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## Assumptive Sequent-Based Argumentation

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#### Abstract

In many expert and everyday reasoning contexts it is very useful to reason on the basis of defeasible assumptions. For instance, if the information at hand is incomplete we often use plausible assumptions, or if the information is conflicting we interpret it as consistently as possible. In this paper sequentbased argumentation, a form of logical argumentation in which arguments are represented by a sequent, is extended to incorporate defeasible assumptions. The resulting assumptive framework is general, in that several other approaches to reasoning with assumptive sequent-based argumentation has many desirable properties. It will be shown that assumptive sequent-based argumentation can easily be extended to a prioritized setting, it satisfies rationality postulates and reasoning with maximally consistent subsets can be represented in it.

## 1 Introduction

Assumptions are an important concept in defeasible reasoning. Often, in both expert and everyday reasoning, the information provided is not complete or it is inconsistent. To derive conclusions in such cases, additional information can be assumed or only consistent subsets can be considered. There are many approaches to reasoning with assumptions within the artificial intelligence literature. One of the earlier and best-known formalisms is that of default logic [4, 59]. Intuitively, a default rule of the form  $\phi : \phi_1, \ldots, \phi_n/\psi$ , represents that the conclusion  $\psi$  can be derived, if  $\phi$  is given and no inconsistencies arise when  $\phi_1, \ldots, \phi_n$  hold.

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A well-known formal method for modeling defeasible reasoning is formal argumentation. The idea is that an argument can only be considered as accepted or warranted, when it is defended from all of its attackers. Argumentation frameworks in abstract argumentation theory, introduced by Dung [36], represent this idea by means of a directed graph. The nodes in the graph represent arguments (which are abstract entities) and the edges represent the attacks (the nature of which is unknown). Abstract argumentation can be instantiated in various ways, resulting in logical (also known as deductive or structured) argumentation. In these approaches the arguments have a specific structure and attacks depend on this structure [24, 25, 54]. For example, in [24] the argumentation machinery is combined with classical logic. In logical argument relation can be defined) and rationality postulates from e.g., [30], such as the consistency of the derived conclusions, can be studied, see also [55].

One such logical argumentation framework is *sequent-based argumentation* [10], in which arguments are represented by sequents, as introduced by Gentzen [39] and well-known in proof theory. Attacks between arguments are formulated by *sequent elimination rules*, which are special inference rules. The resulting framework is generic and modular, in that any logic, with a corresponding sound and complete sequent calculus, can be taken as the deductive base (the so-called *core logic*).

Several extensions and relations to other frameworks for nonmonotonic reasoning have been studied for sequent-based argumentation. A dynamic proof theory was introduced [11, 12] to study argumentation from a proof-theoretic perspective. Furthermore, the relation to reasoning with maximally consistent subsets, a common way to maintain consistency when given an inconsistent set of information [60], was investigated [7, 9]. Sequent-based argumentation was extended to incorporate priorities [8] and hypersequents [27]. The latter are a generalization of Gentzen's sequents [13] and allow to take logics such as the semi-relevance logic RM [3, 14] and the modal logic S5 [38] as the core logic. However, in sequent-based argumentation or any of its generalizations, it is not possible to distinguish between facts and defeasible assumptions. This can result in attacks on arguments that are constructed only from facts. As facts represent knowledge that is known to be true, there should be no conflict between facts, nor should arguments constructed only from facts be attacked, since otherwise one could doubt the known information. Therefore, this paper, an extended version of [26], proposes a further generalization, that allows to distinguish between facts and defeasible assumptions.

The contribution of this paper is twofold. First, sequent-based argumentation is extended. To each sequent a component for assumptions is added, to distinguish between defeasible and strict premises. This way, in addition to the given information, assumptions can be made to reach further conclusions. An assumptive argument can only be attacked in its defeasible assumptions, thus assuring that the facts (the given information or strict premises) always hold. After introducing this assumptive sequent-based argumentation framework, we show how it can be generalized to include priorities, based on the approach from [8]. In human reasoning preferences are a common feature in the process of deriving conclusions. It is therefore beneficial if formal approaches to modeling defeasible reasoning can account for possible preferences. Including priorities in formal argumentation makes it possible to order arguments and accept only the most preferred ones. Then the rationality postulates from [30] are studied, which shows that the introduced framework satisfies some basic desirable properties. Furthermore, the representation of reasoning with maximally consistent subsets is investigated.

Second, instances of the obtained framework are studied. For this, three approaches to reasoning with assumptions from the literature are considered:

- Assumption-based argumentation (ABA): a structured argumentation framework which is also semi-abstract, in that there are only limited assumptions on the underlying deductive system [25, 64]. ABA was introduced to determine a set of assumptions that can be accepted as a conclusion from the given information. One of the aims of ABA is to provide a general framework that can incorporate other frameworks for nonmonotonic reasoning, such as default logic and other default reasoning frameworks.
- Adaptive logics: is a logical framework in which the goal is to interpret information as consistently or as normally as possible [21, 62]. What as consistently or as normally as possible means, depends on the lower limit logic, which can be understood as the core logic of the adaptive logic, and the application. In contrast to the other two approaches, the defeasible assumptions (called abnormalities) are assumed not to hold. A dynamic proof system provides a syntactic way to derive conclusions. Many forms of defeasible reasoning can be expressed by an adaptive logic, (see, e.g., [62], in particular page 86, for an overview).
- Default assumptions: were introduced as one of three ways to turn a monotonic consequence relation nonmonotonic [48]. Nonmonotonicty is obtained by varying the set of assumptions. Maximal sets of assumptions that are consistent with the given set of formulas are added to the consequence relation. A formula is then considered as derived if it is a consequence for each set of assumptions. Due to the maximality requirement on the sets of assumptions,

it is a generalization of the consequence relations from [60].

Each of these three approaches covers instances of defeasible reasoning. Although they are related (see [43]), what makes them interesting to consider separately are their particular designs. For example, the type of framework (e.g., argumentation based or (supra-classical) logic based) and the different notions of assumptions, i.e., positive interpretations (the assumptions are assumed to hold) and negative interpretations (the assumptions are assumed not to hold). A general assumptive argumentation framework, of which the above three cases are instances, will therefore be beneficial in the search for a general framework for defeasible reasoning.

The introduced framework is general and modular. Any Tarskian logic with a corresponding sequent calculus can be taken as the core logic and, as will be shown in Section 4, it incorporates some well-known approaches to nonmonotonic reasoning with assumptions. Furthermore, the framework is well-behaved since, in most cases, the rationality postulates from [30] are satisfied. By means of the here introduced assumptive sequent-based argumentation framework, logics, such as intuitionistic logic, many of the well-known modal logics and several relevance logics, can be equipped with defeasible assumptions. Hence, the results of this paper generalize to many deductive core systems, as long as the Tarskian conditions are fulfilled.

As noted above, this paper is an extension of [26]. The results of [26] are part of this paper, now including full proofs. Additionally, this paper studies the properties of the proposed framework in more detail. That is, the incorporation of priorities and the rationality postulates from [30] are studied and it is shown how reasoning with maximally consistent subsets with assumptions can be represented in it. Moreover, the sections on adaptive logics and default assumptions are new.

The paper is organized as follows. In the next section, we provide preliminaries on the used notation and logical notions, a short introduction to abstract argumentation is given and the main definitions of sequent-based argumentation are recalled. Then, in Section 3, the general framework for assumptive sequent-based argumentation is introduced and generalized to a prioritized setting (Section 3.1), rationality postulates are studied (Section 3.2) and the representation of reasoning with maximally consistent subsets is investigated (Section 3.3). To demonstrate the expressiveness of the assumptive sequent-based framework and how it can be applied, in Section 4 it is shown how some well-known approaches to reasoning with defeasible assumptions can be represented in it: assumption-based argumentation (Section 4.1); adaptive logics (Section 4.2); and default assumptions (Section 4.3). Related work is discussed in Section 5 and we conclude in Section 6.

## 2 Preliminaries

In this section we review some basic notions that will be useful throughout the paper: the basic logical setting, abstract argumentation as introduced in [36] (Section 2.1) and sequent-based argumentation from [5, 10] (Section 2.2).

Throughout the paper only propositional languages are considered, denoted by  $\mathcal{L}$ . Atomic formulas are denoted by p, q, formulas are denoted by  $\gamma, \delta, \phi, \psi$ , sets of formulas are denoted by  $\mathcal{S}, \mathcal{T}$ , and finite sets of formulas are denoted by  $\Gamma, \Delta$ . Later on sets of assumptions are denoted by  $\mathcal{AS}, \mathcal{A}$  and finite sets of assumptions by A. All of these can be primed or indexed.

**Definition 1.** A *logic* for a language  $\mathcal{L}$  is a pair  $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ , where  $\vdash$  is a (Tarskian) consequence relation for  $\mathcal{L}$ , having the following properties:

- reflexivity: if  $\phi \in S$ , then  $S \vdash \phi$ ;
- transitivity: if  $\mathcal{S} \vdash \phi$  and  $\mathcal{S}', \phi \vdash \psi$ , then  $\mathcal{S}, \mathcal{S}' \vdash \psi$ ; and
- monotonicity: if  $\mathcal{S}' \vdash \phi$  and  $\mathcal{S}' \subseteq \mathcal{S}$ , then  $\mathcal{S} \vdash \phi$ .

Furthermore, the following property is assumed:

• non-triviality: there is a non-empty set of  $\mathcal{L}$ -formulas  $\mathcal{S}$  and an  $\mathcal{L}$ -formula  $\phi$  such that  $\mathcal{S} \nvDash \phi$ .

In this section and the next (Section 3) the following connectives will sometimes be considered:

- a negation operator  $(\neg)$ :  $p \not\vdash \neg p$  and  $\neg p \not\vdash p$ , for every atom p,
- a conjunction operator ( $\wedge$ ):  $\mathcal{S} \vdash \phi \land \psi$  iff  $\mathcal{S} \vdash \phi$  and  $\mathcal{S} \vdash \psi$ ,
- a disjunction operator ( $\lor$ ):  $S, \phi \lor \psi \vdash \gamma$  iff  $S, \phi \vdash \gamma$  or  $S, \psi \vdash \gamma$ ,
- an implication operator  $(\supset)$ :  $\mathcal{S}, \phi \vdash \psi$  iff  $\mathcal{S} \vdash \phi \supset \psi$ .

We shall abbreviate  $(\phi \supset \psi) \land (\psi \supset \phi)$  by  $\phi \leftrightarrow \psi$ . Furthermore, we denote by  $\land \Gamma$  (respectively, by  $\lor \Gamma$ ) the conjunction (respectively, the disjunction) of all the formulas in  $\Gamma$  and we let  $\neg S = \{\neg \phi \mid \phi \in S\}$ . In examples based on classical logic (CL), it is assumed that all four connectives are part of the language. In the example instances in Section 4, the properties of possible connectives depend on the underlying deductive base.

**Definition 2.** Let  $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$  be a logic, where  $\mathcal{L}$  contains at least the connectives  $\neg$  and  $\land$ , and let  $\mathcal{T}$  be a set of  $\mathcal{L}$ -formulas.

- The closure of  $\mathcal{T}$  is denoted by  $\mathsf{CN}(\mathcal{T})$  (thus,  $\mathsf{CN}(\mathcal{T}) = \{\phi \mid \Gamma \vdash \phi \text{ for } \Gamma \subseteq \mathcal{T}\}$ ).
- $\mathcal{T}$  is consistent (for  $\vdash$ ), if there are no formulas  $\phi_1, \ldots, \phi_n \in \mathcal{T}$  such that  $\vdash \neg \bigwedge_{i=1}^n \phi_i$ .
- A subset C of T is a minimal conflict of T (w.r.t. ⊢), if C is inconsistent and for any c ∈ C, C \ {c} is consistent. Free(T) denotes the set of formulas in T that are not part of any minimal conflict of T.

#### 2.1 Abstract Argumentation

An *abstract argumentation framework*, as introduced by Dung [36], can be viewed as a directed graph. In this graph nodes represent arguments (which are abstract, i.e., they do not have an internal structure) and the arrows represent attacks between arguments, see Figure 1 for a graphical representation. Formally:

**Definition 3.** An (abstract) argumentation framework is a pair  $\mathcal{AF} = \langle \text{Args}, \mathcal{AT} \rangle$ , where Args is a set of arguments and  $\mathcal{AT} \subseteq \text{Args} \times \text{Args}$  is an attack relation on these arguments.



Figure 1: Abstract argumentation framework

**Example 1.** Consider the abstract argumentation framework from Figure 1. The graph in the figure represents  $\mathcal{AF} = \langle \operatorname{Args}, \mathcal{AT} \rangle$  where  $\operatorname{Args} = \{a_1, a_2, a_3, a_4, a_5\}$  and  $\mathcal{AT} = \{(a_1, a_2), (a_2, a_3), (a_3, a_1), (a_4, a_1), (a_4, a_5), (a_5, a_4)\}.$ 

Given an argumentation framework  $\mathcal{AF}$ , Dung-style semantics [36] can be applied to it, to determine what combinations of arguments (called *extensions*) can collectively be accepted from the framework.

**Definition 4.** Let  $\mathcal{AF} = \langle \operatorname{Args}, \mathcal{AT} \rangle$  be an argumentation framework and let  $S \subseteq$  Args be a set of arguments. It is said that:

- S attacks an argument a if there is an  $a' \in S$  such that  $(a', a) \in \mathcal{AT}$ ;
- S defends an argument a if S attacks every attacker of a;
- S is conflict-free if there are no  $a_1, a_2 \in S$  such that  $(a_1, a_2) \in \mathcal{AT}$ ;
- S is *admissible* if it is conflict-free and it defends all of its elements.

An admissible set that contains all the arguments that it defends is a *complete* extension of  $\mathcal{AF}$ . Below are definitions of some particular complete extensions of  $\mathcal{AF}$ :

- the grounded extension of  $\mathcal{AF}$  is the minimal (with respect to  $\subseteq$ ) complete extension of  $\mathcal{AF}$ ;
- a preferred extension of  $\mathcal{AF}$  is a maximal (with respect to  $\subseteq$ ) complete extension of  $\mathcal{AF}$ ;
- a stable extension of  $\mathcal{AF}$  is a complete extension of  $\mathcal{AF}$  that attacks every argument not in it.

In what follows we shall refer to either complete (cmp), grounded (grd), preferred (prf) or stable (stb) semantics as *completeness-based semantics*. We denote by  $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$  the set of all the extensions of  $\mathcal{AF}$  under the semantics  $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{grd}, \mathsf{prf}, \mathsf{stb}\}$ . The subscript is omitted when this is clear from the context. As shown in [36], the grounded extension is unique for a given framework, we will therefore sometimes identify  $\mathsf{Ext}_{\mathsf{grd}}(\mathcal{AF})$  with its single element.<sup>1</sup>

Throughout the paper we will rely on several properties of the semantics defined above. For example, every stable extension is also a preferred extension, but not vice versa. In fact, the grounded extension always exists and there is always a preferred extension, but there is not necessarily a stable extension. For more details see e.g. [17].

**Example 2.** Recall the setting from Example 1, for the argumentation framework from Figure 1. Here we have that  $a_4$  and  $a_5$  attack each other and both defend themselves. Examples of conflict-free sets are  $\{a_1, a_5\}$  and  $\{a_2, a_4\}$ .

For the extensions, note that the grounded extensions is  $\emptyset$ . Furthermore, there are three complete extensions:  $\emptyset$ ,  $\{a_5\}$  and  $\{a_2, a_4\}$ , the last two of these are also preferred extensions and  $\{a_2, a_4\}$  is stable.

<sup>&</sup>lt;sup>1</sup>Other extensions are discussed, e.g., in [16, 17, 18].

It has been argued that abstract argumentation should be instantiated [55], something which Dung already did in his seminal paper [36]. The study of instantiated abstract argumentation frameworks has resulted in several approaches to structured (also called logical or deductive) argumentation [5, 24, 25, 54]. In this paper we consider sequent-based argumentation [5, 10].

### 2.2 Sequent-Based Argumentation

As usual in logical argumentation (see, e.g., [24, 52, 53, 61]), arguments in this paper will have a specific structure based on the underlying formal language, the so-called *core logic*. In the current setting arguments are represented by the well-known proof-theoretic notion of a *sequent* [39].

**Definition 5.** Let  $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$  be a logic and  $\mathcal{S}$  a set of  $\mathcal{L}$ -formulas.

- An  $\mathcal{L}$ -sequent (sequent for short) is an expression of the form  $\Gamma \Rightarrow \Delta$ , where  $\Gamma$  and  $\Delta$  are finite sets of formulas in  $\mathcal{L}$  and  $\Rightarrow$  is a symbol that does not appear in  $\mathcal{L}$ .<sup>2</sup>
- An L-argument (argument for short) is an  $\mathcal{L}$ -sequent  $\Gamma \Rightarrow \psi$ ,<sup>3</sup> where  $\Gamma \vdash \psi$ .  $\Gamma$  is called the support set of the argument and  $\psi$  its conclusion.
- An L-argument based on S is an L-argument  $\Gamma \Rightarrow \psi$ , where  $\Gamma \subseteq S$ . The set of all the L-arguments based on S will be denoted by  $\operatorname{Arg}_{I}(S)$ .

Given an argument  $a = \Gamma \Rightarrow \psi$ , we denote  $\mathsf{Supp}(a) = \Gamma$  and  $\mathsf{Conc}(a) = \psi$ .

The formal systems used for the constructions of sequents (and so of arguments) for a logic  $L = \langle \mathcal{L}, \vdash \rangle$ , are sequent calculi [39], denoted here by C. In what follows it is assumed that C is sound and complete for  $L = \langle \mathcal{L}, \vdash \rangle$ , i.e.,  $\Gamma \Rightarrow \psi$  is provable in C iff  $\Gamma \vdash \psi$ . One of the advantages of sequent-based argumentation is that any logic with a corresponding sound and complete sequent calculus can be used as the core logic.<sup>4</sup> The construction of arguments from simpler arguments is done by the *inference rules* of the sequent calculus [39]. See Figure 2 for the sequent calculus LK of classical logic (CL).<sup>5</sup>

<sup>&</sup>lt;sup>2</sup>Intuitively, in many sequent calculi, a sequent  $\Gamma \Rightarrow \Delta$  can be understood as: if all formulas in  $\Gamma$  are true, then at least one formula in  $\Delta$  is true.

<sup>&</sup>lt;sup>3</sup>Set signs in arguments are omitted.

<sup>&</sup>lt;sup>4</sup>See [10] for further discussion and advantages of this approach.

<sup>&</sup>lt;sup>5</sup>Note that sequents are defined for sets of formulas. This avoids the need for contraction rules in LK. However, the conclusion of arguments (and later on derivations in single conclusioned calculi) contains at most one formula, i.e.,  $\Gamma \Rightarrow \phi, \psi$  is not allowed.

<b>Axioms:</b> $\phi \Rightarrow \phi$						
Logical rules:						
$\frac{\Gamma, \phi, \psi \Rightarrow \Delta}{\Gamma, \phi \land \psi \Rightarrow \Delta} \ [\land \Rightarrow]$	$\frac{\Gamma \Rightarrow \phi, \Delta  \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \land \psi, \Delta} \ [\Rightarrow \land]$					
$\frac{\Gamma \Rightarrow \phi, \Delta}{\Gamma, \neg \phi \Rightarrow \Delta} \ [\neg \Rightarrow]$	$\frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma \Rightarrow \neg \phi, \Delta} \ [\Rightarrow \neg]$					
$\frac{\Gamma \Rightarrow \phi, \Delta  \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \supset \psi \Rightarrow \Delta} \ [\supset \Rightarrow]$	$\frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \supset \psi, \Delta} \ [\Rightarrow \supset]$					
$\frac{\Gamma, \phi \Rightarrow \Delta  \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \lor \psi \Rightarrow \Delta} \ [\lor \Rightarrow]$	$\frac{\Gamma \Rightarrow \phi, \psi, \Delta}{\Gamma \Rightarrow \phi \lor \psi, \Delta} \ [\Rightarrow \lor]$					
Structural rules:						
$\frac{\Gamma_1 \Rightarrow \phi, \Delta_1  \Gamma_2, \phi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} $ [Cut]	$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta} \text{ [Mon]} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \phi, \Delta} \text{ [Mon]}$					

Figure 2: The sequent calculus LK for classical logic.

In addition to arguments, an argumentation system contains attacks between arguments as well. In our case, attacks are represented by sequent elimination rules. Such a rule consists of an attacking argument (the first condition of the rule), an attacked argument (the last condition of the rule), conditions for the attack (the conditions in between) and a conclusion (the eliminated attacked sequent). The outcome of an application of such a rule is that the attacked sequent is 'eliminated'. The elimination of a sequent  $a = \Gamma \Rightarrow \Delta$  is denoted by  $\Gamma \neq \Delta$ .

**Definition 6.** A sequent elimination rule (or attack rule) is a rule  $\mathcal{R}$  of the form:

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \dots \Gamma_n \Rightarrow \Delta_n}{\Gamma_n \Rightarrow \Delta_n} \quad \mathcal{R} \tag{1}$$

Let  $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$  be a logic,  $\mathsf{C}$  its corresponding sequent calculus and  $\mathcal{S}$  a set of  $\mathcal{L}$ formulas. It is said that a sequent elimination rule  $\mathcal{R}$  is  $Arg_{\mathsf{L}}(\mathcal{S})$ -applicable (with
respect to some substitution  $\theta$ ), applicable for short, if  $\theta(\Gamma_1) \Rightarrow \theta(\Delta_1), \theta(\Gamma_n) \Rightarrow$   $\theta(\Delta_n) \in \operatorname{Arg}_{\mathsf{L}}(\mathcal{S})$  and for each  $1 < i < n, \theta(\Gamma_i) \Rightarrow \theta(\Delta_i)$  is derivable in  $\mathsf{C}$ . It is then
said that  $\theta(\Gamma_1) \Rightarrow \theta(\Delta_1) \mathcal{R}$ -attacks  $\theta(\Gamma_n) \Rightarrow \theta(\Delta_n)$ .

The following example shows some of the possible elimination rules.

**Example 3.** Suppose  $\mathcal{L}$  contains a negation operator  $\neg$  and a conjunction operator  $\land$ . See [10, 63] for a definition of many sequent elimination rules. Below are three of them (assuming that  $\Gamma_2 \neq \emptyset$ ):

A sequent-based framework is now defined as follows:

**Definition 7.** A sequent-based argumentation framework for a set of formulas S based on the logic  $L = \langle \mathcal{L}, \vdash \rangle$  and a set AR of sequent elimination rules, is a pair  $\mathcal{AF}_{L,AR}(S) = \langle \operatorname{Arg}_{L}(S), \mathcal{AT} \rangle$ , where  $\mathcal{AT} \subseteq \operatorname{Arg}_{L}(S) \times \operatorname{Arg}_{L}(S)$  and  $(a_{1}, a_{2}) \in \mathcal{AT}$  iff there is an  $\mathcal{R} \in AR$  such that  $a_{1} \mathcal{R}$ -attacks  $a_{2}$ .

In what follows, to simplify notation, the subscripts L and/or AR are omitted when these are clear from the context or arbitrary.

**Example 4.** Let  $\mathcal{AF}_{\mathsf{CL},\{\mathsf{Ucut}\}}(\mathcal{S})$  be an argumentation framework, with classical logic as its core logic, Undercut as the only attack rule and the set  $\mathcal{S} = \{p, p \supset q, \neg q\}$ . Some of the arguments are:

$$\begin{aligned} a &= p, p \supset q \Rightarrow q \qquad b = \neg q \Rightarrow \neg q \qquad c = p \Rightarrow p \\ d &= \Rightarrow q \lor \neg q \qquad e = p \supset q, \neg q \Rightarrow \neg p. \end{aligned}$$

Note that a attacks b and e since  $\Rightarrow q \leftrightarrow \neg \neg q$  is derivable in LK. Similarly, e attacks a and c, since  $\Rightarrow \neg p \leftrightarrow \neg p$ . The argument d cannot be attacked, since the considered attack rule attacks arguments in their support and d has an empty support set. See Figure 3 for a graphical representation of these arguments and the attacks between them. Note that the figure only shows the five arguments mentioned above. Many other arguments are not shown. However, these five arguments are sufficient to illustrate some of the notions of this section.

A sequent-based argumentation framework  $\mathcal{AF}_{L,AR}(\mathcal{S}) = \langle \operatorname{Arg}_{L}(\mathcal{S}), \mathcal{AT} \rangle$  can be seen as an instance of a Dung-style argumentation framework  $\mathcal{AF} = \langle \operatorname{Args}, \mathcal{AT} \rangle$ , where  $\operatorname{Args} = \operatorname{Arg}_{L}(\mathcal{S})$  (Definition 3). Therefore, Dung-style semantics (Definition 4) can be applied to it.

From this entailment relations that are induced from a given sequent-based argumentation framework and its semantics can be defined.



Figure 3: Part of the sequent-based argumentation framework from Example 4 for  $S = \{p, p \supset q, \neg q\}.$ 

**Definition 8.** Given a sequent-based argumentation framework  $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$ , the semantics as defined in Definition 4 induce corresponding (nonmonotonic) *entailment relations*:

- $\mathcal{S} \sim^{\cap}_{\mathsf{L},\mathsf{sem}} \phi$  iff there is an  $a \in \bigcap \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$ , such that  $\mathsf{Conc}(a) = \phi$ ,
- $\mathcal{S} \models_{\mathsf{L},\mathsf{sem}}^{\cup} \phi$  iff for some  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$ , there is an argument  $\Gamma \Rightarrow \phi \in \mathcal{E}$  where  $\Gamma \subseteq \mathcal{S}$ ,
- $\mathcal{S} \sim_{\mathsf{L},\mathsf{sem}}^{\mathbb{m}} \phi$  iff for every  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$  there is an  $a \in \mathcal{E}$  and  $\mathsf{Conc}(a) = \phi$ .

Since the grounded extension is unique,  $\succ_{\mathsf{L},\mathsf{grd}}^{\cap}$ ,  $\succ_{\mathsf{L},\mathsf{grd}}^{\cup}$  and  $\succ_{\mathsf{L},\mathsf{grd}}^{\mathbb{m}}$  coincide and will be denoted by  $\succ_{\mathsf{L},\mathsf{grd}}$ .

**Example 5.** Consider the framework from Example 4, for  $S = \{p, p \supset q, \neg q\}$  and Undercut as the only attack rule. Recall that only a few of the existing arguments were mentioned in the previous example. Since the argument  $d \Rightarrow q \lor \neg q$  is not attacked it holds that  $S \models_{\mathsf{CL},\mathsf{grd}} q \lor \neg q$ . It can be shown that there are three preferred extensions:  $\mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathsf{L}}(S)) = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$  where  $\mathcal{E}_1 = \mathrm{Arg}_{\mathsf{L}}(\{p, p \supset q\})$ ,  $\mathcal{E}_2 = \mathrm{Arg}_{\mathsf{L}}(\{p, \neg q\})$  and  $\mathcal{E}_3 = \mathrm{Arg}_{\mathsf{L}}(\{p \supset q, \neg q\})$ . Thus, for  $\phi \in S$  we have that  $S \models_{\mathsf{CL},\mathsf{prf}} \phi$  and  $S \models_{\mathsf{CL},\mathsf{prf}} \phi$ . Now consider the formula  $p \lor \neg q$ . Although  $S \models_{\mathsf{CL},\mathsf{prf}} p \lor \neg q$ ,  $S \models_{\mathsf{CL},\mathsf{prf}} p$  and  $S \models_{\mathsf{CL},\mathsf{prf}} \neg q$ , it holds that  $S \models_{\mathsf{CL},\mathsf{prf}} p \lor \neg q$ . This follows since in each  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathsf{L}}(S))$ , there is an argument  $a_p \in \mathcal{E}$  such that  $\mathsf{Conc}(a_p) = p$  and/or there is an argument  $a_q \in \mathcal{E}$  such that  $\mathsf{Conc}(a_q) = \neg q$ . In both cases  $p \lor \neg q$  can be derived from the conclusions of  $\mathcal{E}$ .

## **3** Assumptive Sequent-Based Argumentation

Sometimes deriving conclusions requires making assumptions, for example, because there is simply not enough information given, or the information provided is conflicting. There are many ways in which assumptions are handled in the literature, e.g., default logic [59], assumption-based argumentation [25], default assumptions [48] and adaptive logics [21]. In this section the sequent-based argumentation framework from Section 2.2, is extended to incorporate assumptions.

In what follows we assume that, instead of one set of formulas, the input consists of two sets of  $\mathcal{L}$ -formulas:  $\mathcal{AS}$ , the *defeasible premises*, a set of assumptions, the form of which depends on the application and the logic; and  $\mathcal{S}$ , the *strict premises*, the formulas of which can intuitively be understood as facts. As before, a logic  $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$  is assumed to have a corresponding sequent calculus  $\mathsf{C}$ . This calculus will be adjusted to  $\mathsf{C}'$ , in order to allow for assumptions. Both  $\mathsf{C}$  and  $\mathsf{C}'$  are assumed to be sound and complete for  $\mathsf{L}$ . Furthermore, in the current section,  $\mathcal{L}$  will contain at least a negation operator  $\neg$  and a conjunction operator  $\land$ , as in Section 2.

**Definition 9.** Let  $L = \langle \mathcal{L}, \vdash \rangle$  be a logic, with a corresponding sound and complete sequent calculus C and the corresponding adjusted calculus C', let S be a set of  $\mathcal{L}$ -formulas and  $\mathcal{AS}$  a set of assumptions.

- An assumptive  $\mathcal{L}$ -sequent ((assumptive) sequent for short) is an expression of the form  $A \not> \Gamma \Rightarrow \Delta$ .
- An assumptive L-argument ((assumptive) argument for short) is an assumptive sequent  $A \not \Gamma \Rightarrow \psi$ , that is provable in C'.<sup>6</sup>
- An assumptive L-argument based on S and AS is an assumptive argument  $A \$   $\Gamma \Rightarrow \psi$  such that  $\Gamma \subseteq S$  and  $A \subseteq AS$ . As before, the set of all the assumptive L-arguments based on S and AS is denoted by  $\operatorname{Arg}_{\mathsf{L}}(S, AS)$ .

**Notation 1.** Let  $a = A \ rightharpoondown \Gamma$  be an assumptive argument. Then Ass(a) = A denotes the assumptions of the argument a. As before,  $Supp(a) = \Gamma$  and  $Conc(a) = \psi$ . Furthermore, for S a set of arguments,  $Concs(S) = \{Conc(a) \mid a \in S\}$ ,  $Supps(S) = \bigcup \{Supp(a) \mid a \in S\}$  and  $Ass(S) = \bigcup \{Ass(a) \mid a \in S\}$ . In case that  $A = \emptyset$ , a will sometimes be written as  $\Gamma \Rightarrow \psi$ .

Because of the additional component (the assumptions) in an argument, rules have to be defined that allow for the movement of assumptions around  $\zeta$ .

**Definition 10.** Let  $L = \langle \mathcal{L}, \vdash \rangle$  be a logic, S a set of  $\mathcal{L}$  formulas and  $\mathcal{AS}$  a set of assumptions. The following two rules allow to move assumptions:

$$\frac{A \clubsuit \Gamma, \phi \Rightarrow \psi}{A, \phi \And \Gamma \Rightarrow \psi} \operatorname{AS}^{l}_{\mathcal{AS}} \qquad \qquad \frac{A, \phi \And \Gamma \Rightarrow \psi}{A \And \Gamma, \phi \Rightarrow \psi} \operatorname{AS}^{r}_{\mathcal{AS}} \quad \text{where } \phi \in \mathcal{AS}.$$

<sup>&</sup>lt;sup>6</sup>In this paper, C' will differ from C only in that it is defined in terms of assumptive sequents rather than sequents (as in Definition 5) and that it has rules that allow for assumptions to be moved to and from the left side of  $\xi$ .

**Remark 1.** For a logic  $L = \langle \mathcal{L}, \vdash \rangle$ , a set of  $\mathcal{L}$ -formulas  $\mathcal{S}$  and a set of assumptions  $\mathcal{AS}$ , let  $\Gamma \subseteq \mathcal{S}$  and  $A \subseteq \mathcal{AS}$ , if  $AS^r_{\mathcal{AS}}$  and  $AS^l_{\mathcal{AS}}$  are rules in C' then:  $A \cup \Gamma \Rightarrow \phi$  is derivable in C iff  $A \ \Gamma \Rightarrow \phi$  is derivable in C'.

**Remark 2.** The rules from Definition 10 are necessary to construct assumptive arguments. Note that these rules can only be applied to assumptions (i.e., elements from  $\mathcal{AS}$ ). Thus, although assumptions might occur left and right of  $\boldsymbol{\zeta}$  in a derivation, assumptive sequents (and therefore the arguments in this paper) are such that assumptions only occur on the left side of  $\boldsymbol{\zeta}$ .

An important rule is [Cut] (see Figure 2). In view of Remark 1, the following two versions are admissible when  $AS^{r}_{AS}$  and  $AS^{l}_{AS}$  are part of C' and [Cut] is admissible in C:

$$\frac{A_1 \land \Gamma_1 \Rightarrow \Delta_1, \phi \quad A_2 \land \Gamma_2, \phi \Rightarrow \Delta_2}{A_1, A_2 \land \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \quad [Cut] \qquad \frac{A_1 \land \Gamma_1 \Rightarrow \Delta_1, \phi \quad A_2, \phi \land \Gamma_2 \Rightarrow \Delta_2}{A_1, A_2 \land \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \quad [Cut]$$

Figure 4 shows how the sequent calculus for classical logic LK (from Figure 2) can be extended to LK'. IN view of the discussion above, LK' contains only one cut rule.

**Example 6.** Recall, from Example 4, the set of formulas  $\{p, p \supset q, \neg q\}$ , where CL is the core logic and LK the corresponding calculus. Let now  $S = \{p\}$  and  $\mathcal{AS} = \{p \supset q, \neg q\}$  and take LK' from Figure 4 as the corresponding calculus. The assumptive counterparts of the arguments in Example 4 are then:

$$\begin{aligned} a_{\mathcal{AS}} &= p \supset q \ p \Rightarrow q \qquad b_{\mathcal{AS}} = \neg q \ \Rightarrow \neg q \qquad c_{\mathcal{AS}} = p \Rightarrow p \\ d_{\mathcal{AS}} &= \Rightarrow q \lor \neg q \qquad e_{\mathcal{AS}} = p \supset q, \neg q \ \Rightarrow \neg p. \end{aligned}$$

Arguments are attacked in the set of assumptions. When choosing a (set of) attack rule(s), it is important to note that these reflect the interpretation of an assumption. In the rules below, the interpretation of the assumptions is positive: they are assumed to hold. If the interpretation is negative instead, the negation in the condition(s) of the first two rules should be removed. See Section 4.2 on adaptive logics for a setting with negative assumptions.

**Example 7.** Assume  $A_1 \ \ \Gamma_1 \Rightarrow \phi_1; A_2, \psi \ \ \Gamma_2 \Rightarrow \phi_2 \in \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}) \text{ and } \Delta \subseteq \mathcal{S}.$ Let  $a = A \ \ \Gamma \Rightarrow \phi \in \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ , we continue using  $A \ \ \Gamma \Rightarrow \phi$  to denote that a has been eliminated. Examples of sequent elimination rules for assumptive

<b>Axioms:</b> $\phi \Rightarrow \phi$				
Logical rules:				
$\frac{A \ \Gamma, \phi, \psi \Rightarrow \Delta}{A \ \Gamma, \phi \land \psi \Rightarrow A}$	$\frac{\Delta}{\Delta} [\land \Rightarrow] \qquad \frac{\Delta}{\Delta}$	$\frac{A \ \Gamma \Rightarrow \phi, \Delta  A \ \Gamma \Rightarrow \psi, \Delta}{A \ \Gamma \Rightarrow \phi \land \psi, \Delta} \ [\Rightarrow \land]$		
$\frac{A \clubsuit \Gamma \Rightarrow \phi, \Delta}{A \clubsuit \Gamma, \neg \phi \Rightarrow \Delta}$	- [¬⇒]	$\frac{A \ (\Gamma, \phi \Rightarrow \Delta)}{A \ (\Gamma \Rightarrow \neg \phi, \Delta)} \ [\Rightarrow \neg]$		
$\frac{A \clubsuit \Gamma \Rightarrow \phi, \Delta  A \clubsuit \Gamma,}{A \clubsuit \Gamma, \phi \supset \psi \Rightarrow}$	$\frac{\psi \Rightarrow \Delta}{\Delta} \ [\supset \Rightarrow]$	$\frac{A \left\{ \Gamma, \phi \Rightarrow \psi, \Delta \right.}{A \left\{ \Gamma \Rightarrow \phi \supset \psi, \Delta \right.} [\Rightarrow \supset]$		
$\frac{A \clubsuit \Gamma, \phi \Rightarrow \Delta  A \And \Gamma,}{A \clubsuit \Gamma, \phi \lor \psi \Rightarrow A}$	$\frac{\psi \Rightarrow \Delta}{\Delta} \ [\lor \Rightarrow]$	$\frac{A \ (\Gamma \Rightarrow \phi, \psi, \Delta)}{A \ (\Gamma \Rightarrow \phi \lor \psi, \Delta)} \ [\Rightarrow \lor]$		
Structural rules:				
$\frac{A_1 \left\{ \begin{array}{cc} \Gamma_1 \Rightarrow \Pi, \phi & A_2 \left\{ \begin{array}{c} \Gamma_2, \phi \Rightarrow \Delta \\ \end{array} \right. \\ A_1, A_2 \left\{ \begin{array}{c} \Gamma_1, \Gamma_2 \Rightarrow \Pi, \Delta \end{array} \right\} $ [Cut]				
$\frac{A \ \Gamma, \phi \Rightarrow \psi}{A, \phi \ \Gamma \Rightarrow \psi} \ \mathrm{AS}^{l}_{\mathcal{AS}}$	$\frac{A,\phi \ \Gamma \Rightarrow \psi}{A \ \Gamma,\phi \Rightarrow \psi} \ \mathrm{AS}^{r}_{\mathcal{A}}$	$^{\mathcal{AS}}$ where $\phi \in \mathcal{AS}$		
$\frac{A \ \Gamma \Rightarrow \Delta}{A \ \Gamma, \phi \Rightarrow \Delta} \ [\text{LMon}]$	$\frac{A \land \Gamma \Rightarrow \Pi}{A \land \Gamma \Rightarrow \phi, \Pi} [\text{RM}]$	[on]		

Figure 4: The assumptive sequent calculus LK' for classical logic.

sequent-based argumentation are (see Section 4 for other definitions):

$$\frac{A_{1} \left\{ \Gamma_{1} \Rightarrow \phi_{1} \quad \phi_{1} \Rightarrow \neg \psi \quad A_{2}, \psi \left\{ \Gamma_{2} \Rightarrow \phi_{2} \right\} \right\}}{A_{2}, \psi \left\{ \Gamma_{2} \Rightarrow \phi_{2} \right\}} \operatorname{AT}_{\mathcal{AS}}^{\Rightarrow}$$

$$\frac{A_{1} \left\{ \Gamma_{1} \Rightarrow \phi_{1} \quad \phi_{1} \Rightarrow \neg \psi \quad \neg \psi \Rightarrow \phi_{1} \quad A_{2}, \psi \left\{ \Gamma_{2} \Rightarrow \phi_{2} \right\} \right\}}{A_{2}, \psi \left\{ \Gamma_{2} \Rightarrow \phi_{2} \right\}} \operatorname{AT}_{\mathcal{AS}}^{\Rightarrow}$$

$$\frac{\Delta \Rightarrow \neg \bigwedge A_{1} \quad A_{1} \left\{ \Gamma_{1} \Rightarrow \phi_{1} \right\}}{A_{1} \left\{ \Gamma_{1} \Rightarrow \phi_{1} \right\}} \operatorname{AT}_{\mathcal{AS}}^{\mathsf{Con}}$$

When the superscript is clear from the context or arbitrary, it will be omitted. In the remainder we sometimes write that, for the arguments as in the first two rules,  $A_2, \psi \ \Gamma_2 \Rightarrow \phi_2$  is attacked in  $\psi$  by  $A_1 \ \Gamma_1 \Rightarrow \phi_1$ .

Each of the above rules reflects that an assumptive argument can only be attacked in its assumptions. The rule  $\operatorname{AT}_{\mathcal{AS}}^{\Leftrightarrow}$  can be seen as the assumptive version of the direct undercut rule from Example 3. The  $\operatorname{AT}_{\mathcal{AS}}^{\mathsf{Con}}$  rule can be understood as the assumptive version of consistency undercut. This rule attacks arguments that have an inconsistent set of assumptions (in which case it could be that  $\Delta = \emptyset$ ) or the set of assumptions is inconsistent with the set of facts. In Example 8 below, if  $\operatorname{AT}_{\mathcal{AS}}^{\mathsf{Con}}$ would be part of the attack rules, the argument  $a_6$  would be  $\operatorname{AT}_{\mathcal{AS}}^{\mathsf{Con}}$ -attacked.

**Definition 11.** An assumptive sequent-based argumentation framework for a set of formulas S, set of assumptions  $\mathcal{AS}$ , based on a logic  $L = \langle \mathcal{L}, \vdash \rangle$  and a set AR of sequent elimination rules (such as those from Example 7), is a pair  $\mathcal{AF}_{L,AR}(S, \mathcal{AS}) =$  $\langle \operatorname{Arg}_{L}(S, \mathcal{AS}), \mathcal{AT} \rangle$ , where  $\mathcal{AT} \subseteq \operatorname{Arg}_{L}(S, \mathcal{AS}) \times \operatorname{Arg}_{L}(S, \mathcal{AS})$  and  $(a_{1}, a_{2}) \in \mathcal{AT}$  iff there is a rule  $\mathcal{R} \in AR$  such that  $a_{1} \mathcal{R}$ -attacks  $a_{2}$ .

Note that, although no restrictions are placed on S and AS in the definition above, in Section 3.2 it is shown why S should be consistent. Such a restriction can not be enforced in general, since there are cases where S has to be inconsistent, in order for the argumentation process to be interesting. Section 4.2, on adaptive logics, is an example of such a case.

Like before, when these are clear from the context or arbitrary, the subscripts L and/or AR are omitted. The semantics, as defined in Definition 4 can be applied to assumptive sequent-based argumentation frameworks.

**Example 8.** Let  $\mathcal{AF}_{\mathsf{CL},\{AT_{\mathcal{AS}}^{\leftrightarrow}\}}(\mathcal{S},\mathcal{AS}) = \langle \operatorname{Arg}_{\mathsf{CL}}(\mathcal{S},\mathcal{AS}),\mathcal{AT} \rangle$ , where  $\mathcal{S} = \{p\}$  and  $\mathcal{AS} = \{p \supset q, \neg q\}$ , as in Example 6. Then some of the arguments in  $\operatorname{Arg}_{\mathsf{CL}}(\mathcal{S},\mathcal{AS})$  are:

$$\begin{aligned} a_1 &= p \Rightarrow p & a_2 &= p \supset q \ \Rightarrow p \supset q & a_3 &= \neg q \ \Rightarrow \neg q \\ a_4 &= p \supset q \ p \Rightarrow q & a_5 &= \neg q \ p \Rightarrow p \land \neg q & a_6 &= p \supset q, \neg q \ \Rightarrow \neg p \end{aligned}$$

As in Example 4, these are only a few of the derivable arguments. However, these arguments are sufficient for the purpose of this example and the other arguments do not change the discussion and evaluation below.

Note that  $a_4$  attacks any argument with  $\neg q$  in the assumptions (i.e.,  $a_3$ ,  $a_5$  and  $a_6$ ), since  $\Rightarrow q \leftrightarrow \neg \neg q$  is derivable in LK'. To see why  $a_5$  attacks  $a_2$ ,  $a_4$  and  $a_6$ , take

a look at the following derivations:

$$\begin{array}{c} \frac{p \Rightarrow p}{p \Rightarrow p, q} \left[ \operatorname{Mon} \right] & \frac{q \Rightarrow q}{p, q \Rightarrow q} \left[ \operatorname{Mon} \right] \\ [\supset \Rightarrow] \\ \hline \frac{p, p \supset q \Rightarrow q}{p, \neg q, p \supset q \Rightarrow} \left[ \neg \Rightarrow \right] \\ \hline \frac{p, \neg q \Rightarrow \neg (p \supset q)}{p, \neg q \Rightarrow \neg (p \supset q)} \left[ \Rightarrow \neg \right] \\ \hline \Rightarrow (p \land \neg q) \supset \neg (p \supset q) \right] \left[ \Rightarrow \supset \right] \end{array} \qquad \begin{array}{c} \frac{p \Rightarrow p}{p \Rightarrow p, q} \left[ \operatorname{Mon} \right] & \frac{p \Rightarrow p}{p \Rightarrow \neg q, q} \left[ \Rightarrow \neg \right] \\ \hline \frac{p \Rightarrow p \land \neg q, p \supset q \Rightarrow}{p \Rightarrow \neg q, q} \left[ \Rightarrow \neg \right] \\ \hline \frac{p \Rightarrow p \land \neg q, q}{p \Rightarrow p \land \neg q, q} \left[ \Rightarrow \supset \right] \\ \hline \neg (p \supset q) \Rightarrow p \land \neg q \right] \left[ \Rightarrow \supset \right] \\ \hline \Rightarrow \neg (p \supset q) \supset (p \land \neg q) \left[ \Rightarrow \supset \right] \end{array}$$

See Figure 5 for a graphical representation of the given arguments and the attacks between them.



Figure 5: Part of the assumptive sequent-based argumentation framework from Example 8 for  $S = \{p\}$  and  $\mathcal{AS} = \{p \supset q, \neg q\}$ .

Since  $\operatorname{Ass}(a_1) = \emptyset$ , the argument  $a_1$  cannot be attacked. It follows that  $a_1 \in \bigcap \operatorname{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\mathsf{CL},\{\operatorname{AT}_{\mathcal{AS}}^{\leftrightarrow}\}}(\mathcal{S},\mathcal{AS}))$  and hence  $a_1 \in \operatorname{Ext}_{\mathsf{grd}}(\mathcal{AF}_{\mathsf{CL},\{\operatorname{AT}_{\mathcal{AS}}^{\leftrightarrow}\}}(\mathcal{S},\mathcal{AS}))$ , where  $\operatorname{Ext}_{\mathsf{grd}}(\mathcal{AF}_{\mathsf{CL},\{\operatorname{AT}_{\mathcal{AS}}^{\leftrightarrow}\}}(\mathcal{S},\mathcal{AS}))$  is identified with its single element. There are five admissible sets in the framework from Figure 5:  $\emptyset$ ,  $\{a_1\}$ ,  $\{a_1, a_2, a_4\}$ ,  $\{a_1, a_3, a_5\}$ ,  $\{a_2, a_4\}$  and  $\{a_3, a_5\}$ . Note that  $a_6$  is not part of any admissible set. To see this, note that both  $a_4$  and  $a_5$  have to be attacked and not defended, yet any attacker of  $a_4$  and  $a_5$  is also an attacker of  $a_6$ .

The entailment relations for an assumptive framework  $\mathcal{AF}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$  are defined similarly to those in Definition 8 and are denoted by  $\succ^{\star}_{\mathcal{AS},\mathsf{sem}}$  for  $\star \in \{\cap, \cup, \mathbb{m}\}$  and where  $\mathcal{AS}$  is the set of assumptions. **Example 9.** Recall  $\mathcal{AF}_{\mathsf{CL},\{\mathsf{AT}_{\mathcal{AS}}^{\leftrightarrow}\}}(\mathcal{S},\mathcal{AS})$  from Example 8, where  $\mathcal{S} = \{p\}$  and  $\mathcal{AS} = \{p \supset q, \neg q\}$ . In view of the discussion about the extensions in that example,  $\mathcal{S} \models_{\mathcal{AS},\mathsf{grd}} p$ , since  $a_1$  is not attacked. Moreover,  $\mathcal{S} \models_{\mathcal{AS},\mathsf{sem}}^{\cup} \phi$  for  $\mathsf{sem} \in \{\mathsf{cmp},\mathsf{prf},\mathsf{stb}\}$  and  $\phi \in \mathsf{CN}_{\mathsf{CL}}(\{p \supset q, p\} \cup \{\neg q, p\})$ , but  $\mathcal{S} \models_{\mathcal{AS},\mathsf{sem}}^{\cap} \psi$  for  $\mathsf{sem} \in \{\mathsf{cmp},\mathsf{prf},\mathsf{stb}\}$  and  $\psi \in \{p \supset q, \neg q\}$ . This follows since for each  $\phi \in \{p \supset q, \neg q\}$  there is an argument a with  $\mathsf{Conc}(a) = \phi$  and there is some  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{CL}},\{\mathsf{AT}_{\mathcal{AS}}^{\leftrightarrow}\}(\mathcal{S},\mathcal{AS}))$  such that  $a \in \mathcal{E}$ . However, there is also some  $\mathcal{E}' \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{CL}},\{\mathsf{AT}_{\mathcal{AS}}^{\leftrightarrow}\}(\mathcal{S},\mathcal{AS}))$  such that  $a \notin \mathcal{E}'$ , for  $\mathsf{sem} \in \{\mathsf{cmp},\mathsf{prf},\mathsf{stb}\}$ .

In the next sections we study some properties of assumptive sequent-based argumentation frameworks. First, in Section 3.1 priorities among the assumptions are incorporated. Then, in Section 3.2 the rationality postulates from [30] for the resulting prioritized frameworks are shown. In Section 3.3 we discuss how reasoning with maximally consistent subsets, as introduced in [60], can be generalized to the assumptive setting and can be represented by the here introduced framework. In these sections we assume that the rules from Figure 6 are admissible in the sequent calculus C. This way it is not necessary to choose a specific core logic to prove the results and the proofs can be kept relatively simple (i.e., no case distinctions are necessary to cover different kinds of rules). Note that this requirement does not limit the presented assumptive framework, only the calculi for which the results hold.

