windsor, ON: Informal Logic, 1994.

[80] Philip N. Johnson-Laird, *Mental Models: Towards a Cognitive Science of Language, Inference and Consciousness*, Cambridge, MA: Harvard University Press, 1983.

[81] James M. Joyce, "The development of subjective bayesianism", in Dov M. Gabbay, Stephan Hartmann and John Woods, editors, *Inductive Logic*, volume 10 of Gabbay and Woods, editors, *Handbook of the History of Logic*, pages 415-475, Amsterdam: North-Holland, 2011.

[82] Daniel Kahneman, Paul Slovic and Amos Tversky, *Judgement Under Uncertainty: Heuristics and Biases*, Cambridge: Cambridge University Press, 1982.

[83] Immanuel Kant, *Logic*, translated with an introduction by Robert S. Hartman and Wolfgang Schwartz, New York: Dover, 1974. First published, in German, in 1800.

[84] William and Martha Kneale, *The Development of Logic*, Oxford: Clarendon Press, 1962; paperback in 1974.

[85] Saul A. Kripke, "A completeness theorem in modal logic", *Journal of Symbolic Logic*, 24 (1959), 1-14.

[86] Diogenes Laertius, *Lives of Eminent Philosopher*s. R. D. Hicks, editor and translator, in two volumes, London: Loeb Classical Library 1925.

[87] Mark Lange, *Laws and Lawmakers: Science, Metaphysics and the Laws of Nature*, New York: Oxford University Press, 2009.

[88] Larry Lauden, *Truth, Error and Criminal Law: An Essay in Legal Epistemology*, New York: Cambridge University Press, 2006.

[89] —*The Law's Flaws: Rethinking Trials and Errors*, volume 3 of Law and Society, London: College Publications, 2016.

[90] Clarence Irving Lewis and Cooper H. Langford, *A Survey of Symbolic Logic*, New York: Dover, 1959; originally published in 1932.

[91] Anthony Arthur Long, "Language and thought in Stoicism", in his *Problems in Stoicism*, pages 75-113, London: Duckworth, 1971.

[92] Jan Łukasiewicz, "Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalklüs", *Comptes rendus des séance de la Société des Sciences et des Lettres de Varsovie Cl. III*, 23 (1930). Translated in Storrs McCall editor, Polish Logic 1920-1939, pages 40-65, Oxford: Clarendon Press, 1967.

[93] Storrs McCall, *Aristotle's Modal Syllogisms*, Amsterdam: North-Holland, 1963.

[94] Lorenzo Magnani, *Abductive Cognition: The Epistemological and Eco-Cognitive Dimensions of Hypothetical Reasoning*, volume 3 of Cognitive Systems Monographs, Berlin: Springer, 2009.

[95] —"Naturalizing logic and errors of reasoning vindicated: Logic reapproaches cognitive science," *Journal of Applied Logic*, 13 (2015), 13-36.

[96] —"The urgent need of a naturalized logic", in G. Dodig-Crmkovic, M. J. Schroeder, editors, Contemporary Natural Philosophy and Philosophies, a special guest-edited number of *Philosophies*, 34 (2018),

[97] —*The Abductive Structure of Scientific Creativity: An Essay on the Ecology of Cognition*, volume 37 of Studies in Applied Logic, Epistemology and Rational Ethics, Cham, Switzerland: Springer, 2017.

[98] Benson Mates, *Stoic Logic*, Berkeley and Los Angeles: University of California Press, 1953.

[99] Deborah G. Mayo, *Statistical Inference as Severe Testing: How to Get Beyond the Statistics Wars*, New York: Cambridge University Press, 2018.

[100] *Collected Works of John Stuart Mill*, edited by J. M. Robson, with an Introduction by R. F. McRae, Toronto: University of Toronto Press, 1973 and 1974 respectively.

[101] Amirouche Moktefi, "The social shaping of modern logic" in *Natural Argument: A Tribute to John Woods*, Dov M. Gabbay, Lorenzo Magnani, Ahti-Veikko Pietarinen and Woosuk Park, editors, volume 40 in the Tribute series, pages 503-520, London: College Publications, 2019.

[102] Woosuk Park, *Abduction in Context: The Conjectural Dynamics of Scientific Reasoning*, volume 32 of Studies in Applied Philosophy, Epistemology and Rational Ethics, Cham, Switzerland: Springer, 2017.

[103] Terence Parsons, "The development of supposition theory in the later 12th through 14th centuries", in Dov M. Gabbay and John Woods, editors, *Mediaeval and Renaissance Logic*, volume 2 of Gabbay and Woods, editors, *Handbook of the History of Logic*, pages 156-280, Amsterdam: North Holland, 2008.