#### 3.1 Adding Priorities

Another important and often applied way to distinguish between elements of the premises, is by means of priorities. By assigning priorities to some knowledge, or expressing preferences among the knowledge, the derivation process can be adjusted such that as much as possible of the most preferred knowledge is accepted. Within argumentation, for many frameworks prioritized versions have been studied, including sequent-based argumentation [8]. In the assumptive setting, facts always hold, thus these are preferred over any other premise. But among the assumptions, a user might have preferences.

**Definition 12.** A priority function for a language  $\mathcal{L}$  is a function  $\pi : \mathcal{L} \to \mathbb{N}^+$ . Given a set of  $\mathcal{L}$ -formulas  $\mathcal{S}$ , we denote  $\pi(\mathcal{S}) = {\pi(\phi) \mid \phi \in \mathcal{S}}$ . Moreover,  $\max_{\pi}(\mathcal{S}) = {\phi \in \mathcal{S} \mid \nexists \psi \in \mathcal{S}, \pi(\phi) < \pi(\psi)}$  denotes the set of formulas from  $\mathcal{S}$  with maximal  $\pi$ -value. We let  $\max_{\pi}(\emptyset) = 0$ .

In what follows, it is assumed that a formula  $\phi$  is preferred over a formula  $\psi$  if  $\pi(\phi) \leq \pi(\psi)$ ,  $\phi$  is strictly preferred over  $\psi$  if it is preferred over  $\psi$  and  $\pi(\psi) \not\leq \pi(\phi)$ . Thus, intuitively, a lower  $\pi$ -value means a higher preference.

With this priority function, the attack relation induced by  $AT_{\mathcal{AS}}$  can be refined:

<b>Axioms:</b> $\phi \Rightarrow \phi$					
Logical rules:					
$\frac{A \left\{ \Gamma, \phi, \psi \Rightarrow \Delta \right\}}{A \left\{ \Gamma, \phi \land \psi \Rightarrow \Delta} [\land \Rightarrow] \qquad \frac{A \left\{ \Gamma \Rightarrow \phi, \Pi \right\} A \left\{ \Gamma \Rightarrow \psi, \Pi \right\}}{A \left\{ \Gamma \Rightarrow \phi \land \psi, \Pi} [\Rightarrow \land]$					
$\frac{A \left\{ \Gamma \Rightarrow \phi, \Pi \right\}}{A \left\{ \Gamma, \neg \phi \Rightarrow \Pi \right\}} [\neg \Rightarrow] \qquad \qquad \frac{A \left\{ \Gamma, \phi \Rightarrow \Pi \right\}}{A \left\{ \Gamma \Rightarrow \neg \phi, \Pi \right\}} [\Rightarrow \neg]$					
Structural rules:					
$\frac{A_1 \land \Gamma_1 \Rightarrow \Pi, \phi  A_2 \land \Gamma_2, \phi \Rightarrow \Delta}{A_1, A_2 \land \Gamma_1, \Gamma_2 \Rightarrow \Pi, \Delta}  [Cut]  \frac{A_1 \land \Gamma_1 \Rightarrow \Pi, \phi  A_2, \phi \land \Gamma_2 \Rightarrow \Delta}{A_1, A_2 \land \Gamma_1, \Gamma_2 \Rightarrow \Pi, \Delta}  [Cut]$					
$\frac{A \ \Gamma, \phi \Rightarrow \psi}{A, \phi \ \Gamma \Rightarrow \psi} \ \mathrm{AS}^{l}_{\mathcal{AS}} \qquad \qquad \frac{A, \phi \ \Gamma \Rightarrow \psi}{A \ \Gamma, \phi \Rightarrow \psi} \ \mathrm{AS}^{r}_{\mathcal{AS}}  \text{where } \phi \in \mathcal{AS}$					
$\begin{vmatrix} A & \Gamma \Rightarrow \Delta \\ \overline{A} & \Gamma, \phi \Rightarrow \Delta \end{vmatrix} [LMon] \qquad \frac{A & \Gamma \Rightarrow \Pi}{A & \Gamma \Rightarrow \phi, \Pi} [RMon]$					

Figure 6: Rules that are assumed to be part of (or admissible in) the calculus C (in the case that C is single-conclusioned II should be empty and  $\Delta$  contains at most one formula).

**Definition 13.** Let  $a_1, a_2 \in \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ , it is said that  $a_1 \operatorname{AT}_{\mathcal{AS}}^{\star, \leq \pi}$ -attacks  $a_2$  if and only if  $a_1 \operatorname{AT}_{\mathcal{AS}}^{\star}$ -attacks  $a_2$  in  $\psi$  and  $\max_{\pi}(\operatorname{Ass}(a_1)) \leq \pi(\psi)$ , for  $\star \in \{\Rightarrow, \Leftrightarrow\}$  or  $a_1 \operatorname{AT}_{\mathcal{AS}}^{\mathsf{Con}}$ -attacks  $a_2$ .

**Remark 3.** An  $\operatorname{AT}_{\mathcal{AS}}^{\operatorname{Con},\leq_{\pi}}$ -attack is always successful, since the attacker has an empty set of assumptions, the superscript  $\leq_{\pi}$  will therefore often be omitted from the notation.

**Example 10.** Recall the examples from the previous section, for the assumptive framework  $\mathcal{AF}_{\mathsf{CL},\{\mathsf{AT}_{\mathcal{AS}}^{\leftrightarrow}\}}(\mathcal{S},\mathcal{AS}) = \langle \operatorname{Arg}_{\mathsf{CL}}(\mathcal{S},\mathcal{AS}),\mathcal{AT} \rangle$ , where  $\mathcal{S} = \{p\}, \mathcal{AS} = \{p \supset q, \neg q\}$ . Let  $\pi(p \supset q) = 2$  and  $\pi(\neg q) = 3$ . Then, not all attacks of the flat setting (i.e., the setting without priorities) go through. For example, although  $a_5$  attacks  $a_4$  in the flat setting, this attack goes no longer through given the priority function  $\pi$ . In fact, since  $a_5$  attacks arguments in the assumption  $p \supset q$ , no argument is attacked by  $a_5$  given this priority function.

**Definition 14.** A prioritized assumptive sequent-based argumentation framework for a set of formulas S, set of assumptions  $\mathcal{AS}$ , based on a logic  $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ ,  $\pi$ a priority function on  $\mathcal{L}$  and AR the set of sequent elimination rules, is a triple  $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}^{\leq \pi}(S,\mathcal{AS}) = \langle \operatorname{Arg}_{\mathsf{L}}(S,\mathcal{AS}), \mathcal{AT}, \leq_{\pi} \rangle$ , where  $\mathcal{AT} \subseteq \operatorname{Arg}_{\mathsf{L}}(S,\mathcal{AS}) \times \operatorname{Arg}_{\mathsf{L}}(S,\mathcal{AS})$ and  $(a_1, a_2) \in \mathcal{AT}$  iff there is a rule  $\mathcal{R}^{\leq \pi} \in \mathsf{AR}$  such that  $a_1 \mathcal{R}^{\leq \pi}$ -attacks  $a_2$ .

Like before, the semantics of Definition 4 can be applied to prioritized assumptive sequent-based argumentation frameworks. The corresponding entailment relations are denoted by  $\sim_{\mathcal{AS},\mathsf{sem}}^{\leq \pi,\star}$ , where  $\star \in \{\cap, \cup, \mathbb{m}\}$  and  $\mathcal{AS}$  is the set of assumptions.

**Example 11.** Consider the setting from Example 10, in which CL is the core logic,  $\operatorname{AT}_{\mathcal{AS}}^{\Leftrightarrow,\leq\pi}$  the attack rule,  $\mathcal{S} = \{p\}$ ,  $\mathcal{AS} = \{p \supset q, \neg q\}$ , the priority function  $\pi$  is such that  $\pi(p \supset q) = 2$  and  $\pi(\neg q) = 3$ . As mentioned, not all attacks as presented in Figure 5 go through. For a graphical representation of this prioritized assumptive framework, see Figure 7. Given the priority function  $\pi$ ,  $a_2$  is no longer attacked



Figure 7: Part of the prioritized assumptive sequent-based argumentation framework from Example 11 for  $S = \{p\}$  and  $\mathcal{AS} = \{p \supset q, \neg q\}$ , with  $\pi(p \supset q) = 2$  and  $\pi(\neg q) = 3$ .

and  $a_3$  can no longer be defended from the attack by  $a_4$ . Thus  $\mathcal{S} \models_{\mathcal{AS},\mathsf{grd}}^{\leq \pi} \phi$ , where  $\phi \in \mathsf{CN}_{\mathsf{CL}}(\{p, p \supset q\})$ . On the other hand  $\mathcal{S} \models_{\mathcal{AS},\mathsf{cmp}}^{\leq \pi, \cup} \neg q$ .

In the next section some desirable properties of (prioritized) assumptive sequentbased argumentation frameworks are studied, in terms of the rationality postulates from [30].

#### 3.2 Rationality Postulates

There are many structured argumentation frameworks introduced and studied in the literature. It is therefore important to have an objective measure for the usefulness of such frameworks and to make sure that the resulting extensions satisfy some basic desirable properties. To this end, the rationality postulates from [30] are studied. Before introducing the postulates, the notion of a sub-argument will be useful.

**Definition 15.** Let  $\mathcal{AF}_{\mathsf{L}}^{\leq_{\pi}}(\mathcal{S}, \mathcal{AS}) = \langle \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}), \mathcal{AT}, \leq_{\pi} \rangle$  be an argumentation framework and consider two arguments  $a, a' \in \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$  such that  $a = A \ T \Rightarrow \phi$  and  $a' = A' \ \Gamma' \Rightarrow \phi'$ . Then a' is a *sub-argument* of a if  $\Gamma' \subseteq \Gamma$  and  $A' \subseteq A$ . The set of sub-arguments of a is denoted by  $\operatorname{Sub}(a)$ .

**Definition 16.** Let  $\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S}, \mathcal{AS}) = \langle \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}), \mathcal{AT}, \leq_{\pi} \rangle$  be an assumptive argumentation framework for the logic  $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ , the set  $\mathcal{S}$  of  $\mathcal{L}$ -formulas, the set  $\mathcal{AS}$  of assumptions and some semantics sem.  $\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S}, \mathcal{AS})$  satisfies:

- closure of extensions: iff  $Concs(\mathcal{E}) = CN(Concs(\mathcal{E}))$  for each extension  $\mathcal{E} \in Ext_{sem}(\mathcal{AF}_{L}^{\leq \pi}(\mathcal{S}, \mathcal{AS}));$
- sub-argument closure: iff  $a \in \mathcal{E}$  implies that  $\mathsf{Sub}(a) \subseteq \mathcal{E}$  for all extensions  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S}, \mathcal{AS}));$
- consistency: iff  $Concs(\mathcal{E})$  is consistent for each  $\mathcal{E} \in Ext_{sem}(\mathcal{AF}_{L}^{\leq \pi}(\mathcal{S}, \mathcal{AS}))$ .

**Remark 4.** In [30], there are two different postulates for inconsistency: direct consistency (the consistency postulate above) and indirect consistency. However, in view of the closure of extensions postulate, indirect consistency follows from the consistency postulate in our setting. This is why the above postulates are discussed in the given order.

Furthermore, sub-argument closure was not defined as a postulate, but is shown as a proposition ([30, Proposition 1]). Note that the framework from [30] is different from the one presented here, thus the notion of a sub-argument is also different. However, the definition of sub-arguments as given here, corresponds to that of e.g., [1, 2, 8].

**Remark 5.** In the proofs of the rationality postulates below, it will be assumed that S is consistent. Consider for example  $\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(S, \mathcal{AS}) = \langle \operatorname{Arg}_{\mathsf{L}}(S, \mathcal{AS}), \mathcal{AT}, \leq_{\pi} \rangle$ , for CL the core logic, with LK as calculus and where  $S = \{p, \neg p\}$  and  $\mathcal{AS} = \{q\}$ . Some of the arguments are:

$$a = p \Rightarrow p$$
  $b = \neg p \Rightarrow \neg p$   $c = q \Rightarrow q$   $d = p, \neg p \Rightarrow \neg q$ 

Note that a, b and d cannot be attacked, since  $\operatorname{Ass}(a) = \operatorname{Ass}(b) = \operatorname{Ass}(d) = \emptyset$ . Thus  $a, b, d \in \operatorname{Ext}_{\operatorname{grd}}(\mathcal{AF}_{\mathsf{L}}^{\leq_{\pi}}(\mathcal{S}, \mathcal{AS}))$ . Moreover, d attacks c and c cannot be defended, though one might argue that the conflict of p should not cause q to be excluded from the conclusions. In Lemma 2, it will be shown that, when  $\mathcal{S}$  is inconsistent,  $\operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \emptyset)$  is the only extension.

The next lemma introduces some sequent rules that will be used in the proofs of this section.

**Lemma 1.** Let  $L = \langle \mathcal{L}, \vdash \rangle$  be a logic with corresponding sequent calculus C, in which the rules from Figure 6 are admissible. Then the rules from Figure 8 are admissible as well.

$$\begin{array}{l} \frac{\Gamma, \neg \neg \phi \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta} \ [\neg \neg \not\Rightarrow] & \frac{\Gamma, \phi_1, \dots, \phi_n \Rightarrow \Pi}{\Gamma \Rightarrow \neg (\phi_1 \land \dots \land \phi_n), \Pi} \ [\Rightarrow \neg \land] \\ & \frac{\Gamma \Rightarrow \neg (\phi_1 \land \dots \land \phi_n), \Pi}{\Gamma, \phi_1, \dots, \phi_n \Rightarrow \Pi} \ [\not\Rightarrow \neg \land] \end{array}$$

Figure 8: Admissible rules in the minimal calculus from Figure 6 (in the case that C is single-conclusioned  $\Pi$  should be empty and  $\Delta$  contains at most one formula).

The proof is by means of derivations in the minimal calculus from Figure 6 and can be found in Appendix A.

The next lemma shows that, when S is inconsistent, there is exactly one extension that contains only the arguments with an empty set of assumptions. Moreover, together with Remark 5, it provides the motivation to assume that S is consistent.

**Lemma 2.** Let  $\mathcal{AF}_{\mathsf{L}}^{\leq_{\pi}}(\mathcal{S},\mathcal{AS}) = \langle Arg_{\mathsf{L}}(\mathcal{S},\mathcal{AS}), \mathcal{AT}, \leq_{\pi} \rangle$  be an argumentation framework. If  $\mathcal{S}$  is inconsistent, then  $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}^{\leq_{\pi}}(\mathcal{S},\mathcal{AS})) = \{Arg_{\mathsf{L}}(\mathcal{S},\emptyset)\}$  for each  $\mathsf{sem} \in \{\mathsf{grd},\mathsf{cmp},\mathsf{prf},\mathsf{stb}\}.$ 

Proof. Let  $\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S}, \mathcal{AS}) = \langle \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}), \mathcal{AT}, \leq_{\pi} \rangle$  be an argumentation framework for the logic  $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$  with corresponding calculus  $\mathsf{C}$ , inconsistent set of  $\mathcal{L}$ -formulas  $\mathcal{S}$ , set of assumptions  $\mathcal{AS}$  and  $\pi$  a priority function. Since an attack is always on formulas in the assumptions of an argument, none of the arguments in  $\operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \emptyset)$ can be attacked, thus  $\operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \emptyset) \subseteq \operatorname{Ext}_{\mathsf{grd}}(\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S}, \mathcal{AS})).$ 

By assumption S is inconsistent, thus there are  $\phi_1, \ldots, \phi_n \in S$ , such that  $\vdash \neg \bigwedge_{i=1}^n \phi_i$ . Thus, by the completeness of C for L,  $\Rightarrow \neg \bigwedge_{i=1}^n \phi_i$  and by  $[\not\Rightarrow \neg \land] \phi_1, \ldots, \phi_n \Rightarrow$  are derivable in C. Let  $\psi \in \mathcal{AS}$  be arbitrary, by [RMon], a =

 $\phi_1, \ldots, \phi_n \Rightarrow \neg \psi$  is derivable in C. Note that  $a \in \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \emptyset)$  and a attacks any argument  $b \in \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$  for which  $\psi \in \operatorname{Ass}(b)$ . Since  $\psi \in \mathcal{AS}$  was arbitrary, it follows that  $\operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \emptyset)$  attacks any argument with a non-empty set of assumptions. Hence,  $\operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \emptyset)$  attacks any argument not in it. Therefore  $\operatorname{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S}, \mathcal{AS})) = \{\operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \emptyset)\}$  for each  $\mathsf{sem} \in \{\mathsf{grd}, \mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$ .

For the following lemmas let  $\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S}, \mathcal{AS}) = \langle \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}), \mathcal{AT}, \leq_{\pi} \rangle$  be an argumentation framework. The framework is induced by the logic  $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$  (with corresponding calculus C), the set of  $\mathcal{L}$ -formulas  $\mathcal{S}$ , the set of assumptions  $\mathcal{AS}$ , the priority ordering  $\pi$  on formulas in  $\mathcal{L}$  ( $\leq_{\pi}$  is based on  $\pi$ ) and the attack rules  $\operatorname{AT}_{\mathcal{AS}}^{\star,\leq_{\pi}}$  and  $\operatorname{AT}_{\mathcal{AS}}^{\mathsf{Con}}$ , where  $\star \in \{\Rightarrow, \Leftrightarrow\}$ . Moreover, let  $\mathsf{sem} \in \{\mathsf{grd}, \mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$ . In view of Remark 5 and Lemma 2, suppose that  $\mathcal{S}$  is consistent.

Before proving the rationality postulates for assumptive sequent-based argumentation, two helpful lemmas are considered. The first shows that an argument a is only  $AT_{\mathcal{AS}}^{\mathsf{Con}}$ -attacked if its set of assumptions is inconsistent with the set of facts. The second lemma shows that the set of assumptions from an extension together with the set of facts is always consistent. Together with the rationality postulates, these are good properties to have: the arguments that are accepted in the end should have no assumptions that are conflicting with the facts.

**Lemma 3.**  $a = A \ rightarrow \Gamma \Rightarrow \phi \in Arg_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}) \text{ is } AT_{\mathcal{AS}}^{\mathsf{Con}} \text{-}attacked iff } A \cup \mathcal{S} \text{ is inconsistent.}$ 

*Proof.* Let  $a = A \ rightarrow \Gamma \Rightarrow \phi \in \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$  and

- ⇒ suppose that *a* is  $\operatorname{AT}_{\mathcal{AS}}^{\mathsf{Con}}$ -attacked. Thus there is some  $\Delta \subseteq \mathcal{S}$  such that  $\Delta \Rightarrow \neg \bigwedge A$  is derivable in C. Hence, by  $[\not\Rightarrow \neg \land] A, \Delta \Rightarrow$  is derivable, by  $[\Rightarrow \neg \land]$  it follows that  $\Rightarrow \neg \land (A \cup \Delta)$  is derivable. Thus, by the soundness of C for L,  $\vdash \neg \land (A \cup \Delta)$ . Therefore, by Definition 2,  $A \cup \mathcal{S}$  is inconsistent.
- $\leftarrow \text{ now suppose that } A \cup S \text{ is inconsistent. Then there are } \phi_1, \dots, \phi_n \in A \cup S \text{ such that } \vdash \neg \bigwedge_{i=1}^n \phi_i. \text{ Note that } \{\phi_1, \dots, \phi_n\} \cap A \neq \emptyset, \text{ since } S \text{ is consistent by assumption. Thus, by the completeness of C for L}, \Rightarrow \neg \bigwedge_{i=1}^n \phi_i. \text{ Hence, by } [\not \Rightarrow \neg \land], \phi_1, \dots, \phi_n \Rightarrow \text{ is derivable in C. Let } \{\phi_1, \dots, \phi_n\} \cap S = \Delta. \text{ By } [\text{LMon}], \Delta, A \Rightarrow \text{ is derivable and, by } [\Rightarrow \neg \land], \Delta \Rightarrow \neg \land A \text{ is derivable. Hence } a \text{ is } AT_{AS}^{\text{Con-attacked.}} \square$

**Lemma 4** (Consistency of the assumptions). Let  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}^{\leq_{\pi}}(\mathcal{S}, \mathcal{AS}))$ , then  $\mathsf{Ass}(\mathcal{E}) \cup \mathcal{S}$  is consistent.

*Proof.* Let  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}^{\leq_{\pi}}(\mathcal{S},\mathcal{AS}))$  and suppose, towards a contradiction, that  $\mathsf{Ass}(\mathcal{E}) \cup \mathcal{S}$  is not consistent. Then there is a minimal set of formulas  $\Gamma = \{\phi_1, \ldots, \phi_n\}$ 

 $\phi_n\} \subseteq \mathsf{Ass}(\mathcal{E})$  such that there are formulas  $\psi_1, \ldots, \psi_m \in \mathcal{S}$  for which  $\vdash \neg(\bigwedge_{i=1}^n \phi_i \land \bigwedge_{j=1}^m \psi_j)$ . Note that  $n \ge 1$ , since  $\mathcal{S}$  is consistent by assumption.

By the completeness of C for L, it follows that  $\Rightarrow \neg(\bigwedge_{i=1}^{n} \phi_i \land \bigwedge_{j=1}^{m} \psi_j)$  is derivable in C. By  $[\not\Rightarrow \neg \land] \phi_1, \ldots, \phi_n, \psi_1, \ldots, \psi_m \Rightarrow$  is derivable in C. Let  $\phi_i \in \{\phi_1, \ldots, \phi_n\}$  be such that  $\pi(\phi_i) = \max_{\pi}(\{\phi_1, \ldots, \phi_n\})$ . By  $[\Rightarrow \neg]$  and  $\operatorname{AS}^l_{\mathcal{AS}} a = \phi_1, \ldots, \phi_{i-1}, \phi_{i+1}, \ldots, \phi_n$   $\checkmark \psi_1, \ldots, \psi_m \Rightarrow \neg \phi_i$ . Note that a cannot be  $\operatorname{AT}^{\operatorname{Con}}_{\mathcal{AS}}$  attacked. This follows since  $\operatorname{Ass}(a) = \{\phi_1, \ldots, \phi_{i-1}, \phi_{i+1}, \ldots, \phi_n\}$  and thus  $\operatorname{Ass}(a) \subsetneq \Gamma$ , but  $\Gamma$  was assumed to be minimal. Since  $\phi_1, \ldots, \phi_n \in \operatorname{Ass}(\mathcal{E})$ , any attacker of a is an attacker of some  $a' \in \mathcal{E}$ . Therefore, because  $\mathcal{E}$  is a completeness-based extension,  $a \in \mathcal{E}$ . Recall that  $\phi_i$  was chosen such that  $\max_{\pi}(\operatorname{Ass}(a)) \leq \pi(\phi_i)$ . Since  $\phi_i \in \operatorname{Ass}(\mathcal{E})$ , there is some  $b \in \mathcal{E}$  such that  $\phi_i \in \operatorname{Ass}(b)$ . Thus a attacks b. A contradiction to the conflict-freeness of  $\mathcal{E}$ .

With this the rationality postulates from Definition 16 can be shown.

**Lemma 5** (Closure).  $\mathcal{AF}_{\mathsf{L}}^{\leq_{\pi}}(\mathcal{S}, \mathcal{AS})$  satisfies closure of extensions: for each extension  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}^{\leq_{\pi}}(\mathcal{S}, \mathcal{AS}))$  it holds that  $\mathsf{Concs}(\mathcal{E}) = \mathsf{CN}(\mathsf{Concs}(\mathcal{E}))$ .

*Proof.* ( $\subseteq$ ) This follows immediately by the reflexivity of  $\vdash$ .

 $(\supseteq) \text{ Now suppose that } \phi \in \mathsf{CN}(\mathsf{Concs}(\mathcal{E})). \text{ Thus there are } \phi_1, \ldots, \phi_n \in \mathsf{Concs}(\mathcal{E})$ such that  $\phi_1, \ldots, \phi_n \vdash \phi$  and  $a_i = A_i \ \ \Gamma_i \Rightarrow \phi_i \in \mathcal{E}$  for each  $i \in \{1, \ldots, n\}$ . Since C is complete for L, it follows that  $\phi_1, \ldots, \phi_n \Rightarrow \phi$  is derivable in C. Thus, by [Cut], from the  $a_i$ 's  $a = A_1, \ldots, A_n \ \ \Gamma_1, \ldots, \Gamma_n \Rightarrow \phi$  is derivable in C'. If a is not attacked (e.g., because  $\mathsf{Ass}(a) = \emptyset$ ) it follows immediately that  $a \in \mathcal{E}$  thus that  $\phi \in \mathsf{Concs}(\mathcal{E})$ . Now suppose that a is attacked by some  $b \in \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ . Note that, by Lemma 3, this is not an  $\operatorname{AT}_{\mathcal{AS}}^{\mathsf{Con}}$ -attack, since by Lemma 4,  $\operatorname{Ass}(\mathcal{E}) \cup \mathcal{S}$  is consistent and  $\operatorname{Ass}(a) \subseteq \operatorname{Ass}(\mathcal{E})$ . Thus there is some  $\psi \in A_i$ , for some  $i \in \{1, \ldots, n\}$  such that  $\operatorname{Conc}(b) \Rightarrow \neg \psi$  and  $\max_{\pi}(\operatorname{Ass}(b)) \leq \pi(\psi)$ . It follows immediately that b attacks  $a_i$ as well. Since  $a_i \in \mathcal{E}$  and  $\mathcal{E}$  is complete, it follows that  $a \in \mathcal{E}$  as well. Therefore  $\phi \in \operatorname{Concs}(\mathcal{E})$ .

Rather than showing sub-argument closure directly, a stronger property is shown: an argument constructed from assumptions that other arguments in an extension already contain is also part of that extension.

**Lemma 6** (Assumption inclusion).  $\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S}, \mathcal{AS})$  satisfies assumption inclusion: for  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S}, \mathcal{AS}))$  and  $a \in Arg_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ , if  $\mathsf{Ass}(a) \subseteq \mathsf{Ass}(\mathcal{E})$ , then  $a \in \mathcal{E}$ .

Proof. Let  $a = A \ \mathbf{f} \ \Gamma \Rightarrow \phi \in \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$  such that  $\operatorname{Ass}(a) \subseteq \operatorname{Ass}(\mathcal{E})$ . Suppose there is some  $b = A' \ \mathbf{f} \ \Delta \Rightarrow \psi \in \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$  such that b attacks a (if no such attacker exists it follows immediately that  $a \in \mathcal{E}$ ). By Lemma 3, this is not an  $\operatorname{AT}_{\mathcal{AS}}^{\mathsf{Con}}$ -attack,

since  $\operatorname{Ass}(a) \subseteq \operatorname{Ass}(\mathcal{E})$ , hence by Lemma 4,  $A \cup \mathcal{S}$  is consistent. Thus there is some  $\gamma \in A$  such that  $\psi \Rightarrow \neg \gamma$  and  $\max_{\pi}(A') \leq \pi(\gamma)$ . Since  $\operatorname{Ass}(a) \subseteq \operatorname{Ass}(\mathcal{E})$ , there is some  $c \in \mathcal{E}$  such that  $\gamma \in \operatorname{Ass}(c)$ . Thus *b* attacks *c* as well. Therefore, since  $\mathcal{E}$  is assumed to be complete,  $\mathcal{E}$  defends *c* and thus *a* from the attack by *b*. It follows that  $a \in \mathcal{E}$ .

From the lemma above it follows immediately that an extension is the set of arguments constructed from S and some  $\mathcal{AS}' \subseteq \mathcal{AS}$ .

**Corollary 1.** For any  $\mathcal{E} \in \operatorname{Ext}_{\operatorname{sem}}(\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S}, \mathcal{AS}))$  there is some  $\mathcal{AS}' \subseteq \mathcal{AS}$  such that  $\mathcal{E} = \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}')$ .

Proof. First note that, for any  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S},\mathcal{AS}))$ , there is always some  $\mathcal{AS}' \subseteq \mathcal{AS}$  such that  $\mathcal{E} \subseteq \operatorname{Arg}_{\mathsf{L}}(\mathcal{S},\mathcal{AS}')$ . In particular,  $\mathcal{E} \subseteq \operatorname{Arg}_{\mathsf{L}}(\mathcal{S},\mathsf{Ass}(\mathcal{E}))$ . Now let  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S},\mathcal{AS}))$  and let  $\mathcal{AS}' = \mathsf{Ass}(\mathcal{E})$ . Consider some  $a \in \operatorname{Arg}_{\mathsf{L}}(\mathcal{S},\mathcal{AS}')$ , thus  $\mathsf{Ass}(a) \subseteq \mathcal{AS}'$ . By Lemma 6 it follows immediately that  $a \in \mathcal{E}$ . Hence,  $\mathcal{E} \supseteq \operatorname{Arg}_{\mathsf{L}}(\mathcal{S},\mathcal{AS}')$  as well.

The following lemma is a corollary of the above result:

**Lemma 7** (Sub-argument closure).  $\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S}, \mathcal{AS})$  satisfies sub-argument closure: let  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S}, \mathcal{AS}))$ , then  $a \in \mathcal{E}$  implies that  $\mathsf{Sub}(a) \subseteq \mathcal{E}$ .

**Lemma 8** (Consistency).  $\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S}, \mathcal{AS})$  satisfies consistency: for each extension  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S}, \mathcal{AS}))$  it holds that  $\mathsf{Concs}(\mathcal{E})$  is consistent.

*Proof.* Suppose, towards a contradiction, that  $\mathsf{Concs}(\mathcal{E})$  is not consistent. Then there are  $\phi_1, \ldots, \phi_n \in \mathsf{Concs}(\mathcal{E})$  such that  $\vdash \neg \bigwedge_{i=1}^n \phi_i$ . By the completeness of  $\mathsf{C}$ for  $\mathsf{L}, \Rightarrow \neg \bigwedge_{i=1}^n \phi_i$  is derivable in  $\mathsf{C}$ . Hence, there are arguments  $a_1, \ldots, a_n \in \mathcal{E}$ such that  $a_i = A_i \, \bigl\langle \, \Gamma_i \Rightarrow \phi_i \text{ for } i \in \{1, \ldots, n\}$ . Then, by  $[\not\Rightarrow \neg \land], \phi_1, \ldots, \phi_n \Rightarrow$ is derivable and, by [Cut], so is  $a = A_1, \ldots, A_n \, \bigl\langle \, \Gamma_1, \ldots, \Gamma_n \Rightarrow \, .$  By construction  $\mathsf{Ass}(a) \subseteq \mathsf{Ass}(\mathcal{E})$ . Thus, by Lemma 6,  $a \in \mathcal{E}$ . However, by Remark 1 and  $[\Rightarrow \neg \land],$  $\Rightarrow \neg \bigwedge_{i=1}^n (A_i \cup \Gamma_i)$  is derivable in  $\mathsf{C}$ . A contradiction to Lemma 4. Thus  $\mathsf{Concs}(\mathcal{E})$  is consistent.  $\Box$ 

From these lemmas the next theorem follows.

**Theorem 1.** Let  $L = \langle \mathcal{L}, \vdash \rangle$  be a logic with corresponding sound and complete sequent calculus C in which the rules from Figure 6 are admissible, let S be a consistent set of  $\mathcal{L}$ -formulas,  $\mathcal{AS}$  a set of assumptions and let  $\pi$  be a priority function on the formulas in  $\mathcal{L}$ . Moreover, let  $\mathcal{AF}_{L}^{\leq \pi}(S, \mathcal{AS}) = \langle Arg_{L}(S, \mathcal{AS}), \mathcal{AT}, \leq_{\pi} \rangle$  be the corresponding argumentation framework, with  $AT_{\mathcal{AS}}^{\star,\leq_{\pi}}$  and  $AT_{\mathcal{AS}}^{\mathsf{Con}}$  as the attack rules, where  $\star \in \{\Rightarrow, \Leftrightarrow\}$ . Then  $\mathcal{AF}_{L}^{\leq \pi}(S, \mathcal{AS})$  satisfies closure of extensions, sub-argument closure and consistency for completeness-based semantics.

### 3.3 Maximally Consistent Subsets with Assumptions

In many reasoning contexts, the provided information is inconsistent. A well-known way to maintain consistency when given an inconsistent set of formulas is by means of reasoning with maximally consistent subsets, as introduced in [60]. The representation of reasoning with maximally consistent subsets by means of structured argumentation approaches has been studied in e.g. [2, 32, 40, 41], see [6] for a survey. Moreover, this kind of reasoning has been applied in several areas of artificial intelligence, such as knowledge-based integration systems [15], consistency operators for belief revision [46] and computational linguistics [49]. It is therefore useful to study the representation of reasoning with maximally consistent subsets in assumptive sequent-based argumentation as well. To do so, the notion of a maximally consistent subset has to be adjusted to account for the two sets of premises: facts (S) and assumptions (AS). Following [8], in this section we suppose that both sets are finite. First some basic notions and the entailment relations are recalled.

Notation 2. The set of all maximally consistent subsets of S for the logic L is denoted by  $MCS_{L}(S)$ . The subscript is omitted when arbitrary or clear from the context.

**Definition 17.** Let  $L = \langle \mathcal{L}, \vdash \rangle$  and  $\mathcal{S}$  a set of  $\mathcal{L}$ -formulas. Several entailment relations for reasoning with maximally consistent subsets are defined as follows:

- $\mathcal{S} \mathrel{\sim}^{\cap}_{\mathsf{mcs}} \phi$  iff  $\phi \in \mathsf{CN}(\bigcap \mathsf{MCS}(\mathcal{S}));$
- $\mathcal{S} \mathrel{\sim}^{\cup}_{\mathsf{mcs}} \phi$  iff  $\phi \in \bigcup_{\mathcal{T} \in \mathsf{MCS}(\mathcal{S})} \mathsf{CN}(\mathcal{T});$
- $\mathcal{S} \mathrel{\sim}_{\mathsf{mcs}}^{\widehat{}} \phi$  iff  $\phi \in \bigcap_{\mathcal{T} \in \mathsf{MCS}(\mathcal{S})} \mathsf{CN}(\mathcal{T})$ .

**Example 12.** Consider the set  $S = \{p, p \supset q, \neg q\}$  and core logic CL, as in Example 4. Then there are three maximally consistent subsets:  $MCS(S) = \{\{p, p \supset q\}, \{p, \neg q\}, \{p \supset q, \neg q\}\}$ . Hence  $\bigcap MCS(S) = \emptyset$ . Moreover,  $S \models_{\mathsf{mcs}}^{\cap} \phi$  if and only if  $\phi$  is a CL-tautology. But  $S \models_{\mathsf{mcs}}^{\cup} \psi$  for  $\psi \in S$  (since for each  $\psi \in S$  there is a  $\mathcal{T} \in MCS(S)$  such that  $\psi \in \mathcal{T}$ ) and  $S \models_{\mathsf{mcs}}^{\cap} p \lor \neg q$  (since from each  $\mathcal{T} \in MCS(S)$ ).

Recently it was shown that sequent-based argumentation (as recalled in Section 2.2) is a useful platform to incorporate reasoning with maximally consistent subsets [9].

**Proposition 1** ([9], Propositions 3.8 and 4.3). Let  $\mathcal{AF}_{\mathsf{CL},\{\mathsf{Ucut}\}}(\mathcal{S}) = \langle Arg_{\mathsf{L}}(\mathcal{S}), \mathcal{AT} \rangle$ , take classical logic as core logic, Undercut as attack rule and let  $\mathcal{S}$  be a set of formulas:

- $\mathcal{S} \mathrel{\sim}_{\mathsf{grd}} \phi \ i\!f\!f \, \mathcal{S} \mathrel{\sim}_{\mathsf{prf}}^{\cap} \phi \ i\!f\!f \, \mathcal{S} \mathrel{\sim}_{\mathsf{stb}}^{\cap} \phi \ i\!f\!f \, \mathcal{S} \mathrel{\sim}_{\mathsf{mcs}}^{\cap} \phi$
- $\mathcal{S} \mathrel{\mathop{\succ}}_{\mathsf{prf}}^{\cup} \phi \text{ iff } \mathcal{S} \mathrel{\mathop{\succ}}_{\mathsf{stb}}^{\cup} \phi \text{ iff } \mathcal{S} \mathrel{\mathop{\succ}}_{\mathsf{mcs}}^{\cup} \phi.$

For  $\mathcal{AF}_{\mathsf{CL},\{\mathsf{DUcut}\}}(S) = \langle Arg_{\mathsf{L}}(S), \mathcal{AT} \rangle$ , with Direct Undercut as attack rule, classical logic as core logic and S a set of formulas, it was shown that:

Indeed, the results from Examples 5 and 12 coincide.

Following the previous section, it will be assumed that S is consistent. To allow for assumptions, the set  $MCS_{L}(S, AS)$  is defined, which takes an additional set of formulas (AS) as input. Then  $T \in MCS_{L}(S, AS)$  iff  $T \subseteq AS$ ,  $T \cup S$  is consistent and there is no  $T \subset T' \subseteq AS$  such that  $T' \cup S$  is consistent. Thus,  $MCS_{L}(S, AS)$ is the set of all maximally consistent subsets of AS that are consistent with S. The entailment relations are adjusted as follows:

**Definition 18.** Let  $L = \langle \mathcal{L}, \vdash \rangle$ ,  $\mathcal{S}$  a consistent set of  $\mathcal{L}$ -formulas and  $\mathcal{AS}$  a set of assumptions.

- $\mathcal{S} \models_{\mathsf{mcs}}^{\cap,\mathcal{AS}} \phi \text{ iff } \phi \in \mathsf{CN}(\bigcap \mathsf{MCS}(\mathcal{S},\mathcal{AS}) \cup \mathcal{S});$
- $\mathcal{S} \sim_{\mathsf{mcs}}^{\cup,\mathcal{AS}} \phi$  iff  $\phi \in \bigcup_{\mathcal{T} \in \mathsf{MCS}(\mathcal{S},\mathcal{AS})} \mathsf{CN}(\mathcal{S} \cup \mathcal{T});$
- $\mathcal{S} \sim_{\mathsf{mcs}}^{\mathfrak{m},\mathcal{AS}} \phi$  iff  $\phi \in \bigcap_{\mathcal{T} \in \mathsf{MCS}(\mathcal{S},\mathcal{AS})} \mathsf{CN}(\mathcal{S} \cup \mathcal{T}).$

**Example 13.** Let CL be the core logic,  $S = \{p\}$  and  $\mathcal{AS} = \{p \supset q, \neg q\}$ . Recall that in Example 12, where there was no distinction between facts and defeasible assumptions, there where three maximally consistent subsets. Now, given the distinction, there are two:  $\mathsf{MCS}_{\mathsf{CL}}(S, \mathcal{AS}) = \{\{p \supset q\}, \{\neg q\}\}$ . Therefore,  $S \triangleright_{\mathsf{mcs}}^{\cap, \mathcal{AS}} \phi$  iff  $\phi \in \mathsf{CN}_{\mathsf{CL}}(\{p\})$ , this is the case since p is now a fact and thus should always follow. However,  $S \models_{\mathsf{mcs}}^{\cup, \mathcal{AS}} \psi$ , for  $\psi \in \{p \supset q, \neg q\}$ .

In order to generalize reasoning with maximally consistent subsets to the prioritized setting, we define an ordering on sets of  $\mathcal{L}$ -formulas:

**Definition 19.** Let  $\Gamma, \Delta \subseteq \mathcal{L}$  and let  $\pi$  be a priority function on  $\mathcal{L}$ . Where  $\pi_j(\Gamma) = \{\phi \in \Gamma \mid \pi(\phi) = j\}, \Gamma \preceq_{\pi} \Delta$  if and only if there is some  $i \geq 1$ , such that  $\pi_i(\Gamma) \supseteq \pi_i(\Delta)$  and for each  $j < i, \pi_j(\Gamma) = \pi_j(\Delta)$ . When  $\Delta \not\preceq_{\pi} \Gamma$ , then  $\Gamma \prec_{\pi} \Delta$ .

**Remark 6.** The ordering on sets of formulas from Definition 19 is transitive: if  $S_1 \leq_{\pi} S_2$  and  $S_2 \leq_{\pi} S_3$  then  $S_1 \leq_{\pi} S_3$ .

With this, the set of the  $\leq_{\pi}$ -most preferred maximally consistent subsets can be defined:

**Definition 20.**  $\mathsf{MCS}^{\preceq}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}) = \{\mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}) \mid \nexists \mathcal{T}' \in \mathsf{MCS}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}) \text{ such that } \mathcal{T}' \prec_{\pi} \mathcal{T} \}.$ 

**Example 14.** Consider again CL as the core logic,  $S = \{p\}$  and  $\mathcal{AS} = \{p \supset q, \neg q\}$ . Let  $\pi$  be the priority function from Example 10, where  $\pi(p \supset q) = 2$  and  $\pi(\neg q) = 3$ . Then  $\mathsf{MCS}_{\mathsf{CL}}^{\preceq}(S, \mathcal{AS}) = \{\{p \supset q\}\}$ .

Now consider S = r and  $AS = \{p, q, \neg p \lor \neg q\}$ . There are three maximally consistent subsets:  $MCS_{CL}(S, AS) = \{\{p, q\}, \{p, \neg p \lor \neg q\}, \{q, \neg p \lor \neg q\}\}$ . Consider two cases:

- Let  $\pi(p) = 1$ ,  $\pi(q) = 2$  and  $\pi(\neg p \lor \neg q) = 3$ . Then  $\{p,q\} \prec_{\pi} \{p,\neg p \lor \neg q\} \prec_{\pi} \{q,\neg p \lor \neg q\}$ . Thus  $\mathsf{MCS}_{\mathsf{CL}}^{\prec}(\mathcal{S},\mathcal{AS}) = \{\{p,q\}\}.$
- If  $\pi(p) = \pi(q) = 2$  and  $\pi(\neg p \lor \neg q) = 1$ , then  $\{p, \neg p \lor \neg q\}$  and  $\{q, \neg p \lor \neg q\}$  are incomparable and both are strictly preferred to  $\{p, q\}$ . Thus  $\mathsf{MCS}_{\mathsf{CL}}^{\preceq}(\mathcal{S}, \mathcal{AS}) = \{\{p, \neg p \lor \neg q\}, \{q, \neg p \lor \neg q\}\}$ .

The prioritized counterparts of the entailment relations from Definition 18 are defined as:

**Definition 21.** Let  $L = \langle \mathcal{L}, \vdash \rangle$ ,  $\mathcal{S}$  a consistent set of  $\mathcal{L}$ -formulas,  $\mathcal{AS}$  a set of assumptions and  $\pi$  a priority function on  $\mathcal{L}$ .

- $\mathcal{S} \models_{\mathsf{mcs},\preceq}^{\cap,\mathcal{AS}} \phi \text{ iff } \phi \in \mathsf{CN}(\bigcap \mathsf{MCS}^{\preceq}(\mathcal{S},\mathcal{AS}) \cup \mathcal{S});$
- $\mathcal{S} \models_{\mathsf{mcs},\preceq}^{\cup,\mathcal{AS}} \phi \text{ iff } \phi \in \bigcup_{\mathcal{T} \in \mathsf{MCS}^{\preceq}(\mathcal{S},\mathcal{AS})} \mathsf{CN}(\mathcal{S} \cup \mathcal{T});$
- $\mathcal{S} \sim_{\mathsf{mcs},\preceq}^{\oplus,\mathcal{AS}} \phi$  iff  $\phi \in \bigcap_{\mathcal{T} \in \mathsf{MCS}^{\preceq}(\mathcal{S},\mathcal{AS})} \mathsf{CN}(\mathcal{S} \cup \mathcal{T}).$

**Example 15.** Recall from Example 14, that for  $S = \{p\}$ ,  $\mathcal{AS} = \{p \supset q, \neg q\}$ ,  $\pi$  such that  $\pi(p \supset q) = 2$  and  $\pi(\neg q) = 3$ , there is only one assumptive maximally consistent subset:  $\mathsf{MCS}_{\mathsf{CL}}^{\preceq}(S, \mathcal{AS}) = \{\{p \supset q\}\}$ . It thus follows that  $S \models_{\mathsf{mcs},\preceq}^{\star,\mathcal{AS}} \phi$ , where  $\star \in \{\cup, \cap, \mathbb{M}\}$  iff  $\phi \in \mathsf{CN}_{\mathsf{CL}}(\{p, p \supset q\})$ .

For the last setting from Example 14, where S = r and  $\mathcal{AS} = \{p, q, \neg p \lor \neg q\}$ such that  $\pi(p) = \pi(q) = 2$  and  $\pi(\neg p \lor \neg q) = 1$  note that  $S \mid_{\mathsf{mcs},\preceq}^{\star,\mathcal{AS}} \phi$  for  $\star \in \{\cup, \cap, \mathbb{n}\}$ and  $\phi \in \mathsf{CN}_{\mathsf{CL}}(\{r, \neg p \lor \neg q\})$ . Moreover  $S \mid_{\mathsf{mcs},\preceq}^{\cup,\mathcal{AS}} \phi$  for  $\phi \in \{p, q\}$ .

The next theorem shows that it is no coincidence that the results from the first part of the previous example correspond to that of Example 11. Like in the previous section, in view of Remark 5 and Lemma 2, it will be assumed that S is consistent.
**Theorem 2.** Let  $L = \langle \mathcal{L}, \vdash \rangle$  be such that the rules from Figure 6 are admissible in its corresponding calculus C, S a finite and consistent set of  $\mathcal{L}$ -formulas,  $\mathcal{AS}$ a finite set of assumptions and  $\pi$  a priority function on  $\mathcal{L}$ . For  $\mathcal{AF}_{L}^{\leq \pi}(S, \mathcal{AS}) = \langle Arg_{L}(S, \mathcal{AS}), \mathcal{AT}, \leq_{\pi} \rangle$  an argumentation framework, where  $\mathcal{AT}$  is based on the attack rules  $AT_{\mathcal{AS}}^{*,\leq_{\pi}}$  and  $AT_{\mathcal{AS}}^{\operatorname{Con}}$ , with  $\star \in \{\Rightarrow, \Leftrightarrow\}$ :

- $1. \ \mathcal{S} \models_{\mathsf{mcs},\preceq}^{\cap,\mathcal{AS}} \phi \ i\!f\!f \, \mathcal{S} \models_{\mathcal{AS},\mathsf{grd}}^{\leq\pi} \phi \ i\!f\!f \, \mathcal{S} \models_{\mathcal{AS},\mathsf{prf}}^{\leq\pi,\cap} \phi \ i\!f\!f \, \mathcal{S} \models_{\mathcal{AS},\mathsf{stb}}^{\leq\pi,\cap} \phi$
- 2.  $S \vdash_{\mathsf{mcs},\preceq}^{\cup,\mathcal{AS}} \phi$  iff  $S \vdash_{\mathcal{AS},\mathsf{prf}}^{\leq_{\pi},\cup} \phi$  iff  $S \vdash_{\mathcal{AS},\mathsf{stb}}^{\leq_{\pi},\cup} \phi$

For the next lemmas, needed to prove the above theorem, suppose that the conditions from the theorem statement hold.

The first lemma shows that if there is an attack between two arguments, the union of the assumptions and support sets of these arguments is inconsistent.

**Lemma 9.** Let  $a_1, a_2 \in Arg_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ , if  $a_1 AT_{\mathcal{AS}}^{\star, \leq_{\pi}}$ -attacks  $a_2$ , then  $\mathsf{Ass}(a_1) \cup \mathsf{Ass}(a_2) \cup \mathsf{Supp}(a_1) \cup \mathsf{Supp}(a_2)$  is inconsistent.

Proof. Let  $a_1, a_2 \in \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$  and suppose that  $a_1 = A \ \mathsf{C} \ \Gamma \Rightarrow \phi \ \operatorname{AT}_{\mathcal{AS}}^{\star,\leq\pi}$ -attacks  $a_2$ , thus  $\phi \Rightarrow \neg \psi$ , for some  $\psi \in \operatorname{Ass}(a_2)$ . Thus, by [Cut]  $A \ \mathsf{C} \ \Gamma \Rightarrow \neg \psi$  is derivable in  $\mathsf{C}'$ . By  $[\neg\Rightarrow]$  and  $[\neg\neg\neq]$  it follows that  $A \ \mathsf{C} \ \Gamma, \psi \Rightarrow$  is derivable. Then, by Remark 1 and  $[\Rightarrow\neg\wedge]$  the sequent  $\Rightarrow \neg \wedge (A \cup \Gamma \cup \{\psi\})$  is derivable in  $\mathsf{C}$ . Hence, by the soundness of  $\mathsf{C}$  for  $\mathsf{L}$  it follows that  $\vdash \neg \wedge (A \cup \Gamma \cup \{\psi\})$ . Therefore  $\operatorname{Ass}(a_1) \cup \operatorname{Ass}(a_2) \cup \operatorname{Supp}(a_1) \cup \operatorname{Supp}(a_2)$  is inconsistent.  $\Box$ 

The next lemma shows that for any maximally consistent subset of assumptions (i.e, any member of  $MCS_{L}^{\leq}(\mathcal{S}, \mathcal{AS})$ ), no consistent set of assumptions can be strictly preferred over it.

**Lemma 10.** Let  $\mathcal{T} \in \mathsf{MCS}^{\preceq}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ , if  $\mathcal{AS}' \subseteq \mathcal{AS}$  is such that  $\mathcal{AS}' \cup \mathcal{S}$  is consistent, then  $\mathcal{AS}' \not\prec_{\pi} \mathcal{T}$ .

Proof. Let  $\mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}^{\preceq}(\mathcal{S}, \mathcal{AS})$  and  $\mathcal{AS}' \subseteq \mathcal{AS}$  such that  $\mathcal{AS}' \cup \mathcal{S}$  is consistent. Then there is some  $\mathcal{AS} \supseteq \mathcal{AS}^* \supseteq \mathcal{AS}'$  such that  $\mathcal{AS}^* \in \mathsf{MCS}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ . Since  $\mathcal{AS}' \subseteq \mathcal{AS}^*$ ,  $\mathcal{AS}' = \mathcal{AS}^*$  or there is an  $i \ge 1$  such that  $\pi_i(\mathcal{AS}^*) \supseteq \pi_i(\mathcal{AS}')$  and  $\pi_j(\mathcal{AS}^*) = \pi_j(\mathcal{AS}')$ for each j < i and thus  $\mathcal{AS}^* \preceq_{\pi} \mathcal{AS}'$ . By definition of  $\mathsf{MCS}_{\mathsf{L}}^{\preceq}(\mathcal{S}, \mathcal{AS})$ ,  $\mathcal{AS}^* \not\prec_{\pi} \mathcal{T}$ .

With help from the above two lemmas, the next three lemmas show how maximally consistent subsets of assumptions are related to grounded (Lemma 11), stable (Lemma 12) and preferred (Lemma 13) extensions. **Lemma 11.** If  $A \subseteq \bigcap \mathsf{MCS}_{\mathsf{L}}^{\prec}(\mathcal{S}, \mathcal{AS})$  and  $a = A \ {} \quad \Gamma \Rightarrow \phi \in Arg_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ , for some  $\Gamma \subseteq \mathcal{S}$ , then  $a \in \mathsf{Ext}_{\mathsf{grd}}(\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S}, \mathcal{AS}))$ .

Proof. Let  $\mathcal{E} \in \operatorname{Ext_{cmp}}(\mathcal{AF}_{\mathsf{L}}^{\leq_{\pi}}(\mathcal{S},\mathcal{AS}))$  and  $A \subseteq \bigcap \operatorname{MCS}_{\mathsf{L}}^{\leq}(\mathcal{S},\mathcal{AS})$  and suppose that a = A  $\begin{subarray}{l} \Gamma \Rightarrow \phi \in \operatorname{Arg}_{\mathsf{L}}(\mathcal{S},\mathcal{AS})$  for some  $\Gamma \subseteq \mathcal{S}$ . Suppose that there is some  $b \in \operatorname{Arg}_{\mathsf{L}}(\mathcal{S},\mathcal{AS})$ , such that b attacks a. Note that, since  $A \subseteq \bigcap \operatorname{MCS}_{\mathsf{L}}^{\leq}(\mathcal{S},\mathcal{AS})$ ,  $A \cup \mathcal{S}$  is consistent. Thus, by Lemma 3, b cannot  $\operatorname{AT}_{\mathcal{AS}}^{\mathsf{Con}}$ -attack a. By Lemma 9, it follows that  $A \cup \Gamma \cup \operatorname{Ass}(b) \cup \operatorname{Supp}(b)$  is inconsistent. Also since  $A \subseteq \bigcap \operatorname{MCS}_{\mathsf{L}}^{\leq}(\mathcal{S},\mathcal{AS})$ ,  $\Gamma \cup \operatorname{Ass}(b) \cup \operatorname{Supp}(b)$  is inconsistent. Therefore, there are  $\psi_1, \ldots, \psi_n \in \Gamma \cup \operatorname{Ass}(b) \cup \operatorname{Supp}(b)$ , such that  $\vdash \neg(\psi_1 \wedge \ldots \wedge \psi_n)$ . Note that, since  $\Gamma, \operatorname{Supp}(b) \subseteq \mathcal{S}$  and  $\mathcal{S}$  is consistent by assumption  $\operatorname{Ass}(b) \cap \{\psi_1, \ldots, \psi_n\} \neq \emptyset$ .

Suppose, wlog., that  $\Delta = \{\psi_1, \ldots, \psi_l\} \subseteq S$  and  $\{\psi_{l+1}, \ldots, \psi_n\} \subseteq \mathsf{Ass}(b)$ . Then, by the completeness of C for L and  $[\not\Rightarrow \neg \land], \Delta, \psi_{l+1}, \ldots, \psi_n \Rightarrow$  is derivable in C. By [LMon], and  $[\Rightarrow \neg \land] c = \Delta \Rightarrow \neg \land \mathsf{Ass}(b)$  is derivable in C. Since  $\mathsf{Ass}(c) = \emptyset$  and  $\Delta \subseteq S$  it follows that  $c \in \operatorname{Arg}_{\mathsf{L}}(S, \mathcal{AS})$ . This also means that c cannot be attacked, therefore  $c \in \mathcal{E}$ . From this it follows that b is  $\operatorname{AT}_{\mathcal{AS}}^{\operatorname{Con}}$ -attacked by  $\mathcal{E}$ . Thus  $\mathcal{E}$  defends a from any attacker. Moreover, since  $\mathcal{E}$  was arbitrarily chosen, a is part of any complete extension. Recall that the grounded extension is the  $\subseteq$ -minimal complete extension. Therefore  $a \in \operatorname{Ext}_{\operatorname{grd}}(\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(S, \mathcal{AS}))$ .

The proofs of the following two lemmas are based on proofs in [45].

**Lemma 12.** If  $\mathcal{T} \in \mathsf{MCS}^{\leq}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ , then  $Arg_{\mathsf{L}}(\mathcal{S}, \mathcal{T}) \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}^{\leq \pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$ .

*Proof.* Let  $\mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}^{\leq}(\mathcal{S}, \mathcal{AS})$  and let  $\mathcal{E} = \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T})$ . In what follows we show that  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{stb}}\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S}, \mathcal{AS})$ , by showing that  $\mathcal{E}$  is conflict-free and stable.<sup>7</sup>

 $\mathcal{E}$  is conflict-free. Suppose towards a contradiction, that  $\mathcal{E}$  is not conflict-free. Then there are  $a_1, a_2 \in \mathcal{E}$  such that  $a_1 = A_1 \land \Gamma_1 \Rightarrow \phi_1$ ;  $a_2 = A_2 \land \Gamma_2 \Rightarrow \phi_2$  and  $a_1 \operatorname{AT}_{\mathcal{AS}}^{\star, \leq \pi}$ -attacks  $a_2$ , for  $\star \in \{\Rightarrow, \Leftrightarrow, \mathsf{Con}\}$ . Since  $A_2 \subseteq \mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}^{\leq}(\mathcal{S}, \mathcal{AS}), A_2 \cup \mathcal{S}$  is consistent. Thus, by Lemma 3, this is not an  $\operatorname{AT}_{\mathcal{AS}}^{\mathsf{Con}}$  attack. However then, by Lemma 9,  $\operatorname{Ass}(a_1) \cup \operatorname{Ass}(a_2) \cup \operatorname{Supp}(a_1) \cup \operatorname{Supp}(a_2)$  is inconsistent, a contradiction to the assumption that  $\operatorname{Ass}(a_1), \operatorname{Ass}(a_2) \subseteq \mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}^{\leq}(\mathcal{S}, \mathcal{AS})$ . Thus  $\mathcal{E}$  is conflict-free.

 $\mathcal{E}$  is stable. Now suppose that there is some  $b = A' \ \Gamma' \Rightarrow \phi' \in \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}) \setminus \mathcal{E}$ and  $\mathcal{E}$  does not attack b. Thus, since  $\mathcal{E} = \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T})$  and  $b \notin \mathcal{E}$ , there is some  $\phi \in \operatorname{Ass}(b)$  such that  $\phi \notin \mathcal{T}$ . Suppose first that  $A' \cup \mathcal{S}$  is inconsistent. Then b is  $\operatorname{AT}_{\mathcal{AS}}^{\operatorname{Con}}$ -attacked by an argument that has an empty assumptions set and thus cannot be attacked itself. It follows immediately that  $\mathcal{E}$  attacks b, a contradiction. Now

<sup>&</sup>lt;sup>7</sup>The statements " $\mathcal{E}$  conflict-free and stable" (i.e.,  $\mathcal{E}$  is conflict-free and attacks all arguments not in it) and " $\mathcal{E}$  is complete and stable" are equivalent [17, Proposition 3.39].

suppose that  $A' \cup S$  is consistent. Since  $\mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}^{\prec}(S, \mathcal{AS})$  (i.e.,  $\mathcal{T}$  is maximally consistent w.r.t. S) and  $\phi \notin \mathcal{T}, \mathcal{T} \cup S \cup \{\phi\}$  is inconsistent. Let  $\mathcal{C}_1, \mathcal{C}_2, \ldots \subseteq \mathcal{T}$  be all the minimal subsets of  $\mathcal{T}$  such that  $\mathcal{C}_i \cup S \cup \{\phi\}$  is inconsistent. Thus, for each i, there are  $\psi_1^i, \ldots, \psi_{n_i}^i \in \mathcal{C}_i \cup S$  such that  $\vdash \neg(\psi_1^i \wedge \ldots \wedge \psi_{n_i}^i \wedge \phi)$ . By the completeness of  $\mathsf{C}$  for  $\mathsf{L}, \Rightarrow \neg(\psi_1^i \wedge \ldots \wedge \psi_{n_i}^i \wedge \phi)$  is derivable in  $\mathsf{C}$ . By  $[\not\Rightarrow \neg \wedge]$  and  $[\Rightarrow \neg], \psi_1^i, \ldots, \psi_{n_i}^i \Rightarrow \neg \phi$  is derivable in  $\mathsf{C}$ . Let  $A_i = \{\psi_1^i, \ldots, \psi_{n_i}^i\} \cap \mathcal{AS}$  and  $\Gamma_i = \{\psi_1^i, \ldots, \psi_{n_i}^i\} \cap S$ . Note that, by assumption,  $\mathcal{AS} \cap \mathcal{S} = \emptyset$  and  $\{\psi_1^i, \ldots, \psi_{n_i}^i\} \subseteq \mathcal{AS} \cup \mathcal{S}$ , hence  $A_i \cup \Gamma_i = \{\psi_1^i, \ldots, \psi_{n_i}^i\}$ . Thus, by  $\mathrm{AS}_{\mathcal{AS}}^l, a_i = A_i \ \binom{\binom{\mathcal{C}}{\Gamma}} \Rightarrow \neg \phi \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T})$ . However, since  $a_i$  does not attack  $b, \max_{\pi}(A_i) \not\leq \pi(\phi)$ , for all i.

Let  $\Pi \subseteq \bigcup_{i\geq 1} \max_{\pi}(\mathcal{C}_i)$  be such that it contains at least one member of each  $\{\psi \in \mathcal{C}_i \mid \pi(\psi) = \max_{\pi}(\mathcal{C}_i)\}$  and let  $\Theta = (\mathcal{T} \setminus \Pi) \cup \{\phi\}$ . Note that, since  $\max_{\pi}(\mathcal{C}_i) > \pi(\phi)$  for each i,  $\min_{\pi}(\Pi) > \pi(\phi)$ , thus  $\Theta \prec_{\pi} \mathcal{T}$ . Suppose first that  $\Theta \cup \mathcal{S}$  is not consistent. Then there are  $\Theta' \subseteq \Theta$  and  $\Delta \subseteq \mathcal{S}$  such that  $\vdash \neg \wedge (\Theta' \cup \Delta)$ . Note that  $\phi \in \Theta'$ , since  $\Theta' \setminus \{\phi\} \subseteq \mathcal{T}$ . Therefore, there is some i such that  $\Theta' \setminus \{\phi\} \supseteq \mathcal{C}_i$ . However, by construction, there is some  $\psi \in \mathcal{C}_i$  such that  $\psi \notin \Theta$ . A contradiction. Therefore,  $\Theta \cup \mathcal{S}$  is consistent. Hence, by Lemma 10  $\Theta \not\prec_{\pi} \mathcal{T}$ . Also a contradiction. Therefore,  $\mathcal{E}$  attacks b, from which it follows that  $\mathcal{E}$  is stable.

**Lemma 13.** Let  $\mathcal{E} \in \text{Ext}_{prf}(\mathcal{AF}_{L}^{\leq_{\pi}}(\mathcal{S},\mathcal{AS}))$ , then there is some  $\mathcal{T} \in \text{MCS}_{L}^{\leq}(\mathcal{S},\mathcal{AS})$  such that  $\mathcal{E} = Arg_{L}(\mathcal{S},\mathcal{T})$ .

Proof. Suppose, for a contradiction, that there is some  $\mathcal{E} \in \operatorname{Ext}_{\operatorname{prf}}(\mathcal{AF}_{\operatorname{L}}^{\leq \pi}(\mathcal{S}, \mathcal{AS}))$ such that there is no  $\mathcal{T} \in \operatorname{MCS}_{\operatorname{L}}^{\leq}(\mathcal{S}, \mathcal{AS})$  for which  $\mathcal{E} = \operatorname{Arg}_{\operatorname{L}}(\mathcal{S}, \mathcal{T})$ . Note that, by Corollary 1, there is some  $\mathcal{AS}' \subseteq \mathcal{AS}$  such that  $\mathcal{E} = \operatorname{Arg}_{\operatorname{L}}(\mathcal{S}, \mathcal{AS}')$ . If  $\mathcal{AS}' \cup \mathcal{S}$  would be inconsistent we have an immediate contradiction with Lemma 4. Hence, there is some  $\mathcal{T}' \in \operatorname{MCS}_{\operatorname{L}}(\mathcal{S}, \mathcal{AS})$  such that  $\mathcal{AS}' \subseteq \mathcal{T}'$ . If  $\mathcal{T}' \in \operatorname{MCS}_{\operatorname{L}}^{\leq}(\mathcal{S}, \mathcal{AS})$ , then by Lemma 12  $\operatorname{Arg}_{\operatorname{L}}(\mathcal{S}, \mathcal{T}') \in \operatorname{Ext}_{\operatorname{stb}}(\mathcal{AF}_{\operatorname{L}}^{\leq \pi}(\mathcal{S}, \mathcal{AS}))$  and therefore (by [36, Lemma 15] any stable extension is a preferred extension)  $\operatorname{Arg}_{\operatorname{L}}(\mathcal{S}, \mathcal{T}') \in \operatorname{Ext}_{\operatorname{prf}}(\mathcal{AF}_{\operatorname{L}}^{\leq \pi}(\mathcal{S}, \mathcal{AS})),$ a contradiction with the assumption that no such set exists. Therefore there is some  $\mathcal{T} \in \operatorname{MCS}_{\operatorname{L}}^{\leq}(\mathcal{S}, \mathcal{AS})$  for which  $\mathcal{T} \prec_{\pi} \mathcal{T}'$ . It follows that there is some i, such that  $\pi_i(\mathcal{T}) \supseteq \pi_i(\mathcal{T}')$  and for each  $j < i, \pi_j(\mathcal{T}) = \pi_j(\mathcal{T}')$ . Let  $\Delta = \pi_i(\mathcal{T}) \setminus \pi_i(\mathcal{T}')$  and let  $\operatorname{S} = \{a \in \operatorname{Arg}_{\operatorname{L}}(\mathcal{S}, \mathcal{AS}' \cup \Delta) \mid \Gamma \Rightarrow \neg \bigwedge \operatorname{Ass}(a)$  is derivable for some  $\Gamma \subseteq \mathcal{S}\}.$ 

Since  $\operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}') = \mathcal{E} \in \operatorname{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S}, \mathcal{AS}))$ , by Lemma 3 none of the arguments in  $\mathcal{E}$  are  $\operatorname{AT}_{\mathcal{AS}}^{\operatorname{Con}}$ -attacked thus  $\mathcal{E} \cap \mathsf{S} = \emptyset$ . Note that, since  $\pi_j(\mathcal{AS}' \cup \Delta) = \pi_j(\mathcal{T})$  for  $j \leq i$  and there is some  $\psi \in \pi_i(\mathcal{AS}' \cup \Delta) \setminus \pi_i(\mathcal{AS}')$ ,  $\operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}' \cup \Delta) \setminus \mathsf{S} \neq \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}')$  thus  $\psi \diamondsuit \oplus \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}' \cup \Delta)$ . We show that  $\mathcal{E}' = \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}' \cup \Delta) \setminus \mathsf{S}$  is admissible.

 $\mathcal{E}'$  is conflict-free. To see this, note first that  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S}, \mathcal{AS}))$  and  $\operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \Delta) \subseteq \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T}) \in \operatorname{Ext}_{\mathsf{stb}}(\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S}, \mathcal{AS}))$  are conflict-free. Suppose, for

some arguments  $a_1, a_2 \in \mathcal{E}'$ , that  $a_1$  attacks  $a_2$ . By the definition of  $\mathcal{E}'$  this is not an  $\operatorname{AT}_{\mathcal{AS}}^{\mathsf{Con}}$  attack. Thus  $\operatorname{Conc}(a_1) \Rightarrow \neg \psi$  for  $\psi \in \operatorname{Ass}(a_2)$  and  $\max_{\pi}(\operatorname{Ass}(a_1)) \leq \pi(\psi)$ . Suppose first that  $\psi \in \Delta$ . Since  $\max_{\pi}(\operatorname{Ass}(a_1)) \leq \pi(\psi) = i$ , it follows that  $\operatorname{Ass}(a_1) \subseteq \mathcal{T}$ . But then  $a_1 \in \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T})$ , a contradiction, since  $a_1$  attacks any argument with  $\psi$ in the set of assumptions and  $\operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T})$  is a stable extension and thus conflict-free. Let now  $\psi \in \mathcal{AS}'$ . Then there is some  $a_3 \in \mathcal{E}$  such that  $a_1$  attacks  $a_3$  as well. Thus  $a_1 \notin \mathcal{E}$ , since  $\mathcal{E}$  is conflict-free. Since  $a_3 \in \mathcal{E}$ , there is an  $a_4 \in \mathcal{E}$ , such that  $a_4$  attacks  $a_1$  in some formula  $\psi' \in \operatorname{Ass}(a_1) \cap \Delta$ . Hence  $\max_{\pi}(\operatorname{Ass}(a_4)) \leq \pi(\psi') = i$ . But then  $a_4 \in \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T})$ , again a contradiction. Thus  $\mathcal{E}'$  is conflict-free.

 $\mathcal{E}'$  is admissible. Note that, since  $\mathcal{E} \in \operatorname{Ext}_{\operatorname{prf}}(\mathcal{AF}_{\operatorname{L}}^{\leq \pi}(\mathcal{S},\mathcal{AS}))$ , any attack in a formula in  $\mathcal{AS}'$  is defended by  $\mathcal{E}$ . Let  $a = A \ \mathbf{I} = \mathcal{F} \Rightarrow \phi \in \mathcal{E}'$  be such that it is attacked by some  $b = A' \ \mathbf{I} \Rightarrow \phi' \in \operatorname{Arg}_{\operatorname{L}}(\mathcal{S},\mathcal{AS})$  in  $\gamma \in A \cap \Delta$ . Thus  $\max_{\pi}(A') \leq \pi(\gamma)$ . By Lemma 12,  $\operatorname{Arg}_{\operatorname{L}}(\mathcal{S},\mathcal{T}) \in \operatorname{Ext}_{\operatorname{stb}}(\mathcal{AF}_{\operatorname{L}}^{\leq \pi}(\mathcal{S},\mathcal{AS}))$ . Moreover, since  $\gamma \in \mathcal{T}$ , there is some  $a' \in \operatorname{Arg}_{\operatorname{L}}(\mathcal{S},\mathcal{T})$ , such that  $\gamma \in \operatorname{Ass}(a')$ . Hence there is some  $c \in \operatorname{Arg}_{\operatorname{L}}(\mathcal{S},\mathcal{T})$  such that c attacks b in some  $\gamma' \in A'$ . Thus  $\max_{\pi}(\operatorname{Ass}(c)) \leq \pi(\gamma)$  and since  $\gamma' \in A'$ ,  $\max_{\pi}(\operatorname{Ass}(c)) \leq \pi(\gamma) = i$ . Therefore  $c \in \mathcal{E}'$  as well. It follows that  $\mathcal{E}'$  defends itself against all attackers. Hence  $\mathcal{E}'$  is admissible. Since  $\mathcal{E}' \supseteq \mathcal{E}$  this is a contradiction to the assumption that  $\mathcal{E} \in \operatorname{Ext}_{\operatorname{prf}}(\mathcal{AF}_{\operatorname{L}}^{\leq \pi}(\mathcal{S},\mathcal{AS}))$ .

Therefore  $\mathcal{E} \subseteq \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T})$ , for some  $\mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}^{\leq}(\mathcal{S}, \mathcal{AS})$ . By Lemma 12 it follows that  $\operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T}) \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S}, \mathcal{AS}))$ , thus  $\mathcal{E} = \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T})$ .

With the above lemmas Theorem 2 can be proven:

*Proof.* Let  $\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S}, \mathcal{AS}) = \langle \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}), \mathcal{AT}, \leq_{\pi} \rangle$  be an assumptive sequent-based argumentation framework for the logic  $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ , such that the rules from Figure 6 are admissible in the corresponding calculus  $\mathsf{C}$ . Let  $\mathcal{S}$  be a finite and consistent set of  $\mathcal{L}$ -formulas,  $\mathcal{AS}$  a finite set of assumptions, let  $\phi$  be an  $\mathcal{L}$ -formula and suppose that  $\pi$  is a priority function on  $\mathcal{L}$ . Furthermore, let  $\mathcal{AT}$  be based on the attack rules  $\operatorname{AT}_{\mathcal{AS}}^{\star,\leq_{\pi}}$  and  $\operatorname{AT}_{\mathcal{AS}}^{\operatorname{Con}}$ , where  $\star \in \{\Rightarrow, \Leftrightarrow\}$ .

• ( $\Rightarrow$ ) Note that  $S \mid_{\mathcal{AS}, \mathsf{grd}}^{\leq \pi} \phi$  implies  $S \mid_{\mathcal{AS}, \mathsf{prf}}^{\leq \pi, \cap} \phi$  implies  $S \mid_{\mathcal{AS}, \mathsf{stb}}^{\leq \pi, \cap} \phi$ . Suppose that  $S \mid_{\mathsf{Mcs}, \preceq}^{\circ, \mathcal{AS}} \phi$ , thus there are  $A \subseteq \bigcap \mathsf{MCS}_{\mathsf{L}}^{\preceq}(\mathcal{S}, \mathcal{AS})$  and  $\Gamma \subseteq \mathcal{S}$ , such that  $A \cup \Gamma \vdash \phi$ . By the completeness of  $\mathsf{C}$  for  $\mathsf{L}$  and Remark 1,  $A \ \Gamma \Rightarrow \phi \in \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ . By Lemma 11, it follows that  $A \ \Gamma \Rightarrow \phi \in \operatorname{Ext}_{\mathsf{grd}}(\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S}, \mathcal{AS}))$ . Therefore  $\mathcal{S} \mid_{\mathcal{AS}, \mathsf{grd}}^{\leq \pi} \phi$  and thus  $\mathcal{S} \mid_{\mathcal{AS}, \mathsf{prf}}^{\leq \pi, \cap} \phi$  and  $\mathcal{S} \mid_{\mathcal{AS}, \mathsf{stb}}^{\leq \pi, \cap} \phi$  as well.

<sup>&</sup>lt;sup>8</sup>The statements " $\mathcal{E}$  is a  $\subseteq$ -maximal admissible set of  $\mathcal{AF}$ " and " $\mathcal{E}$  is a  $\subseteq$ -maximal complete extension of  $\mathcal{AF}$ " (the definition of preferred extensions in Definition 4) are equivalent [17, Proposition 3.35].

( $\Leftarrow$ ) Suppose that  $S \models_{\mathcal{AS},\mathsf{stb}}^{\leq_{\pi},\cap} \phi$ . Then there is an  $a \in \bigcap \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S},\mathcal{AS}))$ such that  $\mathsf{Ass}(a) \subseteq \mathcal{AS}$ ,  $\mathsf{Supp}(a) \subseteq \mathcal{S}$  and  $\mathsf{Conc}(a) = \phi$ . By Lemma 12, for each  $\mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}^{\preceq}(\mathcal{S},\mathcal{AS}), a \in \operatorname{Arg}_{\mathsf{L}}(\mathcal{S},\mathcal{T})$ . Thus  $\mathsf{Ass}(a) \subseteq \bigcap \mathsf{MCS}_{\mathsf{L}}^{\preceq}(\mathcal{S},\mathcal{AS})$ . From a, by Remark 1 and the soundness of  $\mathsf{C}$  for  $\mathsf{L}$ ,  $\mathsf{Ass}(a) \cup \mathsf{Supp}(a) \vdash \phi$ . Therefore,  $\mathcal{S} \models_{\mathsf{mcs},\preceq}^{\cap,\mathcal{AS}} \phi$ .

• ( $\Rightarrow$ ) Note that  $S \models_{\mathcal{AS},\mathsf{stb}}^{\leq \pi,\cup} \phi$  implies  $S \models_{\mathcal{AS},\mathsf{prf}}^{\leq \pi,\cup} \phi$ . Suppose that  $S \models_{\mathsf{mcs},\preceq}^{\cup,\mathcal{AS}} \phi$ . Then, there is some  $\mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}^{\leq}(\mathcal{S},\mathcal{AS})$  such that  $\mathcal{T} \cup \mathcal{S} \vdash \phi$ . By the completeness of  $\mathsf{C}$  for  $\mathsf{L}$  and Remark 1, there are  $A \subseteq \mathcal{T}$  and  $\Gamma \subseteq \mathcal{S}$  such that  $A \clubsuit \Gamma \Rightarrow \phi \in \operatorname{Arg}_{\mathsf{L}}(\mathcal{S},\mathcal{T})$ . By Lemma 12,  $\operatorname{Arg}_{\mathsf{L}}(\mathcal{S},\mathcal{T}) \in \operatorname{Ext}_{\mathsf{stb}}(\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S},\mathcal{AS}))$ . Thus there is some stable extension  $\mathcal{E}$ , such that  $\phi \in \operatorname{Concs}(\mathcal{E})$ . Therefore  $\mathcal{S} \models_{\mathcal{AS},\mathsf{stb}}^{\leq \pi,\cup} \phi$  and thus  $\mathcal{S} \models_{\mathcal{AS},\mathsf{prf}}^{\leq \pi,\cup} \phi$ .

( $\Leftarrow$ ) Suppose that  $S \models_{\mathcal{AS},\mathsf{prf}}^{\leq_{\pi},\cup} \phi$ . Then there is some  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathsf{L}}^{\leq_{\pi}}(\mathcal{S},\mathcal{AS}))$ such that there is some  $A \ \mathbf{f} \ \Gamma \Rightarrow \phi$  where  $A \subseteq \mathcal{AS}$  and  $\Gamma \subseteq \Gamma$ . Hence, by Remark 1 and the soundness of  $\mathsf{C}$  for  $\mathsf{L}, A \cup \Gamma \vdash \phi$ . By Lemma 13, there is some  $\mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}^{\preceq}(\mathcal{S},\mathcal{AS})$  such that  $\mathcal{E} = \operatorname{Arg}_{\mathsf{L}}(\mathcal{S},\mathcal{T})$ . Thus  $A \subseteq \mathcal{T}$ . Hence,  $\mathcal{S} \models_{\mathsf{mcs},\preceq}^{\cup,\mathcal{AS}} \phi$ .

• ( $\Rightarrow$ ) Note that  $S \models_{\mathcal{AS},\mathsf{prf}}^{\leq \pi, \widehat{\square}} \phi$  implies  $S \models_{\mathcal{AS},\mathsf{stb}}^{\leq \pi, \widehat{\square}} \phi$ . Suppose that  $S \models_{\mathcal{AS},\mathsf{prf}}^{\leq \pi, \widehat{\square}} \phi$ , then there is some  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S}, \mathcal{AS}))$  such that there is no  $a \in \mathcal{E}$  with  $\mathsf{Conc}(a) = \phi$ . By Lemma 13, there is some  $\mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}^{\leq}(\mathcal{S}, \mathcal{AS})$  such that  $\mathcal{E} = \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T})$ . Hence, there is no  $A \subseteq \mathcal{T}$  and  $\Gamma \subseteq \mathcal{S}$ , such that  $A \cup \Gamma \vdash \phi$ . Therefore  $\phi \notin \mathsf{CN}_{\mathsf{L}}(\mathcal{S} \cup \mathcal{T})$ . Thus  $\mathcal{S} \models_{\mathsf{mcs},\preceq}^{\in, \mathcal{AS}} \phi$ .

 $(\Leftarrow) \text{ Now suppose that } \mathcal{S} \not\models^{\textcircled{m},\mathcal{AS}}_{\mathsf{mcs},\preceq} \phi. \text{ Thus there is some } \mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}^{\preceq}(\mathcal{S},\mathcal{AS}), \text{ such that there are no } A \subseteq \mathcal{T}, \Gamma \subseteq \mathcal{S}, \text{ for which } A \cup \Gamma \Rightarrow \phi. \text{ Hence there is no } a \in \operatorname{Arg}_{\mathsf{L}}(\mathcal{S},\mathcal{T}) \text{ such that } \operatorname{Conc}(a) = \phi. \text{ By Lemma 12, it follows that } \operatorname{Arg}_{\mathsf{L}}(\mathcal{S},\mathcal{T}) \in \operatorname{Ext}_{\mathsf{stb}}(\mathcal{AF}_{\mathsf{L}}^{\leq \pi}(\mathcal{S},\mathcal{AS})). \text{ Therefore } \mathcal{S} \not\models^{\leq \pi, \textcircled{m}}_{\mathcal{AS},\mathsf{stb}} \phi \text{ and thus also } \mathcal{S} \not\models^{\leq \pi, \textcircled{m}}_{\mathcal{AS},\mathsf{prf}} \phi.$ 

**Remark 7.** As can be seen from the results above, the preferred and stable extensions coincide, when the rules from Figure 6 are admissible in the calculus of the core logic. In fact, by Lemmas 12 and 13  $\operatorname{Ext}_{prf}(\mathcal{AF}_{L}^{\leq \pi}(\mathcal{S}, \mathcal{AS})) = \{\operatorname{Arg}_{L}(\mathcal{S}, \mathcal{T}) \mid \mathcal{T} \in \operatorname{MCS}_{L}^{\leq}(\mathcal{S}, \mathcal{AS})\} = \operatorname{Ext}_{stb}(\mathcal{AF}_{L}^{\leq \pi}(\mathcal{S}, \mathcal{AS}))$ . Although it is possible that no stable extension exists in abstract argumentation (see [36]), assumptive sequent-based argumentation is not the only approach to logical argumentation in which the preferred and stable extensions coincide. For example, this is the case for instances of ASPIC<sup>+</sup> (see [51]), simple contrapositive assumption based argumentation (see [41]) and sequent-based argumentation (see [9]). For an overview see [6].

When stable and preferred extensions do not coincide in abstract argumentation, this is because of odd cycles in the argumentation framework. In, for example, ASPIC<sup>+</sup>, such cycles may also exist, since the contrariness function might be onesided. However, given the assumptions made to prove the results (i.e., because  $[\Rightarrow\neg], [\neg\Rightarrow]$  and [Cut] are admissible), such cycles do not cause these problems in the setting of the theorem. For example, a possible odd cycle might exist when  $p \land \neg p \in \mathcal{AS}$ , since then  $p \land \neg p$   $\Rightarrow \neg (p \land \neg p)$  would be derivable with the rules from Figure 6. However, this cycle is attacked by  $\Rightarrow \neg (p \land \neg p)$ , which cannot be attacked.

In the next section the general framework defined here will be applied to several well-known approaches to nonmonotonic reasoning with assumptions.

## 4 Some Assumptive Approaches and Their Properties

We will consider three well-known frameworks for nonmonotonic reasoning with assumptions. Assumption-based argumentation in Section 4.1, adaptive logics in Section 4.2 and default assumptions in Section 4.3. For each of these approaches the representation by the introduced assumptive sequent-based approach, maximally consistent subsets, as well as the rationality postulates from [30] are discussed.

In this paper only the flat approaches are considered. On the one hand, because the objective of this paper is just to show that the presented assumptive frameworks are expressive enough to represent other approaches to reasoning with assumptions and, on the other hand, because there are often several possibilities to introduce priorities, for assumption-based argumentation see e.g., [35, 42] and for adaptive logics see e.g., [57, 58].

#### 4.1 Assumption-Based Argumentation

Assumption-based argumentation (ABA) was introduced in [25], see [37, 64] for an introduction and an overview. In contrast to the other two examples that will be discussed, ABA is already defined in terms of argumentation frameworks. It takes as input a formal deductive system, a set of assumptions and a contrariness mapping for each assumption. There are only few requirements placed on each of these, keeping the framework semi-abstract on the one hand, while the arguments have a formal structure and the attacks are based on the latter. We first consider some of the most important definitions of the ABA-framework, from [25]:

**Definition 22.** A *deductive system* is a pair  $\langle \mathcal{L}, \mathcal{R} \rangle$ , where:

•  $\mathcal{L}$  is a formal language;

•  $\mathcal{R}$  is a set of rules of the form  $\phi_1, \ldots, \phi_n \to \phi$ , for  $\phi_1, \ldots, \phi_n, \phi \in \mathcal{L}$  and  $n \ge 0$ .

**Definition 23.** A deduction from a theory  $\Gamma$  is a sequence  $\psi_1, \ldots, \psi_m$ , where m > 0, such that for all  $i = 1, \ldots, m, \psi_i \in \Gamma$ , or there is a rule  $\phi_1, \ldots, \phi_n \to \psi_i \in \mathcal{R}$  with  $\phi_1, \ldots, \phi_n \in \{\psi_1, \ldots, \psi_{i-1}\}$ . A deduction from  $\Gamma$  using rules in  $\mathcal{R}$  is denoted by  $\Gamma \vdash^{\mathcal{R}} \psi_m$ .

Clearly, a deductive system is not necessarily based on a logic in the sense of Section 2, thus the possible connectives do not necessarily have the properties they were assumed to have in the previous sections. However, in this section, the examples will be based on classical logic, in which the connectives have the properties as discussed after Definition 1.

**Example 16.** An example of a deductive system is classical logic, such that  $\phi_1, \ldots, \phi_n \to \phi \in \mathcal{R}_{\mathsf{CL}}$  if and only if  $\phi_1, \ldots, \phi_n \vdash_{\mathsf{CL}} \phi$ . Thus,  $\Gamma \vdash^{\mathcal{R}} \psi$  if and only if  $\Gamma \vdash_{\mathsf{CL}} \psi$ .

From this ABA argumentation frameworks can be defined:<sup>9</sup>

**Definition 24.** An *ABA-framework* is a tuple  $\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A}) = \langle \mathcal{L}, \mathcal{R}, \mathcal{S}, \mathcal{A}, \overline{\cdot} \rangle$  where:

- $\langle \mathcal{L}, \mathcal{R} \rangle$  is a deductive system;
- $S \subseteq \mathcal{L}$  a set of formulas, that satisfies non-triviality (there is some  $\mathcal{L}$ -formula  $\phi$  such that  $S \nvDash^{\mathcal{R}} \phi$ );
- $\mathcal{A} \subseteq \mathcal{L}$  a non-empty set of *assumptions* for which  $\mathcal{S} \cap \mathcal{A} = \emptyset$ ; and
- $\overline{\cdot}$  a mapping from  $\mathcal{A}$  into a set of  $\mathcal{L}$ -formulas, where  $\overline{\phi}$  is the set of the *contrary* formulas of  $\phi$ .

In the remainder, if a set of formulas S satisfies non-triviality, it is said that S is non-trivializing.

A simple way of defining contrariness in the context of classical logic is by  $\overline{\phi} = \{\neg\phi\}$ . In what follows, by  $A', \Gamma \vdash^{\mathcal{R}} \overline{\phi}$  it is meant that there is some  $\psi \in \overline{\phi}$  such that  $A', \Gamma \vdash^{\mathcal{R}} \psi$ . Moreover, to avoid clutter, the superscript  $\mathcal{R}$  in  $\vdash^{\mathcal{R}}$  is sometimes omitted.

The consistency notions from Definition 2 can be defined in terms of a contrariness function as well, in order to avoid confusion with the previously defined notion, we will refer to (maximally) contrary-consistent sets of assumptions:

# **Definition 25.** Given an ABA-framework $\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A})$ , a set $A \subseteq \mathcal{A}$ is:

<sup>&</sup>lt;sup>9</sup>Note that not in all the literature on ABA the set of facts (in the notation of this paper S) is part of the framework. Rather, these are special rules (so-called domain oriented rules), denoted by  $\rightarrow \phi$  for  $\phi \in S$ . Thus, one could understand S such that  $\phi \in S$  in our setting iff  $\rightarrow \phi \in \mathcal{R}$  if a set of facts is not part of the framework.

- contrary-consistent if and only if there is no  $\phi \in A$  such that  $A', \Gamma \vdash^{\mathcal{R}} \overline{\phi}$  for some  $A' \subseteq A$  and some  $\Gamma \subseteq S$ ;
- maximally contrary-consistent, denoted by  $A \in MCS(S, A)$ , if and only if A is contrary-consistent and there is no contrary-consistent A' such that  $A \subset A' \subseteq A$ .

The closure of  $\mathcal{T} \subseteq \mathcal{L}$  is defined as  $\mathsf{CN}(\mathcal{T}) = \{ \phi \mid \Gamma \vdash^{\mathcal{R}} \phi \text{ for } \Gamma \subseteq \mathcal{T} \}.$ 

ABA-arguments are defined in terms of deductions and an attack is on the assumptions of the attacked argument. Following [37], arguments are not required to be contrary-consistent.

**Definition 26.** Let  $\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A}) = \langle \mathcal{L}, \mathcal{R}, \mathcal{S}, \mathcal{A}, \overline{\cdot} \rangle$ . An *ABA-argument* for  $\phi \in \mathcal{L}$  is a deduction  $A \cup \Gamma \vdash^{\mathcal{R}} \phi$ , where  $A \subseteq \mathcal{A}$  and  $\Gamma \subseteq \mathcal{S}$ . The set  $\operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R} \rangle}^{\mathsf{ABA}}(\mathcal{S}, \mathcal{A})$  denotes the set of all ABA-arguments for  $\mathcal{S}$  and  $\mathcal{A}$ .

**Definition 27.** Let  $\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A}) = \langle \mathcal{L}, \mathcal{R}, \mathcal{S}, \mathcal{A}, \overline{\cdot} \rangle$ . An argument  $A \cup \mathcal{S} \vdash^{\mathcal{R}} \phi$ attacks an argument  $A' \cup \mathcal{S} \vdash^{\mathcal{R}} \phi'$  iff  $\phi \in \overline{\psi}$  for some  $\psi \in A'$ .

**Example 17.** Recall the deductive system  $\mathcal{R}_{\mathsf{CL}}$  for classical logic, described in Example 16 and let  $\overline{\phi} = \{\neg\phi\}$ . Consider the sets  $\mathcal{S} = \{s\}$  and  $\mathcal{A} = \{p, q, \neg p \lor \neg q, \neg p \lor r, \neg q \lor r\}$ . Some of the arguments of  $\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A})$  are:

$$\begin{aligned} a &= s \vdash s \qquad b = p, \neg p \lor \neg q \vdash \neg q \\ c &= q, \neg p \lor \neg q \vdash \neg p \qquad d = p, q, \neg p \lor r, \neg q \lor r \vdash r. \end{aligned}$$

Note that a cannot be attacked, since the set of assumptions of a is empty. For the other arguments, b attacks c and d, and c attacks b and d.

Semantics are defined as usual, see Definition 4. From this the corresponding entailment relations can be defined:

**Definition 28.** Let  $\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A}) = \langle \mathcal{L}, \mathcal{R}, \mathcal{S}, \mathcal{A}, \overline{\cdot} \rangle$  and sem  $\in \{\mathsf{grd}, \mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$ . Then:

- $\mathcal{A} \cup \mathcal{S} \vdash_{\mathsf{ABA},\mathsf{sem}}^{\cup} \phi$  iff for some  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A}))$  there is an argument  $A \cup \Gamma \vdash^{\mathcal{R}} \phi \in \mathcal{E}$ .
- $\mathcal{A} \cup \mathcal{S} \mathrel{\sim}^{\cap}_{\mathsf{ABA},\mathsf{sem}} \phi$  iff there is an  $a \in \bigcap \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A}))$ , where  $a = A \cup \Gamma \vdash^{\mathcal{R}} \phi$ .
- $\mathcal{A} \cup \mathcal{S} \mathrel{\sim}^{\mathbb{M}}_{\mathsf{ABA},\mathsf{sem}} \phi$  iff for every  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A}))$  there is an  $a \in \mathcal{E}$  with  $\mathsf{Conc}(a) = \phi$ .

**Example 18.** Recall the setting from Example 17, where the deductive system was  $\mathcal{R}_{\mathsf{CL}}$  from Example 16,  $\mathcal{S} = \{s\}$  and  $\mathcal{A} = \{p, q, \neg p \lor \neg q, \neg p \lor r, \neg q \lor r\}$ . It can be shown that  $\mathcal{A} \cup \mathcal{S} \models_{\mathsf{ABA},\mathsf{sem}}^{\star} s$ , for  $\star \in \{\cap, \cup, \mathbb{m}\}$ ,  $\mathsf{sem} \in \{\mathsf{grd}, \mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$ , this follows since s is a fact. Furthermore,  $\mathcal{A} \cup \mathcal{S} \models_{\mathsf{ABA},\mathsf{sem}}^{\cup} \phi$ , but  $\mathcal{A} \cup \mathcal{S} \models_{\mathsf{ABA},\mathsf{sem}}^{\cap} \phi$  and  $\mathcal{A} \cup \mathcal{S} \models_{\mathsf{ABA},\mathsf{sem}}^{\cap} \phi$  for  $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$  and  $\phi \in \{p, q, \neg p \lor \neg q\}$ , to see this, note that for each formula  $\phi \in \{p, q, \neg p \lor \neg q\}$  there is an extension from which  $\phi$  can be derived.

Based on the above notions from assumption-based argumentation, a corresponding sequent-based ABA-framework can be defined:

**Definition 29.** Let  $\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A}) = \langle \mathcal{L}, \mathcal{R}, \mathcal{S}, \mathcal{A}, \overline{\cdot} \rangle$  be an ABA-framework. The corresponding *sequent-based ABA-framework* is defined as a pair $\mathcal{AF}^{\mathsf{ABA}_{\Rightarrow}}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(\mathcal{S}, \mathcal{A}) = \langle \operatorname{Arg}^{\mathsf{ABA}_{\Rightarrow}}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(\mathcal{S}, \mathcal{A}), \mathcal{AT} \rangle$ , where:

- $\mathcal{R}_{\Rightarrow}$  is defined as follows:
  - if  $\langle \mathcal{L}, \mathcal{R} \rangle$  is a logic in the sense of Definition 1 with corresponding sound and complete sequent calculus C in which [Cut] is admissible,  $\mathcal{R}_{\Rightarrow} = C \cup \{AS_{ABA}\}$  such that:

$$\frac{A \ (\Gamma, \phi \Rightarrow \psi)}{A, \phi \ (\Gamma \Rightarrow \psi)} AS_{ABA} \qquad \frac{A, \phi \ (\Gamma \Rightarrow \psi)}{A \ (\Gamma, \phi \Rightarrow \psi)} AS_{ABA} \qquad \text{where } \phi \in \mathcal{A}$$

- otherwise  $\mathcal{R}_{\Rightarrow} = \{\mu(r) \mid r \in \mathcal{R}\} \cup \{AS_{ABA}, [Cut], [id]\}$  where, for each  $r = \phi_1, \dots, \phi_n \to \phi \in \mathcal{R}$ 

$$\overline{\phi_1, \dots, \phi_n \Rightarrow \phi} \ \mu(r)$$
 and  $\overline{\phi \Rightarrow \phi} \ [id]$ 

- $a = A \ (\Gamma \Rightarrow \phi \in \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R} \Rightarrow \rangle}^{\mathsf{ABA} \Rightarrow} (\mathcal{S}, \mathcal{A}) \text{ for } A \subseteq \mathcal{A}, \Gamma \subseteq \mathcal{S} \text{ iff there is a derivation of } a \text{ using rules in } \mathcal{R}_{\Rightarrow}.$
- $(a_1, a_2) \in \mathcal{AT}$  iff  $a_1 \mathcal{R}$ -attacks  $a_2$  as defined in Definition 7, for  $\mathsf{AR} = \{\mathsf{AT}_{\mathsf{ABA}}\}$ and:

$$\frac{A_1 \land \Gamma_1 \Rightarrow \overline{\phi} \quad A_2, \phi \land \Gamma_2 \Rightarrow \psi}{A_2, \phi \land \Gamma_2 \Rightarrow \psi} \text{ AT}_{\mathsf{ABA}}$$
(2)

**Remark 8.** Similar to Remark 1, since the rules  $AS_{ABA}$  are part of the calculus of any sequent-based ABA-framework:  $A \cup \Gamma \Rightarrow \phi$  is derivable iff  $A \ \Gamma \Rightarrow \phi$  is derivable.

In the next example we show how classical logic, with corresponding sequent calculus LK can be taken as underlying deductive system.

**Example 19.** Let  $\mathsf{CL} = \langle \mathcal{L}, \vdash \rangle$ , where  $\overline{\phi} = \{\neg \phi\}$  and  $\mathcal{R}_{\Rightarrow} = \mathsf{LK}$ . According to Definition 9  $A \ \Gamma \Rightarrow \phi \in \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{A})$  iff  $\Gamma \cup A \Rightarrow \phi$  is derivable in  $\mathsf{LK}$ , for some finite  $A \subseteq \mathcal{A}$  and  $\Gamma \subseteq \mathcal{S}$ . Since  $\mathcal{R}_{\Rightarrow} = \mathsf{LK} \cup \{\mathsf{AS}_{\mathsf{ABA}}\}$  it follows immediately that  $A \cup \Gamma \Rightarrow \phi$  is derivable in  $\mathcal{R}_{\Rightarrow}$  iff it is derivable in  $\mathsf{LK}$ .

The next proposition formalizes the representation of ABA in assumptive sequent-based argumentation, via the above described translation.

**Proposition 2.** Let  $\langle \mathcal{L}, \mathcal{R} \rangle$  be a deductive system,  $S \subseteq \mathcal{L}$  a non-trivializing set of formulas and  $\mathcal{A} \subseteq \mathcal{L}$  a set of assumptions, such that  $\Gamma \subseteq S$  and  $A \subseteq \mathcal{A}$  are finite and  $\mathcal{A} \cap S = \emptyset$ . Let  $\mathcal{AF}^{\mathsf{ABA}_{\Rightarrow}}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(S, \mathcal{A}) = \langle \operatorname{Arg}^{\mathsf{ABA}_{\Rightarrow}}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(S, \mathcal{A}), \mathcal{AT} \rangle$  be a sequent-based ABA-framework that corresponds to the ABA-framework  $\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \rangle}(S, \mathcal{A}) = \langle \mathcal{L}, \mathcal{R}, \mathcal{S}, \mathcal{A}, \overline{\cdot} \rangle$ .  $\mathcal{A} \cup S \models_{\mathsf{ABA},\mathsf{sem}}^{\star} \phi$  iff  $S \models_{\mathcal{A},\mathsf{sem}}^{\star} \phi$  for  $\mathsf{sem} \in \{\mathsf{grd}, \mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$  and  $\star \in \{\cup, \cap, \mathbb{n}\}$ .

The above proposition is a corollary of the following two lemmas. Suppose that the conditions from the proposition statement hold:

**Lemma 14.** 
$$A \cup \Gamma \vdash^{\mathcal{R}} \phi \in Arg^{\mathsf{ABA}}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A}) \text{ iff } A \ (\Gamma \Rightarrow \phi \in Arg^{\mathsf{ABA}}_{\langle \mathcal{L}, \mathcal{R} \Rightarrow \rangle}(\mathcal{S}, \mathcal{A}).$$

*Proof.* If the deductive system is a logic  $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ , with corresponding sound and complete sequent calculus  $\mathsf{C}$ , it follows that  $A \cup \Gamma \vdash^{\mathcal{R}} \phi \in \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R} \rangle}^{\mathsf{ABA}}(\mathcal{S}, \mathcal{A})$  iff  $A \cup \Gamma \vdash_{\mathsf{L}} \phi$ . By the soundness and completeness of  $\mathsf{C}$  for  $\mathsf{L}$  we have that  $A \cup \Gamma \Rightarrow \phi$  is derivable iff  $A \cup \Gamma \vdash_{\mathsf{L}} \phi$ . And by Remark 8 it follows that, since  $A \subseteq \mathcal{A}$  and  $\Gamma \subseteq \mathcal{S}$ ,  $A \cup \Gamma \Rightarrow \phi$  is derivable in  $\mathsf{C}$  iff  $A \ \ \Gamma \Rightarrow \phi \in \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R} \Rightarrow \rangle}^{\mathsf{ABA}}(\mathcal{S}, \mathcal{A})$ .