[104] Volker Peckhaus, "Schröder's logic" in Dov M. Gabbay and John Woods, editors, *The Rise of Modern Logic: From Leibniz to Frege*, volume 3 of Gabbay and Woods, editors,

Handbook of the History of Logic, pages 557-609, Amsterdam: North-Holland, 2004.

[105] Charles Sanders Peirce, *Reasoning and the Logic of Things: The Cambridge Conference Lectures of 1898*, Kenneth Laine Kettner, editor, with an introduction by Kettner and Hilary Putnam, Cambridge, MA: Harvard University Press, 1992.

[106] Henri Poincaré, *Science and Method*, London: Thomas Nelson, 1914.

[107] Alexander Pope, *An Essay on Man*, London: Printed for J. Wilford, 1733.

[108] Emil L. Post, "Introduction to a general theory of elementary propositions", in van Heijenoort, pages 265-283. Kurt Gödel, "Die Vollstandigkeit der Axioms des logischen Funktionkalküls", *Mondatshefte for Mathematik und Physik*, 37 (1930), 349-360; Translated in van Heijenoort (1967), pages 582-591.

[109] Lawrence H. Powers, *Non-contradiction*, with a Foreword by Hans V. Hansen, volume 39 of Studies in Logic, London: College Publications, 2012.

[110] Henry Prakken, *Logical Tools for Modelling Legal Argument*, Dordrecht: Kluwer, 1997

[111] Henry Prakken And Giovanni Sartor, "A dialectical model of assessing conflicting arguments in legal reasoning", *Artificial Intelligence and Law*, 4 (1996), 331-368.

[112] Graham Priest and Stephen Read, "The formalization of Ockham's theory of supposition", *Mind* 86 (1977), 109-113.

[113] Dag Prawitz, *Natural Deduction: A Proof-Theoretical Study*, Stockholm: Almqvist Wiksell, 1965.

[114] —"Remarks on some approaches to the concept of logical consequence", *Synthese*, 62 (1985), 153-171.

[115] Hilary Putnam, "The logic of quantum mechanics", in his *Mathematics, Matter and Method: Philosophical Papers*, vol. 1, pages 174-197, Cambridge: Cambridge University Press, 1975.

[116] William Van Orman Quine, *Methods of Logic*, New York: Holt, 1950.

[117] —"Epistemology naturalized" in his *Ontological Relativity and Other Essays*, pages 69-90, New York: Columbia University Press, 1969.

[118] Raymond Reiter, "On closed world bases", *Journal of Logic and Data Bases*, (1978), 55-76, and "A Logic for default reasoning", *Artificial Intelligence*, 12 (1980), 81-132.

[119] Nicholas Rescher, *Plausible Reasoning: An Introduction to the Theory and Practice of Plausible Inference*, Assen: Van Gorcum, 1976.

[120] Adriane Rini, *Aristotle's Modal Proofs: Prior Analytics A8-22 in Predicate Logic*, Dordrecht: Springer, 2011.

[121] Richard Routley, Val Plumwood, Robert K. Meyer and Ross Brady, *Relevant Logics and Their RIvals*, Atascadero; Ridgeview, 1984.

[122] Stephen Read, *Relevant Logic*, Oxford: Blackwell, 1988.

[123] Stewart Shapiro, *Foundations Without Foundationalism: A Case for Second-Order Logic*, New York: Oxford University Press, 1991.

[124] D. J. Shoesmith and Timothy John Smiley, *Multiple-Conclusion Logic*, Cambridge: Cambridge University Press, 1978.

[125] Hartley Slater, *Logic is not Mathematics*, volume 35 of Studies in Logic, London: College Publications, 2011.

[126] Fred Sommers, *The Logic of Natural Language*, Oxford: Clarendon Press, 1987.

[127] John F. Sowa, "The role of logic and ontology in language and reasoning", in Roberto Poli & Johanna Seibt, editors, *Theory and Applications of Ontology: Philosophical Perspectives*, pages 231-263, Berlin: Springer, 2014.

[128] Wolfgang Spohn, *The Laws of Belief: Ranking Theory and its Philosophical Applications*, New York: Oxford University Press, 2012

[129] Alfred Tarski, "Contributions to the theory of models I", *Indagationes Mathematicae*, 16 (1954), 572-581.

[130] —"The concept of truth in formalized languages", in his *Logic, Semantics, Metamathematics: Papers from 1923-1938*, J. H. Woodger, translator, Oxford: Oxford University Press; second edition, John Corcoran, editor, pages 152-278, Indianapolis: Hackett, 1983.