For other types of deductive systems, consider both directions:

- ⇒ Assume that  $A \cup \Gamma \vdash^{\mathcal{R}} \phi \in \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R} \rangle}^{\mathsf{ABA}}(\mathcal{S}, \mathcal{A})$ . Then there is a deduction from the theory  $A \cup \Gamma$  for the formula  $\phi$ . By Definition 23, there is a sequence  $\psi_1, \ldots, \psi_m$  ( $\psi_m = \phi$ ), such that for each  $i = 1, \ldots, m, \psi_i \in A \cup \Gamma$  or there is a rule  $\phi_1, \ldots, \phi_n \to \psi_i = r \in R$  and  $\phi_1, \ldots, \phi_n \in \{\psi_1, \ldots, \psi_{i-1}\}$ . We proceed by induction on m, showing that for each  $\psi_i$ , there is a sequent  $s_i = A_i \cup \Gamma_i \Rightarrow \psi_i$ :
  - m=1 Then either  $\psi_1 \in A \cup \Gamma$  and thus  $\psi_1 \Rightarrow \psi_1$  is derivable in  $\mathcal{R}_{\Rightarrow}$ , by [id]. Or there is a rule  $\rightarrow \psi_1 \in R$ . Hence  $\Rightarrow \psi_1 \in \mathcal{R}_{\Rightarrow}$  for  $A \cup \Gamma = \emptyset$ . Since  $\psi_1 = \psi_m = \phi, A \cup \Gamma \Rightarrow \phi$  is derivable.
  - m=k+1 Assume that for sequences up to  $k \ge 1$ , for each  $\psi_i$  there is a sequent  $s_i = A_i \cup \Gamma_i \Rightarrow \psi_i$ . Now consider  $\psi_{k+1}$ . Then  $\psi_{k+1} \in A \cup \Gamma$ , from which it follows immediately that  $A \cup \Gamma \Rightarrow \psi_{k+1}$  is derivable in  $\mathcal{R}_{\Rightarrow}$ , or there

is a rule  $\phi_1, \ldots, \phi_n \to \psi_{k+1} = r \in \mathcal{R}$  and  $\phi_1, \ldots, \phi_n \in \{\psi_1, \ldots, \psi_k\}$ . By Definition 29,  $\phi_1, \ldots, \phi_n \Rightarrow \psi_{k+1} \in \mathcal{R}_{\Rightarrow}$ . Furthermore, by induction hypothesis, for each  $\psi_i \in \{\psi_1, \ldots, \psi_k\}$ , there is a sequent  $s_i = A_i \cup \Gamma_i \Rightarrow \psi_i$ . Hence,  $\phi_1, \ldots, \phi_n \in \{\mathsf{Conc}(s_1), \ldots, \mathsf{Conc}(s_k)\}$ . By applying [Cut] a sequent  $s_{k+1} = A_{k+1} \cup \Gamma_{k+1} \Rightarrow \psi_{k+1}$  is obtained.

Hence, there is a sequence of sequents  $s_1, \ldots, s_m$ , such that  $s_i$  is derived from  $s_1, \ldots, s_{i-1}$  by applying rules in  $\mathcal{R}_{\Rightarrow}$  and  $s_m = A \cup \Gamma \Rightarrow \phi$ . That  $A \not \Gamma \Rightarrow \phi \in \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\mathsf{ABA}_{\Rightarrow}}(\mathcal{S}, \mathcal{A})$  follows by Remark 8.

- $\leftarrow \text{ Now suppose that } a = A \ \ \Gamma \Rightarrow \phi \in \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\mathsf{ABA}_{\Rightarrow}}(\mathcal{S}, \mathcal{A}). \text{ By Remark 8, } \mathcal{A} \cup \Gamma \Rightarrow \phi \\ \text{ is derivable in } \mathcal{R}_{\Rightarrow} \text{ as well. Then there is a derivation via a sequence of sequents } \\ s_1, \ldots, s_m, \text{ where } s_i = A_i \cup \Gamma_i \Rightarrow \psi_i \text{ for each } i \in \{1, \ldots, m\} \text{ is the result of } \\ \text{ applying rules from } \mathcal{R}_{\Rightarrow} \text{ to sequents in } \{s_1, \ldots, s_{i-1}\} \text{ and } s_m = \mathcal{A} \cup \Gamma \Rightarrow \phi. \\ \text{ Again by induction on the length of the derivation } m, \text{ for each } s_i, \text{ there is a } \\ \text{ deduction } \mathsf{Ass}(s_i) \cup \mathsf{Supp}(s_i) \vdash^{\mathcal{R}} \mathsf{Conc}(s_i) \text{ via the sequence } \Phi_i = \psi_1^i, \ldots, \psi_m^i: \end{cases}$ 
  - m=1 Then  $\phi \in A \cup \Gamma$  in which case  $s_m = \phi \Rightarrow \phi$  or there is a  $\mu(r) \in \mathcal{R}_{\Rightarrow}$ such that  $\mu(r) = \Rightarrow \phi$  and thus, by Definition 29,  $r = \rightarrow \phi \in \mathcal{R}$ . Hence  $A \cup \Gamma \vdash^{\mathcal{R}} \phi$ .
  - m=k+1 Now assume that for derivations up to length  $k \ge 1$ , for each  $s_i$ , there is a deduction from  $\mathsf{Ass}(s_i) \cup \mathsf{Supp}(s_i)$  for  $\mathsf{Conc}(s_i)$  via the sequence  $\Phi_i$ . That  $s_m$  is derivable implies that  $\mathsf{Conc}(s_m) \in \mathsf{Ass}(s_m) \cup \mathsf{Supp}(s_m)$ , in which case  $s_m = \mathsf{Conc}(s_m) \Rightarrow \mathsf{Conc}(s_m)$ , from which it follows immediately that there is a deduction  $\mathsf{Ass}(s_m) \cup \mathsf{Supp}(s_m) \vdash^{\mathcal{R}} \mathsf{Conc}(s_m)$  or  $s_m$  is the result of applying a rule to sequents in  $\{s_1, \ldots, s_k\}$ :
    - \* suppose that [Cut] was applied to  $s_{j_1}, s_{j_2} \in \{s_1, \ldots, s_k\}$ . By induction hypothesis, there are deductions  $\mathsf{Ass}(s_{j_1}) \cup \mathsf{Supp}(s_{j_1}) \vdash^{\mathcal{R}} \mathsf{Conc}(s_{j_1})$  and  $\mathsf{Ass}(s_{j_2}) \cup \mathsf{Supp}(s_{j_2}) \vdash^{\mathcal{R}} \mathsf{Conc}(s_{j_2})$  via the sequence  $\Phi_{j_1}$  respectively  $\Phi_{j_2}$ . Then  $\mathsf{Ass}(s_m) \cup \mathsf{Supp}(s_m) \vdash^{\mathcal{R}} \mathsf{Conc}(s_m)$  is obtained via the sequence  $\Phi_m = \Phi_{j_1} \circ_{\mathsf{Conc}(s_{j_1})} \Phi_{j_2}$ , where  $\Phi^1 \circ_{\psi} \Phi^2$  denotes the concatenation of  $\Phi^1$  with  $\Phi^2$  such that all occurrences of  $\psi$  in  $\Phi^2$  are taken out.
    - \* suppose that  $s_m$  is the result of applying  $\overline{\phi_1, \ldots, \phi_n \Rightarrow \phi} \ \mu(r) \in \mathcal{R}_{\Rightarrow}$ . By construction,  $\phi_1, \ldots, \phi_n \rightarrow \phi = r \in \mathcal{R}$  such that  $\phi_j \in \{\psi_1, \ldots, \psi_k\}$  is obtained via a sequence  $\Phi'_j$ , for each  $j \in \{1, \ldots, n\}$ . Therefore,  $\mathsf{Ass}(s_m) \cup \mathsf{Supp}(s_m) \vdash^{\mathcal{R}} \mathsf{Conc}(s_m)$ .

Thus, for the derivation of a, of any length m, via the sequence of sequents,  $s_1, \ldots, s_m$ , there is a deduction from  $A \cup \Gamma$  via the sequence  $\Phi_m$ , for  $\phi$ . Hence  $A \cup \Gamma \vdash^{\mathcal{R}} \phi \in \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R} \rangle}^{\operatorname{ABA}}(\mathcal{S}, \mathcal{A}).$ 

**Lemma 15.** Let  $a, b \in Arg^{ABA}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A})$  and a', b' their corresponding ABA-sequent arguments, thus  $a', b' \in Arg^{ABA_{\Rightarrow}}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(\mathcal{S}, \mathcal{A})$ .<sup>10</sup> Then a attacks b in  $\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A})$  iff a' attacks b' in  $\mathcal{AF}^{ABA_{\Rightarrow}}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(\mathcal{S}, \mathcal{A})$ .

*Proof.* Consider the  $\Rightarrow$ -direction, the  $\Leftarrow$ -direction is similar and left to the reader.

Let  $a, b \in \operatorname{Arg}_{(\mathcal{L},\mathcal{R})}^{\mathsf{ABA}}(\mathcal{S},\mathcal{A})$  and assume  $a = A \cup \Gamma \vdash^{\mathcal{R}} \phi$  attacks  $b = A' \cup \Gamma' \vdash^{\mathcal{R}} \phi'$ . Then, by Definition 27,  $\phi \in \overline{\psi}$  for  $\psi \in A'$ . By Lemma 14,  $a' = A \clubsuit \Gamma \Rightarrow \phi$  and  $b' = A' \clubsuit \Gamma' \Rightarrow \phi'$  are arguments in  $\mathcal{AF}_{(\mathcal{L},\mathcal{R}_{\Rightarrow})}^{\mathsf{ABA}_{\Rightarrow}}(\mathcal{S},\mathcal{A})$   $(a',b' \in \operatorname{Arg}_{(\mathcal{L},\mathcal{R}_{\Rightarrow})}^{\mathsf{ABA}_{\Rightarrow}}(\mathcal{S},\mathcal{A}))$ . Since  $\phi \in \overline{\psi}$  for  $\psi \in A'$ , it follows that  $a' \operatorname{AT}_{\mathsf{ABA}}$ -attacks b'.

With this Proposition 2 can be shown:

*Proof.* Let  $\langle \mathcal{L}, \mathcal{R} \rangle$  be a deductive system,  $S \subseteq \mathcal{L}$  a non-trivializing set of formulas and  $\mathcal{A} \subseteq \mathcal{L}$  a set of assumptions, such that  $\Gamma \subseteq S$  and  $\mathcal{A} \subseteq \mathcal{A}$  are finite and  $\mathcal{A} \cap S = \emptyset$ . Let  $\mathcal{AF}^{\mathsf{ABA}_{\Rightarrow}}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(S, \mathcal{A}) = \langle \operatorname{Arg}^{\mathsf{ABA}_{\Rightarrow}}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(S, \mathcal{A}), \mathcal{AT} \rangle$  be a sequent-based ABA-framework that corresponds to the ABA-framework  $\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \rangle}(S, \mathcal{A}) = \langle \mathcal{L}, \mathcal{R}, S, \mathcal{A}, \overline{\cdot} \rangle$ . We show only some cases, leaving the others to the reader. First note that:

- 1. if  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A}))$  then  $\{a' \mid a \in \mathcal{E} \text{ and } a' \text{ corresponds to } a \text{ as in Lemma } 14\} = \mathcal{E}' \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \Rightarrow \rangle}^{\mathsf{ABA}_{\Rightarrow}}(\mathcal{S}, \mathcal{A}));$
- 2. if  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}^{\mathsf{ABA}\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}\Rightarrow \rangle}(\mathcal{S}, \mathcal{A}))$  then  $\{a' \mid a \in \mathcal{E} \text{ and } a' \text{ corresponds to } a \text{ as in Lemma } 14\} = \mathcal{E}' \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A})).$

We show only the first item, leaving the second item to the reader. Let  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A}))$  and let  $\mathcal{E}' = \{a' \mid a \in \mathcal{E} \text{ where } a' \text{ corresponds to } a \text{ as in Lemma 14}\}$ . To show that  $\mathcal{E}'$  is complete.

 $\mathcal{E}'$  is conflict-free. Since  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A}))$  it follows immediately that  $\mathcal{E}$  is conflict-free. By the construction of  $\mathcal{E}'$  and Lemma 15 it follows that  $\mathcal{E}'$  is conflict-free as well.

 $\mathcal{E}'$  defends itself. Suppose  $a' \in \mathcal{E}'$  is attacked by some  $b' \in \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\mathsf{ABA}_{\Rightarrow}}(\mathcal{S}, \mathcal{A})$ . By the construction of  $\mathcal{E}'$  and Lemma 14 there exist  $a, b \in \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R} \rangle}^{\mathsf{ABA}}(\mathcal{S}, \mathcal{A})$  corresponding

<sup>&</sup>lt;sup>10</sup>That a' and b' exist follows from Lemma 14.

to a' and b' respectively, such that  $a \in \mathcal{E}$ . Moreover, by Lemma 15, b attacks a. Since  $a \in \mathcal{E}$ , there is some  $c \in \mathcal{E}$  such that c defends a. By the construction of  $\mathcal{E}'$  it follows that  $c' \in \mathcal{E}'$  and by Lemma 15 it defends a' against the attack from b'. Thus  $\mathcal{E}'$  defends a'.

 $\mathcal{E}'$  contains the arguments it defends. Suppose that  $a' \in \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\operatorname{ABA}_{\Rightarrow}}(\mathcal{S}, \mathcal{A})$  is defended by  $\mathcal{E}'$ . Then there is some  $b' \in \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\operatorname{ABA}_{\Rightarrow}}(\mathcal{S}, \mathcal{A})$  such that b' attacks a' and there is some  $c' \in \mathcal{E}'$  such that c' attacks b'. By the construction of  $\mathcal{E}'$  and Lemma 14, there are corresponding arguments  $a, b, c \in \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R} \rangle}^{\operatorname{ABA}}(\mathcal{S}, \mathcal{A})$  such that a is attacked by b, b is attacked by c and  $c \in \mathcal{E}$ . Thus c defends a against the attack by b. Since  $\mathcal{E}$  is complete  $a \in \mathcal{E}$ . Hence, by the construction of  $\mathcal{E}'$  it follows that  $a' \in \mathcal{E}'$ .

Therefore we have that  $\{a' \mid a \in \mathcal{E} \text{ and } a' \text{ corresponds to } a \text{ as in Lemma 14} \} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}^{\mathsf{ABA}}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(\mathcal{S}, \mathcal{A}))$ . It remains to show that  $\mathcal{A} \cup \mathcal{S} \triangleright_{\mathsf{ABA},\mathsf{sem}}^{\star} \phi$  iff  $\mathcal{S} \models_{\mathcal{A},\mathsf{sem}}^{\star} \phi$  for  $\star \in \{\cup, \cap, \mathbb{m}\}$  and completeness-based semantics sem. We show the case for  $\star = \cup$  and  $\mathsf{sem} = \mathsf{cmp}$ .

- $\Rightarrow \text{ Let } \mathcal{A} \cup \mathcal{S} \mathrel{\mathop{\textstyle \bigvee}}_{\mathsf{ABA,cmp}}^{\cup} \phi. \text{ Then there is some } \mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A})) \text{ such that there is some } a \in \mathcal{E} \text{ where } a = A \cup \Gamma \vdash^{\mathcal{R}} \phi \text{ for some } A \subseteq \mathcal{A} \text{ and } \Gamma \subseteq \mathcal{S}.$ By Lemma 14  $a' = A \ \mathbf{A} \cap \mathsf{F} \Rightarrow \phi \in \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R} \Rightarrow \rangle}^{\mathsf{ABA}}(\mathcal{S}, \mathcal{A}).$ Moreover, by the first item above it follows that there is some  $\mathcal{E}' \in \operatorname{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \Rightarrow \rangle}^{\mathsf{ABA}}(\mathcal{S}, \mathcal{A})) \text{ and } a' \in \mathcal{E}'.$ Therefore  $\mathcal{S} \mathrel{\mathop{\textstyle \bigvee}}_{\mathcal{A},\mathsf{cmp}}^{\cup} \phi.$
- $\leftarrow \text{ Let } \mathcal{S} \triangleright_{\mathcal{A}, \mathsf{cmp}}^{\cup} \phi. \text{ Then there is some } \mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A})) \text{ such that there is some } a \in \mathcal{E} \text{ where } a = A \ \mathbf{I} = \phi \text{ for some } A \subseteq \mathcal{A} \text{ and } \Gamma \subseteq \mathcal{S}. \text{ By the second item above it follows that there is some } \mathcal{E}' \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\mathsf{ABA}_{\Rightarrow}}(\mathcal{S}, \mathcal{A})) \text{ such that } a' \in \mathcal{E}' \text{ where } a' \text{ corresponds to } a \text{ as in Lemma 14, thus } a' = A \cup \Gamma \vdash^{\mathcal{R}} \phi. \text{ Hence } \mathcal{A} \cup \mathcal{S} \succ_{\mathsf{ABA},\mathsf{cmp}}^{\cup} \phi.$

**Example 20.** Recall the setting from Example 18, in which  $S = \{s\}$ ,  $A = \{p, q, \neg p \lor \neg q, \neg p \lor r, \neg q \lor r\}$  and classical logic is the core logic. Let  $\mathcal{R}_{\Rightarrow} = \mathsf{LK}$ . Some of the arguments of the sequent-based ABA-framework  $\mathcal{AF}^{\mathsf{ABA}_{\Rightarrow}}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(S, \mathcal{A}) = \langle \operatorname{Arg}^{\mathsf{ABA}_{\Rightarrow}}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(S, \mathcal{A}), \mathcal{AT} \rangle$  are:

$$\begin{aligned} a &= s \Rightarrow s \qquad b = p, \neg p \lor \neg q \diamondsuit \Rightarrow \neg q \\ c &= q, \neg p \lor \neg q \diamondsuit \Rightarrow \neg p \qquad d = p, q, \neg p \lor r, \neg q \lor r \diamondsuit \Rightarrow r. \end{aligned}$$

Note that a cannot be attacked, since  $Ass(a) = \emptyset$ . Thus  $\mathcal{S} \sim^{\star}_{\mathcal{A},sem} s$  for  $sem \in \{grd, cmp, prf, stb\}$  and  $\star \in \{\cup, \cap, \mathbb{m}\}$ . However, the argument d is attacked by

both b and c. Moreover b attacks c and c attacks b. It can be shown that, for  $\phi \in \{p, q, \neg p \lor \neg q\}, S \not\models_{\mathcal{A}, sem}^{\star} \phi$  for sem  $\in \{\mathsf{grd}, \mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$  and  $\star \in \{\cap, \mathbb{n}\}$  but also  $S \models_{\mathcal{A}, \mathsf{sem}'}^{\cup} \phi$  for sem'  $\in \{\mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}.$ 

We will now turn to the representation of reasoning with maximally consistent subsets in the here presented framework.

**Remark 9.** In this section maximally consistent subsets are defined as in Definition 25. The corresponding entailment relations are defined in the same way as those in Definition 18, now with respect to the definition of contrary-consistent sets. We continue using the notation  $\succ_{\mathsf{mcs}}^{\star,\mathcal{AS}}$  for  $\star \in \{\cap, \cup, \mathbb{n}\}$ .

The relations between ABA and reasoning with maximally consistent subsets and between sequent-based argumentation and maximally consistent subsets have been studied before, see [7, 9, 41]. In addition to the two entailment relations in [41] (in the notation of this paper  $\succ_{mcs}^{\square,\mathcal{AS}}$  and  $\succ_{mcs}^{\cup,\mathcal{AS}}$ ), we will also consider the entailment relation  $\vdash_{mcs}^{\square,\mathcal{AS}}$ . For the proof of these relations, like in [41], it is assumed that  $\vdash^{\mathcal{R}}$ is contrapositive:

**Definition 30.**  $\vdash^{\mathcal{R}}$  is said to be *contrapositive for assumptions* if for any  $\phi \in A$  and any  $\psi \in \mathcal{A}$  it holds that  $A \cup \Gamma \vdash^{\mathcal{R}} \overline{\psi}$  if and only if  $(A \setminus \{\phi\}) \cup \{\psi\} \cup \Gamma \vdash^{\mathcal{R}} \overline{\phi}$ .

Similar to the assumption made in the previous section, that the rules from Figure 6 are admissible in the sequent calculus of the core logic, requiring that  $\vdash^{\mathcal{R}}$  is contrapositive restricts the generality of the result, not the above introduced representation.

The proofs of Proposition 3 and the lemmas necessary for it are partially based on proofs in [9]. For similar reasons as those in the previous section we will assume that S is non-trivializing.

**Proposition 3.** Let  $\mathcal{AF}^{ABA_{\Rightarrow}}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(\mathcal{S}, \mathcal{A})$  be a sequent-based ABA-framework for a deductive system  $\langle \mathcal{L}, \mathcal{R} \rangle$ ,  $\mathcal{S} \subseteq \mathcal{L}$  a non-trivializing set of formulas and  $\mathcal{A}$  a set of assumptions. Suppose that  $\vdash^{\mathcal{R}}$  is contrapositive for assumptions. Then:

$$\begin{split} 1. \ \mathcal{S} & \mathrel{\triangleright}^{\cap}_{\mathcal{A}, \mathsf{prf}} \phi \ iff \ \mathcal{S} \; \mathrel{\triangleright}^{\cap}_{\mathcal{A}, \mathsf{stb}} \phi \ iff \ \mathcal{S} \; \mathrel{\triangleright}^{\cap, \mathcal{A}}_{\mathsf{mcs}} \phi; \\ 2. \ \mathcal{S} & \mathrel{\triangleright}^{\cup}_{\mathcal{A}, \mathsf{prf}} \phi \ iff \ \mathcal{S} \; \mathrel{\triangleright}^{\cup}_{\mathcal{A}, \mathsf{stb}} \phi \ iff \ \mathcal{S} \; \mathrel{\triangleright}^{\cup, \mathcal{A}}_{\mathsf{mcs}} \phi; \\ 3. \ \mathcal{S} & \mathrel{\triangleright}^{\bigcap}_{\mathcal{A}, \mathsf{prf}} \phi \ iff \ \mathcal{S} \; \mathrel{\triangleright}^{\bigcap}_{\mathcal{A}, \mathsf{stb}} \phi \ iff \ \mathcal{S} \; \mathrel{\triangleright}^{\bigcap, \mathcal{A}}_{\mathsf{mcs}} \phi. \end{split}$$

As in Section 3.3 we first consider two lemmas that will be useful in the proofs of the above proposition. The first shows that for any maximally consistent subset of assumptions  $\mathcal{T}$ , if some assumption  $\phi$  is not part of  $\mathcal{T}$ , there is some argument a such that the conclusion of a is a contrary of  $\phi$ . The second shows that the set of assumptions from which the arguments in a complete extension is constructed, are contrary-consistent.

For the next proofs, suppose that the conditions of the statement of the proposition hold.

**Lemma 16.** For each set  $\mathcal{T} \subseteq \mathcal{A}$ : if  $\mathcal{T} \in \mathsf{MCS}(\mathcal{S}, \mathcal{A})$  then for each  $\phi \in \mathcal{A} \setminus \mathcal{T}$ , there is some finite  $A \subseteq \mathcal{T}$  and some finite  $\Gamma \subseteq \mathcal{S}$  such that  $A \ \mathbf{\zeta} \Gamma \Rightarrow \overline{\phi} \in Arg^{\mathsf{ABA}_{\Rightarrow}}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(\mathcal{S}, \mathcal{A}).$ 

*Proof.* Assume that  $\mathcal{T} \in \mathsf{MCS}(\mathcal{S}, \mathcal{A})$  and consider some  $\phi \in \mathcal{A} \setminus \mathcal{T}$ . By Definition 25, there is some  $A' \subseteq \mathcal{T} \cup \{\phi\}$  and some  $\Gamma \subseteq \mathcal{S}$  such that  $A' \cup \Gamma \vdash^{\mathcal{R}} \overline{\psi}$  for some  $\psi \in \mathcal{T} \cup \{\phi\}$ . Consider two cases:

- $\psi \in \mathcal{T}$ . By contraposition,  $(A' \setminus \{\phi\}) \cup \{\psi\} \cup \Gamma \vdash^{\mathcal{R}} \overline{\phi}$ .
- $\psi = \phi$ . Then  $A' \subseteq \mathcal{T}$ .

In both cases there is an  $A \subseteq \mathcal{T}$  and a  $\Gamma \subseteq \mathcal{S}$  such that  $A \cup \Gamma \vdash^{\mathcal{R}} \overline{\phi} \in \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R} \rangle}^{\mathsf{ABA}}(\mathcal{S}, \mathcal{A})$ . Hence, by Lemma 14,  $A \ \Gamma \Rightarrow \overline{\psi} \in \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R} \Rightarrow \rangle}^{\mathsf{ABA}}(\mathcal{S}, \mathcal{A})$ .

**Lemma 17.** The set  $Ass(\mathcal{E})$ , for any  $\mathcal{E} \in Ext_{cmp}(\mathcal{AF}^{ABA_{\Rightarrow}}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(\mathcal{S}, \mathcal{A}))$  is contrary-consistent.

Proof. Assume, towards a contradiction, that  $\operatorname{Ass}(\mathcal{E}) = \{\phi_1, \ldots, \phi_n\}$  is not contraryconsistent. Then, by Definition 25 there are  $A \subseteq \operatorname{Ass}(\mathcal{E})$  and  $\Gamma \subseteq \mathcal{S}$  such that  $A, \Gamma \vdash^{\mathcal{R}} \overline{\phi_i}$  for some  $\phi_i \in \operatorname{Ass}(\mathcal{E})$ . By Lemma 14,  $a = A \ \mathbf{f} \ \Gamma \Rightarrow \overline{\phi_i}$  is derivable. Note that, if a is not attacked,  $a \in \mathcal{E}$ . Suppose that a is attacked by an argument  $b = A' \ \mathbf{f}' \Rightarrow \psi \in \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\operatorname{ABA}_{\Rightarrow}}(\mathcal{S}, \mathcal{A})$ . Then  $\psi \in \overline{\psi'}$  for some  $\psi' \in \mathcal{A}$ . Hence  $\psi' \in \operatorname{Ass}(\mathcal{E})$ . Thus b attacks some argument  $a' \in \mathcal{E}$  as well. Since  $a' \in \mathcal{E}$ , there is an argument  $c \in \mathcal{E}$  which defends a' and thus a from the attack by b. Since  $\mathcal{E}$  is complete,  $a \in \mathcal{E}$ . Thus whether a is attacked or not,  $a \in \mathcal{E}$ . However, a attacks each  $a_j \in \mathcal{E}$  with  $\phi_i \in \operatorname{Ass}(a_j)$ . A contradiction with the conflict-freeness of the complete extension  $\mathcal{E}$ .

The next two lemmas show how maximally consistent subsets relate to stable (Lemma 18) and preferred (Lemma 19) extensions.

# $\textbf{Lemma 18.} \ \textit{If $\mathcal{T} \in \mathsf{MCS}(\mathcal{S},\mathcal{A})$ then $Arg^{\mathsf{ABA}_{\Rightarrow}}_{\langle \mathcal{L},\mathcal{R}_{\Rightarrow} \rangle}(\mathcal{S},\mathcal{T}) \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}^{\mathsf{ABA}_{\Rightarrow}}_{\langle \mathcal{L},\mathcal{R}_{\Rightarrow} \rangle}(\mathcal{S},\mathcal{A}))$.}$

*Proof.* Assume that  $\mathcal{T} \in \mathsf{MCS}(\mathcal{S}, \mathcal{A})$  and let  $\mathcal{E} = \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R} \Rightarrow \rangle}^{\mathsf{ABA}}(\mathcal{S}, \mathcal{T})$ . We show that  $\mathcal{E}$  is conflict-free and stable.

 $\mathcal{E}$  is conflict-free. Suppose, towards a contradiction, that  $\mathcal{E}$  is not conflict-free. Then there are arguments  $a_1 = A_1 \$   $\Gamma_1 \Rightarrow \phi_1$  and  $a_2 = A_2 \$   $\Gamma_2 \Rightarrow \phi_2$ , such that  $a_1, a_2 \in \mathcal{E}$  and  $a_1$  attacks  $a_2$ . Thus  $\phi_1 \in \overline{\psi}$  for some  $\psi \in A_2$ . However, by assumption  $A_1 \cup A_2 \subseteq \mathcal{T}$ . A contradiction with the assumption that  $\mathcal{T} \in \mathsf{MCS}(\mathcal{S}, \mathcal{A})$ .

 $\mathcal{E} \text{ is stable. Now suppose that } b = A' \ \mathbf{\hat{\zeta}} \ \Gamma' \Rightarrow \phi' \in \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\mathsf{ABA}_{\Rightarrow}}(\mathcal{S}, \mathcal{A}) \setminus \mathcal{E} \text{ for some } \Gamma' \subseteq \mathcal{S} \text{ and } A' \subseteq \mathcal{A}. \text{ Since } b \notin \mathcal{E} = \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\mathsf{ABA}_{\Rightarrow}}(\mathcal{S}, \mathcal{T}), \text{ there is some } \phi \in A' \text{ such that } \phi \notin \mathcal{T}. \text{ Since, by supposition } \mathcal{T} \in \mathsf{MCS}(\mathcal{S}, \mathcal{A}), \text{ from Lemma 16, there are finite } A \subseteq \mathcal{T}, \Gamma \subseteq \mathcal{S} \text{ such that } A \ \mathbf{\hat{\zeta}} \ \Gamma \Rightarrow \overline{\phi} \in \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\mathsf{ABA}_{\Rightarrow}}(\mathcal{S}, \mathcal{A}). \text{ Because } A \subseteq \mathcal{T} \text{ it follows that } A \ \mathbf{\hat{\zeta}} \ \Gamma \Rightarrow \overline{\phi} \in \mathcal{E}. \text{ Hence } b \text{ is attacked by } \mathcal{E}. \text{ Therefore } \mathcal{E} \text{ attacks every argument in } \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\mathsf{ABA}_{\Rightarrow}}(\mathcal{S}, \mathcal{A}) \setminus \mathcal{E} \text{ and thus } \mathcal{E} \in \operatorname{Ext}_{\mathsf{stb}}(\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\mathsf{ABA}_{\Rightarrow}}(\mathcal{S}, \mathcal{A})). \square$ 

**Lemma 19.** If  $\mathcal{E} \in \operatorname{Ext}_{prf}(\mathcal{AF}^{ABA_{\Rightarrow}}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(\mathcal{S}, \mathcal{A}))$  then there is some  $\mathcal{T} \in \operatorname{MCS}(\mathcal{S}, \mathcal{A})$  such that  $\mathcal{E} = \operatorname{Arg}^{ABA_{\Rightarrow}}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(\mathcal{S}, \mathcal{T}).$ 

Proof. Assume, towards a contradiction, that for some  $\mathcal{E} \in \operatorname{Ext}_{\operatorname{prf}}(\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\operatorname{ABA}\Rightarrow}(\mathcal{S}, \mathcal{A}))$ there is no  $\mathcal{T} \in \operatorname{MCS}(\mathcal{S}, \mathcal{A})$  such that  $\mathcal{E} = \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\operatorname{ABA}\Rightarrow}(\mathcal{S}, \mathcal{T})$ . Consider first the case that there is some  $\mathcal{T} \in \operatorname{MCS}(\mathcal{S}, \mathcal{A})$  such that  $\mathcal{E} = \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\operatorname{ABA}\Rightarrow}(\mathcal{S}, \mathcal{T}')$  for  $\mathcal{T}' \subsetneq \mathcal{T}$ . Thus  $\mathcal{E} \subsetneq \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\operatorname{ABA}\Rightarrow}(\mathcal{S}, \mathcal{T})$ . By Lemma 18, it follows that  $\operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\operatorname{ABA}\Rightarrow}(\mathcal{S}, \mathcal{T}) \in$  $\operatorname{Ext}_{\operatorname{stb}}(\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\operatorname{ABA}\Rightarrow}(\mathcal{S}, \mathcal{A}))$ . A contradiction to the assumption that  $\mathcal{E}$  is preferred and thus maximal. Thus if  $\mathcal{T} \in \operatorname{MCS}(\mathcal{S}, \mathcal{A})$  does not exist such that  $\mathcal{E} = \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\operatorname{ABA}\Rightarrow}(\mathcal{S}, \mathcal{T})$ , a  $\mathcal{T}' \subsetneq \mathcal{T}$  for which  $\mathcal{E} = \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\operatorname{ABA}\Rightarrow}(\mathcal{S}, \mathcal{T}')$  does not exist either. Thus, since there is no  $\mathcal{T} \in \operatorname{MCS}(\mathcal{S}, \mathcal{A})$  such that  $\mathcal{E} = \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\operatorname{ABA}\Rightarrow}(\mathcal{S}, \mathcal{T})$ , there is no  $\mathcal{T} \in \operatorname{MCS}(\mathcal{S}, \mathcal{A})$ such that  $\operatorname{Ass}(\mathcal{E}) \subseteq \mathcal{T}$  and hence,  $\operatorname{Ass}(\mathcal{E})$  is contrary-inconsistent. A contradiction with Lemma 17 and the assumption that  $\mathcal{E} \in \operatorname{Ext}_{\operatorname{prf}}(\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\operatorname{ABA}\Rightarrow}(\mathcal{S}, \mathcal{T}))$  is stable (and therefore preferred) and thus  $\mathcal{E} = \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\operatorname{ABA}\Rightarrow}(\mathcal{S}, \mathcal{T})$ . □

We now turn to the proof of Proposition 3:

*Proof.* Let  $\mathcal{AF}^{ABA\Rightarrow}_{\langle \mathcal{L}, \mathcal{R} \Rightarrow \rangle}(\mathcal{S}, \mathcal{A})$  be a sequent-based ABA-framework, where  $\langle \mathcal{L}, \mathcal{R} \rangle$  is a deductive system,  $\mathcal{S}$  is a non-trivializing set of  $\mathcal{L}$ -formulas and  $\mathcal{A}$  is a set of assumptions. Consider each item in both directions:

1. ( $\Rightarrow$ ) Note that  $S \models_{\mathcal{A},\mathsf{prf}}^{\cap} \phi$  implies  $S \models_{\mathcal{A},\mathsf{stb}}^{\cap} \phi$ . Suppose  $S \models_{\mathsf{mcs}}^{\cap,\mathcal{A}} \phi$ , but that there is some finite  $A \subseteq \mathcal{A}$  and some  $\Gamma \subseteq S$  such that  $A \ \ \Gamma \Rightarrow \phi \in \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R} \Rightarrow \rangle}^{\mathsf{ABA} \Rightarrow}(S, \mathcal{A})$ . Now, by assumption,  $A \not\subseteq \cap \mathsf{MCS}(S, \mathcal{A})$ . Hence, there is some  $\phi' \in A \setminus \bigcap \mathsf{MCS}(S, \mathcal{A})$ . From which it follows that there is some  $\mathcal{T} \in \mathsf{MCS}(\mathcal{S}, \mathcal{A})$  such that  $\phi' \notin \mathcal{T}$ . Therefore  $A \ \ \Gamma \Rightarrow \phi \notin \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R} \Rightarrow \rangle}^{\mathsf{ABA} \Rightarrow}(S, \mathcal{T})$ .

By Lemma 18,  $\operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\mathsf{ABA}_{\Rightarrow}}(\mathcal{S}, \mathcal{T}) \in \operatorname{Ext}_{\mathsf{stb}}(\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\mathsf{ABA}_{\Rightarrow}}(\mathcal{S}, \mathcal{A}))$ , thus  $\mathcal{S} \not \sim_{\mathcal{A}, \mathsf{stb}}^{\cap} \phi$ (and thus  $\mathcal{S} \not \sim_{\mathcal{A}, \mathsf{prf}}^{\cap} \phi$ ) as well.

( $\Leftarrow$ ) Suppose that  $S \models_{\mathsf{mcs}}^{\cap,\mathcal{A}} \phi$ . Thus, there are finite  $A \subseteq \bigcap \mathsf{MCS}(\mathcal{S},\mathcal{A})$  and  $\Gamma \subseteq S$  such that  $A \ \Gamma \Rightarrow \phi \in \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R} \Rightarrow \rangle}^{\mathsf{ABA} \Rightarrow}(\mathcal{S}, \mathcal{A})$  is derivable. By Lemma 19  $\operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R} \Rightarrow \rangle}^{\mathsf{ABA} \Rightarrow}(\Gamma, A) \subseteq \bigcap \operatorname{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \Rightarrow \rangle}^{\mathsf{ABA} \Rightarrow}(\mathcal{S}, \mathcal{A}))$ . Hence we have that  $A \ \Gamma \Rightarrow \phi \in \bigcap \operatorname{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \Rightarrow \rangle}^{\mathsf{ABA} \Rightarrow}(\mathcal{S}, \mathcal{A}))$ . From which it follows that  $S \models_{\mathcal{A},\mathsf{prf}}^{\cap} \phi$  and thus  $S \models_{\mathcal{A},\mathsf{stb}}^{\cap} \phi$ .

2. ( $\Rightarrow$ ) Note that  $S \models_{\mathcal{A},\mathsf{stb}}^{\cup} \phi$  implies  $S \models_{\mathcal{A},\mathsf{prf}}^{\cup} \phi$ . Suppose that  $S \models_{\mathcal{A},\mathsf{prf}}^{\cup} \phi$ . Then there is some  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\langle \mathcal{L},\mathcal{R} \Rightarrow \rangle}^{\mathsf{ABA} \Rightarrow}(\mathcal{S},\mathcal{A}))$  such that  $A \clubsuit \Gamma \Rightarrow \phi \in \mathcal{E}$ , for  $A \subseteq \mathcal{A}$ and  $\Gamma \subseteq S$ . From Lemma 19 it follows that there is some  $\mathcal{T} \in \mathsf{MCS}(\mathcal{S},\mathcal{A})$ such that  $\mathcal{E} = \operatorname{Arg}_{\langle \mathcal{L},\mathcal{R} \Rightarrow \rangle}^{\mathsf{ABA} \Rightarrow}(\mathcal{S},\mathcal{T})$  (thus  $A \subseteq \mathcal{T}$ ). Hence, by Definition 25 and Lemma 14,  $\phi \in \mathsf{CN}(\mathcal{T} \cup \mathcal{S})$  it follows that  $\mathcal{S} \models_{\mathsf{mcs}}^{\cup,\mathcal{A}} \phi$ .

 $\begin{array}{l} (\Leftarrow) \text{ Assume that } \mathcal{S} \mid_{\mathsf{mcs}}^{\cup,\mathcal{A}} \phi. \text{ Then there is some } \mathcal{T} \in \mathsf{MCS}(\mathcal{S},\mathcal{A}) \text{ such that } \phi \in \mathsf{CN}(\mathcal{T} \cup \mathcal{S}). \end{array} \\ \text{ Therefore, there is a deduction from } \mathcal{A} \cup \Gamma \subseteq \mathcal{T} \cup \mathcal{S} \text{ for } \phi \ (\mathcal{A} \cup \Gamma \vdash^{\mathcal{R}} \phi \in \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\mathsf{ABA}}(\mathcal{S}, \mathcal{A})) \text{ and thus, by Lemma 14 } \mathcal{A} \ \mathbf{\zeta} \ \Gamma \Rightarrow \phi \in \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\mathsf{ABA}}(\mathcal{S}, \mathcal{A}). \\ \text{ From Lemma 18 it follows that } \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\mathsf{ABA}}(\mathcal{S}, \mathcal{T}) \in \operatorname{Ext}_{\mathsf{stb}}(\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\mathsf{ABA}_{\Rightarrow}}(\mathcal{S}, \mathcal{A})). \\ \text{ Thus } \mathcal{S} \mid_{\mathcal{A}, \mathsf{stb}}^{\cup} \phi \text{ as well.} \end{array}$ 

3.  $S \models_{\mathcal{A},\mathsf{stb}}^{\mathbb{m}} \phi$  implies  $S \models_{\mathsf{mcs}}^{\mathbb{m},\mathcal{A}} \phi$ : suppose that  $S \nvDash_{\mathsf{mcs}}^{\mathbb{m},\mathcal{A}} \phi$ , then there is some  $\mathcal{T} \in \mathsf{MCS}(\mathcal{S},\mathcal{A})$  for which  $\phi \notin \mathsf{CN}(\mathcal{S} \cup \mathcal{T})$ . Hence, there are no  $A \subseteq \mathcal{T}$  and  $\Gamma \subseteq S$  with  $A \clubsuit \Gamma \Rightarrow \phi \in \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R} \Rightarrow \rangle}^{\mathsf{ABA} \Rightarrow}(\mathcal{S}, \mathcal{T})$ . By Lemma 18 it follows that  $\operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R} \Rightarrow \rangle}^{\mathsf{ABA} \Rightarrow}(\mathcal{S}, \mathcal{T}) \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \Rightarrow \rangle}^{\mathsf{ABA} \Rightarrow}(\mathcal{S}, \mathcal{A}))$ , thus  $\mathcal{S} \nvDash_{\mathcal{A}, \mathsf{stb}}^{\mathbb{m}} \phi$ .

 $\mathcal{S} \models_{\mathsf{mcs}}^{\mathbb{n},\mathcal{A}} \phi \text{ implies } \mathcal{S} \models_{\mathcal{A},\mathsf{prf}}^{\mathbb{n}} \phi \text{: suppose that } \mathcal{S} \models_{\mathcal{A},\mathsf{prf}}^{\mathbb{n}} \phi \text{. Then there is some preferred extension } \mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\langle \mathcal{L},\mathcal{R}_{\Rightarrow} \rangle}^{\mathsf{ABA}_{\Rightarrow}}(\mathcal{S},\mathcal{A})) \text{ such that there is no } \mathcal{A} \ \mathsf{F} \Rightarrow \phi \in \mathcal{E} \text{ for } \mathcal{A} \subseteq \mathcal{A} \text{ and } \Gamma \subseteq \mathcal{S}. \text{ From Lemma 19 it follows that there is some } \mathcal{T} \in \mathsf{MCS}(\mathcal{S},\mathcal{A}) \text{ such that } \operatorname{Arg}_{\langle \mathcal{L},\mathcal{R}_{\Rightarrow} \rangle}^{\mathsf{ABA}_{\Rightarrow}}(\mathcal{S},\mathcal{T}) = \mathcal{E} \text{ and } \phi \notin \mathsf{CN}(\mathcal{S} \cup \mathcal{T}). \text{ Thus } \mathcal{S} \models_{\mathsf{mcs}}^{\mathbb{n},\mathcal{A}} \phi.$ 

 $\mathcal{S} \sim^{\mathbb{M}}_{\mathcal{A}, \mathsf{prf}} \phi$  implies  $\mathcal{S} \sim^{\mathbb{M}}_{\mathcal{A}, \mathsf{stb}} \phi$ : this follows immediately since any stable extension is a preferred extension [36, Lemma 15].

**Example 21.** Recall from Example 18 the sets  $S = \{s\}$  and  $\mathcal{A} = \{p, q, \neg p \lor \neg q, \neg p \lor r, \neg q \lor r\}$ . Then  $\mathsf{MCS}(S, \mathcal{A}) = \{\{p, q, \neg p \lor r, \neg q \lor r\}, \{p, \neg p \lor \neg q, \neg p \lor r, \neg q \lor r\}, \{q, \neg p \lor \neg q, \neg p \lor r, \neg q \lor r\}$ . Hence  $\bigcap \mathsf{MCS}(S, \mathcal{A}) = \{\neg p \lor r, \neg q \lor r\}$ . Therefore,  $S \models_{\mathsf{mcs}}^{\cap, \mathcal{A}} \phi$  and  $S \models_{\mathsf{mcs}}^{\oplus, \mathcal{A}} \phi$  for  $\phi \in \mathsf{CN}(\{s, \neg p \lor r, \neg q \lor r\})$  and  $S \models_{\mathsf{mcs}}^{\cup, \mathcal{A}} \phi$  for  $\phi \in \{p, q, \neg p \lor \neg q\}$ . Moreover,

by the results from Proposition 3 it follows that  $S \models_{\mathcal{A},\mathsf{sem}}^{\star} \phi$  for  $\star \in \{\cap, \cup, \cap\}$  and  $\phi \in \{s, \neg p \lor r, \neg q \lor r\}$  and  $S \models_{\mathcal{A},\mathsf{sem}}^{\cup} \phi$  for  $\phi \in \{p, q, \neg p \lor \neg q\}$ , which corresponds indeed to the results from Example 20.

The results presented above are summarized in the following theorem.

**Theorem 3.** Let  $\mathcal{AF}^{\mathsf{ABA}_{\Rightarrow}}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(\mathcal{S}, \mathcal{A})$  be a sequent-based ABA-framework, for  $\langle \mathcal{L}, \mathcal{R} \rangle$  a deductive system,  $\mathcal{S}$  a non-trivializing set of  $\mathcal{L}$ -formulas and  $\mathcal{A}$  a set of assumptions such that  $\mathcal{S} \cap \mathcal{A} = \emptyset$ , then:

1.  $\mathcal{A} \cup \mathcal{S} \models_{\mathsf{ABA},\mathsf{sem}}^{\star} \phi$  iff  $\mathcal{S} \models_{\mathcal{A},\mathsf{sem}}^{\star} \phi$  for  $\mathsf{sem} \in \{\mathsf{grd}, \mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$  and  $\star \in \{\cup, \cap, \mathbb{m}\}$  (*Proposition 2*).

For the following, let  $\vdash^{\mathcal{R}}$  be contrapositive for assumptions:

- 2.  $\mathcal{A} \cup \mathcal{S} \models_{\mathsf{ABA},\mathsf{sem}}^{\cap} \phi$  iff  $\mathcal{S} \models_{\mathcal{A},\mathsf{sem}}^{\cap, \mathcal{A}} \phi$  iff  $\mathcal{S} \models_{\mathsf{mcs}}^{\cap, \mathcal{A}} \phi$ , for  $\mathsf{sem} \in \{\mathsf{prf}, \mathsf{stb}\}$  (Propositions 2 and 3.1).
- 3.  $\mathcal{A} \cup \mathcal{S} \models_{\mathsf{ABA},\mathsf{sem}}^{\cup} \phi$  iff  $\mathcal{S} \models_{\mathcal{A},\mathsf{sem}}^{\cup} \phi$  iff  $\mathcal{S} \models_{\mathsf{mcs}}^{\cup,\mathcal{A}} \phi$  for  $\mathsf{sem} \in \{\mathsf{prf}, \mathsf{stb}\}$  (Propositions 2 and 3.2).
- 4.  $\mathcal{A} \cup \mathcal{S} \models_{\mathsf{ABA},\mathsf{sem}}^{\mathbb{m}} \phi \text{ iff } \mathcal{S} \models_{\mathcal{A},\mathsf{sem}}^{\mathbb{m}} \phi \text{ iff } \mathcal{S} \models_{\mathsf{mcs}}^{\mathbb{m},A} \phi, \text{ for sem} \in \{\mathsf{prf},\mathsf{stb}\} \text{ (Propositions 2 and 3.3).}$

We will now turn to the rationality postulates from [30], see also Section 3.2. For these proofs consider the sequent-based ABA-framework  $\mathcal{AF}^{\mathsf{ABA}\Rightarrow}_{\langle \mathcal{L},\mathcal{R}\Rightarrow\rangle}(\mathcal{S},\mathcal{A}) = \langle \operatorname{Arg}^{\mathsf{ABA}\Rightarrow}_{\langle \mathcal{L},\mathcal{R}\Rightarrow\rangle}(\mathcal{S},\mathcal{A}), \mathcal{AT} \rangle$  for some deductive system  $\langle \mathcal{L},\mathcal{R} \rangle$ , let  $\mathcal{S}$  be a non-trivializing set of  $\mathcal{L}$ -formulas and  $\mathcal{A}$  a set of assumptions, where  $\Gamma \subseteq \mathcal{S}, A \subseteq \mathcal{A}$  are finite and  $\mathcal{S}\cap \mathcal{A} = \emptyset$ . Let sem  $\in \{\mathsf{grd}, \mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$ . Note that, due to the definition of contrary-consistency from Definition 25, the consistency postulate, defined in Definition 16, has to be adjusted:

•  $Concs(\mathcal{E})$  is consistent if and only if there is no  $\phi \in \mathcal{A}$  such that  $\phi, \overline{\phi} \in CN(Concs(\mathcal{E}))$ .

**Lemma 20** (Sub-argument closure).  $\mathcal{AF}^{\mathsf{ABA}\Rightarrow}_{\langle \mathcal{L}, \mathcal{R} \Rightarrow \rangle}(\mathcal{S}, \mathcal{A})$  satisfies sub-argument closure: let  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}^{\mathsf{ABA}\Rightarrow}_{\langle \mathcal{L}, \mathcal{R} \Rightarrow \rangle}(\mathcal{S}, \mathcal{A}))$ , then for all  $a \in \mathcal{E}$ ,  $\mathsf{Sub}(a) \subseteq \mathcal{E}$ .

Proof. Let  $a = A \ for \ rightarrow \phi \in \mathcal{E}$ ,  $a' = A' \ for \ rightarrow \phi' \in \mathsf{Sub}(a)$  and assume  $b \in \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\mathsf{ABA}_{\Rightarrow}}(\mathcal{S}, \mathcal{A})$  attacks a'. Then  $\operatorname{Conc}(b) = \overline{\psi}$  for some  $\psi \in A'$ . By definition of a sub-argument  $A' \subseteq A$ , hence b attacks a as well. Since  $\mathcal{E}$  is complete and  $a \in \mathcal{E}$ , it follows that there is a  $c \in \mathcal{E}$  which defends a and thus a' from the attack by b. Therefore  $a' \in \mathcal{E}$  and hence  $\operatorname{Sub}(a) \subseteq \mathcal{E}$ .

**Lemma 21** (Closure).  $\mathcal{AF}^{\mathsf{ABA}_{\Rightarrow}}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(\mathcal{S}, \mathcal{A})$  satisfies closure of extensions, for each extension  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}^{\mathsf{ABA}_{\Rightarrow}}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(\mathcal{S}, \mathcal{A}))$  it holds that  $\mathsf{Concs}(\mathcal{E}) = \mathsf{CN}(\mathsf{Concs}(\mathcal{E}))$ .

*Proof.* By Definition 25, it follows immediately that  $\mathsf{Concs}(\mathcal{E}) \subseteq \mathsf{CN}(\mathsf{Concs}(\mathcal{E}))$ . Suppose  $\phi \in \mathsf{CN}(\mathsf{Concs}(\mathcal{E}))$ . Then there are arguments  $a_1, \ldots, a_n \in \mathcal{E}$ , with  $\mathsf{Supp}(a_i) = \Gamma_i$ ,  $\mathsf{Conc}(a_i) = \phi_i$ ,  $\mathsf{Ass}(a_i) = A_i$  for  $1 \leq i \leq n$  and  $\phi_1, \ldots, \phi_n \Rightarrow \phi$  is derivable, using rules in  $\mathcal{R}_{\Rightarrow}$ . By [Cut]  $a = A_1, \ldots, A_n \notin \Gamma_1, \ldots, \Gamma_n \Rightarrow \phi$ .

Note that, if a is not attacked,  $a \in \mathcal{E}$ , thus  $\phi \in \mathsf{Concs}(\mathcal{E})$ . Now suppose  $b \in \operatorname{Arg}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}^{\mathsf{ABA}_{\Rightarrow}}(\mathcal{S}, \mathcal{A})$  attacks a. Then  $\mathsf{Conc}(b) = \overline{\psi}$  for some  $\psi \in A_1 \cup \ldots \cup A_n$ . Without loss of generality assume  $\psi \in A_i$ . Then b attacks  $a_i$  as well. Since  $a_i \in \mathcal{E}$  it follows that  $\mathcal{E}$  defends against the attack from b. Therefore  $a \in \mathcal{E}$  as well.  $\Box$ 

**Lemma 22** (Consistency).  $\mathcal{AF}^{\mathsf{ABA}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}}(\mathcal{S}, \mathcal{A})$  satisfies consistency: for each extension  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}^{\mathsf{ABA}_{\Rightarrow}}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(\mathcal{S}, \mathcal{A}))$ , there is no  $\phi \in \mathcal{A}$  such that  $\phi, \overline{\phi} \in \mathsf{CN}(\mathsf{Concs}(\mathcal{E}))$ .

*Proof.* Assume, towards a contradiction, that  $Concs(\mathcal{E})$  is not consistent. Then there are arguments  $a, b \in \mathcal{E}$ , such that  $\overline{Conc}(a) = Conc(b)$  (since  $CN(Concs(\mathcal{E})) = Concs(\mathcal{E})$ ). However, by Lemma 17,  $Ass(\mathcal{E})$  is consistent. Hence, by Definition 25, no such arguments exist.

**Theorem 4.** Let  $\mathcal{AF}^{\mathsf{ABA}\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}\Rightarrow \rangle}(\mathcal{S}, \mathcal{A}) = \langle \operatorname{Arg}^{\mathsf{ABA}\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}\Rightarrow \rangle}(\mathcal{S}, \mathcal{A}), \mathcal{AT} \rangle$  be an sequent-based ABA-framework, for the deductive system  $\langle \mathcal{L}, \mathcal{R} \rangle$ ,  $\mathcal{S}$  a non-trivializing set of  $\mathcal{L}$ -formulas,  $\mathcal{A}$  a set of assumptions and  $\operatorname{AT}_{\mathsf{ABA}}$  the only attack rule. Then the framework  $\mathcal{AF}^{\mathsf{ABA}\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}\Rightarrow \rangle}(\mathcal{S}, \mathcal{A})$  satisfies sub-argument closure, closure under strict rules and consistency.

#### 4.2 Adaptive Logics

Adaptive logics, originally introduced by Batens (see e.g., [21, 62] for an overview), are a logical framework that offer contributions to the research on formalizations of defeasible reasoning forms. It was developed to interpret (possibly) inconsistent theories as consistently as possible. From the perspective of epistemology, the introduction of adaptive logics has been motivated by the lack of a proof-theoretic account that captures the dynamic and defeasible aspects of human reasoning [20]. Adaptive logics have been frequently applied to reasoning forms typical for scientific reasoning (such as handling inconsistencies, inductive generalizations and abductive inferences). From the perspective of nonmonotonic logics, adaptive logics are a subclass of the preferential models known from [47]. Adaptive logics differ from other approaches based on preferential models in that they offer an adequate dynamic proof theory for the resulting nonmonotonic consequence relations. Nowadays adaptive logics cover many application contexts, such as inconsistent knowledge bases, default reasoning and circumscription, abstract argumentation, abduction, fuzzy logic, induction and deontic conflict. The idea is to interpret the premises *as normally as possible*. What this means depends on the logic and the application. The most common form for adaptive logics is the so-called standard format:

**Definition 31.** Adaptive logics in the standard format consist of three elements:

- the lower limit logic (LLL), the logic that is strengthened by the adaptive logic, with:
  - a Tarskian consequence relation (see Definition 1); and
  - a characteristic semantics.
- a set of abnormalities Ω, the form of the abnormalities depends on the lower limit logic and the application; and
- an adaptive strategy, either the reliability strategy which is a more cautious reasoning form or minimal abnormality strategy, which is a more credulous form of reasoning.

 $AL_{LLL}^x$ , where  $x \in \{r, m\}$  is the adaptive logic with lower limit logic **LLL** and strategy x, which can be the reliability strategy (r) or the minimal abnormality strategy (m). When the strategy and/or lower limit logic are arbitrary or clear from the context, the superscript and/or subscript are omitted.

A third strategy, that is not part of the standard format, is the *normal selections* strategy (n), which is even more credulous than the minimal abnormality strategy. In this section we will also consider this third strategy and will therefore also discuss the adaptive logic  $AL_{LLL}^n$ .

In the literature there are many logics that are used as lower limit logic. For example da Costa's  $C_i$  systems [22] and classical modal logics [23, 50] for which interpreting the premises as normally as possible means as non-conflicting as possible. Another example is the logic CLuN, introduced by Batens [19] under the name PI. It is obtained by adding the axioms  $\phi \lor \sim \phi$  to full positive classical logic, as such, it is a very weak paraconsistent logic. For CLuN interpreting the premises as normally as possible means as consistent as possible.

The set of abnormalities, denoted by  $\Omega$ , contains all the formulas of a logical form that depends on the lower limit logic of the adaptive logic and its application. Elements of  $\Omega$  will be denoted by  $!\phi$ , where  $\phi$  is the abnormal formula. In terms of abnormalities, interpreting the premises as normally as possible means that the premises are interpreted in a way that as few abnormalities as possible are validated. **Example 22.** Consider the paraconsistent logic CLuN. Let  $\Omega$  be the set of formulas of the form  $\sim \phi \land \phi$ , where  $\phi$  is a CLuN-formula. Then  $\mathsf{AL}^r_{\mathsf{CLuN}} = \langle \mathsf{CLuN}, \Omega, \mathsf{reliability} \rangle$  is the adaptive logic with lower limit logic CLuN and the reliability strategy.

The following notation will be useful in the definition of the consequence relations and proofs:

**Notation 3.** Let  $Dab(\Pi)$  denote the classical disjunction of members in  $\Pi$ , where  $\Pi$  is a finite subset of  $\Omega$ , then:

- the minimal Dab consequences for a premise set  $\Gamma$  are all the Dab( $\Pi$ ) such that  $\Gamma \vdash_{\mathbf{LLL}} \text{Dab}(\Pi)$  and there is no  $\Pi' \subset \Pi$  such that  $\Gamma \vdash_{\mathbf{LLL}} \text{Dab}(\Pi')$ ;
- if  $\text{Dab}(\Pi_1)$ ,  $\text{Dab}(\Pi_2)$ ,... are the minimal Dab consequences for  $\Gamma$ , then  $U(\Gamma) = \Pi_1 \cup \Pi_2 \cup \ldots$  is called the set of *unreliable abnormalities*; and
- let  $\Sigma(\Gamma) = {\Pi_1, \Pi_2, \ldots}$ , then  $\Phi(\Gamma)$  denotes the set of all minimal choice sets of  $\Sigma(\Gamma)$ .<sup>11</sup>

**Example 23.** Let  $S = \{p, q, \sim p \lor \sim q, \sim p \lor r, \sim q \lor r\}$  and suppose that CLuN is the lower limit logic. Then  $(p \land \sim p) \lor (q \land \sim q)$  is a minimal Dab-consequence of S. When reasoning skeptically, both p and q are considered unreliable, thus intuitively r should not follow. However, when reasoning more credulously, r can follow. To see this, suppose that p is unreliable (it is abnormal), then q could be normal, thus from q and  $\sim q \lor r$ , r follows.

In this paper we define the entailment relations of an adaptive logic semantically, based on [62]. For the dynamic proof theory of adaptive logics see [62, Chapter 2]. In what follows let  $\mathcal{M}_{LLL}(\Gamma)$  denote the set of all LLL-models for the set of formulas  $\Gamma$ .

**Definition 32.** Let *M* be an **LLL**-model, the *abnormal part of M* is then  $Ab(M) = \{\phi \in \Omega \mid M \vDash \phi\}$ .

From the abnormal part of a model a (strict) partial order can be defined on the models of a given premise set  $\Gamma$ :

- $M \sqsubset_{Ab}^{\Gamma} M'$  iff  $Ab(M) \subset Ab(M')$ ;
- $M \sqsubseteq_{Ab}^{\Gamma} M'$  iff  $Ab(M) \subseteq Ab(M')$ .

**Definition 33.** A model  $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$  is a *reliable model of*  $\Gamma$  if  $\mathrm{Ab}(M) \subseteq U(\Gamma)$ . The set of all reliable models of  $\Gamma$  is denoted by  $\mathcal{M}_{\mathsf{AL}^r}(\Gamma)$ .

<sup>&</sup>lt;sup>11</sup>A choice set of  $\Sigma(\Gamma)$  is a set of formulas  $\Delta$ , such that  $\Delta \cap \Pi_i \neq \emptyset$  for each  $\Pi_i \in \Sigma(\Gamma)$ .  $\Delta$  is minimal when there is no choice set  $\Delta'$  of  $\Sigma(\Gamma)$  such that  $\Delta' \subset \Delta$ .

**Definition 34.** A model  $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$  is a minimally abnormal model of  $\Gamma$  when for all other models  $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$  of  $\Gamma$ ,  $\operatorname{Ab}(M') \not\subset \operatorname{Ab}(M)$ . The set of all minimally abnormal models of  $\Gamma$  is denoted by  $\mathcal{M}_{\mathsf{AL}^m}(\Gamma)$ .

Thus, the minimally abnormal models are the minimal elements of the partial order  $\sqsubset_{Ab}^{\Gamma}$ .

**Definition 35.** The entailment relations for the three strategies are then defined by:

- $\Gamma \vdash_{\mathbf{LLL}}^{r,\Omega} \phi$  if and only if for each  $M \in \mathcal{M}_{\mathsf{AL}^r}(\Gamma), M \vDash \phi$ .
- $\Gamma \vdash_{\mathbf{LLL}}^{m,\Omega} \phi$  if and only if for each  $M \in \mathcal{M}_{\mathsf{AL}^m}(\Gamma), M \vDash \phi$ .
- $\Gamma \vdash_{\mathbf{LLL}}^{n,\Omega} \phi$  if and only if there is a model  $M \in \mathcal{M}_{\mathsf{AL}^m}(\Gamma)$  such that for all  $M' \in \mathcal{M}_{\mathsf{LLL}}(\Gamma)$  for which  $\operatorname{Ab}(M) = \operatorname{Ab}(M'), M' \vDash \phi$ .

**Example 24.** Recall the set  $S = \{p, q, \neg q \lor \neg p, \neg q \lor r, \neg p \lor r\}$  from Example 23, where CLuN is the lower limit logic. Three types of models can be considered, they differ in their abnormal parts:  $M_1$  for which  $Ab(M_1) = \{p \land \neg p\}$ ,  $M_2$  for which  $Ab(M_2) = \{q \land \neg q\}$  and  $M_3$  for which  $Ab(M_3) = \{p \land \neg p, q \land \neg q\}$ . As mentioned in Example 23, intuitively it is expected that r follows when reasoning credulously, but not when reasoning skeptically. Indeed,  $S \not|_{\mathsf{CLuN}}^r r$ , while  $S \mid_{\mathsf{CLuN}}^m r$  and  $S \mid_{\mathsf{CLuN}}^n r$ .

In assumptive sequent-based argumentation with a lower limit logic **LLL** as core logic, an inference rule (RC) is added to the sequent calculus **C** of **LLL**. The idea is similar to the rules  $AS_{AS}$  introduced in Definition 10. Let  $\phi$  be a formula in the language of **LLL** and let  $!\phi$  denote the abnormality for the formula  $\phi$ . We consider two variations and will refer in both cases to the *RC*-rule:

$$\frac{\Pi \ r \Rightarrow \Delta, \psi \lor ! \phi}{\Pi, ! \phi \ r \Rightarrow \Delta, \psi} \text{ RC} \qquad \qquad \frac{\Pi \ r \Rightarrow \Delta, ! \phi}{\Pi, ! \phi \ r \Rightarrow \Delta} \text{ RC} \tag{3}$$

For a logic  $L = \langle \mathcal{L}, \vdash \rangle$ , with corresponding sequent calculus C, let  $C' = C \cup \{RC\}$ . AL-sequent arguments are then defined as follows:

**Definition 36.** Let **LLL** be a lower limit logic, with corresponding sound and complete sequent calculus  $\mathsf{C}$ , let  $\mathcal{S}$  be a set of **LLL**-formulas and  $\Omega$  a set of abnormalities. An assumptive **LLL**-argument based on  $\mathcal{S}$  and  $\Omega$  (*AL*-(sequent )argument for short) is an assumptive sequent  $\Pi \ \mathbf{S} \ \Gamma \Rightarrow \psi$ , provable in  $\mathsf{C}'$ , where  $\Pi \subseteq \Omega$  and  $\Gamma \subseteq \mathcal{S}$ . Arg<sub>LLL,\Omega</sub>( $\mathcal{S}$ ) denotes the set of all AL-arguments based on  $\mathcal{S}$  and  $\Omega$ . **Definition 37.** The sequent elimination rule for assumptive sequent-based argumentation with adaptive logics is defined as, where  $\Pi \ \Gamma \Rightarrow \phi, \ \Theta, \phi \ \Delta \Rightarrow \psi \in \operatorname{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S})$ :

$$\frac{\Pi \ \mathbf{f} \ \Gamma \Rightarrow \phi \quad \Theta, \phi \ \mathbf{f} \ \Delta \Rightarrow \psi}{\Theta, \phi \ \mathbf{f} \ \Delta \Rightarrow \psi} \ \mathrm{AT}_{\mathsf{AL}} \tag{4}$$

An assumptive sequent-based argumentation framework for adaptive logics is now defined as:

**Definition 38.** An adaptive logic sequent-based argumentation framework ((sequentbased) AL-framework for short) for the lower limit logic  $\mathbf{LLL} = \langle \mathcal{L}, \vdash \rangle$ , with corresponding sequent calculus C, set of abnormalities  $\Omega$ , set of formulas S and  $\operatorname{AT}_{\mathsf{AL}}$ as sequent elimination rule, is a pair  $\mathcal{AF}_{\mathsf{LLL},\Omega}(S) = \langle \operatorname{Arg}_{\mathsf{LLL},\Omega}(S), \mathcal{AT} \rangle$ . Where  $\operatorname{Arg}_{\mathsf{LLL},\Omega}(S)$  is the set of AL-arguments based on S and  $\Omega$ ,  $\mathcal{AT} \subseteq \operatorname{Arg}_{\mathsf{LLL},\Omega}(S) \times \operatorname{Arg}_{\mathsf{LLL},\Omega}(S)$  and  $(a_1, a_2) \in \mathcal{AT}$  iff  $a_1 \operatorname{AT}_{\mathsf{AL}}$ -attacks  $a_2$ .

**Example 25.** Consider again the set  $S = \{p, q, \sim q \lor \sim p, \sim q \lor r, \sim p \lor r\}$  and let  $\mathcal{AF}_{\mathsf{CLuN},\Omega}(S) = \langle \operatorname{Arg}_{\mathsf{CLuN},\Omega}(S), \mathcal{AT} \rangle$ , where  $\mathcal{AT}$  is based on  $\operatorname{AT}_{\mathsf{AL}}$ . Note that  $!\psi \in \Omega$  if and only if  $\psi$  is a  $\mathsf{CLuN}$ -formula and  $!\psi = \psi \land \sim \psi$ . Some of the arguments in  $\operatorname{Arg}_{\mathsf{CLuN},\Omega}(S)$  are:

$$\begin{aligned} a &= p \Rightarrow p \qquad b = q \Rightarrow q \qquad c = \sim q \lor \sim p \Rightarrow \sim q \lor \sim p \\ d &= p, \sim p \lor \sim q \Rightarrow \sim q \lor ! p \qquad e = q, \sim p \lor \sim q \Rightarrow \sim p \lor ! q \\ f &= !p \clubsuit \mathcal{S} \Rightarrow ! q \qquad g = !q \clubsuit \mathcal{S} \Rightarrow ! p \qquad h = !p \clubsuit \mathcal{S} \Rightarrow r \qquad k = !q \clubsuit \mathcal{S} \Rightarrow r. \end{aligned}$$

As in previous sections, these are only a subset of the available arguments. See Figure 9 for a graphical representation.



Figure 9: Part of the AL-framework of Example 25 for  $S = \{p, q, \sim q \lor \sim p, \sim q \lor r, \sim p \lor r\}$ .

The consequence relation corresponding to an adaptive logic sequent-based argumentation framework  $\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S})$  is denoted by  $\succ_{\Omega,\mathsf{sem}}^{\star}$  for each semantics and  $\star \in \{\cap, \cup, \mathbb{n}\}$ . Similar to Proposition 2 the following representational theorem can be shown:

**Theorem 5.** Let  $\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}) = \langle Arg_{\mathbf{LLL},\Omega}(\mathcal{S}), \mathcal{AT} \rangle$  be a sequent-based argumentation framework for the lower limit logic  $\mathbf{LLL} = \langle \mathcal{L}, \vdash \rangle$ , with corresponding sequent calculus  $\mathsf{C}$ , set of abnormalities  $\Omega$  and set of  $\mathcal{L}$ -formulas  $\mathcal{S}$ .

- 1.  $\mathcal{S} \models_{\mathbf{LLL}}^{m,\Omega} \phi$  if and only if  $\mathcal{S} \models_{\Omega,\mathsf{prf}}^{\mathbb{n}} \phi$ .
- 2.  $\mathcal{S} \models_{\mathbf{LLL}}^{r,\Omega} \phi$  if and only if  $\mathcal{S} \models_{\Omega,\mathsf{prf}}^{\cap} \phi$ .
- 3.  $\mathcal{S} \models_{\mathbf{LLL}}^{n,\Omega} \phi$  if and only if  $\mathcal{S} \models_{\Omega,\mathsf{prf}}^{\cup} \phi$ .

Due to the requirement of further notation and many technical details, the proof of the above theorem is placed in Appendix B.

For adaptive logic sequent-based argumentation frameworks, the representation of reasoning with maximally consistent subsets (recall Section 3.3) follows from the results in [56], in which it was shown that the consequence relations of adaptive logics are directly related to those of default assumptions, discussed in the next section. We therefore refer to Corollary 2 on page 53.

**Example 26.** Recall the setting from Example 25, for the sequent-based ALframework  $\mathcal{AF}_{\mathsf{CLuN},\Omega}(\mathcal{S}) = \langle \operatorname{Arg}_{\mathsf{CLuN},\Omega}(\mathcal{S}), \mathcal{AT} \rangle$ ,  $\mathcal{S} = \{p, q, \sim q \lor \sim p, \sim q \lor r, \sim p \lor r\}$ and nine arguments were introduced. Two preferred extensions can be considered:  $\mathcal{E}_1 \supseteq \{a, b, c, d, e, f, h\}$  and  $\mathcal{E}_2 \supseteq \{a, b, c, d, e, g, k\}$ . Hence  $\mathcal{S} \models_{\Omega, \mathsf{prf}}^{\cap} r$  but  $\mathcal{S} \not\models_{\Omega, \mathsf{prf}}^{\cap} r$ .

The above example shows that the consistency postulate (Definition 16) does not hold in sequent-based AL-frameworks. This is the case since S is not necessarily consistent. In fact, applying argumentation to a set of formulas S is only interesting when it is inconsistent, since otherwise the consequences would be the same as the conclusions that are already derivable with the lower limit logic. However, we will show below that the other two postulates (i.e., closure and sub-argument closure) can be shown for adaptive logic sequent-based argumentation.

In what follows let  $\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}) = \left\langle \operatorname{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S}), \mathcal{AT} \right\rangle$  be a sequent-based ALframework for  $\mathcal{S}$  a set of formulas,  $\Omega$  a set of assumptions and  $\operatorname{AT}_{\mathsf{AL}}$  the attack rule.

**Lemma 23** (Sub-argument closure). Let  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}))$ , if  $a \in \mathcal{E}$  then  $\mathsf{Sub}(a) \subseteq \mathcal{E}$ .

*Proof.* Assume  $a \in \mathcal{E}$ . Let  $a' \in \mathsf{Sub}(a)$  and assume  $b \in \operatorname{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S})$  attacks a'. Thus  $\mathsf{Conc}(b) \in \mathsf{Ass}(a')$ . Since, by definition,  $\mathsf{Ass}(a') \subseteq \mathsf{Ass}(a)$ , it follows that b attacks a as well. Therefore, there is some  $c \in \mathcal{E}$ , which defends a, and thus a' from the attack by b. Hence,  $a' \in \mathcal{E}$ .

**Lemma 24** (Closure). Let  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}))$ , then  $\mathsf{Concs}(\mathcal{E})$  is closed under strict rules.

*Proof.* To show Concs(*E*) = CN(Concs(*E*)). Note that Concs(*E*) ⊆ CN(Concs(*E*)) by the reflexivity of ⊢, it remains to show that Concs(*E*) ⊇ CN(Concs(*E*)). Suppose  $\phi \in CN(Concs(\mathcal{E}))$ . Then there are arguments  $a_1, \ldots, a_n \in \mathcal{E}$  such that Conc( $a_i$ ) =  $\phi_i$ , Supp( $a_i$ ) =  $\Gamma_i$  and Ass( $a_i$ ) =  $\Pi_i$  for  $1 \le i \le n$  and  $\phi_1, \ldots, \phi_n \vdash \phi$ . By the completeness of C and applying [Cut] it follows that  $a = Ass(a_1), \ldots, Ass(a_n)$   $\Gamma_1, \ldots, \Gamma_n \Rightarrow \phi$  is derivable. Note that any attacker of a is an attacker of one of the arguments  $a_1, \ldots, a_n$ . Since  $\mathcal{E} \in Ext_{cmp}(\mathcal{AF}_{LLL,\Omega}(\mathcal{S}))$ , it follows that  $a \in \mathcal{E}$  as well. Therefore Concs( $\mathcal{E}$ ) = CN(Concs( $\mathcal{E}$ )).

As noted after Example 26, the consistency postulate does not hold for sequentbased AL-frameworks since S can be inconsistent.

**Theorem 6.** Let  $\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}) = \langle Arg_{\mathbf{LLL},\Omega}(\mathcal{S}), \mathcal{AT} \rangle$  for  $\mathcal{S}$  a set of formulas,  $\Omega$  a set of abnormalities and  $AT_{\mathsf{AL}}$  the attack rule.  $\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S})$  satisfies sub-argument closure and closure under strict rules under completeness-based semantics. But it does not satisfy consistency.

#### 4.3 Default Assumptions

In [48], Makinson presents three ways of turning a classical consequence relation nonmonotonic. The first of which uses additional background assumptions, called *default assumptions*. The resulting nonmonotonic consequence relation is directly related to the assumptive maximally consistent subset consequence relations from Definition 18, as well as to the adaptive consequence relation for minimal abnormality  $\triangleright_{\mathbf{LLL}}^{m,\Omega}$  from Definition 35, see [56]. Because of the relations between the different approaches, default assumptions are used in this section to show how adaptive sequent-based argumentation as introduced in the previous section is related to reasoning with (assumptive) maximally consistent subsets.

In addition to the default assumption consequence relation introduced in [48]  $(\succ_{\mathsf{mcs}}^{\oplus,\mathcal{AS}} \text{ in Section 3.3})$ , the two other relations from Definition 18 (i.e.,  $\succ_{\mathsf{mcs}}^{\cap,\mathcal{AS}}$  and  $\succ_{\mathsf{mcs}}^{\cup,\mathcal{AS}}$ ) will be considered as well. For the remainder of this section, it is assumed that  $\mathcal{L}$  contains at least a negation operator  $\neg$  as introduced in Section 2.

**Example 27.** Let CL be the core logic and, as in Example 18,  $S = \{s\}$  and  $\mathcal{AS} = \{p, q, \neg p \lor \neg q, \neg p \lor r, \neg q \lor r\}$ . Then  $\mathsf{MCS}(S, \mathcal{AS}) = \{\{p, \neg p \lor \neg q, \neg p \lor r, \neg q \lor r\}, \{q, \neg p \lor \neg q, \neg p \lor r, \neg q \lor r\}, \{q, \neg p \lor r, \neg q \lor r\}, \{p, q, \neg p \lor r, \neg q \lor r\}\}$ . Clearly  $S \models_{\mathsf{mcs}}^{\star, \mathcal{AS}} s$ , additionally  $S \models_{\mathsf{mcs}}^{\star, \mathcal{AS}} \neg p \lor r$  and  $S \models_{\mathsf{mcs}}^{\star, \mathcal{AS}} \neg q \lor r$  for  $\star \in \{\mathbb{n}, \cap\}$ . Furthermore,  $S \models_{\mathsf{mcs}}^{\cup, \mathcal{AS}} \phi$  for  $\phi \in S \cup \mathcal{AS}$ .

Recall the entailment relations  $\succ_{\mathbf{LLL}}^{r,\Omega}$  and  $\succ_{\mathbf{LLL}}^{m,\Omega}$  from Definition 35. For  $\mathcal{S}$  a set of formulas, **LLL** a monotonic logic,  $\Omega$  a set of abnormalities and  $\mathcal{AS}$  a set of default assumptions, in [56] it is shown that, where the maximally consistent subsets are taken with respect to the core logic **LLL**:

•  $S \vdash_{\mathbf{LLL}}^{m,\Omega} \phi$  iff  $S \vdash_{\mathsf{mcs}}^{\mathfrak{m},\neg\Omega} \phi$  and similarly  $S \vdash_{\mathsf{mcs}}^{\mathfrak{m},\mathcal{AS}} \phi$  iff  $S \vdash_{\mathbf{LLL}}^{m,\neg\mathcal{AS}} \phi$ •  $S \vdash_{\mathbf{LLL}}^{r,\Omega} \phi$  iff  $S \vdash_{\mathsf{mcs}}^{\Omega,\neg\Omega} \phi$  and similarly  $S \vdash_{\mathsf{mcs}}^{\Omega,\mathcal{AS}} \phi$  iff  $S \vdash_{\mathbf{LLL}}^{r,\neg\mathcal{AS}} \phi$ .

Let  $\succ_{\Omega, \mathsf{prf}}^{\star, \mathrm{AL}}$  for  $\star \in \{\cap, \mathbb{N}, \cup\}$  denote the consequence relation corresponding to an adaptive logic sequent-based argumentation framework, as defined in the previous section. The following corollary is obtained from the results in [56], Theorem 5 and Proposition 2.

**Corollary 2.** Let  $\mathcal{AF}_{\mathsf{L},\mathcal{K}}(\mathcal{S}) = \langle \operatorname{Arg}_{\mathsf{L},\mathcal{K}}(\mathcal{S}), \mathcal{AT} \rangle$ , where  $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$  is a monotonic logic with corresponding sequent calculus  $\mathsf{C}, \mathcal{S}$  is a set of formulas and  $\mathcal{K}$  is a set of default assumptions. Then:

1. 
$$S \models_{\mathsf{mcs}}^{\mathbb{m},\mathcal{AS}} \phi \text{ iff } S \models_{\mathcal{AS},\mathsf{prf}}^{\mathbb{m}} \phi \text{ iff } S \models_{\mathsf{L}}^{m,\neg\mathcal{AS}} \phi \text{ iff } S \models_{\neg\mathcal{AS},\mathsf{prf}}^{\mathbb{n},AL} \phi.$$
  
2.  $S \models_{\mathsf{mcs}}^{\cap,\mathcal{AS}} \phi \text{ iff } S \models_{\mathcal{AS},\mathsf{prf}}^{\cap} \phi \text{ iff } S \models_{\mathsf{L}}^{r,\neg\mathcal{AS}} \phi \text{ iff } S \models_{\neg\mathcal{AS},\mathsf{prf}}^{\cap,AL} \phi.$ 

## 5 Related Literature

That one framework can be expressed by another (and vice versa), is nothing new. Relations between different formal approaches to nonmonotonic reasoning have been studied in the literature. As mentioned in the introduction, default logic is an instance of ABA [25]. The results in [56] were used in Section 4.3, to relate reasoning with maximally consistent subsets and the presented adaptive logic setting. In [43], ABA in relation to adaptive logics and vice versa, and ASPIC<sup>+</sup> to ABA were studied. Furthermore, reasoning with maximally consistent subsets and the related consequence relations are studied for other structured argumentation frameworks [2, 7, 9, 32, 41, 65], see [6] for a survey. By introducing assumptive sequentbased argumentation, a first step was made into the study of how sequent-based argumentation fits within this group of nonmonotonic reasoning systems. Although different approaches to formal argumentation can be expressed by one another, one way of making a distinction between them is by their level of abstraction. Abstract argumentation (see Dung [36] and recall Section 2.1) is the most abstract and, as mentioned, it has been argued that it should be instantiated [55]. When looking at some approaches to logical argumentation (i.e., ABA, (assumptive) sequent-based argumentation and ASPIC<sup>+</sup> mentioned below), we can distinguish different levels of abstraction. ASPIC<sup>+</sup> [51, 54] is the most fine-grained perspective, where arguments come with a full proof structure. On the other hand, ABA is the most abstract of the three, since the semantics are applied to sets of sets of assumptions and the derivation of a conclusion is completely abstract. (Assumptive) sequent-based argumentation lies between these two approaches, it is less abstract than ABA, since an argument consists of a support set and a conclusion (and in the case of assumptive sequent-based argumentation, it is clear which strict and defeasible assumptions were used in the construction of an argument), but the exact derivation of the argument is not part of the argument itself.

In Section 4 we have only studied three of the well-known approaches to reasoning with defeasible assumptions. Two other well-known approaches were not mentioned here: ASPIC<sup>+</sup> [51, 54] and default logic [4, 59]. The first, like ABA, is an approach to structured argumentation, in which a distinction is made between axioms (the strict premises in the setting of this paper) and ordinary premises (the assumptions in this paper) and there are two types of rules: strict and defeasible ones. Moreover, an extensive study into the use of preferences was done in [51]. The result is an expressive structured argumentation system.

Research on  $ASPIC^+$  has focused on applications and on the enrichment of the expressive power of the underlying language (such as the addition of preferences and having strict and defeasible rules) to be able to model different aspects of human reasoning. Research on sequent-based argumentation, which was introduced in the tradition of instantiating abstract argumentation with Tarskian logics (see also [24, 40], has focused on studying logical properties of the resulting entailment relations and semantic extensions of an argumentation framework. As pointed out in, e.g., [29], defining argumentation frameworks with a robust meta-theory (e.g., satisfying the rationality postulates from [30, 31]), is not only interesting from a theoretical point of view, but is also beneficial for practical purposes. However, because of the many components from which an argumentation framework is constructed, this has been challenging for  $ASPIC^+$  in case the set of strict rules is sufficiently rich (e.g., when these are based on a Tarskian logic) [29]. The only instantiation that satisfies all standard rationality postulates is  $ASPIC^{\ominus}$ , see [44]. In contrast, sequent-based argumentation has been studied with these challenges in mind. Classes of frameworks, instantiated with Tarskian logics, have been identified that satisfy all rationality postulates and other logical properties. Moreover, dynamic derivations ([11, 12]), introduced for sequent-based argumentation, provide a proof-theoretic approach to formal argumentation with which Gentzen-type sequent calculi can be applied to study the reasoning process of argumentation. Thus, while  $ASPIC^+$  frameworks are very expressive and many possible applications have been studied, sequent-based argumentation has mainly been investigated to obtain a clear view of its meta-theoretic properties. How  $ASPIC^+$  and (assumptive) sequent-based argumentation relate remains a question for future work. A good starting point for this investigation are the results in [28], where it is shown that both, in a setting without priorities, can be translated in a very simple argumentation setting.

The second approach, default logic, was already shortly mentioned in the introduction as one of the best-known approaches to reasoning with defeasible rules. There are however several specific additional problems one faces when representing default logic in sequent-based argumentation, besides the handling of default assumptions. One is that, although default logic has CL as underlying deductive system, classical connectives are not handled in a standard way when they occur in default rules. For example, disjunction does not allow for reasoning by cases and negation does not allow for contraposition. This is shown in the following example.

**Example 28.** Recall from the introduction that a default rule is of the form  $\phi : \phi_1, \ldots, \phi_n/\psi$ , which represents that  $\psi$  can be derived, if  $\phi$  is given and no inconsistencies arise when  $\phi_1, \ldots, \phi_n$  hold. Intuitively, one could expect that such a default rule can be translated into an assumptive sequent:  $\phi_1, \ldots, \phi_n \ \phi \Rightarrow \psi$ . Suppose that  $\mathcal{AF}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}) = \langle \operatorname{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}), \mathcal{AT} \rangle$ , where CL is the core logic, the sequent calculus is  $\mathsf{LK}'$  with in addition the sequents obtained by translating the rules from  $\mathcal{D}$  and  $\mathcal{AS}$  contains the assumptions from the rules in  $\mathcal{D}$  (i.e.,  $\phi_1, \ldots, \phi_n \in \mathcal{AS}$  if the rule above is part of  $\mathcal{D}$ ).

- Let  $S = \{\neg q\}$  and let  $\mathcal{D} = \left\{\frac{\emptyset:p}{q}\right\}$ . This rule would be translated into  $p \not \Rightarrow q$ . However, then by  $\operatorname{AS}_{\mathcal{AS}}^r$ ,  $[\Rightarrow\neg]$  and  $[\neg\Rightarrow]$  the sequent  $\neg q \Rightarrow \neg p$  is derivable. Moreover, since  $\neg q \in S$ ,  $\neg q \Rightarrow \neg p$  is an argument that cannot be attacked: its set of assumptions is empty. Therefore  $\neg p \in \operatorname{Concs}(\bigcap \operatorname{Ext}_{\operatorname{sem}}(\mathcal{AF}_{\mathsf{L}}(S,\mathcal{AS})))$ , for any of the considered semantics. Yet  $\neg p$  is not a default conclusion.
- Now suppose that  $S = \emptyset$  and let  $\mathcal{D} = \left\{\frac{\emptyset:p}{q\vee t}, \frac{\emptyset:q}{v}\right\}$ . These default rules are translated into the sequents  $p \right\} \Rightarrow q \lor t$  and  $q \right\} \Rightarrow v$ . From these, by applying  $[\Rightarrow \lor], [\lor \Rightarrow], [\operatorname{Cut}]$  and weakening,  $p \right\} \Rightarrow v \lor t$  can be derived. Since there are no attackers,  $v \lor t \in \operatorname{Concs}(\bigcap \operatorname{Ext}_{\operatorname{sem}}(\mathcal{AF}_{\mathsf{L}}(S, \mathcal{AS})))$ . However, in deault logic,  $v \lor t$  is not a consequence.

In the first case, the problem arises because of the application of the sequent rules for negation. Similarly, in the second case, the rules for disjunction make it possible to derive  $v \lor t$ .

Because of examples such as the ones above, there is an asymmetry when reasoning classically with the consequences of applications of defaults where all connectives have their standard meaning, and reasoning with the defaults themselves. This asymmetry poses an additional challenge for a representation of default logic within the presented framework. As a solution for this, in the representation of default logic in ABA (see [25, §2.3]), classical logic cannot be applied to the assumptions in the default rule. However, one of the advantages of (assumptive) sequent-based argumentation, is the modularity of the approach (any logic with corresponding sequent calculus can be taken as the deductive base) and the availability of dynamic proofs [12], which allow for the automatic derivation of arguments. In light of this, the representation of default logic in assumptive sequent-based argumentation without such restrictions is left for future work.

# 6 Conclusion

In order to incorporate defeasible assumptions, sequent-based argumentation was extended to assumptive sequent-based argumentation. An additional component was added to each sequent, to contain the defeasible assumptions. As in sequentbased argumentation, any logic with a corresponding sound and complete sequent calculus can be taken as the core logic. It was shown how the assumptive framework can be generalized to a prioritized setting and several desirable properties were investigated. Furthermore, three well-known and much researched approaches to reasoning with assumptions were investigated in the context of assumptive sequentbased argumentation. It was shown that assumption-based argumentation (ABA), adaptive logics and default assumptions can be embedded in the here introduced framework.

Due to its generic and modular setting (only few requirements are placed on the logic and its corresponding calculus) assumptive sequent-based argumentation is a very general approach to reasoning with assumptions. In addition, the presented proofs do not rely on specific properties of the logic and only a few rules are assumed to be admissible in the calculus. This paper therefore paves the way to equip many well-known logics (e.g., intuitionistic logic and many modal logics) with defeasible assumptions. Moreover, although we required the logic to be Tarskian in this paper (recall Definition 1), this is not strictly necessary for the general definitions of assumptive sequent-based argumentation. It would therefore be possible to take a substructural logic, often characterized in terms of sequent calculi, as the core logic of an assumptive sequent-based argumentation framework. This would, for example, allow to incorporate a non-transitive system such as ST, which has been applied to study paradoxes [33, 34]. Note that such a system cannot be represented by a deductive system underlying ABA, since these are assumed to be transitive.

Though relations to other forms of reasoning with defeasible assumptions have been discussed in detail, it was not the objective of this paper to show how various approaches relate to each other, but instead to introduce a general logical argumentation framework, that allows for reasoning with assumptions in different settings. For example, situations in which assumptions are supposed to hold (such as in ABA) or supposed not to be satisfied (such as in adaptive logics), different core logics, such that different settings can be modeled, allowing for a priority function as additional input and with different mechanisms (Dung-style semantics and maximally consistent subsets). For the three approaches that were taken as example in Section 4, it was shown that the resulting sequent-based framework satisfies the rationality postulates from [30] (except for consistency in the case of adaptive logic). Therefore, assumptive sequent-based argumentation is a very general and flexible framework (it allows for many instances and can easily be adjusted to the requirements of different situations), that is also well-behaved (it satisfies some desirable properties).

The presented assumptive sequent-based argumentation framework can be extended to include other research on sequent-based argumentation. For example, the notion of a sequent can be generalized to a hypersequent, as in [27]. This way further core logics and calculi can be taken as the deductive base and the results of the extensive studies on sequent calculi in proof theory can be benefited from in formal argumentation. Furthermore, the dynamic proof theory from [12] can be adjusted to the assumptive setting presented here, thus extending this proof-theoretic approach to formal argumentation to account for defeasible assumptions. The availability of first-order sequent calculi opens up the possibility to investigate nonmonotonic systems such as circumscription. Though these extensions are left for future work, they will further strengthen the benefits of the assumptive sequent-based approach to formal argumentation. In addition, it would be interesting to know if assumptive sequent-based argumentation is more expressive than ABA, adaptive logics and/or default assumptions, or if they are equivalent. Therefore, in future work, we will investigate instances of the example frameworks, to see if these can express (assumptive) sequent-based argumentation.

#### Borg

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## A Admissible Rules in the Minimal Calculus

In this appendix we show that the rules from Figure 8 are indeed derivable in any (single conclusioned) sequent calculus in which the rules from Figure 6 are admissible. We show this by sequent derivations in the minimal calculus from Figure 6.

**Lemma 1.** Let  $L = \langle \mathcal{L}, \vdash \rangle$  be a logic with corresponding sequent calculus C, in which the rules from Figure 6 are admissible. Then the rules from Figure 8 are admissible as well.

*Proof.* Let  $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$  be a logic with corresponding sequent calculus  $\mathsf{C}$  in which the rules from Figure 6 are admissible. We show that the rules from Figure 8 are admissible. Recall that  $\Pi$  is empty if  $\mathsf{C}$  is a single conclusioned calculus and  $\Delta$  contains at most one formula. We consider each of the axioms and rules in turn, note that each of the derivations can also be done in a single conclusioned calculus.

 $[\Rightarrow \land \land]$  First a useful derivation, that shows that  $\phi_1, \ldots, \phi_n \Rightarrow \phi_1 \land \ldots \land \phi_n$  is derivable. We show the case for n = 3, the cases for other values of n are similar.

$$\begin{array}{c} \frac{\phi_1 \Rightarrow \phi_1}{\phi_1, \phi_2 \Rightarrow \phi_1} \begin{bmatrix} \text{LMon} \end{bmatrix} & \frac{\phi_2 \Rightarrow \phi_2}{\phi_2, \phi_3 \Rightarrow \phi_2} \begin{bmatrix} \text{LMon} \end{bmatrix} & \frac{\phi_3 \Rightarrow \phi_3}{\phi_2, \phi_3 \Rightarrow \phi_3} \begin{bmatrix} \text{LMon} \end{bmatrix} \\ \frac{\phi_1, \phi_2, \phi_3 \Rightarrow \phi_1}{\phi_1, \phi_2, \phi_3 \Rightarrow \phi_1 \land \phi_2} \begin{bmatrix} \text{Mon} \end{bmatrix} & \frac{\phi_3 \Rightarrow \phi_3}{\phi_1, \phi_2, \phi_3 \Rightarrow \phi_3} \begin{bmatrix} \text{LMon} \end{bmatrix} \\ \frac{\phi_1, \phi_2, \phi_3 \Rightarrow \phi_1 \land \phi_2}{\phi_1, \phi_2, \phi_3 \Rightarrow \phi_1 \land \phi_2 \land \phi_3} \begin{bmatrix} \text{Mon} \end{bmatrix} \\ \frac{\phi_3 \Rightarrow \phi_3}{\phi_1, \phi_2, \phi_3 \Rightarrow \phi_3} \begin{bmatrix} \text{LMon} \end{bmatrix} \\ \frac{\phi_1, \phi_2, \phi_3 \Rightarrow \phi_1 \land \phi_2 \land \phi_3}{\phi_1, \phi_2, \phi_3 \Rightarrow \phi_1 \land \phi_2 \land \phi_3} \begin{bmatrix} \text{LMon} \end{bmatrix} \\ \frac{\phi_3 \Rightarrow \phi_3}{\phi_1, \phi_2, \phi_3 \Rightarrow \phi_3} \begin{bmatrix} \text{LMon} \end{bmatrix} \\ \frac{\phi_3 \Rightarrow \phi_3}{\phi_1, \phi_2, \phi_3 \Rightarrow \phi_3} \begin{bmatrix} \text{LMon} \end{bmatrix} \\ \frac{\phi_3 \Rightarrow \phi_3}{\phi_1, \phi_2, \phi_3 \Rightarrow \phi_3} \begin{bmatrix} \text{LMon} \end{bmatrix} \\ \frac{\phi_1, \phi_2, \phi_3 \Rightarrow \phi_1 \land \phi_2 \land \phi_3}{\phi_1, \phi_2, \phi_3 \Rightarrow \phi_1 \land \phi_2 \land \phi_3} \begin{bmatrix} \text{LMon} \end{bmatrix} \\ \frac{\phi_1, \phi_2, \phi_3 \Rightarrow \phi_3}{\phi_1, \phi_2, \phi_3 \Rightarrow \phi_1 \land \phi_2 \land \phi_3} \begin{bmatrix} \text{LMon} \end{bmatrix} \\ \frac{\phi_1, \phi_2, \phi_3 \Rightarrow \phi_3}{\phi_1, \phi_2, \phi_3 \Rightarrow \phi_1 \land \phi_2 \land \phi_3} \begin{bmatrix} \text{LMon} \end{bmatrix} \\ \frac{\phi_1, \phi_2, \phi_3 \Rightarrow \phi_3}{\phi_1, \phi_2, \phi_3 \Rightarrow \phi_1 \land \phi_2 \land \phi_3} \begin{bmatrix} \text{LMon} \end{bmatrix} \\ \frac{\phi_1, \phi_2, \phi_3 \Rightarrow \phi_3}{\phi_1, \phi_2, \phi_3 \Rightarrow \phi_1 \land \phi_2 \land \phi_3 & \phi_$$

$$\begin{array}{l} [\Rightarrow \neg \wedge] \\ [\Rightarrow \neg \wedge] \\ [\Rightarrow \neg \wedge] \\ \hline \begin{array}{c} \frac{\Gamma, \phi_1, \dots, \phi_n \Rightarrow \Pi}{\vdots} \quad [\wedge \Rightarrow] \\ \hline \Gamma, \phi_1 \wedge \dots \wedge \phi_n \Rightarrow \Pi} \quad [\wedge \Rightarrow] \\ \hline \Gamma \Rightarrow \neg (\phi_1 \wedge \dots \wedge \phi_n), \Pi \quad [\Rightarrow \neg] \\ \hline \hline \hline \phi_1, \dots, \phi_n \Rightarrow \phi_1 \wedge \dots \wedge \phi_n \quad [\Rightarrow \wedge \wedge] \\ \hline \hline \phi_1, \dots, \phi_n, \neg (\phi_1 \wedge \dots \wedge \phi_n) \Rightarrow \quad [\neg \Rightarrow] \\ \hline \Gamma, \phi_1, \dots, \phi_n \Rightarrow \Pi \end{array}$$

#### **B** Representation Adaptive Logics

In this appendix we turn to the proof of the following theorem:

**Theorem 5.** Let  $\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}) = \langle Arg_{\mathbf{LLL},\Omega}(\mathcal{S}), \mathcal{AT} \rangle$  be a sequent-based argumentation framework for the lower limit logic  $\mathbf{LLL} = \langle \mathcal{L}, \vdash \rangle$ , with corresponding sequent calculus  $\mathsf{C}$ , set of abnormalities  $\Omega$  and set of  $\mathcal{L}$ -formulas  $\mathcal{S}$ .

- 1.  $S \models_{\mathbf{LLL}}^{m,\Omega} \phi$  if and only if  $S \models_{\Omega,\mathsf{prf}}^{\mathbb{n}} \phi$ .
- 2.  $\mathcal{S} \models_{\mathbf{LLL}}^{r,\Omega} \phi$  if and only if  $\mathcal{S} \models_{\Omega,\mathsf{prf}}^{\cap} \phi$ .
- 3.  $\mathcal{S} \models_{\mathbf{LLL}}^{n,\Omega} \phi$  if and only if  $\mathcal{S} \models_{\Omega,\mathsf{prf}}^{\cup} \phi$ .

In order to prove the above theorem, some further notation, facts and lemmas are necessary. Let  $\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}) = \langle \operatorname{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S}), \mathcal{AT} \rangle$  be a sequent-based argumentation framework as defined in Definition 38, with as the core logic the lower limit logic **LLL**, the corresponding sequent calculus C, where  $\Omega$  is a set of abnormalities and  $\mathcal{S}$  is a set of formulas.

**Notation 4.** Let  $\Psi \in \Phi(\mathcal{S})$ , define  $\operatorname{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S}) =_{\mathrm{df}} \{\Pi \ \ \Gamma \Rightarrow \psi \mid \Pi \subseteq \Omega \setminus \Psi \text{ and } \Gamma \subseteq \mathcal{S} \}.$ 

The following result from [62, Lemma 5.5.1] will be useful in the proof of Theorem 5.

**Lemma 25.** Let  $\Xi$  be a set of finite subsets of S, and let  $\mathsf{CS}(\cdot)$  denote the function that returns the set of all the choice sets of a set of sets. Let  $\Psi = \{\phi_i \mid i \in \mathbb{N}^+\} \in \mathsf{CS}(\Xi)$  and define  $\widehat{\Psi} = \bigcap_{i>0} \Psi_i$  where  $\Psi_0 = \Psi$  and (where  $i + 1 \leq n$ ):

$$\Psi_{i+1} = \begin{cases} \Psi_i & \text{if there is a } \Delta \in \Xi \text{ such that } \Psi_i \cap \Delta = \{\phi_{i+1}\} \\ \Psi_i \setminus \{\phi_{i+1}\} & \text{else} \end{cases}$$
we have:  $\widehat{\Psi} \in \min_{\subset} (\mathsf{CS}(\Xi)).$ 

### Corollary 3.

- 1. For each choice set  $\Psi$  there is a minimal choice set  $\Psi'$  such that  $\Psi' \subseteq \Psi$ .
- 2. Let  $\Psi \in \Phi(\mathcal{S})$ , then for each  $\phi \in \Psi$  there is a  $\Pi \in \Sigma(\mathcal{S})$  such that  $\Psi \cap \Pi = \{\phi\}$ .

*Proof.* Consider both items:

- 1. This follows immediately from Lemma 25.
- 2. Let  $\Psi \in \Phi(S)$  and suppose that there is some  $\phi \in \Psi$ , such that there is no  $\Pi \in \Sigma(S)$  for which  $\Psi \cap \Pi = \{\phi\}$ . Since  $\Psi$  is a choice set of  $\Sigma(S)$ , there must be some  $\Pi \in \Sigma(S)$  such that  $\phi \in \Pi$ . Therefore for each  $\Pi \in \Sigma(S)$  such that  $\phi \in \Pi, \Psi \cap \Pi \supseteq \{\phi\}$ . However, then  $\Psi \setminus \{\phi\}$  would also be a choice set of  $\Sigma(S)$ . A contradiction to the minimality of  $\Psi$ .  $\Box$

**Fact 1.** Let  $\Gamma \subseteq S$  and  $\Pi \subseteq \Omega$  be finite. Moreover, let  $\Psi \in \Phi(S)$ . Then:

- 1. For each  $\phi \in \Psi$ ,  $\Pi \setminus \{\phi\} \ \Gamma \Rightarrow \phi \in Arg_{\mathbf{LLL}} \overline{\Psi}(\mathcal{S})$ .
- 2.  $\operatorname{Concs}(\operatorname{Arg}_{\operatorname{\mathbf{LLL}},\overline{\Psi}}(\mathcal{S})) \supseteq \Psi.$
- 3. Let  $\Pi \ \ \Gamma \Rightarrow \phi \in Arg_{\mathbf{LLL},\Omega}(\mathcal{S}) \text{ and } \mathsf{S} \subseteq Arg_{\mathbf{LLL},\Omega}(\mathcal{S}).$  If  $\Pi \cap \mathsf{Concs}(\mathsf{S}) \neq \emptyset$  then  $\mathsf{S}$  attacks  $\Pi \ \ \Gamma \Rightarrow \phi.$
- 4. Let  $\Pi \ \ \Gamma \Rightarrow \phi \in Arg_{\mathbf{LLL},\Omega}(\mathcal{S})$ . If  $\Pi \cap \Psi \neq \emptyset$  then  $Arg_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$  attacks the argument  $\Pi \ \ \Gamma \Rightarrow \phi$ .
- 5. Let  $\phi \in \Omega$ , then for any  $\Pi \ rightarrow \Gamma \Rightarrow \phi \in Arg_{\mathbf{LLL},\Omega}(\mathcal{S})$ , there is some  $\Pi' \subseteq \Pi \cup \{\phi\}$  such that  $\Pi' \in \Sigma(\mathcal{S})$ .