[131] Christopher W. Tindale, *Fallacies and Argument Appraisal*, Cambridge: Cambridge University Press, 2007.

[132] Stephen E. Toulmin, *Philosophy of Science*, London: Hutchins, 1953.

[133] —*The Uses of Argument*, Cambridge: Cambridge University Press, 1958.

[134] Johan van Benthem, "Dynamic logic for belief revision", *Journal of Applied Non-classical Logics*, 14 (2004), 1-26.

[135] —*Logic in Games*, Cambridge, MA: MIT Press, 2014.

[136] Frans H. van Eemeren and Rob Grootendorst, *A Systematic Theory of Argumentation: The Pragma-Dialectical Approach*, Cambridge Cambridge University Press, 2004.

[137] Frans H. van Eemeren, Peter Houtlosser and Francesca Snoeck Henkemans, *Argumentative Indicators in Discourse*, Dordrecht: Springer, 2007.

[138] Frans H. van Eemeren, Bart Garsen and Bert Mueffels, *Fallacies and Judgements of Reasonableness: Empirical Research Concerning Pragma-Dialectical*, Dordrecht: Springer, 2009.

[139] Bart Verheij, "Dialectical argumentation with argumentation schemes: an approach to legal logic", *Artificial Logic and Law* 11 (2003), 1-19.

[140] Douglas Walton, Christopher Reed and Fabrizio Macagno, *Argumentation Schemes*, Cambridge: Cambridge University Press, 2008.

[141] Douglas Walton and Thomas F. Gordon, "Critical questions in computational models of legal argument", *Argumentation in Artificial Intelligence and Law*, IAAIL Workshop Series, in P. E. Dunne and T. Bench-Capon, editors, pages 103-111, Nijmegan: Wolf Legal Publishers, 2005.

[142] Douglas Walton and Giovanni Sartor, "Teledogical justifications of argumentation schemes", *Argumentation*, 27 (2013), 111-142.

[143] John Woods, "Fictionality and the logic of relations", *Southern Journal of Philosophy*, 7 (1969), 51-63.

[144] —*The Logic of Fiction: A Sounding of Deviant Logic*; 2nd edition with a Foreword by Nicholas Griffin, volume 23 of Studies in Logic, London: College Publications, 2009. First published by Mouton in 1974.

[145] —"John Stuart Mill (1806-1873)" File of Fallacies, Argumentation, 13 (1999), 317-334.

[146] —"Cognitive economics and the logic of abduction", *Review of Symbolic Logic*, 5 (2012), 148-161.

[147] —*Errors of Reasoning: Naturalizing the Logic of Inference*, volume 45 of Studies in Logic, London: College Publications, 2013; reprinted with corrections in 2014.

[148] —*Aristotle's Earlier Logic*, 2nd edition, Studies in Logic volume 53, London: College Publications, 2014. First published in Paris by Hermes Science.

[149] —"Logic naturalized", in Juan Redmond, Olga Pombo Martins and Ángel Nepomucheno Fernández, editors, *Epistemology, Knowledge and the Impact of Interaction*, pages 403-432, Cham, Switzerland: Springer, 2016.

[150] —"The fragility of argument", in Fabio Paglieri, Laura Bonelli and Silvia Felletti, editors, *The Property of Argument: Cognitive Approaches to Argumentation and Persuasion*, volume 59 in Studies in Logic, pages 99-128, London: College Publications, 2016.

[151] —*Is Legal Reasoning Irrational? An Introduction to the Epistemology of Law*, 2nd edition, revised and expanded, volume 2 of Law and Society, London: College Publications, 2018. First published in 2015.

[152] —"Abduction and inference to the best explanation", in J. Anthony Blair, editor, Studies in Critical Thinking, volume 8 of *Windsor Studies in Argumentation*, pages 405-430, Windsor, ON: WSIA, 2019.

[153] —"What did Frege take Russell to have proved?" *Synthese*, July 22, 2019. DOI 10.1007/s11229-019-02324-4.

[154] —"Evidence, probativity and knowledge: A troubled trio", in Hans V. Hansen, editor, *Proceedings of the 12th Meeting of the Ontario Society for the Study of Argument*, Windsor, ON: WSIA, to appear in 2020.

[155] John Woods and Douglas Walton, *Fallacies: Selected Papers 1972-1982*, 2nd edition, with a Foreword by Dale Jacquette, volume 7 of Studies in Logic, London: College Publications, 2007. First published in Dordrecht by Foris Publications, 1989.

[156] John Woods and Alirio Rosales, *From Logicism's Ashes: The Rise of Mathematical Philosophy*, forthcoming.

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