*Proof.* Let  $\Gamma \subseteq S$ ,  $\Pi \subseteq \Omega$  and  $\Psi \in \Phi(S)$ . Consider each of the items in turn.

- 1. Let  $\phi \in \Psi$ , then, by Corollary 3.2 there is some  $\Pi \in \Sigma(S)$ , such that  $\phi \in \Pi$  and  $\Psi \cap \Pi = \{\phi\}$ . Where Dab( $\Pi$ ) is a minimal Dab consequence of S. Thus  $S \vdash_{\mathbf{LLL}}$  Dab( $\Pi$ ). Hence, by the completeness of C for  $\mathbf{LLL}$  for some  $\Gamma \subseteq S$ ,  $\Gamma \Rightarrow \bigvee \Pi$  is derivable. And thus, by applying RC (several times)  $\Pi \setminus \{\phi\} \ \Gamma \Rightarrow \phi$  is derivable in C'. Since  $\Gamma \subseteq S$ ,  $\Pi \setminus \{\phi\} \ \Gamma \Rightarrow \phi \in \operatorname{Arg}_{\mathbf{LLL},\Omega}(S)$ . Moreover, since  $\Psi \cap \Pi = \{\phi\}, (\Pi \setminus \{\phi\}) \subseteq \Omega \setminus \Psi$ . Therefore  $\Pi \setminus \{\phi\} \ \Gamma \Rightarrow \phi \in \operatorname{Arg}_{\mathbf{LLL},\overline{\Psi}}(S)$ .
- 2. Suppose that  $\phi \in \Psi$ . Then, by the previous item  $\phi \in \mathsf{Concs}(\operatorname{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S}))$ .

- 3. Let  $\Pi \ (\Gamma \Rightarrow \phi \in \operatorname{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S}) \text{ and } \mathsf{S} \subseteq \operatorname{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S}) \text{ and suppose that } \Pi \cap \operatorname{\mathsf{Concs}}(\mathsf{S}) \neq \emptyset$ . Then there is some  $a \in \mathsf{S}$ , such that  $\operatorname{\mathsf{Conc}}(a) \in \Pi$ . By definition of the attack rule  $\operatorname{AT}_{\mathsf{AL}}$ , it follows that a attacks  $\Pi \ (\Gamma \Rightarrow \phi)$ .
- 4. Let  $\Pi \ \ \Gamma \Rightarrow \phi \in \operatorname{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S})$  and suppose that  $\Pi \cap \Psi \neq \emptyset$ . Thus there is some  $\psi \in \Pi$ , such that  $\psi \in \Psi$ . By Item 2,  $\psi \in \operatorname{Concs}(\operatorname{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S}))$ . Thus, by the previous item,  $\operatorname{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$  attacks  $\Pi \ \ \Gamma \Rightarrow \phi$ .
- 5. Let  $\phi \in \Omega$  and  $a = \Pi \ rightarrow \Gamma \Rightarrow \phi \in \operatorname{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S})$ , since a is derivable in  $\mathbf{C}'$ ,  $a' = \Gamma \Rightarrow \phi \lor \bigvee \Pi$  is derivable in  $\mathbf{C}$  as well. Thus, by the soundness and monotonicity of  $\mathbf{C}$  for  $\mathbf{LLL} \ \mathcal{S} \vdash \phi \lor \bigvee \Pi$ . Hence, by the definition of minimal Dab consequences (Notation 3), there is some  $\Pi' \subseteq \Pi \cup \{\phi\}$  such that  $\operatorname{Dab}(\Pi')$ is a minimal Dab consequence for  $\Gamma$  and thus  $\Pi' \in \Sigma(\Gamma)$ .  $\Box$

The following facts can be found in [21, 62]:

#### Fact 2.

- 1.  $\mathcal{S} \succ_{\mathbf{LLL}}^{r,\Omega} \phi$  iff there is a (finite) set of abnormalities  $\Pi \subseteq \Omega \setminus U(\mathcal{S})$  such that  $\mathcal{S} \vdash_{\mathbf{LLL}} \phi \lor Dab(\Pi)$  [21, Theorem 7].
- 2.  $\mathcal{S} \vdash_{\mathbf{LLL}}^{m,\Omega} \phi$  iff for all  $\Psi \in \Phi(\mathcal{S})$  there is a  $\Pi \subseteq \Omega \setminus \Psi$  such that  $\mathcal{S} \vdash_{\mathbf{LLL}} \phi \lor Dab(\Pi)$  [21, Theorem 8].
- 3.  $S \vdash_{\mathbf{LLL}}^{n,\Omega} \phi$  iff there is a  $\Pi \subseteq \Omega$  such that  $S \vdash_{\mathbf{LLL}} \phi \lor Dab(\Pi)$  and for some  $\Psi \in \Phi(S), \ \Psi \cap \Pi = \emptyset$  [62, Theorem 2.8.3].

4.  $U(S) = \bigcup \Phi(S)$  [21, Theorem 11.5].

**Fact 3.** If  $\Gamma \vdash_{\mathbf{LLL}} \phi \lor Dab(\Pi)$ , where  $\Gamma \subseteq S$  there is some  $\Gamma' \subseteq \Gamma$  such that  $\Pi \ \Gamma' \Rightarrow \phi \in Arg_{\mathbf{LLL},\Omega}(S)$ .

*Proof.* Suppose that  $\Gamma \vdash_{\mathbf{LLL}} \phi \lor \operatorname{Dab}(\Pi)$  and that  $\Gamma \subseteq S$ . By the completeness of C for  $\mathbf{LLL}$ ,  $\Gamma' \Rightarrow \phi \lor \operatorname{Dab}(\Pi)$  is derivable in C for some  $\Gamma' \subseteq \Gamma$ . Thus, by applying RC (several times),  $\Pi \ \Gamma' \Rightarrow \phi$  is derivable in C. Since  $\Gamma \subseteq S$ ,  $\Pi \ \Gamma' \Rightarrow \phi \in \operatorname{Arg}_{\mathbf{LLL},\Omega}(S)$ .

Before proving the theorem, we first show how preferred extensions relate to minimal Dab consequences. In particular, we show that preferred extensions are closely related to the above defined set of arguments  $\operatorname{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$ .

**Lemma 26.** Let  $\Psi \in \Phi(\mathcal{S})$ , then  $Arg_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S}) \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}))$ .

*Proof.* Let  $\Psi \in \Phi(\mathcal{S})$  and let  $\mathcal{E} = \operatorname{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$ . We show that  $\mathcal{E}$  is admissible and maximal.

 $\mathcal{E} = \operatorname{Arg}_{\operatorname{LLL},\overline{\Psi}}(\mathcal{S})$  is admissible. Let  $a = \Theta \, \, \diamond \, \Delta \Rightarrow \psi \in \mathcal{E}$ , and assume  $b = \Pi \, \, \diamond \, \Lambda \Rightarrow \phi \in \operatorname{Arg}_{\operatorname{LLL},\Omega}(\mathcal{S})$  attacks a. By Definition 37, it follows that  $\phi \in \Theta$ . Note that, since  $\phi \in \Theta, \phi \in \Omega$ . By Fact 1.5, there is some  $\Pi' \subseteq \Pi \cup \{\phi\}$  such that  $\Pi' \in \Sigma(\mathcal{S})$ . Since  $\phi \notin \Psi$  (by assumption  $a \in \mathcal{E}$ ) and  $\Psi \cap \Pi' \neq \emptyset$  (by Corollary 3.2, recall that  $\Pi' \in \Sigma(\mathcal{S})$ ), also  $\Psi \cap \Pi \neq \emptyset$ . Therefore,  $b \notin \mathcal{E}$ . From which it follows that  $\mathcal{E}$  is conflict-free and since  $\Psi \cap \Pi \neq \emptyset$ , it follows by Fact 1.4 that  $\mathcal{E}$  is admissible.

 $\mathcal{E}$  is maximally admissible. Assume that there is an argument  $\Pi \land \Rightarrow \gamma \in \operatorname{Arg}_{\operatorname{LLL},\Omega}(\mathcal{S}) \setminus \mathcal{E}$  such that  $\mathcal{E} \cup \{\Pi \land \Lambda \Rightarrow \gamma\}$  is admissible. By Fact 1.4 it follows that  $\Pi \cap \Psi = \emptyset$ . Hence  $\Pi \land \Lambda \Rightarrow \gamma \in \mathcal{E}$ :  $\mathcal{E}$  is maximally admissible.  $\Box$ 

**Lemma 27.** Let  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}))$  and  $\Pi \in \Sigma(\mathcal{S})$ , then  $\mathsf{Concs}(\mathcal{E}) \cap \Pi \neq \emptyset$ .

Proof. Let  $\mathcal{E} \in \operatorname{Ext}_{\operatorname{prf}}(\mathcal{AF}_{\operatorname{\mathbf{LLL}},\Omega}(\mathcal{S}))$ . Let  $\Sigma_{\mathcal{E}}$  denote all sets  $\Pi'$  in  $\Sigma(\mathcal{S})$  for which  $\operatorname{Concs}(\mathcal{E}) \cap \Pi' = \emptyset$ . Assume towards a contradiction that  $\Sigma_{\mathcal{E}} \neq \emptyset$ . Let  $\Psi$  be a minimal choice set over  $\Sigma_{\mathcal{E}}$ . That  $\Psi$  exists follows from Corollary 3.1. From Corollary 3.2 it is known that for each  $\phi \in \Psi$  there is a  $\Pi_{\phi} \in \Sigma_{\mathcal{E}}$  such that  $\Psi \cap \Pi_{\phi} = \{\phi\}$ . Since  $\Pi_{\phi} \in \Sigma_{\mathcal{E}} \subseteq \Sigma(\mathcal{S})$ , there is some  $\Lambda \subseteq \mathcal{S}$  such that  $\Lambda \vdash_{\operatorname{\mathbf{LLL}}} \operatorname{Dab}(\Pi_{\phi})$ . By the completeness of  $\mathsf{C}$  for  $\mathsf{L}, \Lambda \Rightarrow \bigvee \Pi_{\phi}$  is derivable and thus, by (several) application(s) of RC, so is  $\Pi_{\phi} \setminus \{\phi\} \$   $\Lambda \Rightarrow \phi$ . Let  $\mathcal{E}' = \mathcal{E} \cup \{\Pi_{\phi} \setminus \{\phi\} \$   $\Lambda \Rightarrow \phi \in \operatorname{Arg}_{\operatorname{\mathbf{LLL}},\Omega}(\mathcal{S}) \mid \phi \in \Sigma_{\mathcal{E}}\}$ . It can be shown that  $\mathcal{E}'$  is admissible:

 $\mathcal{E}'$  is conflict-free. Suppose  $a = \Pi_{\phi} \setminus \{\phi\} \$   $\Lambda' \Rightarrow \phi$  attacks  $\mathcal{E}$ . By assumption  $\mathcal{E}$  is admissible, hence there is an argument  $a' \in \mathcal{E}$  such that a' attacks a. From this it follows that  $\mathsf{Concs}(\mathcal{E}) \cap (\Pi_{\phi} \setminus \{\phi\}) \neq \emptyset$ , which is a contradiction with the assumptions that  $\Pi_{\phi} \in \Sigma_{\mathcal{E}}$  and  $\mathsf{Concs}(\mathcal{E}) \cap \Pi = \emptyset$  for each  $\Pi \in \Sigma_{\mathcal{E}}$ . For the same reason, no argument  $b \in \mathcal{E}$  attacks  $\Pi_{\phi} \setminus \{\phi\} \$   $\Lambda^* \Rightarrow \phi$ , for any  $\Lambda^* \subseteq \mathcal{S}$ . Now suppose that  $\Pi_{\phi} \setminus \{\phi\} \$   $\Lambda \Rightarrow \phi$  attacks  $\Pi_{\psi} \setminus \{\psi\} \$   $\Lambda' \Rightarrow \psi$ . By definition  $\phi \in \Pi_{\psi}$ , which is a contradiction with the assumption that  $\phi \in \Psi$  and  $\Psi \cap \Pi_{\psi} = \{\psi\}$ . Hence  $\mathcal{E}'$  is conflict-free.

 $\mathcal{E}'$  defends its arguments. Suppose, for some argument  $b = \Theta \land \Delta \Rightarrow \psi \in$ Arg<sub>LLL,\Omega</sub>( $\mathcal{S}$ ) \ $\mathcal{E}'$ , that b attacks  $\Pi_{\phi} \setminus \{\phi\} \land \Lambda \Rightarrow \phi$  and  $\mathcal{E}$  does not attack b. Since  $\mathcal{E}'$  is conflict-free it follows that  $\mathsf{Concs}(\mathcal{E}) \cap (\{\psi\} \cup \Theta) = \emptyset$ . Note that, by Definition 37  $\psi \in \Pi_{\phi} \setminus \{\phi\} \subseteq \Omega$ , thus by Fact 1.5, there is a  $\Pi \in \Sigma(\mathcal{S})$  such that  $\Pi \subseteq \{\psi\} \cup \Theta$ . Hence  $\Pi \in \Sigma_{\mathcal{E}}$ . By the construction of  $\mathcal{E}'$ , for each  $\gamma \in \Pi$  and any  $\Delta' \subseteq \mathcal{S}$  such that  $c = \Pi_{\gamma} \setminus \{\gamma\} \land \Delta' \Rightarrow \gamma$  is derivable in  $\mathsf{C}', c \in \mathcal{E}'$ . Note that  $\gamma \neq \psi$ , since it was shown above that  $\mathcal{E}'$  is conflict-free and otherwise  $\mathcal{E}'$  would attack  $\Pi_{\phi} \setminus \{\phi\} \land \Lambda \Rightarrow \phi$ . Thus  $\gamma \in \Theta$ . Therefore  $\Pi_{\gamma} \setminus \{\gamma\} \land \Delta' \Rightarrow \gamma$  attacks b, and thus  $\mathcal{E}'$  is admissible.

However, since  $\mathcal{E}'$  attacks b and, by assumption,  $\mathcal{E}$  does not,  $\mathcal{E} \subsetneq \mathcal{E}'$ . This is a contradiction with  $\mathcal{E}$  being a preferred extension.

**Lemma 28.** If  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}))$ , then there is a  $\Psi \in \Phi(\mathcal{S})$  such that  $\mathcal{E} = Arg_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$ .

*Proof.* Suppose  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}))$ . By Lemma 27 and Corollary 3.1 it follows that  $\mathsf{Concs}(\mathcal{E}) \supseteq \Psi$  for some  $\Psi \in \Phi(\mathcal{S})$ . By Fact 1.3, for all arguments  $\Pi \ \Lambda \Rightarrow \phi \in \mathrm{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S})$ , with  $\Pi \cap \Psi \neq \emptyset$ ,  $\Pi \ \Lambda \Rightarrow \phi \notin \mathcal{E}$ . Hence  $\mathcal{E} \subseteq \mathrm{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$ , with Lemma 26 it thus follows that  $\mathcal{E} = \mathrm{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$ .  $\Box$ 

From Lemmas 26 and 28 it follows that:

**Corollary 4.**  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}))$  iff  $\mathcal{E} = Arg_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$  for some  $\Psi \in \Phi(\mathcal{S})$ .

With this Theorem 5 can be proven:

*Proof.* Let  $\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}) = \langle \operatorname{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S}), \mathcal{AT} \rangle$  be a sequent-based AL-framework for the lower limit logic  $\mathbf{LLL} = \langle \mathcal{L}, \vdash \rangle$ , with corresponding set of abnormalities  $\Omega$ and  $\mathcal{S}$  a set of  $\mathcal{L}$ -formulas. Consider each strategy, in both directions.

1. Start with minimal abnormality.

 $(\Rightarrow) \text{ Suppose that } \mathcal{S} \mid_{\mathbf{LLL}}^{m,\Omega} \phi. \text{ By Fact 2.2 and Fact 3, for all } \Psi \in \Phi(\mathcal{S}) \text{ there} \\ \text{ is a } \Pi \subseteq \Omega \setminus \Psi \text{ such that } \Pi \ \textbf{S} \ \Gamma \Rightarrow \phi \in \operatorname{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S}), \text{ for some } \Gamma \subseteq \mathcal{S}. \text{ By} \\ \text{ Corollary 4, for each preferred extension } \mathcal{E} \text{ there is a } \Psi \in \Phi(\mathcal{S}) \text{ such that} \\ \mathcal{E} = \operatorname{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S}). \text{ From this it follows that for each preferred extension } \mathcal{E} \\ \text{ there is an argument } \Pi' \ \textbf{S} \ \Gamma' \Rightarrow \phi \in \mathcal{E} \text{ for some } \Gamma' \subseteq \mathcal{S} \text{ and } \Pi' \subseteq \Omega \setminus \Psi. \\ \text{ Therefore } \phi \in \operatorname{Concs}(\mathcal{E}) \text{ for each } \mathcal{E} \in \operatorname{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S})). \text{ Hence } \mathcal{S} \mid_{\Omega,\mathsf{prf}}^{\infty} \phi. \\ \end{array}$ 

( $\Leftarrow$ ) Now suppose that  $S \hspace{0.2em} \sim \hspace{-0.2em} \mid \hspace{0.2em} \hspace{0.2em} \hspace{0.2em} \phi$ . Let  $\Psi \in \Phi(S)$  be arbitrary. Then, by Corollary 4, there is an  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathsf{LLL},\Omega}(S))$ , such that  $\mathcal{E} = \operatorname{Arg}_{\mathsf{LLL},\overline{\Psi}}(S)$ . Hence, there is an argument  $\Pi \ \ \Gamma \Rightarrow \phi \in \mathcal{E}$ , for some  $\Gamma \subseteq S$ , from which it follows that  $\Pi \subseteq \Omega \setminus \Psi$ . Thus, by Definition 36 and the definition of the sequent RC-rule,  $\Gamma \Rightarrow \phi \lor \operatorname{Dab}(\Pi)$  is derivable in C'. Hence, by soundness of C and monotonicity of  $\mathsf{LLL}$ ,  $S \vdash_{\mathsf{LLL}} \phi \lor \operatorname{Dab}(\Pi)$ . Since  $\Psi \in \Phi(S)$  is arbitrary, for each such  $\Psi$ , such a  $\Pi$  exists. Therefore, by Fact 2.2 it follows that  $S \mid \sim_{\mathsf{LLL}}^{m,\Omega} \phi$ .

2. The reliability strategy.

(⇒) Suppose that  $\mathcal{S} \models_{\mathbf{LLL}}^{r,\Omega} \phi$ . By Fact 2.1 and Fact 2.4, there is a set  $\Pi \subseteq \Omega \setminus \bigcup \Phi(\mathcal{S})$  of abnormalities, such that  $\mathcal{S} \models_{\mathbf{LLL}} \phi \lor \operatorname{Dab}(\Pi)$ . By Fact 3 for some  $\Gamma \subseteq \mathcal{S}$  it follows that  $\Pi \ \Gamma \Rightarrow \phi \in \operatorname{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S})$ . Furthermore, by the construction of  $\operatorname{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$  and Corollary 4,  $\Pi \ \Gamma \Rightarrow \phi \in \mathcal{E}$ , for every  $\mathcal{E} \in \operatorname{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}))$ . Hence  $\mathcal{S} \models_{\Omega,\mathsf{prf}}^{\cap} \phi$ .

( $\Leftarrow$ ) Now suppose that  $\mathcal{S} \mid_{\Omega, \mathsf{prf}}^{\cap} \phi$ . By assumption there is an argument  $a = \Pi$  $\Gamma \Rightarrow \phi$  for some  $\Gamma \subseteq \mathcal{S}$  such that for all  $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathsf{LLL},\Omega}(\mathcal{S})), a \in \mathcal{E}$ . By Corollary 4 and the construction of  $\operatorname{Arg}_{\mathsf{LLL},\overline{\Psi}}(\mathcal{S})$ , it follows that  $\Pi \cap \Psi = \emptyset$ , for every  $\Psi \in \Phi(\mathcal{S})$ . Hence,  $\Pi \subseteq \Omega \setminus \bigcup \Phi(\mathcal{S})$ . By the soundness of  $\mathsf{C}$  and the RCrule that is available in  $\mathsf{C}'$ , for some  $\Gamma \subseteq \mathcal{S}$ , we have that  $\Gamma \vdash_{\mathsf{LLL}} \phi \lor \operatorname{Dab}(\Pi)$ . Hence, by Fact 2.1 and the monotonicity and soundness of  $\mathsf{LLL} \mathcal{S} \mid_{\mathsf{LLL}}^{r,\Omega} \phi$ .

3. The normal selections strategy.

(⇒) Suppose that  $\mathcal{S} \models_{\mathbf{LLL}}^{n,\Omega} \phi$ . By Fact 2.3, there is a  $\Pi \subseteq \Omega$  such that (a)  $\mathcal{S} \models_{\mathbf{LLL}} \phi \lor \operatorname{Dab}(\Pi)$  and (b) for some  $\Psi \in \Phi(\mathcal{S}), \Psi \cap \Pi = \emptyset$ . From (a) and Fact 3,  $a = \Pi \clubsuit \Gamma \Rightarrow \phi \in \operatorname{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S})$  for some  $\Gamma \subseteq \mathcal{S}$ . By construction of  $\operatorname{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$ , since by (b)  $\Psi \cap \Pi = \emptyset$ ,  $a \in \operatorname{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$ . Thus, by Corollary 4,  $a \in \mathcal{E}$ , for some  $\mathcal{E} \in \operatorname{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}))$ . Therefore  $\mathcal{S} \models_{\Omega,\mathsf{prf}}^{\cup} \phi$ .

( $\Leftarrow$ ) Now assume that  $S \models_{\Omega, \mathsf{prf}}^{\cup} \phi$ . Then there is an  $a = \Pi \ \square \ \square \Rightarrow \phi \in \operatorname{Arg}_{\mathbf{LLL},\Omega}(S)$ , with  $\Gamma \subseteq S$ , such that  $a \in \mathcal{E} \in \operatorname{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathbf{LLL},\Omega}(S))$ . By Corollary 4, there is a  $\Psi \in \Phi(S)$ , such that  $\mathcal{E} = \operatorname{Arg}_{\mathbf{LLL},\overline{\Psi}}(S)$ . Hence, by construction of  $\operatorname{Arg}_{\mathbf{LLL},\overline{\Psi}}(S)$ ,  $\Psi \cap \Pi = \emptyset$ . Moreover, by adjusting the derivation of a, such that RC is never applied, the sequent  $a' = \Gamma \Rightarrow \phi \lor \bigvee \Pi$  is derived. By soundness and monotonicity of C it follows that  $S \vdash_{\mathbf{LLL}} \phi \lor \operatorname{Dab}(\Pi)$ . Thus, by Fact 2.3  $S \models_{\mathbf{LLL}}^{n,\Omega} \phi$ .

# INTRODUCING ABSTRACT ARGUMENTATION WITH MANY LIVES

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#### Abstract

Our starting point is to view argumentation networks (of the form (S, R)) as representing a survival game. The players are the elements of S and the relation R is the attack relation. The various traditional Dung semantics for subset of S can be viewed as defining extensions in the form of possible survival groups  $E \subset S$ . The survival sets E (which are the traditional extensions) are groups of players which are conflict free and able to protect themselves. So far we have a different point of view on extensions which is compatible with the traditional Dung formal mathematical machinery. However, given the survival point of view we can generalise and add additional features to the traditional argumentation networks:

- 1. The new features are:
  - (a) We can add to each x in S a many lives value M(x), meaning how many live attackers are needed to force x to be out (i.e. x to become dead).
  - (b) We associate with each attack pair (y, x) in R a value K(y, x), meaning how many lives are taken out of M(x) should the attack of y on x be successful (i.e. y is alive). The value K(y, x) may be, or may not be, correlated or even related to the number of lives M(y) which y has.
  - (c) The traditional concept of conflict free set is that of a set whose members do not attack one another. With many lives available we look at "living together" sets, using a concept of being able to stay alive together. Members can attack but not able to kill one another. In fact we could introduce different strengths of attack, one when attacking inside a "living together" set and possibly another when a "living together" set protects itself.

- (d) We can now investigate semantics for such systems (S, R, M, K).
- 2. The ideas of adding M and K arise from our research into the argumentation/logic behaviour of multiple complaints. Thus the semantics and additional features of argumentation that we study are inspired by real life applications.

In fact, to protect an alleged offender x against attacks from a group of complainers/victims E, x needs to present much stronger counter attacks, and furthermore the public will tolerate a little bit of inconsistencies among E (i.e. E need not be completely conflict free). This observation led us to the idea that to present a formal argumentation system we need to define three types of attacks,  $\alpha_{\mathbf{a}}, \alpha_{\mathbf{d}}$ , and  $\alpha_{\mathbf{p}}$ , in increasing strength. For E to attack x we use the  $\alpha_{\mathbf{a}}$  attack. For Z to protect x, Z must use the  $\alpha_{\mathbf{p}}$  attack and for E (resp. for Z) to be considered conflict free its members must not  $\alpha_{\mathbf{d}}$  attack one another (though we may tolerate them  $\alpha_{\mathbf{a}}$  attacking one another). Furthermore, the attacks can be defined using the basic attack relation R in a more complex manner. For example  $z \alpha_{\mathbf{a}}$ attacking x can be defined as  $(zRx \land ((\forall u)(uRz \rightarrow zRu)))$ .

3. We discuss our results and compare with other papers on the numerical and ranking aspects of argumentation.

### 1 Orientation: The many lives idea

Our starting point is an argumentation network (S, R), where S is a non empty set (of arguments) and R is a binary relation on S. When  $(x, y) \in R$  holds we say that x (geometrically) attacks y. Dung [6] (see Section 3) introduced several concepts related to (S, R), among them the concept of:

- D1. A subset E of S attacks a node  $y \in S$  iff (for some  $e \in E$  we have eRy).<sup>1</sup>
- D2. A subset E of S is conflict free iff (for no  $e_1, e_2$  in E do we have  $e_1Re_2$ ).
- D3. A subset E of S protects a node  $x \in S$  iff (for all y, if yRx then E attacks y).
- D4. A subset E of S is admissible iff E is conflict free and it protects all its members.
- D5. A subset E is a complete extension iff E is admissible and contains all nodes it protects.

<sup>&</sup>lt;sup>1</sup>The perceptive mathematical reader will see that D1 is not used in the following D2-D5. It is included here for reasons of Socratic exposition. See for example the next item DM1 and Remark 2.4.

The above concepts were defined by Dung using the geometrical single attack, between x (the attacker) and y (the target), namely  $(x, y) \in R$ .

Our generalisation to to the above is to change D1. We introduce a function M(x), for  $x \in S$ , giving a natural number value  $\geq 0$ , for each x, and using M to introduce the new notion of many lives argumentation network, as the system (S, R, M) and modifying the definition D1 into the new DM1 below:

DM1. A subset E of S attacks a node  $y \in S$  iff (for some  $e_i \in E$  we have  $e_i Ry$ , where i = 1, ..., M(y) and where  $i \neq j$  implies  $e_i \neq e_j$ , for all  $0 \leq i, j \leq M(y)$ ).

The function M(x) gives the many lives of x, meaning how many live attackers of x we need in order to kill x.

The change of DM to DM1 necessitates changes in the other DM clauses. In other words, we need to define new corresponding clauses DM2–DM5.

To give our readers an idea of the nature of this paper and the relation of its contribution to formal argumentation, we answer some questions:

**Question 1:** Is traditional Dung network a special case of our new networks?

Answer to question 1: Yes, because we can let M(x) = 1, for all x and define the semantics options for M in such a way that they agrees with the Dung semantics options. However, we must be careful how to define DM2–DM5, so that they also conform to the special case. It may be, however, that we will judge that it is more natural not to force restrictions on M and try to get the Dung semantics as a special case but rather to allow us to depart from the Dung semantics options even in the case that M(x) is always 1. This decision may depend on the needs of the multiple complaints offender application area and on general mathematical smoothness properties which it can offer.

**Questions 2:** What happens with the concept conflict freeness? When arguments have many lives they may be attacked but still be alive, so in what sense can a set of arguments be conflict free? Consider a single point e which attacks itself and has 2 lives, i.e. we have  $S = \{e\}$ .  $R = \{(e, e)\}$  and M(e) = 2. e is not dead because it suffers only one attack and it takes 2 attacks to kill it.

- is  $\{e\}$  conflict free?
- How many lives does *e* have (after the attack)?

Answer to Question 2: Let us move carefully here. We have that e geometrically attacks itself (that is  $(e, e) \in R$ ) but cannot kill itself. Of course we can set up our system to allow e to repeatedly attack itself again and again, in which case e will kill itself after two rounds, but we may choose to allow attackers only one only one attempt at attacking. In this case no matter how we look at it, e cannot be dead or undecided. It is alive with one life left. To overcome this lack of clarity, let us talk about "geometrical attack" and "successful attack" of a set E on a node x. The set E geometrically attacks x if for some y in E we have  $(y, x) \in R$ , (R being the graph "geometry" on S). The set E successfully attacks x if it manages (according to our agreed definition of this notion) to reduce the many lives of x to 0. Given a subset E, we can also talk about the old Dung concept E as being "geometrical conflict free" and introduce a new concept of E as being "at peace" or as "able to survive together".

So according to these new concepts  $E = \{e\}$  with M(e) = 2, does geometrically attack e, but e is able to survive together with itself because it cannot kill itself. We can also reasonably say that e has one life left now after having attacked itself.

**Question 3:** What is a complete extension? The example in Question 2 creates a problem because we get a new network with e geometrically attacking itself, where e has one life (one life left). Why don't we allow e to carry on attacking?

Answer to question 3: We are therefore forced to say that the new concept of a complete extension of any one network is another network. Section 4 discusses how it is identified. So to be clear, given a network (S, R, M, K) and a notion of "semantics" for such networks, the output of this notion is a family of networks of the same type. In comparison for the case of Dung networks of the form (S, R) the output of a semantics is a family of subsets E of S. Note that since any such E is conflict free we can regard it as a network with the empty attack relation,  $(E, \emptyset)$ . So the network  $N = (\{e\}, \{(e, e)\}, M(e) = 2)$ , has the single complete extension which is the network  $N' = (\{e\}, \{(e, e)\}, M(e) = 1)$  which has the traditional complete extension  $N'' = \emptyset = \{e = \text{ undecided}\}$ . We immediately ask: Is this concept compatible with the old Dung concept of extension?. The answer is yes, it is.

**Question 4:** How do we view our paper?

- i Is it a contribution to the area of Numerical Argumentation (by introducing the functions M and K)?
- ii Is it a contribution to Ranking of arguments? something comparable to Grossi and Modgil [8, 24]?

iii Is it part of the Equational Approach [9]?

iv Is it arising from some application area? If indeed it is connected with an application area and is not a purely technical paper, then we further ask: Does it model some part of the application area or does it just draw ideas from the application area and offers another formal argumentation system to (i) or to (ii) which can approximate some features of the application area ?

Answer to question 4: The paper draws ideas from several application areas, as described below, which have the many lives feature in common, and is inspired to formulate a sample formal argumentation theory which connects with the formal areas of numerical argumentation and of ranking. The formal systems suggested are good and flexible enough to be adapted to modelling more accurately any of the application areas which inspired them.

The old Dung concept of a complete extension E is a set but E can be can be viewed as another network because it is conflict free so it is a network with the empty attack relation. So the new concept contains the old concept. This is OK.

In familiar everyday life we have many examples of the many lives/tolerance/ resilience function M(x) of x. These include:<sup>2</sup>

- 1. How may complaints against x can be tolerated/covered-up/ignored before action needs to be taken
- 2. How many applications/demonstrations/hints/pressure/repeated nuisance, can be tolerated before compliance/giving-in.
- 3. How many witnesses are needed legally to establish a fact in law
- 4. How many violations are sufficient to cross a legal threshold to the next legal level.

There have been many cases in the UK where public figures and celebrities were accused by several complainers of alleged misconduct. All these cases and accusations had a similar pattern.

Let x be the accused. First a  $y_1$  would come forward with allegations against x. Naturally x would deny any wrong doing and dismiss  $y_1$ 's accusations. Then more and more accusers come forward, say  $y_2, y_3, \ldots, y_n$ . At some point, say at accuser n, the public perception will change and action/response is taken. It usually starts

<sup>&</sup>lt;sup>2</sup>The idea of many lives actually arose from our argumentation modelling of sex offender's Therapy [1, 2].

with increased activity in social networks and may end up in social pressure on the accused to resign or pressure on the police to investigate the complaints and press charges. The next scenarios can vary from case to case; they include:

- 1. The public figure x resigns and disappears from the news and that is the end of the story.
- 2. The police investigates and the accused might end up with a prison sentence
- 3. Any outcome between the outcomes (1) and (2) above.

We now address items (i)–(iv).

It is true that the function M(x) associates a numerical value with each node xand it is also true that this value is seen in relation to the number of geometrical attackers of x. So on the face of it, there seems to be a connection with Numerical argumentation and the Ranking of arguments. However, the way we use this number M(x) in producing extensions is different. It is metal-level. We want at least M(x) live ("in") attackers of x in order for x to be "out".

If we look at Figure 2, the node y has two lives but its attack on x is counted as one attack. Furthermore geometrically x has two attackers (so its ranking is 2) but in order to be considered "out" in an extension these two attackers must be live ("in").

So the use of these values is different.

As for item (iii), the equational approach is itself meta-level. It derives from the annotated graph a system of equations, solves the equations and derives the extensions from the solutions. This procedures can be done for our many lives graphs as well. We need to find and motivate the right equations.

As for item (iv), we confirm that we looked at various ways of dealing with multiple complaints and devised the system of this paper as generic, showing what kind of features and technical moves to expect, and allow the system to be adapted/refined/expanded for modelling the more specific complaints application areas.

We conclude our answer to question 4 by directing the reader to Remark 2.3 in Section 2.



Figure 1: General attack formation, where "----" denotes attack.

## 2 Semi-formal discussion of the many lives idea in the complaints context

This section presents the many lives idea in a slightly more precise (semi-formal) way, in order to prepare the readers from the informal argumentation community for the later formal sections. We assume such readers have some minimal background in Mathematics.

Our readers from formal argumentation theory can skip this section, after reading the next formal Definition 2.1, of what is a many lives network.

**Definition 2.1** (Many lives network). A general many lives network has the form (S, R, M, K), where S is a non-empty set of arguments, R is the binary attack relation on S, M is a function giving each x in S a natural number of how many lives it has, (including possibly 0) and K is a strength function defined on ER, giving a positive natural number for each (y, x) in R, such that it is at most M(y) + 1.

We now discuss and motivate Definition 628-DJ1.

**Remark 2.2** (Motivating M). Let us first focus on the number M(x) = n of the many number of lives of the node x and consider it as the resilience of x to attacks.

M(x) = how many live attackers does it take to kill  $x^{3}$ 

Figure 1 indicates this basic situation. Figure 1 is a general schematic description and Figure 2 is a particular case of it. Note that is this paper double headed arrows " $\rightarrow$ " denote attacks.

In this figure we assume that each node z has a value M(z) of number of lives and that  $y_1, \ldots, y_k$  attack x. If we have that all the  $y_1, \ldots, y_k$  are alive then x would

<sup>&</sup>lt;sup>3</sup>We use here the informal words "live", "dead" and "kill". We ask the reader to understand them intuitively in this motivating section. Formal definitions will be given later in the formal sections.



Figure 2

be dead if  $M(x) \leq k$ . In fact we can write a formula for the new value  $M^*(x)$ , which is obtained after the attack of  $y_1, \dots, y_k$  is carried out. The value is

 $M^*(x) = M(x) - k$ , if k < M(x) and 0 otherwise.

In particular for mathematical reasons we are going to allow the M function to give values 0. This would force us to say that M(z) = 0 means that z is "dead" for any z.

So in Figure 2 the node x has 2 lives. If for example the node x had 4 lives then it could survive the attack of the nodes  $y_1$  and  $y_2$ , but its number of lives would have been reduced from 4 to 2, because it withstood the attacks of 2 live attackers. Note that although the attacker  $y_1$  has two lives in Figure 2, its attack on x reduces x's number of lives by 1 life only. The number of lives of  $y_1$  indicates how many attacks can kill it, not how strongly  $y_1$  can attack others. Note also that we allow  $y_1$ and  $y_2$  to attack only once and not to attack again and again. This is reasonable if you think of the attack as a complaint on an alleged offender. Repeating the same complaint again and again is still the same attack.

We note the first two principles we are adopting here:

- **PP1**: Every element x has a number M(x) of lives (including possibly the value 0). To really kill x you need to kill it M(x) times.<sup>4</sup> In particular non-attacked elements retain all their many lives intact and have the capability of attacking other elements (reducing the target's number of lives ) if their value is not 0.
- **PP2**: Although an element y may have M(y) lives, when attacking any x it can kill only one of x's lives.

**Remark 2.3** (Motivating K). The reader may wonder at the strong over-simplification of principle **PP2**. Surely even if an alleged offender like say a minister or a

<sup>&</sup>lt;sup>4</sup>For example the case of Israeli minister Sylvan Shalom 2015 (see [26]). Apparently he had 6 lives. After 6 complaints of alleged offences he resigned. We have never heard his name in public since.

president would normally require maybe 6 or seven complaints to be "killed" (i.e. to create enough of a public pressure to force resignation or prosecution), a particularly nasty complaint may reduce the number from 6 to much less! Our answer is that we are simplifying for the sake of simpler mathematics. We are not completely modelling reality in this introductory paper but we are just approximating it. We admit that in real examples of complaints y against alleged offender x (namely  $y \rightarrow x$ ), the strength of attack is not necessarily only 1 (i.e. killing only one of the lives of x).

The perceptive reader might feel that we are simplifying too much. Two strong complaints can kill maybe 3 lives. We can perhaps agree to a more realistic model and allow the annotation for y in the model to be of two numbers,

- 1. M(y) the number of lives which y has.
- 2. K(y), the strength of attack of y or in other words, how many lives does y take when attacking. K(y) can be related to M(y). The rationale being that if M(y) is higher then y is stronger, because y is harder to kill, therefore its attack is stronger. The notation K(y) assumes that the strength of attack of y is the same, no matter whom y attacks. This is still a simplification. We realise that K should also be dependent on the x attacked. If the attack on x, for example is a complaint of y against x, then y might feel more strongly about x than about another x', therefore its attack on x will be stronger than its attack on x'. If we want to make the strength of attack also depend on the target of y, we need to make K a function of the pairs (y, x) where y attacks x. We can write it as  $K(y \rightarrow x)$  or K(y, x), for  $(y, x) \in R$ .

So according to this model, Figure 1 will become Figure 3 and the new value  $M^*(x)$  of x after the attack from all  $y_i$  would be

$$M^*(x) = M(x) \div \sum_{i=1}^k K(y_i, x).$$

Where the symbol "-" is truncated substraction, namely.

$$\alpha \dot{-} \beta = \begin{cases} \alpha - \beta, & \text{if } \alpha \ge \beta \\ 0, & \text{if } \alpha \le \beta \end{cases}$$

So for example in Figure 4, we have that y has attack strength 1, z has 2 and u has 3.

The number of lives of x is 7. So after the attack the new number of lives of x in Figure 4 is  $M^*(x)$ .

$$M^*(x) = 7 - (1 + 2 + 3) = 7 - 6 = 1.$$



Note that we get a new network which is with the same geometrical graph as that of Figure 4, and the same strengths and lives for the top nodes y, z and u but for node x the number of lives is 1.

We hope the reader with experience in dealing with multiple complaints (e.g. student's complaints about a lecturer), can see that there is no end to our improving the model, getting the mathematics more and more complicated. Surely, we can further refine our model, saying, for example, that having one attack of strength 3 should be weaker than 3 attacks of strength 1. We must give extra bonus in recognition that there are more complaints (attacks) on x. We thus can continue and further agree to deduct one more life if the number of attacks is more than 2. Let us show the reader what it would look like to write this formula.

Let k be a natural number. Define  $\beta(k)$  ( $\beta$  for bonus) to be

$$\beta(k) = \begin{cases} 1, & \text{if } k > 2\\ 0, & \text{otherwise} \end{cases}$$

The calculation for  $M^*_{\beta}(x)$  of Figure 4 with bonus  $\beta$  is

$$M^*_{\beta}(x) = 7 - (1 + 2 + 3 + 1) = 0.$$

Thus with the bonus we get that x is dead. Let us go on and further improve the model and get the mathematics even more complicated.

We further remark that we have not addressed in detail the question of a node x attacking more than one other node. For example x may attack node y and also node z. We associated the strength of attack to node x, so the attack of x on y will have the same strength as the attack on the node z. This is not true for all possible applications. In many other complaints contexts, the strength of the complaints of x against y may not be as solid and strong as the complaints on z. This means that the strength of attack needs to be associated not with the node x itself but with the attack arrows emanating from x, giving possibly different strengths to different arrows.

There are examples where the strength of attack is done by associating a number with x itself. In a survival game where the attack is done by shooting a gun, then x has a gun and x shoots always the same strength.

We can now, for the time being, formulate our new principle for the case of strength attached to nodes:

**PP2 new:** Given a network of the form (S, R, M, K) and a node x in S with k live attackers  $y_1, \ldots, y_k$  of x with attack strength  $K(y_1), \ldots, K(y_k)^5$  respectively, i.e. we have  $M(y_i) > 0$  for  $i = 1, \ldots, k$  and m dead attackers  $z_1, \ldots, z_m$ , with  $M(z_j) = 0$ , for  $j = 1, \ldots, m$ , and given M(x) as the number of lives of x, then the new number of lives  $M^*(x)$  after the attack is given by the formula<sup>6</sup>

$$M_{\beta}^{*}(x) = M(x) \div [\beta(k) + \sum_{i=1}^{k} K(y_{i})]$$
(\*)

**Remark 2.4.** 1. We now discuss the idea of a different strength of attack required for protection. Assume for example that student y attacks professor xby accusing x of being verbally abusive to y. Assume z comes to the protection of x by accusing y of being a liar and a cheat and that y is attacking x for leftist political reasons. In this context we might expect the attack of z on yto be extra strong. Had y attacked x for something else, say for lack of clear course notes, then, without the verbal abuse context, the attack of z on y might not have been expected to be as strong.

Let us sum up and say that it is possible to take the view that to protect a node x against the attack from node y you need much stronger killing attacks then just an ordinary attack to kill.

<sup>&</sup>lt;sup>5</sup>If the strength of attack is associated with arrows we replace "K(y)" by " $K(y \rightarrow x)$ ".

<sup>&</sup>lt;sup>6</sup>The formal definition we give in Section 4 is slightly more general. See Definition 4.1.

- 2. Similarly suppose we have a set E of elements trying to live together. The concept of conflict free is that the different elements of E do not attack to kill one another. To be on the safe side we might take the view that to be sure that the elements of E can indeed live together then even if the attack available is stronger than ordinary attack, (but still weaker or equal the protective attack), then still, the elements of E cannot kill one another.
- 3. We thus have 3 types of attacks, which we call  $\alpha_{\mathbf{a}}, \alpha_{\mathbf{p}}$  and  $\alpha_{\mathbf{d}}$ , meaning respectively,

 $\alpha$ (ordinary attack),  $\alpha$ (attack to protect),  $\alpha$ (attacks used in the context of living together), with the restriction that  $\alpha_{\mathbf{p}}$  is stronger or equal to  $\alpha_{\mathbf{d}}$  which is stronger or equal to  $\alpha_{\mathbf{a}}$ . So for example of three such attacks we can have:

- (a) A set of nodes  $Y\alpha_{\mathbf{a}}$  attacks a node z if for at least one element  $y \in Y$  attacks z
- (b) A set of nodes  $Y\alpha_{\mathbf{d}}$  attacks a node z if for at least two different elements  $y \in Y$  attack z
- (c) A set of nodes  $Y\alpha_{\mathbf{p}}$  attack a node z if for at least three pairwise different element  $y \in Y$  attack z.

## 3 Background and concepts from abstract argumentation with additional methodological remarks

This section presents, for the convenience of the reader, some basic concepts of what we called traditional argumentation theory. Such systems contain attacks only. We refer to such system as Dung Argumentation with Attack only (see [6]). We shall then add methodological remarks and explain in what way the systems developed for this paper depart from the traditional ones.

There are two traditional ways to present the semantics for the traditional Dung argumentation with attack, the traditional set theoretical approach and the Caminada labelling approach.<sup>7</sup> For the mapping connections between the two approaches, see [7]. Let us briefly quote the traditional set theoretic approach:

### Definition 3.1.

 $<sup>^{7}</sup>$  Actually there are more ways of calculating the extensions

<sup>3.</sup> The equational approach of Gabbay [9]

<sup>4.</sup> The algorithmic approach, see [1]

- 1. We begin with a pair (S, R), where S is a nonempty set of points (arguments) and R is a binary relation on S (the "attack" relation).
- 2. Given (S, R), a subset E of S is said to be conflict free if for no x, y in E do we have xRy.
- 3. E protects an element  $a \in S$ , if for every x such that xRa, there exists a  $y \in E$  such that yRx holds.
- 4. E is admissible if E is conflict free and protects all of its elements.
- 5. E is a complete extension if E is admissible and contains every element which it protects.

Various different semantics (types of extensions) can be defined by identifying different properties of E. For example we might define that E is a stable extension if E is a complete extension and for each  $y \notin E$  there exists  $x \in E$  such that xRy or the grounded extension as the unique minimal extension or a preferred extension, being a maximal (with respect to set inclusion) complete extension. The above properties give rise to corresponding semantics (stable semantics, grounded semantics and preferred semantics).

It can be proved that extensions satisfying items (1)-(5) of Definition 3.1 do exist. The proof is set- theoretical using fixed points. It is easy to see how the above conditions on extensions E can be interpreted as defining a survival group. The members of the group do not attack one another and attack anyone who attacks one of them. The group also adds to itself all candidates it can protect. This is a group of nodes taking a maximal defensive position.

**Remark 3.2.** Definition 3.1 uses geometrical properties (the "attack" arrow  $\rightarrow$ , to define survival concepts. Since later we are going to generalise the concept of one life to many lives, it is helpful already at this point to rewrite Definition 3.1 in survival terms.

The clause numbers here correspond to the clause numbers in Definition 3.1

- 1. Given (S, R), where S is a nonempty set of points and R is a binary relation on S, a subset E of S is said to attack a point x in S if for some y in E we have that yRx holds.
- 2. A subset E of S is said to be able to survive together, if for no subset Y of E and no point x in E do we have that Y attacks x.

3. E protects an element a in S if whenever a set X attacks a, then the set X - Y does not attack a, where Y is the set

 $Y = \{y | y \text{ in } X \text{ and } E \text{ attacks } y\}.$ 

4. E is admissible if E is able to survive together and E protects all of its elements.

Note for example that if we allow for many lives then if  $a \rightarrow b$  and  $b \rightarrow a$ and each of  $\{a, b\}$  have two lives, then the set  $\{a, b\}$  is able to survive together, because neither of its elements can kill the other.

5. E is a complete extension if E is admissible and contains every element which it protects.

We can also present the complete extensions of A = (S, R), using the Caminada labelling approach, see [7].

**Definition 3.3.** A Caminada labelling of S is a function  $\lambda : S \mapsto \{in, out, und\}$  such that the following holds.

- (C1)  $\lambda(x) = in$ , if for all y attacking  $x, \lambda(y) = out$ .
- (C2)  $\lambda(x) = out$ , if for some y attacking  $x, \lambda(y) = in$ .
- (C3)  $\lambda(x) = und$ , if for all y attacking  $x, \lambda(y) \neq in$ , and for some z attacking  $x, \lambda(z) = und$ .

A consequence of (C1) in Definition 3.3 is that if x is not attacked at all, then  $\lambda(x) = \text{in.}$  Any Caminada labelling yields a complete extension and vice versa. Any  $\{\text{in, out}\}$  Caminada labelling (i.e. with no "und" value) yields a stable extension and vice versa. Set theoretic minimality or maximality conditions on extensions E correspond to the respective conditions on the "in" parts of the corresponding Caminada labellings, see [7].

**Remark 3.4.** Let us summarise the comparison of the Caminada  $\lambda$  function (and hence the notion of the traditional Dung extension which is equivalent to it) with the many lives function M(x):

We can understand the Caminada labelling function  $\lambda(x)$  a partially defined function M, giving values in  $\{0,1\}$  satisfying certain restrictions. If we write M(x) =undefined when M is not defined on x, and write M(x) = in, to mean M(x) = 1and M(x) = out to mean M(x) = 0, then the conditions (C1), (C2) and (C3) of Definition 3.3 become the restrictions on M. This observation is of methodological importance. We are offering a new many lives system and we need to show how the traditional Dung system fits in as a special case. We have just shown that if we allow M to be partial function and put conditions on M in terms of R we can get the traditional Caminada Dung semantics as a special case.

### 4 Formal set theoretic semantics

This section formally defines the notion of many lives networks for our paper and, following Dung [6], develops set theoretic semantics for it.

**Definition 4.1** ( $MK\beta$  annotation for a network). Let  $(S, R, M, K, \beta_m)$  be an annotated network as follows:

- 1. (S, R) is a network with  $S \neq \emptyset$  and  $R \subseteq S \times S$ .
- 2. *M* is a function on *S* giving for each  $x \in S$  a natural number in  $\{0, 1, 2, 3, ...\}$  called the number of lives of *x*.
- 3. K(y,x) is a function giving each attack  $(y,x) \in R$  a natural number value in  $\{1,2,3,\ldots\}$  called the strength of the attack.
- 4.  $\beta_m$  for m a natural number or  $\infty$  is a function k we have  $\beta_m(k) = 0$  if  $k \leq m$ and  $\beta_m(k) = 1$  if k > m.
- 5. Let  $\delta(x)$  be Kronecker  $\delta$  function, namely

$$\delta(x) = \begin{cases} 0, & \text{if } x = 0\\ 1, & \text{if } x \neq 0 \end{cases}$$

- 6. Let Attack(x), for  $x \in S$  and subsets E of S be the set  $\{y|yRx\}$ . Let E be any subset of S and let Attack(E, x) be  $\{y|y \in E \land yRx\}$ .
- 7. Let E be a non-empty subset of S. Let  $M^*(E, x)$  be defined for  $x \in S$  as the function derived from M, satisfying the implicit equation (\*) for any subset E of S and any  $x \in S$ .<sup>8</sup>

<sup>&</sup>lt;sup>8</sup>The perceptive reader might ask why we have  $M^*$  in the right had side of the equation in item 7. We explain this by example. Take the network of Figure 15 with nodes  $\{a, b, c\}$  and attacks  $a \rightarrow b$  and  $b \rightarrow c$ . Let K and  $\beta$  play no role. Let  $E = \{a, b, c\}$ . So we have only M, and let M(a) = 2, M(b) = 1 and M(c) = 2.

$$M^{*}(E,x) = M(x) \div \left[\beta_{m}\left(\sum_{y \in E \land yRx} \delta(M^{*}(E,y))\right) + \sum_{y \in E \land yRx} \left(\delta(M^{*}(E,y))K(y,x)\right)\right]$$
(\*)

8. Let (S, R, M, K) be a system with K(x, y) = 1 for all (x, y) in R and  $M(x) \leq 1$  for all x in S and no  $\beta$  present. We say that this system has a numerically balanced M labelling iff  $M^* = M$ .<sup>9</sup>

**Definition 4.2.** Given an  $MK\beta$  network as in Definition 4.1, with a set S of nodes and a relation R on S, let us define the notion of a non-empty subset E attacking a node x, Notation  $\alpha_{\mathbf{a}}(E, x)$ , as follows:

(#)  $\alpha_{\mathbf{a}}(E, x)$  holds iff by definition  $M^*(E, x) = 0$ , where  $M^*(E, x)$  is as defined in item 7 of Definition 4.1.

It is very important to note that for any E, E' and x we have:

- E attacks x and E is a subset of E' then E' attacks x.
- E does not attack x and E' is a subset of E then E' does not attack x.
- Note that the attack  $\alpha_{\mathbf{a}}$ , is defined using item 7 of Definition 4.1, and is therefore dependent on M and on K. If we use another many lives function N and another strength of attack function L, we will get a different attack relation., which we can call for example by the name  $\alpha_{\mathbf{p}}$ . Note further that if for all x, y we have that  $M(x, y) \leq N(x, y)$ , and/or  $L(x, y) \leq K(x, y)$  then  $\alpha_{\mathbf{p}}$  is a stronger attack than  $\alpha_{\mathbf{a}}$ , namely if E can  $\alpha_{\mathbf{p}}$  kill x then E can  $\alpha_{\mathbf{a}}$  kill x.

Test the equation of item 7 on this network. We get

$$M^*(E, a) = 2$$
  

$$M^*(E, b) = M(b) - \delta(M^*(E, a)) = 1 - 1 = 0)$$
  

$$M^*(E, c) = M(c) - \delta(M^*(E, b)) = 2 - 0 = 2).$$

If we do not put  $M^*$  on the right hand side we get for  $M^*(b)$  the value 1. The definition of  $M^*$  is to yield the many lives values of the nodes following the the propagation of the attacks.

<sup>9</sup>The conditions on (M, K) of item 8 makes the network practically a traditional network with "in" and "out" annotation. If the network is acyclic a numerically balanced labelling exists. Note that we allow M(x) = 0 even for x which is not attacked (i.e. even when all attackers are nonexistent or have M value 0). If we insist that M(x) = 1 in such cases (note this is one of the Caminada conditions), then M will still be numerically balanced but M will yield the grounded stable extension in the acyclic case. Consider a three point acyclic network of Figure 15, with  $S = \{a, b, c\}$  and  $R = \{(a, b), (b, c)\}$ , (that is the network  $a \rightarrow b \rightarrow c$ ). Consider the numerically balanced M(a) = 1, M(b) = M(c) = 0. This M does not give rise to a Dung grounded extension but M' with M'(a) = M'(c) = 1 and M'(b) = 0, which is also numerically balanced does give a Dung grounded extension. **Definition 4.3.** Let (S, R) be a given geometrical network. Imagine we have several possible functions of the form M(x), K(x, y) and  $\beta$  defined on (S, R). We can use different functions  $M, K, \beta$  to define different kinds of attacks as done in Definition 4.2.

Let  $\alpha_{\mathbf{a}}, \alpha_{\mathbf{d}}$ , and  $\alpha_{\mathbf{p}}$ , be three such attacks as defined in Definition 4.2. Assume the relative strength of these attacks is as follows:

- (s1) If  $Y\alpha_{\mathbf{p}}$  attacks z then  $Y\alpha_{\mathbf{d}}$  attacks z
- (s2) If  $Y\alpha_{\mathbf{d}}$  attacks z then  $Y\alpha_{\mathbf{a}}$  attacks z.
  - 1. We say that E is at peace iff for no Y, a in E do we have  $\alpha_{\mathbf{d}}(Y, a)$  holds ("at peace" means "able to live/survive together" where the attack does not kill, compare with Definition 4.7 and Remark 2.4).
  - 2. E protects x if for every Y such that  $\alpha_{\mathbf{a}}(Y, x)$  holds we have that for some subset Y' of Y the protecting set E successfully  $\alpha_{\mathbf{p}}$  attacks all elements of Y' and that the remaining elements of Y, namely the set Y - Y', does not successfully  $\alpha_{\mathbf{a}}$  attack x.
  - 3. E is  $(\mathbf{a}, \mathbf{p}, \mathbf{d})$  admissible if E is at peace and protects its elements

**Lemma 4.4.** If E admissible and protects x then  $E \cup \{x\}$  protects itself.

*Proof.* This is true because E protects all elements of  $E \cup \{x\}$  so  $E \cup \{x\}$  does it (i.e. protects) as well because of the monotonicity condition.

**Lemma 4.5.** If E is at peace and protects its elements and E protects x then  $E \cup \{x\}$  is at peace.

*Proof.* Assume that  $E \cup \{x\}$  is not at peace, get a contradiction. We immediately see that x is not in E.

Let  $Y \subseteq E \cup \{x\}, z \in E \cup \{x\}$  be such Y successfully  $\alpha_{\mathbf{d}}$ -attacks z. Then by our assumptions Y also successfully  $\alpha_{\mathbf{a}}$  attacks z. We distinguish several cases:

**Case 1.**  $x \notin Y, x \neq z$ . This case contradicts *E* at peace.

**Case 2.**  $x \notin Y, z = x$ . We have Y successfully  $\alpha_{\mathbf{d}}$  attacks x and therefore also successfully  $\alpha_{\mathbf{a}}$ -attacks x. Since  $E \alpha_{\mathbf{p}}$ -protects x, E must successfully  $\alpha_{\mathbf{p}}$ -attack some elements  $y_1, \ldots, y_k$  such that  $Y - \{y_1, \ldots, y_k\}$  does not successfully  $\alpha_{\mathbf{a}}$ -attack x. Since Y does successfully  $\alpha_{\mathbf{a}}$ -attack x, there must be at least one  $y_1$  in Y (and

therefore  $y_1$  is not x) such that E successfully  $\alpha_{\mathbf{p}}$ -attacks  $y_1$ . Since by our assumptions say that  $\alpha_{\mathbf{d}}$  attacks are stronger than  $\alpha_{\mathbf{p}}$  attacks (this is assumption (s1)), we get that  $E \alpha_{\mathbf{d}}$  attack  $y_1$ . Thus we have found a  $y_1$  in E which is successfully  $\alpha_{\mathbf{d}}$  attacked by E, a contradiction.

**Case 3.**  $x \in Y$  and x is different from z. Let  $Y_o$  be a subset of E and assume that  $Y = Y_o \cup \{x\}$ . So we have that  $Y_o \cup \{x\}$  successfully  $\alpha_d$ -attacks z and  $z \neq x$ . Since  $z \in E$ ,  $E \alpha_d$ -attacks elements of  $Y_o \cup \{x\}$ . E cannot attack any elements from  $Y_0$  so E attacks x but this is now case 2, which is impossible.

**Case 4.**  $x \in Y, z = x$ . so we have  $Y_o \cup \{x\} \alpha_d$  attacks x. Therefore it  $\alpha_a$  attacks x. Since E protects x, E attacks  $Y_0 \cup \{x\}$  but E cannot attack any of its elements.  $\Box$ 

**Lemma 4.6.** There exists an admissible set  $E \subseteq S$  s.t. E = all elements it protects.

*Proof.* Start with  $\emptyset$ . It protects its elements and is at peace. Suppose  $\emptyset$  protects x then  $\{x\}$  protects x and is at peace.

Continue to increase the set using Lemma 4.4, until we reach a maximal st. This is the set E we need.

**Definition 4.7.** Let  $(S, R, M, K, \beta)$  be the network defined in Definition 4.1, and assume that we have the notion of  $\alpha_{\mathbf{a}}, \alpha_{\mathbf{p}}$ , and  $\alpha_{\mathbf{d}}$  -attack to go with it. Using the notion of such attacks we can identify the family of sets E which are admissible and are equal to the set of all the elements E protects. Let E be such a set. E may  $\alpha_{\mathbf{d}}$ attack some of its elements but such attacks are not successful. This is why E is at peace, precisely because the attacks of E on its elements,  $x \in E$ , are not successful, i.e. these attacks cannot reduce to 0 the many lives M(x) of x. We can now use the notion of  $\alpha_{\mathbf{d}}$ -attack to update the number of lives of each element x in E. Let x be any element x in E such that  $E \alpha_{\mathbf{d}}$ -attacks x. Let the new annotation of xbe  $M^*(E, x)$  of item 7 of Definition 4.1. If x is not  $\alpha_{\mathbf{d}}$  attacked by E, leave its annotation unchanged.

Let  $M_E$  be the new annotation on E. We refer to the system  $(E, R \text{ restricted to } E, M_E \text{ restricted to } E)$  together with the **a** respective attacks restricted to E, as an E complete extension of the original system.

We thus can define the set of all *E*-complete extensions of the original system.

**Example 4.8.** Let us illustrate the concepts of Definition 4.7 using the network of Figure 4. In the network of this figure, (with nodes  $\{y, z, u, x\}$  and where y has attack strength 1, z has 2 and u has 3 and the number of lives of x is 7). In the



Figure 5

network of this figure, the entire set E of nodes is at peace with itself, because y, zand u are not attacked and although they all attack x, x can survive the attack with 1 life left. Thus here  $M_E$  is actually the calculated  $M^*$ .

**Remark 4.9.** We make a few key points related to the definitions of this section.

- 1. Note what the network with  $M, K, \beta$  looks like when M is a fixed number m > 1for all nodes (say m = 2) and K is always 1 and  $\beta = 0$ . This means we have a network where all nodes require 2 attackers to be dead. To appreciate this case, consider the simple Figure 5. Assume m = 2. So we are giving each node in Figure 5 two lives. In this case we get that  $\{a, a', c, c'\}$  are "in" with two lives each and  $\{b, b'\}$  are "out" with zero lives each. If we let m = 3 we get a new network with the same graph figure but with different lives distribution. a, a' have 3 lives, and b, b', c', c' have 1 life each.
- 2. Suppose in item (1) above we adopt a geometrical point of view and say that to be killed we need 3 geometrical attackers. We do not assign life to nodes, just say to be "out" you need 3 "in" attackers. This is in the spirit of [1]. In this case we simply get that all are "in". We do not subtract the number of live attacks from the many lives of the target. When the target is attacked, it is either killed or if not enough live attackers are present, then it stays as is.
- 3. The reader might ask why in Definition 4.2 we were talking about  $\alpha_{\mathbf{a}}$ -attacking, what is the role of the index " $\mathbf{a}$ "? We have this index because we might have more than one type of attack, say we might have also another kind of attack which we might call  $\alpha_{\mathbf{p}}$ . The two notions of attacks,  $\alpha_{\mathbf{a}}$  and  $\alpha_{\mathbf{p}}$  might play different roles in calculating extensions. For example the nodes  $\{a, a'\}$  are geometrically protecting the nodes  $\{c, c'\}$  because they are geometrically attacking the nodes  $\{b, b'\}$  which are the geometrical attackers of  $\{c, c'\}$ . We might make

a distinction and say that attacks are usually  $\alpha_{\mathbf{a}}$  attacks, but when protecting elements we must use  $\alpha_{\mathbf{p}}$  attacks.

For this to work smoothly we need to require that the relationships of Definition 4.3 to hold between  $\alpha_{\mathbf{p}}$  and  $\alpha_{\mathbf{a}}$ . In fact we can also add a another notion of  $\alpha_{\mathbf{d}}$  attacks and say that for a set E to be considered conflict free (at peace) we want that no subset  $Y \subseteq E$  can  $\alpha_{\mathbf{d}}$  attack any  $e \in E$ . To work smoothly we need condition (s1) of Definition 4.3 to hold for  $\alpha_{\mathbf{d}}$  and  $\alpha_{\mathbf{p}}$ .

- Note that Definition 4.3 and the Lemmas and proofs following it, do not use the exact definitions of the attacks but only their relative strengths, being conditions (s1) and (s2). This allows us to give possibly completely different definitions of attacks in our paper [38].
- 5. The ideas of different types of attacks was introduced in Section 8.3 of [1, p. 1855]. The reader can see more discussion in item (2) of the comparison with the literature Section and in the methodological and concluding Section 7.

## 5 Comparison with the literature

We compare several related papers.

(1). Comparison with the universal distortion paper [1]. This paper deals with thinking distortions of sex offenders in particular and of general thinking distortions in general. Part of this paper is the observation that the idea of many lives can be used in argumentation. A simple model is given in the paper and some semantics is described. The full analysis and study of many lives was postponed to the present paper and other papers [27].

(2). Comparison with graded acceptability of arguments paper [8] and [24]. These papers (among other results) propose a framework with a view of distinguishing between nodes that are out because of, say, two successful attacks, as opposed to nodes that are out because of, say, one successful attack. So for example, in Figure 6 which describes a traditional network, d is more "dead" than c because d is attacked by two living attackers while c is attacked by only one.

The authors are trying to bring this difference out by defining a predicate  $d_n^m(X)$ , where X is a set of nodes (intended to be an admissible set) and  $d_n^m(X)$  is the set which X protects. For the purpose of comparison with our own paper, we use the definition for the case  $d_2^1(X)$ , because this is sufficient to bring out the differences



Figure 6

with our paper and our notion of many lives. So X is a set of nodes and  $d_2^1(X)$  defines the set of points which X protects.

We now quote and rewrite Definition 5 of [8] for the case m = 1, n = 2

$$d_2^1(X) = \{x | \neg \exists_{\geq 1} y([y \twoheadrightarrow x \land \neg \exists_{\geq 2} z(z \to y \land z \in X)]\}$$

We can rewrite the above as the following:

$$d_2^1(X) = \{ x | \forall y [(y \twoheadrightarrow x) \to \exists_{\geq 2} z (z \twoheadrightarrow y \land z \in X)] \}.$$

We can again rewrite as the final version  $\sharp$ :

$$d_2^1(X) = \{ x | \forall y [(y \twoheadrightarrow x) \to \exists z_1, z_2(z_1 \neq z_2 \land (z_1 \twoheadrightarrow y) \land (z_2 \twoheadrightarrow y) \land z_1 \in X \land z_2 \in X) ] \}.$$

$$(\sharp)$$

This formula says that x is protected by X iff every attacker y of x, that is, $(y \rightarrow x)$  is itself attacked by two different members of X.

The above formula  $d_2^1(X)$ , which describes how X can protect a node x, looks very related to our two lives concept. However, it is not the same as a 2 lives. To see this, consider Figure 15. Let us apply  $d_2^1(\emptyset)$  to a. This will determine whether a is alive or not. Substituting a for x in the rewritten formula, we find that  $d_2^1(\emptyset)$ holds for a because a has no attackers. Thus a is alive.

Let us now consider the node b. b attacks c. In order for c to be defended, b, being the attacker of c, must be attacked by two live attackers. Such attackers are not available in the figure.

However, b is being attacked by a and to get b dead it is enough to have one live attacker of b which cannot be defended.

The important point here is that we cannot assign a simple number of lives to b. For b the attacker of c, b has number of lives 2. For b the victim being attacked, the number of lives is 1. This is why  $d_2^1$  has two indices "1" and "2". We can, however, assign two numbers to b, one for it being attacked as a victim and one for being attacked as an attacker. This is what we discussed in item 3 of Remark 4.9.

Further note that in our paper [38] we study the notion of forward looking attacks and semantics. We prove in [38] that the many lives semantics is forward looking, while the Grossi Modgil semantics/attack is not.

We now summarise the comparison of our many lives approach with the Grossi and Modgil approach of [8, 24]:

- 1. From the technical mathematical point of view, given a network (S, R) where S is the set of arguments and R is the geometrical attack relation, we can simulate the system  $d_n^m$  of [8, 24] (with n greater or equal m) using two many lives functions M and N, and two types of attacks  $\alpha_{\mathbf{a}}$  (to kill you need M(x) = m live attackers) and  $\alpha_{\mathbf{p}}$  (to protect you need to attack the attacker with n live attackers). We can even add as a bonus another many lives function G(x) = k, with k greater or equal to n, and define  $\alpha_{\mathbf{d}}$  attacks. We can thus do a triple index Grossi-Modgil geometrical function d[m, n, k].
- 2. Note that the attack  $\alpha_{\mathbf{p}}$  can be required to have different strengths for different nodes. Our machinery naturally allows for this. So in the terminology of [8, 24], the function  $d_n^m$  can be different for each of the nodes involved.
- 3. From the conceptual point of view the two approaches, the many lives approach and the Grossi Modgil ranking approach, S are independent and have different origins and goals. The many lives idea comes from, and is inspired by, the offender/complaints/ survival point of view and is to be tested by its ability to adapt and serve its intended application areas. The  $d_n^m$  approach of [8, 24] comes from the geometrical ranking approach of pure formal argumentation, catering for the intuition of
- (\*) x being more "in" or more "out" than y.

The extent to which [8, 24] succeed in addressing this intuition is not relevant to our comparison in this paper (it is discussed, however, in our paper [10]). However, it is relevant to the question of to what extent many lives can also be applied to the same ranking question (\*).

4. We appreciate the fact that implicit in the Grossi and Modgil attempt in [8] to address (\*) is the idea that to protect we can ask for a stronger attack, compatible with our developing networks with a progression of connected attacks. Note that the idea of different types of attacks also appears in Section 8.3 of [1, p. 1855].



Figure 8

We shall study [8, 24] critically elsewhere, see [10] and Section 8.3 of [1].

(3). Comparison with joint (also called Collective) attacks [17] and [4, Chapter 7]. The idea of joint attacks introduced in [17] and also studied in [4, Chapter 7] is explained in Figure 7. In this figure the set  $Y = \{y_1, \ldots, y_k\}$  jointly attacks the node x. The meaning is that only when all  $\{y_i\}$  are live (in) do we have that x is dead (out). Nielsen and Parsons in [17] use a set to point relation  $\mathbb{R}$  for such an attack. So they consider networks of the form  $(S, \mathbb{R} \text{ where } \mathbb{R} \subseteq 2^S \times S$ . In Figure 7 we have  $(Y, x) \in \mathbb{R}$ . The notation of Figure 7 is used by [4, Chapter 7], who also allows for disjunctive attacks of the form of Figure 8.

This means that if w is alive (in), then one of  $Z = \{z_1, \ldots, z_k\}$  must be out. We can also have conjunctive-disjunctive attacks of the form of Figure 9

This means that if all of  $\{y_i\}$  are in then one of  $\{z_j\}$  must be out. See [18]. This can be written as a relation between sets Y and Z.

It is important to realise that the attacks of sets E on nodes e in this paper are not joint attacks but an aggregation of single attacks. Not all members of the set



Figure 10

E need to mount a successful attacks. This is why we have the monotonicity rule, that if E mounts successful attack on e so does any superset of E. The connection with m lives is explained in Figures 10 and 11.

In Figure 10,  $a_1$ ,  $a_2$  and  $a_3$  attack b. b has 2 lives and so for b to be out, at least two of its attackers must be in. Now suppose that only  $a_1$  is in and  $a_2$  and  $a_3$  are out. What do we say now about b?

We say two statements

- 1. b is not out, b is still in.
- 2. The number of lives of b is 1 (reduced by 1).

It is statement (2) about b in Figure 10 which cannot be properly captured/ translated/reduced, by using joint attacks. Statement 1 can be translated into conjunctive attacks as shown in Figure 11 with  $\{a_1, a_2\}\mathbb{R}b, \{a_1, a_3\}\mathbb{R}b$  and  $\{a_3, a_2\}\mathbb{R}b$ , but statement (2) is not represented in Figure 11. There is a more severe way of bringing out the difference. Joint attacks still operate within the framework that each node x is either in or out (or undecided). It may take a joint attack of m > 1



Figure 11

live/in nodes to kill x, but still m cannot be reduced to m-1 if we have only single live attack.

Furthermore, the translation from Figure 10 to Figure 11 does succeed in translating statement (1) about b when b has only 2 lives, but what do we do with the case of b having 4 lives? What do we write? There are not enough attacking nodes to make any distinctions.

We are fairly confident that in general we cannot (prove a theorem that we can always) translate a many lives network into a single life network with joint attacks. In other words we believe the many lives concept cannot be reduced to the concept of joint attacks. Note that the network of Figure 10 can be reduced/translated to a network with joint attacks only, namely Figure 11 with b with one life only. We ask however, how would a translation go if b were to have 10 lives in Figure 10?

Let us summarise as follows:

The semantics for many lives networks is to yield other many lives networks whose nodes have less lives. The semantics of networks with joint attacks is to yield subsets of nodes which are in or out (the rest being undecided).

Let us now ask about the other direction. Can the many lives model simulate joint attacks?

Consider Figure 12. This is a figure with two types of joint attacks.

For the attack of  $a_3$  to succeed, we need b to have one life. For  $a_1$  or  $a_2$  alone not to succeed we need b to have two lives. The problem is that the joint attacks can be mixed, with different joint attacks having a different number of attackers. We can perhaps compensate by adding strength of attack to  $a_3 \rightarrow b$  and get Figure 13

This may work in this case but the reader can see that the two ideas, joint attacks and many lives are different intuitions.<sup>10</sup>

 $<sup>^{10}</sup>$  One of the referees made the following remark, we quote:



Figure 13

"The idea of many lives in not new since in the literature it has been somehow captured by collective attacks. Of course, the approach followed was quite different but the purpose is the same. Personally, I prefer the encoding via collective attacks. The reason is that the number M(x) is independent from the attackers, namely from their strength, their relations to each others, their relevance to the target, etc. This may lead to counter-intuitive results. Assume for instance that N(x) = 3 and surprisingly x is attacked by 3 non-attacked arguments  $y_1, y_2, y_3$ . According to the formalism proposed in the paper x will be rejected independently of  $y_i$ . Suppose that  $y_i$  are all similar (or logically equivalent). x would be rejected while it should not."

Our answer to this remark is as follows:

- 1. The default assumption in argumentation is that different letters for arguments denote completely independent arguments. Otherwise we have the same problem in ordinary argumentation. For example if we have  $x \twoheadrightarrow a, y \twoheadrightarrow a$  and  $b \twoheadrightarrow x$ , then we never ask if x is equivalent to y and so perhaps then b protects a? The default is that x is independent of y.
- 2. We already remarked that apart from the fact that the idea of many lives in different from that of joint/collective attacks, we do not believe that technically we can reduce the machinery of many lives to that of joint attacks.

Such a reduction is possible if we base argumentation on linear logic and use the attack as information input, see [40].



Figure 14

**Example 5.1** (Can many lives be reduced to joint attacks?). Consider Figure 14. We have  $a_2, a_2, a_3$  attacking b which has 2 lives. Assume this figure is part of a much larger network and so it is not known what values  $\{a_1, a_2, a_3\}$  get, ("in" or "out"). We further assume that  $\{a_i\}$  are the only attackers of b and that the larger network is finite acyclic and so has only the ground extension and it is stable (no undecided). Assume also that b is the only argument in the entire network which has two lives.

Our objective is to represent this figure within the traditional framework of Dung. We try to do that using common sense and see what happens. The traditional framework cannot represent numbers, so let us duplicate b and introduce  $b_1, b_2$  and write in the meta-level that  $b_1 = b_2$ . This gives b two lives.

Then we replace  $\{b\}$  by  $\{b_1, b_2\}$  in the larger network.

The problem is that we have  $a_1, a_2, a_3$  each attacking b and we need to say how they attack  $\{b_1, b_2\}$ .

We know that

- 1. If only one of  $\{a_i\}$  is in then b is not out, but its life is reduced from 2 to 1.
- 2. If two of  $\{a_i\}$  are in then b is out.

So if we split/replace b to  $b_1, b_2$  we must then satisfy:

(1\*) If only one of  $\{a_i\}$  is in then only one of  $\{b_1, b_2\}$  is out

 $(2^*)$  If two of  $\{a_i\}$  are in then both  $\{b_1, b_2\}$  are out.

To implement the above we face a technical problem:

• What attack arrows do we draw from the  $a_i$  to the  $b_j$ ?

Let us perform a detailed analysis of our options.

- (p1) We do not draw any attack from a<sub>1</sub>.
  This is not possible because if a<sub>1</sub> = in and a<sub>2</sub> = a<sub>3</sub> = out then we need exactly one of {b<sub>i</sub>} to be out.
- (p2) OK. Let  $a_1 \rightarrow b_1$ . Clearly we will not have that  $a_1$  also attacks  $b_2$  because then if  $a_1 = in$  and  $a_2 = a_3 = out$  both  $b_1, b_2$  will be out.
- (p3) How about  $a_2$ ?

 $a_2$  must attack one of the  $b_i$ , otherwise if  $a_2 = in$  and  $a_1 = a_2 = out$  then none of  $b_i$  would be out. OK, then  $a_2$  must attack  $b_2$ , i.e.  $a_2 \rightarrow b_2$  (it cannot attack  $b_1$ ).

(p4) Now we have an impossibility. What does  $a_3$  attack? If it does not attack at all, then if  $a_3 = in$  and  $a_2 = a_3 = out$ , the none of  $\{b_i\}$  is out. This is wrong.

If  $a_3$  does attack say  $a_3 \twoheadrightarrow b_1$  and not attack  $b_2$ , then if  $a_1 = a_3 = in$  then only  $b_1$  will be out and not both of  $\{b_i\}$ . Again not good.

If  $a_3$  also attacks  $b_2$  then if  $a_3 = in$  and  $a_1 = a_2 = out$  then we get that both  $\{b_i\}$  are out, again not correct.

OK, so what do we do now? It is natural to follow a continuation idea.

Let us form a set  $\{b_1, b_2\}$  and let all  $a_1, a_2, a_3$  each attack the set.

Namely

$$a_1 \twoheadrightarrow \{b_1, b_2\}$$
$$a_2 \twoheadrightarrow \{b_1, b_2\}$$
$$a_3 \twoheadrightarrow \{b_1, b_2\}$$

Let us say that to attack a set is to attack one of the members. We again have a problem.

• If each  $a_i$  says explicitly which member it attacks we are back to the previous dilemma.

- If  $a_i$  does not say which member it attacks then we cannot prevent all  $a_i$  attacking  $b_1$  and we gain nothing.
- If we say all attackers must attack separate members and otherwise (if there are no more un-attacked members) not attack, then this is a fancy language basically repeating the much simpler original numerical b : 2 representation.

What we want is a better representation for the traditional Dung network, which with a small change will represent the many lives generalisation.

To summarise:

### • We need an inspiration

This is a problem for another paper. It is possible to do using linear logic and using ideas from paper [40].

(4). Comparison with abstract dialectical framework (ADF) [14]–[16], [23] and [39]. ADF is a powerful system which can express practically anything you can throw at it. It can express the joint attacks easily. For example the condition on b of Figure 12 can be written as

$$b \leftrightarrow (\neg a_1 \lor \neg a_2) \land \neg a_3.$$

So our adding many lives to ordinary Dung style networks can be easily added to ADF. Take for example Figure 13. ADF simply uses its functions to give the kind of semantics required. The simplest way of doing it is to allow ADF to talk about nodes with numerical annotations, (in other words the basic units are pairs (node, number)), and allow it to associate conditions on the combinations of attacking nodes again with their numerical annotations. So we can say (see Figure 13):

• The new annotation of node (b:2) is (b:99), if its attackers all have annotations which are not prime numbers.

Why prime numbers? Well ADF is mathematical. In mathematics we only increase what we can do and not try to insist on things we should not be allowed to do!. If we insist that ADF uses only in , out, undecided) annotation, then it cannot simulate the many lives annotation. The discussion of Figure 10 in comparison (3) above holds in this case as well.

More interesting is the other direction, can the many lives approach simulate ADF or fragments of it? The answer is that the many lives model is monotonic, namely



• E attacks x and  $E \subseteq E'$  then E' attacks x.

ADF does not have this restriction. We can write in ADF an acceptance condition which is not monotonic, for example:

• E attacks x if the number of elements in E is even.

(5). Comparison with papers with the idea of graduality, e.g. [11]-[13]. These papers and many others like them want to pay attention to the number of attackers on x and the number of attackers on attackers, etc. Paying attention to such distinctions allows us to say that some nodes are "more in" than other nodes. For example in Figure 15, a is "more in" than c. This is a different theme but we can use many lives as another instrument to measure this feature. This is best explained by an example. Consider Figure 15. If we give all nodes one life we get:

**One life:** a = 1, b = 0, c = 1

**Two lives:** a = 2, b = 1, c = 1

Three lives: a = 3, b = 2, c = 2.

So if the network has n nodes, go in sequence up to n lives and see what you get. The differences will show in the sequence. Like the difference between a and c in the sequence for Figure 15.

(6) Comparison with numerical argumentation. We really need not compare our paper with any of the purely technical numerical argumentation publications. Our paper is not intended as such. The relevant papers for comparison are really the ones described in items 1 and 2 above. There is a comparison from the numerical point of view in our paper [10]. Nevertheless, we are including some comparative discussion (of papers [19, 20, 21, 23, 25, 33], and [34] in Appendix 2).

## 6 Methodological discussion and conclusion

This section summarises what is going on and indicates what is more to be done. Given a geometric network of the form (S, R) with  $S \neq \emptyset$  and  $R \subseteq S \times S$ , we propose that we view it as a base/a carrier to be used to define an argumentation network for a target application. We should not be locked into the view that R is the network attack relation. R could be some important relation in the target application from which we derive the relevant attacks and supports. Once we accept this view about a network (S, R), we can turn it into an argumentation network in many different ways, by adding extra structure to it and defining the basic argumentation notions on top of the structure. Our paper [38] discusses some such specific examples obtained by using the relation R, meanwhile, let us give an example generalising the many lives approach.

Let us add a numerical function

$$\mathbf{f}: S \mapsto [0,1]$$

i.e. for each  $x \in S$ ,  $\mathbf{f}(x)$  is a real number  $1 \ge \mathbf{f}(x) \ge 0$ .

The system  $(S, R, \mathbf{f})$  is very general.  $\mathbf{f}$  can be interpreted i many ways. It can be:

- Fuzzy value, introducing argumentation networks.
- Probability function, introducing probabilistic argumentation
- Measure of strength, introducing numerical argumentation
- A [0, 1] solution to some equations in the equation approach generating a many lives function

$$M(x) = \frac{x}{1-x} = \frac{1}{1-1/x}$$

 $\mathbf{f}(x) = 0$  means 0 lives (M(x) = 0).

 $\mathbf{f}(x) = 1$  means immortal lives  $M(x) = \infty$ .

Further note that we can add other types of functions, not necessarily numerical. For example we can add a structured logic function  $\Delta(x)$ , giving for each  $x \in S$ , a logical theory or a formula  $\Delta(x)$  from some logic **L**. **L** could be classical logic or intuitionistic logic of logic programming or some nonmonotonic logic. If we do that we need to define what it means for one logical theory to attach another logical theory, this can lead to systems like Aspic<sup>+</sup> or Assumption based argumentation or argumentation as information input, etc.

Let us continue with the numerical function  $\mathbf{f}$  and let us turn to the system  $(S, R, \mathbf{f})$  into an argumentation network. We need to define some additional basic notions. Let us elaborate using a question and answer dialogue:

**Q1.** What can be attacked?
#### Answer 1:

- Individuals  $x \in S$  or
- subsets  $Y \subseteq S, Y \neq \emptyset$
- Geometrical arrows (i.e. elements of R, giving rise to a higher level attacks)

**Q2:** Who are the attackers?

#### Answer 2:

- Subsets  $E \subseteq S \ E \neq \emptyset$  or
- individuals  $x \in S$ .
- Geometrical arrows (i.e. elements of R, giving rise to a higher level attacks attacking other attacks)

**Q3:** What is the nature of the attacks?

**Answer 3:** Let us take three types of attacks  $\alpha_{\mathbf{a}}, \alpha_{\mathbf{p}}, \alpha_{\mathbf{d}}$ , such that the following holds:

- If  $Y \subseteq S, \alpha_{\mathbf{p}}$  attacks x then  $Y\alpha_{\mathbf{d}}$  attacks x.
- If  $Y \subseteq S$ ,  $\alpha_{\mathbf{d}}$  attacks x then  $Y\alpha_{\mathbf{a}}$  attacks x.

Note to Answer 3: Note hat we can define the attacks in many different ways. Some we already did using the many lives functions M, N, G. We can define many other types of attacks using the function **f**. We shall give different examples in our paper [38], namely Geometrical Attacks, (these are attacks defined in first or higher order logic of the language of (S, R)).

The attacks on x make  $\mathbf{f}(x)$  smaller.

**Q4:** What corresponds to the notion of protecting?

**Answer 4:** We use the attack  $\alpha_{\mathbf{p}}$ . *E* protects *x* if *E*  $\alpha_{\mathbf{p}}$  attacks any set *Y* which  $\alpha_{\mathbf{a}}$  attacks *x*. There can be more variations on this, see Appendix 2.

**Q5:** What corresponds to the notion of conflict freeness of a set *E*?

**Answer 5:** We use  $\alpha_{\mathbf{d}}$ . *E* is at peace (i.e.  $\alpha_{\mathbf{d}}$  conflict free) if it does not  $\alpha_{\mathbf{d}}$  attack its elements.

**Q6:** How do we define semantics for  $(S, R, \mathbf{f})$ ?

**Answer 6:** The general notion of semantics is a function  $\mathbb{F}$ , giving for a given system  $(S, R, \mathbf{f})$  a family of new systems  $\{(E_i, R_i, \mathbf{f}_i)\}$ . There are four main methods of defining semantics:

- 1. The Dung like set theoretical method.
- 2. Translating into classical, intuitionistic, modal or some other logic and taking suitable models (in the semantics of that logic) and translating back.
- 3. Using the equational approach. Generating equations from  $(S, R, \mathbf{f})$  and solving them and the solutions generating semantics.
- 4. Giving direct algorithms on  $(S, R, \mathbf{f})$ , using the attacks to run around (S, R) and redefining new  $(E_i, R_i, \mathbf{f}_i)$ . This is the Algorithmic Approach.

Given a family of  $\{(E_i, R_i, \mathbf{f}_i\})$  we an seek to prove completeness theorems, answering the question of which methods from (1)–(4) can produce this family.

For example we can ask for the case of traditional Dung extensions, which equations and which algorithms can yield exactly all the preferred extensions?

**Q7:** What does this paper do?

Answer 7: We are inspired by the many lives phenomena in the complaints area to define a function  $\mathbf{f}$  and look at suitable and compatible networks  $(S, R, \mathbf{f})$  with attack functions and give a set theoretical semantics. As a result of looking at the multiple complaints application area we reached the conclusion that an argumentation network must have 3 different attack relations (all geometrically defined using R) in increasing strength, all participating in defining extensions.

We compared with other related papers in the literature.

**Q8:** What is your opinion of the significance of this paper?

**Answer 8:** I think there are two aspects to our contribution:

- 1. Introducing many lives with a view of applications to the "complaints" areas of applications as well as many other notions motivated by the many lives idea.
- 2. It seems that it is time to introduce some methodological order to the chaotic jungle of formal argumentation publications. The recent publication of volume 1 of the *Handbook of Argumentation* and the planned material for volume 2 together (I personally believe) the methodological view of this section and the stimulus generated by the Grossi and Modgil papers is the starting trigger point.
- **Q9:** What is your next paper in this area?

We completed paper [27], dealing with the temporal aspects of mul-Answer 9: tiple victims complaining one after another as the case develops in the media. We are writing [31]. We observed that when an alleged offender is attacked by one or two complaints, all of a sudden many more complaints come forward. See the story of [32] for a very famous example. Our paper [27] cannot deal with that. Ordinary traditional temporal logic or any variations based on it cannot deal with sudden avalanche of simultaneous triggered changes. The proper way of modelling this is via Reactive Attacks. We envisage a set S of offenders and victims and a binary relation R on S of inactive attacks from victims to offenders. When a victim xcomes forward to complain about offender y, the attack xRy becomes active. The other attacks are dormant and are activated when several attacks coming to life. Exactly how many complaints trigger the activation of all the others depends on circumstances. Temporal logic is not the right way to handle this. What is happening is that because of the first one or two brave attackers coming forward all the other gain courage to join and attack.

For the idea of Reactivity and its applications see [29]. For the details of reactive grammars we want to use to handle this type of formal argumentation see [30]. The paper itself will be [31].

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## Appendices

## A Appendix 1: Discussion of d[m, n, k]

It is useful to introduce a familiar story as an example for d[m, n, k], the story of the party. To help us appreciate the story let us distinguish three types of attacks for a traditional network (S, R) illustrating d[1, 2, 2]. The examples deals with traditional attack and protect but changes the notion of conflict free:

**Definition A.1.** Let (S, R) be an argumentation network, We define two notions of attack of a subset Y of S on a node x using R as follows:

- Yα<sub>a</sub>-attacks x of a subset Y of S on a node x yRx. This is the traditional Dung attack notion.
- $Y\alpha_{\mathbf{d}}$  (respectively  $Y\alpha_{\mathbf{p}}$ )-attacks x iff for two different  $y_1$  and  $y_2$  in Y we have that  $y_1Rx$  and  $y_2Rx$  (respectively same condition for  $Y\alpha_{\mathbf{p}}$ ).

**Example A.2.** We are planning a party and we have a set S which is the maximal set of all relatives friends, colleagues, etc. who can be invited to the party. The problem is that some of them do not get along/hate some others. So we have a relation R, where xRy (which we might denote by  $x \rightarrow y$ ) means that if x is invited, y must not be invited. Let us also assume that for the sake of fairness, if any candidate has no people objecting to him, the candidate should be invited. For example if the party is a diplomatic event, then certainly all diplomats should be invited unless there is a problem. With this view the problem becomes an ecological kind of network (if you are not attacked you are alive). With this understanding we get here a traditional argumentation network with attack relation R. The complete extensions are possible groups of people we can invite.

These are the traditional Dung extensions obtained by using the **a**-attack notion. If the party is a wedding, we can invite whom we please. So even if someone is not objected to, we can choose not to invite him. So if we have  $S = \{a, b, c\}$  with  $a \rightarrow b \rightarrow c$ , we can invite b and not invite a and c, using the symmetrical closure of the given notion of attack. We cannot get this  $\{b\}$  extension if we use the **a** notion of attack the R attack only. We do need its symmetric closure. However, other problems may arise at a wedding scenario. Let  $S = \{a, b, c, u\}$  with R = $\{(a, b), (b, a), (a, u), (b, u), (u, c)\}$ . Think of xRy to mean x Hates y. So we have a married couple/parents  $\{a, b\}$  who hate each other and hate the uncle u and a child c who is hated by the uncle u. The wedding is of the child. See Figure 16: Common sense wants to invite the child c and the parents a, b.

Traditional Dung extensions semantics can invite only one parent from  $\{a, b\}$ and invite c. Even if we use the Grossi Modgil d[m, n] approach, the set  $\{a, b\}$  is not conflict free. If we use our proposed approach with the attacks of Definition A.1, the parents set will be at peace, the uncle u will be out and c will be invited.

# B Appendix 2: Comparison with argumentation networks with numerical annotations

Our starting point is a network (S, R), with additional annotations to the nodes  $L(x), x \in S$ , where L(x) is any labelling function on S giving values from some labelling domain.



Figure 16

The labels L(x) can be used in the computation of extensions. The labels can restrict or supplement R in some way, or can be transferred to extensions. The following are some examples:

- 1. L(x) can be either "high" or "low" with the understanding that any xRy is disregarded if L(x) = "low" and L(y) = "high". (See [37].)
- 2. L(x) can be a numerical value (say in [0,1]) and we can disregard any set  $E \subseteq S$  provided that  $\Sigma_{x \in E} L(x) \leq \varepsilon$  where  $0 \leq \varepsilon < 1$ . (See [21].)
- 3. L(x) can be the many lives function M(x) as it is used as in this paper.
- 4. L(x) can be a probability distribution on S and it can be transferred as a probability distribution to extensions.
- 5. Note that in examples 2 and 3 the numerical value is used to modify the attack relation R and is not used as strength of attack. We could however use L(x)as part of the attack itself. Assume we have the nodes  $y_1, \ldots, y_k$  attacking the node x. The nodes z in S are annotated with a numerical number L(z). In the general case L(z) can be any real number. It is convenient to assume that the number L(z) is in the interval [0, 1]. We can thus adopt the view that L(z)represents the strength of attack which z can generate. 1 is full strength and 0 is no strength at all.
- 6. The most general point of view of numerical attacks was taken in papers [33] and [34]. In these papers we considered the most general uninterpreted case of numerical attacks. Assume that y attacks x, (that is  $y \rightarrow x$ ) with L(y) = a and

L(x) = b. Mathematically what we have here is two numbers, with one number a attacking another number b, namely  $a \rightarrow b$  and we asked what number b' we get as a result of this attack. The papers discuss various possibilities, among them the obvious b' = b(1-a). If we interpret a = 1 as true (or full strength) an a = 0 as false, we get compatibility with Dung like attacks. There is a connection with fuzzy logic and t-norms [35].

- 7. Note that the number annotation L(z) is used in the attack. Compared with the many lives approach, the number M(z) is not used in the attack. So if y has many lives M(y) = 100 and y is attacking x with M(x) = 2 then the attack strength of y on x is just 1. The number M(y) reflects resilience to attacks on y and not strength of y as an attacker on x.
- 8. We can imagine a prosecutor trying to decide, given the complaints of  $y_1, \ldots, y_k$ on x, whether to press charges on x by putting forward as evidence all the complaints of  $y_1, \ldots, y_k$  or perhaps (expecting a counter attack from x) by putting forward only the more resilient y's (with M(y) a large number). In this case we are using M(y) as a weight and we exclude the attack of y if M(y)does not pass a threshold. This is what paper [21] does, as described in item (2) above. Paper [21] also contains a comparison with papers [33] and [36].
- 9. The papers [33, 34] and [20, 21, 36] all use the numerical annotation in the attack. The qualitative difference between these papers and our current many lives paper is manifested technically in the handling of loops. Loops are welcome and are easier to handle in numerical attacks context, as they naturally lead to fixed point equations. Consider the situation of item (6) above, namely where  $y \rightarrow x$  and assume also that  $x \rightarrow y$ . Thus the attack relation is symmetrical. We use symmetrical R in Ecologies, where different species attack one another and we seek to identify states of equilibrium. (For analysis of networks with symmetrical R, see [19]). Also note that when an offender x is attacked by a victim y, the offender immediately attacks back, so the relation is always symmetrical). When numerical strength annotations are present, we solve equations. For example for x and y above we solve the two equations

$$b' = b(1 - a')$$
 and  $a' = a(1 - b')$ 

we get that the new equilibrium labels are

$$b' = b(1-a)/(1-ab)$$
 and  $a' = a(1-b)/(1-ba)$ 

10. The annotations can be quite complex, as in the paper [25]. We must be careful, however, to keep our systems closely related to application areas and

not embark on pure mathematical extensions. I think there is a connection with [31].

# HILBERT ALGEBRAS IN A NON-CLASSICAL FRAMEWORK: HILBERT ALGEBRAS WITH APARTNESS

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#### Abstract

In this article we observe and analyze Hilbert algebras in a non - classical principled - philosophical - logical framework: We establish the properties of Hilbert algebras within the frames of Bishop's constructive mathematics by considering the carriers of these algebraic structures as sets with apartness relation. In addition to redefining the concept of Hilbert algebras in this environment, we introduce and analyze the concepts of some specific substrates of these algebras, such as co-ideals and co-filters. The concept of co-congruence has been introduced and is associated with co-ideals and quotient algebras, also.

### 1 Introduction

If X is a non-empty set and  $w: X \times X \longrightarrow X$  is an internal binary operation in X, then the system (X, w) is a (simplest) algebraic structure recognizable by its name 'groupoid'. Recall that it is implied that an equality relation is present at X. Thus, the carrier of algebraic construction is the system (X, =). Since w is a total function on  $X \times X$ , that w must be extensive with respect to equality, i.e. the following

$$(\forall x, y, u, v \in X)((x, y) = (u, v) \Longrightarrow w(x, y) = w(u, v))$$

holds. Recall that every predicate  $\mathcal{P}$  determined on X must also be extensive with respect to equality, i.e. the following

$$(\forall x, y \in A)((\mathcal{P}(x) \land x = y) \Longrightarrow \mathcal{P}(y))$$

must be a valid formula. If the operation in X has some additional properties, then many questions arise about the characteristics of this more complex algebraic structure. An example of such a complex algebraic structure is Hilbert algebra. Since there are various modifications to the definition of Hilbert algebra, we will use the following determination: **Definition 1.1.** A *Hilbert algebra* is a triplet  $H = (H, \cdot, 1)$  where H is a nonempty set, ' $\cdot$ ' is a binary operation and '1' is a fixed element of H such that the following axioms hold:

(H-1)  $(\forall x, y \in H)(x \cdot (y \cdot x) = 1),$ (H-2)  $(\forall x, y, z \in H)(x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1),$  and (H-3)  $(\forall x, y \in H)((x \cdot y = 1 \land y \cdot x = 1) \Longrightarrow x = y).$ 

The concept of Hilbert algebra was introduced in early 50-ties by L. Henkin [15] and T. Skolem [36] for some investigations of implication in intuicionistic and other non-classical logics. In 60-ties, these algebras were studied especially by A. Horn [17] and A. Diego from algebraic point of view. A. Diego proved [12] that Hilbert algebras form a variety which is locally finite. Hilbert algebras were treated by D. Busneag [3, 4, 5], S. Celani [6], S. Celani and D. Montangie [7], W. A. Dudek [13, 14], J. B. Jun and K. H. Kim [20], Y. B. Jun, J. W. Nam and S. M. Hong [16, 18, 19] and A. S. Nasab and A. B. Saeid [22] for example.

I. Chajda and R. Halas introduced in [8] the concept of ideal in Hilbert algebra and described connections between such ideals and congruences. In article [13] W. A. Dudek described connections between such ideals and deductive systems. Besides, in [14] it is shown that the class of all Hilbert algebras can embedded into the class of all BCK-algebras.

Our first step is to introduce the notion 'Hilbert algebra with apartness'. In that intention we will transform the Definition 1.1 in the following definition equivalent to the first.

**Definition 1.2.** For a system  $((H, =), \cdot, 1)$ , we say that it is a *pre-Hilbert algebra* with apartness if the following formulas are axioms:

 $(\text{H-1}) \ (\forall x, y \in H)(x \cdot (y \cdot x) = 1),$ 

(H-2)  $(\forall x, y, z \in H)((x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1)$ , and (H-3<sup>\neq</sup>)  $(\forall x, y \in H)(x \neq y \implies (x \cdot y \neq 1 \lor y \cdot x \neq 1)).$ 

Let's change now the logical framework within which we look at the construction of this algebraic structure. We will assume that the logical environment of algebraic constructions is the Intuitionistic Logic IL (see, for example [35, 37]) instead of the Classic Logic CL. In addition, in the observation and analysis of the algebraic constructions determined by the axioms of Definition 1.2 we will take into account the principled-philosophical orientation of the Bishop's constructive mathematics **Bish** (see, for example [1, 2, 21, 31]). By accepting these commitments, the relation  $' \neq '$  is not the negation of the equality relation ' = ' but now it is a particular relation in set H. Therefore,  $(H, =, \neq)$  should be regarded as a relational system. For a relation  $\neq$  on a set H we say that is a *difference relation* if the following holds

 $(\forall x \in H) \neg (x \neq x)$  (consistent) and

 $(\forall x, y \in H)(x \neq y \implies y \neq x)$  (symmetry).

For the difference relation  $\neq$  on H, it is called that is an *apartness* if

 $(\forall x, y, z \in H) (x \neq z \implies (x \neq y \lor y \neq z))$  (co-transitivity)

holds also. In what follows we will treat algebraic forms built on such relational system  $(H, =, \neq)$  as carrier of these constructions. Previous analysis is justification for using the term 'algebra with apartness'. Additionally, all the relations and operations that appear in this text will be strongly extensional with respect to the apartness. Therefore, the internal binary operation mentioned in the Definition 1.2 is strong extensional in the following sense

 $(\forall x, y, x', y' \in H)(x \cdot y \neq x' \cdot y' \implies (x \neq x' \lor y \neq y')).$ 

The preceding implication is equivalent to the following formula

 $(\text{H-0}) \ (\forall x, y, z \in H)((x \cdot z \neq y \cdot z \lor z \cdot x \neq z \cdot y) \Longrightarrow x \neq y).$ 

Our commitment to replace the axiom (H-3) with the axiom  $(H-3^{\neq})$  makes it impossible to obtain validity of formulas by deductions using this first axiom. To illustrate, since formulas (1) - (4) in Proposition 2.1 are obtained by applying axioms (H-3), the question arises as to their validity in Hilbert algebras with apartness. The problems we are facing now are:

- How to determine the concept of Hilbert algebras with apartness?

- How to determine objects in Hilbert algebras with apartness?

- Link the newly constructed objects with the corresponding objects that exist in these algebras in the classical case.

- Describe objects in Hilbert algebras with apartness that do not have their counterpart in the classical case.

The change in the logical environment and the acceptance of the principled - philosophical orientation of the Bishop's constructive mathematics considerably change the observation aspects of the Hilbert algebras. In addition, the new framework of these algebraic constructions is based on sets with apartness. Finally, let's point out that the rules of deductions are changing as well. It is completely acceptable within the philosophical paradigm and logically it is justified to be interested in perceiving, understanding and describing the consequences that these mentioned changes produce.

In the next proposition we show that if the apartness is a tight relation, i.e. if

 $(\forall \, x,y \in H)(\neg(x \neq y) \implies x = y)$ 

holds, then  $(H-3^{\neq})$  implies (H-3).

**Proposition 1.1.** Let  $((H, =, \neq), \cdot, 1)$  be a pre-Hilbert algebra with a tight apartness. Then  $(H-3^{\neq})$  implies (H-3).

*Proof.* Let  $x, y \in H$  be arbitrary elements such that  $x \cdot y = 1$  and  $y \cdot x = 1$ . If there were  $x \neq y$ , we would have  $\neg(x \cdot y = 1) \land \neg(y \cdot x = 1)$  by  $(H-3^{\neq})$ , which is in contradiction with the hypothesis. Therefore, it must be  $\neg(x \neq y)$ . From here it follows x = y because the relation  $\neq$  is a tight relation.  $\Box$ 

Since, in general case, the apartness relation is not a tight relation in H, we conclude that the algebraic structures  $((H, =, \neq), \cdot, 1)$  and  $(H, =, \cdot, 1)$  differ in the formulas in which appears the axiom  $(H-3^{\neq})$  instead of the axiom (H-3), or they are the result of deduction using this axiom.

Algebraic structures developed on sets with apartness relation have been in the focus of many authors for a long time (see, for example [9, 10, 11, 34]). While authors S. Crvenković, M. Mitrović and D. A. Romano investigate semigroups with apartness in the articles [10, 11], A. Cherubini and A. Frigeri deal with inverse semigroups with apartness in the article [9]. D. A. Romano has deals with one particular class of substructures in semigroups with aprtness - a class of co-filters in article [34]. A brief recapitulation of his research on the various classes of algebraic structures with apartness relation it can be found in [31].

In this article the author describe insights into the internal structure of Hilbert algebras with apartness. The specificity of the logical background in which the substructures in Hilbert algebra with apartness are observed and analyzed, provides the identification, determination and insight into the properties of some special substructures and processes that do not have the counterparts in the classical case.

Should there be an academic interest in exploring such algebraic structures with apartness relation? - is a completely natural question that is posed in itself. The second question posed by itself is: Would the results obtained be interesting to a number of mathematicians?

The answer to the first question should always be affirmative. For the academic community, it should be interesting to find out how from some given suppositions deduce consequences in the framework of a chosen logical system in an acceptable way.

This research is a continuation of the author research initiated by the investigation of BCC-algebra with apartness in [33].

# 2 Preliminaries

This section presents some well-known definitions of concepts and processes in Hilbert algebras, and several claims about the properties of these objects and processes so that they can be compared with objects and processes that will be designed later in this article. This algebraic structure has the following properties

**Proposition 2.1** ([12]). Let  $(H, \cdot, 1)$  be a Hilbert algebra. Then:

- (1)  $(\forall x \in H)(x \cdot x = 1),$ (2)  $(\forall x \in H)(1 \cdot x = x),$
- $(2) (\forall x \in H)(1 \quad x = x),$
- $(3) \ (\forall x \in H)(x \cdot 1 = 1),$
- (4)  $(\forall x, y, z \in H)(x \cdot (y \cdot z) = y \cdot (x \cdot z)).$

It is easily checked that in a Hilbert algebra H the relation  $\leq$  defined by

 $(\forall x, y \in H) (x \leqslant y \iff x \cdot y = 1)$ 

is a partial order on H with 1 as the largest element.

**Proposition 2.2** ([4, 12, 13]). Let H be a Hilbert algebra. Then

 $(5) (\forall x \in H)(x \leq x),$   $(6) (\forall x \in H)(x \leq 1),$   $(7) (\forall x, y \in H)(x \leq y \cdot x),$   $(8) (\forall x, y \in H)(x \leq (x \cdot y) \cdot x),$   $(9) (\forall x, y, z \in H)(x \cdot y \leq (y \cdot z) \cdot (x \cdot z)),$   $(10) (\forall x, y, z \in H)(y \cdot z \leq (x \cdot y) \cdot (x \cdot z)),$   $(11) (\forall x, y, z \in H)(x \leq y \Longrightarrow (z \cdot x \leq z \cdot y \land y \cdot z \leq x \cdot z)).$ 

**Definition 2.1.** Let H be a Hilbert algebra and S be a subset of H.

The subset S is a subalgebra of H if the following axioms are satisfied: (Sub1)  $1 \in S$  and (Sub2)  $(\forall x, y \in H)((x \in S \land y \in S) \Longrightarrow x \cdot y \in S)$ . (a) The subset S is an *ideal* of H if the following axioms are satisfied: (J1)  $1 \in S$  and (J2)  $(\forall x, y \in H)((x \in S \land x \cdot y \in S) \Longrightarrow y \in S)$ . (b) The subset S is a *filter* of H if the following axioms are satisfied: (J1)  $1 \in S$  and

(F2)  $(\forall x, y \in H)((y \in S \land x \cdot y \in S) \Longrightarrow x \in S).$ 

**Proposition 2.3.** Let H be a Hilbert algebra.

(i) If J is an ideal of H, then
(12) (∀x, y ∈ H)((x ≤ y ∧ x ∈ J) ⇒ y ∈ J).
(ii) If F is a filter of H, then
(13) (∀x, y ∈ H)((x ≤ y ∧ y ∈ F) ⇒ x ∈ F).

**Definition 2.2.** If  $H_1$  and  $H_2$  are Hilbert algebras, then  $f : H_1 \longrightarrow H_2$  is called morphism of Hilbert algebras if

$$(\forall x, y \in H_1)(f(x \cdot y) = f(x) \cdot f(y)).$$

The following definition ends this section.

**Definition 2.3.** Let  $(H, \cdot, 1)$  be a Hilbert algebra and  $\theta$  be an equivalence on the set H.  $\theta$  is a *congruence* on H if the following

$$(\forall x, y, u \in H)((x, y) \in \theta \implies ((x \cdot u, y \cdot u) \in \theta \land (u \cdot x, u \cdot y) \in \theta))$$

is a valid formula.

### **3** Properties of Hilbert algebras with apartness

First, let us specify one commitment in Bishop's constructive orientation algebra.

Let X be an object (or process) in some classical algebraic structure S determined by a predicate  $\mathcal{P}$ . The dual Y of this concept (or process) in the algebraic structure  $(S, =, \neq)$  with apartness is an object (or process) designed so that the strong compliment  $Y^{\triangleleft}$  of Y in S also satisfies the predicate  $\mathcal{P}$ .

In this case, the objects X and Y are said to be *associated* (or *mutually consistent*). Of course, it is not rare that a classically defined object and its constructive dual are not associates.

Second, let's explain the terms used. For subsets A and B of structure S, we write  $A \bowtie B$  if  $(\forall a \in A)(\forall b \in B)(a \neq b)$  holds. In particular, for  $A = \{x\}$ , we write  $x \triangleleft B$  instead of  $\{x\} \bowtie B$ . Also, we write  $x \neq y$  instead of  $\{x\} \bowtie \{y\}$ . The subset  $\{x \in S : x \triangleleft B\}$  is a strong complement of B in S and it is denoted by  $B^{\triangleleft}$ .

Let us add two explanations to the previous one.

A predicate  $\mathcal{P}$  on structure  $(S, =, \neq)$  is strongly extensional if the following

$$(\forall x, y \in S)(\mathcal{P}(x) \Longrightarrow (y \neq x \lor \mathcal{P}(y)))$$

holds. For example, A is a strongly extensional subset in S, if

$$(\forall x, y \in S)(x \in A \implies (y \in A \lor y \neq x))$$

holds.

Let  $f: A \times ... \times A \longrightarrow B$  be a function.

- f is a strongly extensional if holds

$$(\forall x_1, ..., x_n, y_1, ..., y_n \in A)(f(x_1, ..., x_n) \neq f(y_1, ..., y_n) \implies \bigvee_{i=1}^n (x_i \neq y_i)).$$

Hereinafter it is referred to as 'se-mapping' or 'se-function'.

- f is an *embedding* if holds

$$(\forall x_1, ..., x_n, y_1, ..., y_n \in A) (\bigvee_{i=1}^n (x_i \neq y_i) \implies f(x_1, ..., x_n) \neq f(y_1, ..., y_n)).$$

All definitions of newly introduced concepts and processes with them in this section are the result of this author's own reflections. Also, all claims made in this section are designed and proven by the author.

#### 3.1 Concept of co-subalgebras

**Definition 3.1.** A strongly extensional subset S of a pre-Hilbert algebra with apartness H is a *co-subalgebra* if the following

(S1)  $\neg (1 \in S)$  and

 $(S2) (\forall x, y \in H) (x \cdot y \in S \implies (x \in S \lor y \in S))$ 

are valid.

Speaking language of classical algebra, the set S is a co-subalgebra in H if it is a consistent subset in H.

**Proposition 3.1.** If S is an inhabited co-subalgebra in a pre-Hilbert algebra with apartness H, then  $S^{\triangleleft}$  is a subalgebra in H.

*Proof.* Since S is an inhabited strongly extensional subset of H, then

$$u \in S \implies (u \neq 1 \lor 1 \in S)$$

holds. The second option is impossible because of the hypothesis  $\neg(1 \in S)$ . So  $1 \neq u \in S$  for every  $u \in S$ . This means  $1 \triangleleft S$ , i.e.  $1 \in S^{\triangleleft}$ .

Let  $x, y, u \in H$  be elements such that  $x \in S^{\triangleleft}$ ,  $y \in S^{\triangleleft}$  and  $u \in S$ . Then  $x \cdot y \neq u \in S$  or  $x \cdot y \in S$  by strongly extensionality of S in H. The second option gives  $x \in S \lor y \in S$  which contradicts the hypothesis. Thus, there must be  $x \cdot y \neq u$  for each  $u \in S$ . This proves  $x \cdot y \in S^{\triangleleft}$ .

This proves that  $S^{\triangleleft}$  is a subalgebra in the pre-Hilbert algebra H.

In the previous proposition, it is shown that the concepts of subalgebra and co-subalgebras in a pre-Hilbert algebra with apartness are mutual consistent.

It is clear that the sets H and  $\emptyset$  are co-subalgebras in H. Thus, the family  $\mathfrak{S}(H)$  of all co-subalgebras of algebra H is not empty. The assertion of the following theorem can be proved by direct verification.

**Theorem 3.1.** The family  $\mathfrak{S}(H)$  forms a complete lattice.

*Proof.* Let  $\{S_i\}_{i \in I}$  be a family of co-subalgebras in a pre-Hilbert algebra H. Clearly that  $1 \triangleleft \bigcup_{i \in I} S_i$  and  $1 \triangleleft \bigcap_{i \in I} S_i$  valid.

Let  $x, y \in H$  be elements such that  $x \cdot y \in \bigcup_{i \in I} S_i$ . The there is an index  $k \in I$ such that  $x \cdot y \in S_k$ . Thus  $x \in S_k \subseteq \bigcup_{i \in I} S_i$  or  $y \in \bigcup_{i \in I} S_i$  according to (S2). Since  $\bigcup_{i \in I} S_i$  satisfies conditions (S1) and (S2), therefore  $\bigcup_{i \in I} S_i$  is a co-subalgebra in H.

Let  $\mathfrak{X}$  be a family of all co-subalgebra of the pre-Hilbert algebra H contained in  $\bigcap_{i \in I} S_i$ . Then  $\cup \mathfrak{X}$  is the maximum co-subalgebra included in  $\bigcap_{i \in I} S_i$  according to the second part of this evidence.

If we put  $\sqcup_{i \in I} S_i = \bigcup_{i \in I} S_i$  and  $\sqcap_{i \in I} S_i = \bigcup \mathfrak{X}$ , then  $(\mathfrak{S}(H), \sqcup, \sqcap)$  is a complete lattice.

**Corollary 3.1.** For any subset of A of a pre-Hilbert algebra with apartness H, there is the maximal co-subalgebra contained in A.

*Proof.* The evidence of this corollary follows from the second part of the proof of Theorem 3.1.  $\hfill \Box$ 

**Corollary 3.2.** For any element a of a pre-Hilbert algebra with apartness H, there is the maximal co-subalgebra  $H_a$  such that  $a \triangleleft H_a$ .

*Proof.* The proof of this Corollary follows immediately from the previous Corollary if we take  $A = \{x \in H : x \neq a\}$ .

#### 3.2 Co-order relation in pre-Hilbert algebra with apartness

Co-order relation in sets with apartness was introduced in 1996 in [25] by the author. The relation  $' \leq '$  on the set with apartness H is a co-order in H if it is consistent, co-transitive and linear in the following sense:

 $\begin{array}{l} (\forall x, y \in H)(x \leqslant y \implies x \neq y) \text{ (consistency)} \\ (\forall x, y, z \in H)(x \leqslant z \implies (x \leqslant y \lor y \leqslant z)) \text{ (co-transitivity) and} \\ (\forall x, y \in H)(x \neq y \implies (x \leqslant y \lor y \leqslant x)) \text{ (linearity).} \end{array}$ 

If a set with apartness is supplied with a co-order relation, then it is said to be 'ordered under co-order' or it is 'co-ordered set'.

**Definition 3.2.** For a co-order relation  $' \notin '$  on a pre-Hilbert algebra with apartness  $((H, =, \neq), \cdot, 1)$  says that it is *compatible* with the internal operation in H if

(14)  $(\forall x, y, z \in H)(x \cdot z \leq y \cdot z \Longrightarrow x \leq y)$  (right compatibility)

(15)  $(\forall x, y, z \in H)(z \cdot x \leq z \cdot y \Longrightarrow x \leq y)$  (left compatibility)

are valid formulas.

Speaking by the classical algebra language, the co-order relation in is compatible with the operation if the operation is left and right cancellative with respect to the co-order.

**Lemma 3.1.** Formulas (14) and (15) are equivalent to the following formula (16)  $(\forall x, y, z, u \in H)(x \cdot z \notin y \cdot u \Longrightarrow (x \notin y \lor z \notin u)).$ 

*Proof.* (14)  $\wedge$  (15)  $\implies$  (16). Let  $x, y, z, u \in H$  be arbitrary elements such that  $x \cdot z \notin y \cdot u$ . Then  $x \cdot z \notin y \cdot z$  or  $y \cdot z \notin y \cdot u$  by co-transitivity of  $\notin$ . Thus  $x \notin y \vee z \notin u$  by (14) and (15).

 $(16) \implies (14) \land (15)$ . If we put u = z in (16), we immediately obtain (14). If we put x = z, z = x, y = z and u = y in (16), we immediately obtain (15).

In the following definition, we introduce the relation  $' \nleq '$  on a pre-Hilbert algebra with apartness  $((H, =, \neq), \cdot, 1)$ .

**Definition 3.3.** The relation  $' \notin '$  on a pre-Hilbert algebra with apartness H we introduce by the following formula

$$(\forall x, y \in H) (x \leq y \iff x \cdot y \neq 1).$$

The claims in the following Lemma are obvious

Lemma 3.2. Let H be a pre-Hilbert algebra. Then

(17)  $(\forall x, y \in H) \neg (x \cdot y = 1 \land x \cdot y \neq 1)$ , and

(18)  $(\forall x, y \in H)(x \cdot y = 1 \implies \neg(x \notin y)).$ 

Lemma 3.3. Let H be a pre-Hilbert algebra. Then

(19)  $(\forall x, y \in H) \neg (x \leq y \cdot x)$ , and

(20)  $(\forall x, y, z \in H)(\neg (x \cdot (y \cdot z) \leq (x \cdot y) \cdot (x \cdot z))).$ 

*Proof.* Claims (19) and (20) follow from the previous Lemma with reference to Definition 3.3 and (H-1) and (H-2) respectively.  $\Box$ 

Analysis 3.1. Our intention with relation  $' \leq '$ , determined on this way, is that it should be a co-order relation H compatible with the operation in H.

The axiom  $(H-3^{\neq})$  immediately gives  $x \neq y \implies (x \notin y \lor y \notin x)$ . So, the linearity condition is satisfied for this relation.

For the condition of consistency to be valid, it should be  $x \leq y \implies x \neq y$  for any  $x, y \in H$ . In order to obtain the consistency of the relation  $' \leq '$ , it is sufficient that the following formula

(H-4)  $(\forall x \in H)(x \cdot x = 1)$ 

be a valid formula in a pre-Hilbert algebra with apartness H.

To obtain the co-transitivity property for the relation  $' \notin '$ , we should have  $x \notin z \implies (x \notin y \lor y \notin z)$  for any elements  $x, y, z \in H$ . Suppose that

(H-5)  $(\forall x \in H)(1 \cdot x = x)$  and

(H-6)  $(\forall x \in H)(x \cdot 1 = 1)$ 

are valid formulas in the structure  $(H, =, \neq), \cdot, 1$ ). Then, for any  $x, y, z \in H$  such that  $x \notin z$ , i.e. such that  $x \cdot z \neq 1$ , we have

 $1 \cdot (x \cdot z) \neq (x \cdot y) \cdot (z \cdot z) \lor (x \cdot y) \cdot (z \cdot z) \neq (x \cdot y) \cdot 1.$ 

by co-transitivity of apartness relation and with respect to (H-5) and (H-6). Thus

 $1 \neq x \cdot y \lor x \cdot z \neq 1$ 

by (H-0). So, we have  $x \leq y \lor y \leq z$ . Therefore, accepting hypotheses (H-5) and (H-6), we get that  $' \leq '$  is a co-transitive relation in H.

**Lemma 3.4.** Let *H* be a pre-Hilbert algebra with apartness. Then (H-1) and (H-5) implies (H-4).

*Proof.* If we put y = 1 in (H-1), we get (H-4) with respect to (H-5).

**Lemma 3.5.** Let H be a pre-Hilbert algebra with apartness. Then (H-1) and (H-5) implies (H-6).

*Proof.* If we put x = 1 and y = x in (H-1), we get (H-6) with respect to (H-5).

Summarizing the previous analysis and results of Lemma 3.4 and Lemma 3.5, we conclude:  $' \leq '$  is a co-order relation on a structure of pre-Hilbert algebra with apartness  $((H, =, \neq), \cdot, 1)$  if (H-4), (H-5) and (H-6) are valid formulas in that structure. The foregoing justifies the following definition

**Definition 3.4.** A pre-Hilbert algebra with apartness  $((H, =, \neq), \cdot, 1)$  is a *Hilbert algebra with apartness* if it additionally satisfies the axiom (H-5).

**Proposition 3.2.** Let H be a Hilbert algebra with apartness. Then

 $\begin{array}{l} (5^{\neq}) \ (\forall x \in H)(x \leqslant^{\triangleleft} x); \\ (6^{\neq}) \ (\forall x \in H)(x \notin^{\triangleleft} 1); \\ (7^{\neq}) \ (\forall x, y \in H)(x \notin^{\triangleleft} y \cdot x); \\ (8^{\neq}) \ (\forall x \in H)(x \notin^{\triangleleft} (x \cdot y) \cdot x); \\ (10^{\neq}) \ (\forall x, y, z \in H)(y \cdot z \notin^{\triangleleft} (x \cdot y) \cdot (x \cdot z)). \end{array}$ 

*Proof.*  $(5^{\neq})$  Let  $x, u, v \in H$  be arbitrary elements such that  $u \notin v$ . Then  $u \notin x \lor x \notin v$  by co-transitivity of  $\notin$ . Thus  $u \neq x \lor x \neq v$  by consistency of  $\notin$  and  $(x, x) \neq (u, v) \in \notin$ . Hence  $x \notin^{\triangleleft} x$ .

 $(6^{\neq})$  The proof is derived by same way as the proof for  $(5^{\neq})$ .

 $(7^{\neq})$  Let  $x, y, u, v \in H$  be arbitrary elements such that  $u \notin v$ . Then

$$u \not\leqslant x \lor x \not\leqslant y \cdot x \lor y \cdot x \notin v$$

by co-transitivity of  $\leq$ . Since the second option is impossible by (19), we have  $(x, y \cdot x) \neq (u, v) \in \leq$ . Hence  $x \leq \forall y \cdot x$ .

 $(8^{\neq})$  is obtained directly from  $(7^{\neq})$  if we put  $y = x \cdot y$ .

 $(10^{\neq})$  Let  $x, y, z, u, v \in H$  be arbitrary element such that  $u \notin v$ . Then

$$u \nleq y \cdot z \lor y \cdot z \nleq x \cdot (y \cdot z) \lor x \cdot (y \cdot z) \nleq (x \cdot y) \cdot (x \cdot z) \lor (x \cdot y) \cdot (x \cdot z) \notin v$$

by co-transitivity of  $\leq$ . Since the second and third options are impossible by (19) and (20), we have  $u \neq y \cdot z$  or  $(x \cdot y) \cdot (x \cdot z) \neq v$  by consistency of  $\leq$ . Hence  $y \cdot z \leq^{\triangleleft} (x \cdot y) \cdot (x \cdot z)$ .

In the following proposition we show that the relation  $' \nleq '$  is left compatible with the operation in a Hilbert algebra with apartness H.

**Proposition 3.3.** Let H be a Hilbert algebra with apartness. Then (21)  $(\forall x, y, z \in H)(z \cdot x \leq z \cdot y \implies x \leq y).$ 

*Proof.* Let  $x, y, z \in H$  be such that  $z \cdot x \nleq z \cdot y$ . Then  $(z \cdot x) \cdot (z \cdot y) \neq 1$ . Thus

$$(z \cdot x) \cdot (z \cdot y) \neq (z \cdot (x \cdot y) \cdot ((z \cdot x) \cdot (z \cdot y))) \lor (z \cdot (x \cdot y) \cdot ((z \cdot x) \cdot (z \cdot y)) \neq 1.$$

The second option is impossible because of (H-2). From the first option in the form

$$1 \cdot ((z \cdot x) \cdot (z \cdot y)) \neq (z \cdot (x \cdot y)) \cdot ((z \cdot x) \cdot (z \cdot y)),$$

we get  $1 \neq z \cdot (x \cdot y)$ . Thus  $z \cdot 1 = 1 \neq z \cdot (x \cdot y)$  and  $1 \neq x \cdot y$ . Hence  $x \notin y$ .

Our intention is for the relation  $' \leq '$  to be right compatible with the operation in any Hilbert algebra with apartness H. To this end, we will assume that (4) is a valid formula in a Hilbert algebra with apartness H. With such a supplemented request for a Hilbert algebra with apartness, we have

**Proposition 3.4.** Let H be a Hilbert algebra with apartness in which (4) is a valid formula. Then

(22)  $(\forall x, y, z \in H)(y \cdot z \notin x \cdot z \Longrightarrow x \notin y).$ 

*Proof.* Let  $x, y, z \in H$  be elements such that  $y \cdot z \nleq x \cdot z$ . Then  $(y \cdot z) \cdot (x \cdot z) \neq 1$ . Thus

$$(y \cdot z) \cdot (x \cdot z) \neq (x \cdot y) \cdot ((y \cdot z) \cdot (x \cdot z)) \quad \lor \ (x \cdot y) \cdot ((y \cdot z) \cdot (x \cdot z)) \neq 1.$$

From the second option  $(x \cdot y) \cdot ((y \cdot z) \cdot (x \cdot z)) \neq 1$  we have  $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) \neq 1$ by (4). This means  $y \cdot z \notin (x \cdot y) \cdot (x \cdot z)$ . From here it follows

$$y \cdot z \not\leqslant x \cdot (y \cdot z) \ \lor \ (x \cdot (y \cdot z)) \notin (x \cdot y) \cdot (x \cdot z)$$

according to the co-transitivity of  $\leq$  relation. Part of  $y \cdot z \leq x \cdot (y \cdot z)$  is impossible according to  $(7^{\neq})$ . The second part means  $(x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) \neq 1$  which is also impossible according to (H-2). In both of these cases we have a contradiction. So  $(y \cdot z) \cdot (x \cdot z) \neq (x \cdot y) \cdot ((y \cdot z) \cdot (x \cdot z))$  must be valid. We can write it in the form

$$1 \cdot ((y \cdot z) \cdot (x \cdot z)) \neq (x \cdot y) \cdot ((y \cdot z) \cdot (x \cdot z))$$

in accordance with (H-5). From here we get  $1 \neq x \cdot y$ . Finally, we got  $x \leq y$ , which was to prove.

If we combine the results obtained in Analysis 3.1, Lemma 3.4, Lemma 3.5, Proposition 3.3 and Proposition 3.4, we obtain

**Theorem 3.2.** If H is a Hilbert algebra with apartness in which (4) is a valid gormula, then  $' \leq '$  is a co-order relation in H left compatible and right inverse compatible with respect to the internal operation in H.

*Proof.* The relation  $' \notin '$ , defined by Definition 3.3, is a co-order relation on the set  $(H, =, \neq)$  by comment before Definition 3.4. The relation  $' \notin '$  is a left compatible co-order with respect to the internal operation in H according to Proposition 3.3. Finally,  $' \notin '$  is a co-order relation in H and it is inverse right compatible with respect to the internal operation in H provided that (4) is a valid formula in H, according to Proposition 3.4.

#### 3.3 Concept of co-ideals

The idea of co-ideals in algebraic structures with apartness relation first appeared in Ruitenburg's doctoral dissertation [35]. This idea can also be found in Chapter 8: 'Algebra' of the famous book [37]. The analysis of concepts of co-ideals in semigroups, groups and rings with apartness was in the focus of this author, also (for example [23, 24, 30, 34]). An interested reader about this concepts in algebraic structures with apartness can be found in review article [31].

The following definition introduces the concept of co-ideals in Hilbert algebra with apartness.

**Definition 3.5.** Let H be a Hilbert algebra with apartness. A subset K of H is a *co-ideal* in H if

- (K1)  $1 \triangleleft K$ ; and
- (K2)  $(\forall x, y \in H)(y \in K \implies (x \in K \lor x \cdot y \in K)).$

**Lemma 3.6.** A co-ideal K a Hilbert algebra with apartness H is a strongly extensional subset in H.

*Proof.* Let  $x, y \in H$  be elements such that  $y \in K$ . Then  $x \in K$  or  $x \cdot y \in K$ . The second option give us  $x \cdot y \neq 1 = y \cdot y$  because  $1 \triangleleft K$  and (H-4). Thus  $x \in K \lor x \neq y$  by (H-0).

**Proposition 3.5.** If K is a co-ideal of a Hilbert algebra with apartness H, then the set  $K^{\triangleleft}$  is an ideal in H.

*Proof.* Let  $x, y, u \in H$  be arbitrary elements such that  $x \in K^{\triangleleft}$ ,  $x \cdot y \in K^{\triangleleft}$  and  $u \in K$ . Then  $u \neq y \lor y \in K$  by Lemma 3.6. From the second option we have  $x \cdot y \in K \lor x \in K$ . Hence  $x \neq u \in K$  because the both options  $x \cdot y \in K$  and  $x \in K$  are impossible by hypothesis. So, we conclude that  $t \in K^{\triangleleft}$ . Therefore, the subset  $K^{\triangleleft}$  satisfies (J1) and (J2) in the Definition 2.1.

**Proposition 3.6.** Let K be a co-ideal of a Hilbert algebra with apartness H. Then (23)  $(\forall x, y \in H)(y \in K \implies (x \leq y \lor x \in K)).$ 

*Proof.* Let  $x, y \in H$  be such  $y \in K$ . Then  $x \cdot y \in K$  or  $x \in K$  by (K2). Thus, the option give us  $x \cdot y \neq 1 = y \cdot y$  by (K1) and (H-4). So, this Proposition is proved.  $\Box$ 

Since sets  $\emptyset$  and  $\{x \in H : x \neq 1\}$  are co-ideals in any Hilbert algebra with apartness H, the family  $\mathfrak{K}(H)$  of all co-ideals of H is not empty.

**Theorem 3.3.** The family  $\mathfrak{K}(H)$  of all co-ideals of any Hilbert algebra with apartness H forms a complete lattice.

*Proof.* Let  $\{K_i\}_{i\in I}$  be a family of co-ideal of H. It is obvious that  $1 \triangleleft \bigcup_{i\in I} K_i$ because  $1 \triangleleft K_i$  for any  $i \in I$ . Let  $x, y \in H$  be such  $y \in \bigcup_{i\in I} K_i$ . Then there exists an index  $j \in I$  such that  $y \in K_j$ . Thus  $x \cdot y \in K_j \subseteq \bigcup_{i\in I} K_i$  or  $x \in K_j \subseteq \bigcup_{i\in I} K_i$ because  $K_j$  is a co-ideal of H.

Also, it is clear that  $1 \triangleleft \bigcap_{i \in I} K_i$ . If  $\mathfrak{T}$  is the family of all co-ideals in H contained in  $\bigcap_{i \in I} K_i$ , then  $\cup \mathfrak{T}$  is the maximal co-ideal of H contained in  $\bigcap_{i \in I} K_i$  according to the first part of this proof.

If we put  $\sqcup_{i \in I} K_i = \bigcup_{i \in I} K_i$  and  $\sqcap_{i \in I} K_i = \bigcup \mathfrak{T}$ , then  $(\mathfrak{K}(H), \sqcup, \sqcap)$  is a complete lattice.

#### 3.4 Concept of co-filters

The concept of co-filters in co-ordered sets and algebraic structures with apartness has been introduced and analyzed by this author in several of his articles (for example: [25, 30, 31, 33]).

The following definition introduces the notion of co-filters in Hilbert algebras with apartness.

**Definition 3.6.** Let H be a Hilbert algebra with apartness. A subset G of H is a *co-filter* of H if the following hold

(G1)  $1 \triangleleft G$ , and

(G2)  $(\forall x, y \in H)(x \in G \implies (x \cdot y \in G \lor y \in G)).$ 

**Lemma 3.7.** A co-filetr in a Hilbert algebra with apartness H is a strongly extensional subset in H.

*Proof.* Let  $x, y \in H$  be elements such that  $x \in G$ . Then  $y \in K$  or  $x \cdot y \in G$ . The second option give us  $x \cdot y \neq 1 = y \cdot y$  because  $1 \triangleleft K$  and (H-4). Thus  $x \in K \lor x \neq y$  by (H-0).

**Proposition 3.7.** If G is a co-filter of a Hilbert algebra with apartness H, then the set  $G^{\triangleleft}$  is a filter of H.

*Proof.* It is clear that  $1 \in G^{\triangleleft}$ . Let  $x, y, u \in H$  be elements such that  $y \in G^{\triangleleft}$ ,  $x \cdot y \in G^{\triangleleft}$  and  $u \in G$ . From  $u \in G$  it follows  $u \neq x \lor x \in G$  by by consistency of G in H. Thus from the second option  $x \in G$ , we have  $x \cdot y \in G$  or  $y \in G$ . Both obtained cases contradict to the hypotheses  $y \in G^{\triangleleft}$  and  $x \cdot y \in G^{\triangleleft}$ . So, have to be  $x \neq u \in G$ . This means  $x \in G^{\triangleleft}$ . Thus, the set  $G^{\triangleleft}$  satisfies conditions (F1) and (F2) in Definition 2.1(b).

**Proposition 3.8.** Let G be a co-filter of a Hilbvert algebra with apartness H. Then (24)  $(\forall x, y \in H)(x \in G \implies (x \leq y \lor y \in G)).$ 

*Proof.* Let  $x, y \in H$  be such  $x \in G$ . Then  $x \cdot y \in G$  or  $y \in G$  by (G2). The first option gives  $x \cdot y \neq 1$  by (G1). Thus  $x \notin y \lor y \in G$ .

The following theorem can be proved in an analogous way as the proof of Theorem 3.3, so we will therefore omit its evidence.

**Theorem 3.4.** The family  $\mathfrak{G}(H)$  of all co-filters a Hilbert algebra with apartness H form a complete lattice.

#### 3.5 se-homomorphisms of Hilbert algebras with apartness

**Definition 3.7.** Let  $f : ((H, =_H, \neq_H), \cdot_H, 1_H) \longrightarrow ((M, =_M, \neq_M), \cdot_M, 1_M)$  be a se-mapping between Hilbert algebras with apartness. f is a se-homomorphism of Hilbert algebras with apartness if the following hold

(se-h-1)  $f(1_H) =_M 1_M$ ; and (se-h-2)  $(\forall x, y \in H)(f(x \cdot_H y) =_M f(x) \cdot_M f(y)).$ 

Remark 3.1. From (se-h-2) immediately it follows (se-h-1). Indeed. If we put  $y =_H x$  in (se-h-2) we get  $f(x \cdot_H x) =_M f(x) \cdot_M f(x)$  and hence  $f(1_H) =_M 1_M$  with respect to (H-4).

**Lemma 3.8.** Let f be a a se-homomorphism between Hilbert algebras with apartness. Then:

- (i) the set f(H) is a subalgebra of M;
- (ii) the set Ker(f) is an ideal of H; and

(iii) the set  $Coker(f) = \{x \in H : f(x) =_M 0_M\}$  is a co-ideal of H.

*Proof.* (i) is obvious.

(ii) Let  $x, y \in H$  be such  $x \in Ker(f)$  and  $x \cdot_H y \in Ker(f)$ . Then  $f(x) =_M 1_M$ and

$$1 =_M f(x \cdot_H y) =_M f(x) \cdot_M f(y) =_M 1_N \cdot_M f(y) =_M f(y).$$

Thus  $y \in Ker(f)$ . So, the set Ker(f) satisfies conditions (J1) and (J2).

(iii) Let  $x, y \in H$  be such that  $y \in Coker(f)$ . Then  $f(y) \neq_M 1_M$ . Thus  $1_M \cdot_M f(y) \neq_M f(x \cdot_H y) =_M f(x) \cdot_M f(y)$  or  $f(x \cdot_H y) \neq_M 1_M$  by co-transitivity of apartness in M. From the first option we get  $1_M \neq_M f(x)$  while from the second option we get  $c \cdot_H y \in Coker(f)$ . Since  $1_H \triangleleft Coker(f)$  is obvious, we have proved that the set Coker(f) satisfies conditions (K1) and (K2).

**Theorem 3.5.** Let  $f : H \longrightarrow M$  be a se-homomorphism between Hilbert algebras with apartness. Then

- (i) If K be a co-ideal of M, then  $f^{-1}(K)$  is a co-ideal of H; and
- (ii) If G is a co-filter of M, then  $f^{-1}(G)$  os a co-filter of H.

Proof. Let S be a co-ideal (co-filter) of M. Then  $1_M \triangleleft S$ . If  $u \in f^{-1}(S)$ , then  $f(u) \in S$ . Thus  $f(u) \neq 1_M =_M f(1_H)$  or  $1_M =_M f(1_H) \in S$  by strongly extensionality of S in M. From the first option we have  $u \neq_H 1_H$  because f is a se-mapping. The second option gives  $1_M \in S$ . As the second option leads to contradiction, it has to be  $1_H \neq_H u \in f^{-1}(S)$ . So,  $1_H \triangleleft f^{-1}(S)$ .

Let S be a co-ideal of M and let  $x, y \in H$  be elements such that  $y \in f^{-1}(S)$ . Then  $f(y) \in S$ . Thus  $f(x) \in S$  or  $f(x \cdot_H y) =_M f(x) \cdot_M f(y) \in S$  by (K2). Hence  $x \in f^{-1}(S)$  or  $x \cdot_H y \in f^{-1}(S)$ . This proves that the set  $f^{-1}(S)$  since satisfying conditions (K1) and (K2), is a co-ideal of H.

If S is a co-filter of M, then it can be analogously proved that it satisfies condition (F2). So,  $f^{-1}(S)$  is a co-filter of H.

#### 3.6 Co-congruence on Hilbert algebras with apartness

In this subsection, we will omit the affiliation indices unless this leads to imprecision.

In the following theorem, we describe the properties of relation  $\theta_f$ , defined by

$$(\forall x, y \in H)((x, y) \in \theta_f \iff f(x) \neq f(y)).$$

In this intention, we need some new notions.

A relation  $\theta$  on the set  $(H, =, \neq)$  is a *co-equivalence* on H if it is consistent, symmetric, and co-transitive in the following sense

 $\begin{array}{ll} (\forall x, y \in H)((x, y) \in \theta \implies x \neq y) & (\text{consistency}) \\ (\forall x, y \in H)((x, y) \in \theta \ (y, x) \in \theta) & (\text{symmetry}) \text{ and} \\ (\forall x, y, z \in H)((x, z) \in \theta \implies ((x, y) \in \theta \lor (y, z) \in \theta)) \text{ (co-transitivity).} \end{array}$ 

A co-equivalence  $\theta$  on an algebraic structure  $((H, =, \neq), \cdot)$  is *compatible* with the internal operation in H if the following hold

(c)  $(\forall x, y, z \in H)((x \cdot z, y \cdot z) \in \theta \implies (x, y) \in \theta)$  (right compatibility) and

(d)  $(\forall x, y, z \in H)((z \cdot x, z \cdot y) \in \theta \implies (x, y) \in \theta)$  (left compatibility).

It can be verified without difficulty that  $(a) \wedge (b)$  are equivalent to the following formula

(e)  $(\forall x, y, u, v \in H)((x \cdot u, y \cdot v) \in \theta \implies ((x, y) \in \theta \lor (u, v) \in \theta)).$ 

The foregoing analysis is justification for the following definition.

**Definition 3.8.** A co-equality relation  $\theta$  on a Hilbert algebra with apartness H is a *co-congruence* on H it is satisfies the condition (e).

The notion of co-equivalence on the set with apartness was first introduced and analyzed by the author in his dissertation in 1995 [23], and further developed in articles [25, 32]. Also, the term co-congruence on commutative rings with apartness was first introduced and analyzed by the author in his thesis [23] and article [24]. **Theorem 3.6.** Let  $f; H \longrightarrow T$  be a se-homomorphism of Hilbert algebras with apartness. The relation  $\theta_f$  on H is a co-congruence on H.

*Proof.* Clearly, the relation  $\theta_f$  is a co-equivalence on the set H.

Let's show that the relation  $\theta$  is compatible with the internal operation in H. Let  $x, y, u, v \in H$  be arbitrary elements such that  $(x \cdot u, y \cdot v) \in \theta_f$ . Then  $f(x \cdot u) \neq f(y \cdot v)$ . Thus  $f(x) \cdot f(u) \neq f(y) \cdot f(v)$ . Hence  $f(x) \neq f(y) \lor f(u) \neq f(v)$ . So,  $(x, y) \in \theta_f \lor (u, v) \in \theta_f$ . It is shown that the relation  $\theta_f$  satisfies condition (e).  $\Box$ 

**Proposition 3.9.** If  $\theta$  is a co-congruence on a Hilbert algebra with apartness H, then the relation  $\theta^{\triangleleft}$  is a congruence on H.

*Proof.* If  $\theta$  is a co-congruence on a Hilbert algebra with apartness H, then the relation  $\theta^{\triangleleft}$  is an equivalence on the set H by Proposition 1.1 in [31]. It remains to show that the relation  $\theta^{\triangleleft}$  is compatible with the internal operation in H. Let  $x, y, u, v, t, s \in H$  be elements such that  $(x, y) \triangleleft \theta$ ,  $(u, v) \triangleleft \theta$  and  $(y, s) \in \theta$ . Then

$$(t, x \cdot y) \in \theta \ \lor \ (x \cdot u, y \cdot v) \in \theta \ \lor \ (y \cdot v, s) \in \theta$$

by co-transitivity of  $\theta$ . Thus

$$t \neq x \cdot y \lor (x, y) \in \theta \lor (u, v) \in \theta \lor y \cdot v \neq s$$

by (e) and by consistency of  $\theta$ . As the options  $(x, y) \in \theta$  and  $(u, v) \in \theta$  contradict to the hypothesis, we get  $(x \cdot u, y \cdot v) \neq (t, s) \in \theta$ . Therefore,  $(x \cdot u, y \cdot v) \in \theta^{\triangleleft}$ .  $\Box$ 

It is well known that congruence forms classes of congruence with known traits. Also, a co-congruence forms classes whose traits have been described by this author in several of his texts (for example, see [25, 31, 33]).

**Lemma 3.9.** If  $\theta$  is a co-congruence on a Hilbert algebra with apartness H, then any class of  $\theta$  is a strongly extensional subset of  $H \times H$ .

*Proof.* Let  $x, y, z \in H$  be elements such that  $z \in \theta x$ . Then  $(x, z) \in \theta$ . Thus  $(x, y) \in \theta \lor (y, z) \in \theta \subseteq \neq$  by co-transitivity of  $\theta$ . Hence,  $y \in \theta x \lor y \neq z$ .  $\Box$ 

**Proposition 3.10.** If  $\theta$  is a co-congruence on a Hilbert algebra with apartness H, then the class  $1\theta$  is a co-ideal of H.

*Proof.* If  $u \in 1\theta$ , then  $(1, u) \in \theta$  gives  $1 \neq u$  by consistency of  $\theta$ . This means  $1 \triangleleft 1\theta$ . Let  $x, y \in H$  be arbitrary elements such that  $y \in 1\theta$ . Then  $(1, y) \in \theta$  gives  $(1, x \cdot y) \in \theta \lor (x \cdot y, 1 \cdot y) \in \theta$  by co-transitivity of  $\theta$  and (H-5). Thus  $x \cdot y \in 1\theta \lor (x, 1) \in \theta$  by (c). Finally, we have  $x \cdot y \in 1\theta \lor x \in 1\theta$ . This proves that  $1\theta$  is a co-ideal of H.

In what follows, we need the term 'co-quasiorder relation' on a set with apartness. A relation  $\neq$  on set with apartness H is a *co-quasiorder* on H if it is a consistent and co-transitive relation on the set H. The concept of co-quasiorder relations on sets (and on algebraic structures) with apartness was introduced and analyzed by this author in several of his papers (see: [25, 26, 27, 28, 29, 30, 31]).

**Proposition 3.11.** Let K be a co-ideal of a Hilbert algerba with apartness H. Then the relation  $\preceq$ , defined by

$$(\forall x, y \in H) (x \not\preceq y \iff x \cdot y \in K)$$

is a co-quasiorder relation on H left compatible and right reverse compatible with the operation in H.

*Proof.* Let  $x, y \in H$  be such  $x \not\preceq y$ . Then  $x \cdot y \in K$ . Thus  $x \cdot y \neq 1 = y \cdot y$  by (K1) and (H-5). Hence  $x \neq y$ .

Let  $x, y, z \in H$  be arbitrary elements such that  $x \not\preceq z$ . Then  $x \cdot z \in K$ . Thus  $x \cdot y \in K \lor (x \cdot y) \cdot (x \cdot z) \in K$  by (K2). From the second option follows  $y \cdot z \not\leq (x \cdot y) \cdot (x \cdot z) \lor y \cdot z \in K$  by (23). Since  $y \cdot z \not\leq (x \cdot y) \cdot (x \cdot z)$  is impossible due to  $(10^{\neq})$ , we have  $x \cdot y \in K \lor y \cdot z \in K$  in the ending. Therefore, we got  $x \not\preceq y \lor y \not\preceq z$ .

Let  $x, y, z \in H$  be elements such that  $z \cdot x \not\preceq z \cdot y$ . Then  $(z \cdot x) \cdot (z \cdot y) \in K$ . Thus  $x \cdot y \not\leq (z \cdot x) \cdot (z \cdot y)$  or  $x \cdot y \in K$  by (23). Since the first option contradicts to  $(10^{\neq})$ ,  $x \cdot y \in K$  is left as a valid formula. Therefore,  $x \not\leq y$ .

Suppose that for elements  $x, y, z \in H$ , it holds  $y \cdot z \not\preceq x \cdot z$ . Then  $(y \cdot z) \cdot (x \cdot z) \in K$ . Thus  $x \cdot y \notin (y \cdot z) \cdot (x \cdot z) \lor x \cdot y \in K$  by (23). From the first option, we get

$$x \cdot y \nleq z \cdot (x \cdot y) \lor z \cdot (x \cdot y) \nleq (y \cdot z) \cdot (x \cdot z)$$

by co-transitivity of  $\leq$ . The first option is impossible due to  $(7^{\neq})$  while the second option contradicts with (H-2). So  $xy \in K$  must be valid. hence  $x \neq y$ .

**Corollary 3.3.** If K is a co-ideal of a Hilbert algebra with apartness H, then the relation  $\theta_K = \not\leq \cup \not\leq^{-1}$  is a co-congruence on H with  $1\theta_K = K$ .

*Proof.* Since it is clear that  $\theta_K$  is a co-congruence on H, let us show that  $1\theta_K = K$ . If  $y \in 1\theta_K$ , then  $(y, 1) \in \theta_K$ . thus  $1 = y \cdot 1 \in K \lor y = 1 \cdot y \in K$ . Hence  $y \in K$  by (H-6), (H-5) and (K1).

Conversely, let  $y \in K$ . Then  $1 \cdot y = y \in K$  by (H-5). Thus  $(y, 1) \in \theta_K$  and  $y \in 1\theta_K$ .

Since  $\theta_K^{\triangleleft}$  and  $\theta_K$  are an associate pair of congruence and co-congruence on Hilbert algebra with apartness H, then we can construct the family  $H/(\theta_K^{\triangleleft}, \theta_K) = \{[x] : x \in \mathbb{R}\}$ 

*H*} and the family  $[H : \theta_K] = \{x\theta_K : x \in H\}$  where the equality and the apartness are defined as follows:

$$(\forall x, y \in H)([x] = [y] \iff (x, y) \lhd \theta_K \land [x] \neq [y] \iff (x, y) \in \theta_K)$$

and

$$(\forall x, y \in H)(x\theta_K = y\theta_K \iff (x, y) \lhd \theta_K \land x\theta_K \neq y\theta_K \iff (x, y) \in \theta_K).$$

This last mentioned family has no a counterpart in classical theory of Hilbert algebras.

To show that this quotient structure  $H/(\theta_K^{\triangleleft}, \theta_K)$  is a Hilbert algebra with apartness, we need the following lemma.

**Lemma 3.10.** Let K be a co-ideal of a Hilbert algebra with apartness H. If we define an operation ' \* ' on  $H/(\theta_K^{\triangleleft}, \theta_K)$  in the following way

$$(\forall x, y \in H)([x] * [y] = [x \cdot y]),$$

then it is a well-defined an internal binary operation on  $H/(\theta_K^{\triangleleft}, \theta_K)$ .

**Theorem 3.7.** Let K be a co-ideal of a Hilbert algebra with apartness H. Then the family  $H/(\theta_K^{\triangleleft}, \theta_K)$  is a Hilbert algebra with apartness, where the co-order  $\nleq$  is defined by

 $(\forall x, y \in H)([x] \nleq [y] \iff x \not\preceq y).$ 

*Proof.* Since axioms (H-1), (H-2), and (H-5) are simply verified, we will prove that the axiom  $(H-3^{\neq})$  is a valid formula in  $H/(\theta_K^{\triangleleft}, \theta_K)$ . If  $[x] \neq [y]$ , then  $(x, y) \in \theta_K$ . This means  $x \cdot y \in K \lor y \cdot x \in K$ . Hence  $x \not\leq y \lor y \not\leq x$ . So,  $[x] \not\leq [y] \lor [y] \not\leq [x]$ . On the other hand, we have  $(x \cdot y, 1) \in \theta_K \lor (y \cdot x, 1) \in \theta_K$ . Therefore,  $[x] * [y] \neq$  $[1] \lor [y] * [x] \neq [1]$ .

To show that the family  $[H : \theta_K]$  is also a Hilbert algebra with apartness, we need the following lemma.

**Lemma 3.11.** Let K be a co-ideal of a Hilbert algebra with apartness H. If we define an operation  $' \star '$  on  $[H : \theta_K]$  in the following way

$$(\forall x, y \in H)(x\theta_K \star y\theta_K = (x \cdot y)\theta_K),$$

then it is a well-defined an internal binary operation on  $[H: \theta_K]$ .

**Theorem 3.8.** Let K be a co-ideal of a Hilbert algebra with apartness H. Then the family  $[H : \theta_K]$  is a Hilbert algebra with apartness, where the co-order  $\nleq$  is defined by

$$(\forall x, y \in H)(x\theta_K \nleq y\theta_K \iff x \nleq y).$$

*Proof.* Since axioms (H-1), (H-2), and (H-5) are simply verified, we will prove that the axiom  $(H-3^{\neq})$  is a valid formula in  $[H : \theta_K]$ . If  $x\theta_K \neq y\theta_K$ , then  $(x, y) \in \theta_K$ . This means  $x \cdot y \in K \lor y \cdot x \in K$ . Hence  $x \not\leq y \lor y \not\leq x$ . So,  $x\theta_K \not\leq y\theta_K \lor y\theta_K \not\leq x\theta_K$ . On the other hand, we have  $(x \cdot y, 1) \in \theta_K \lor (y \cdot x, 1) \in \theta_K$ . Therefore,  $x\theta_K \star y\theta_K \neq 1\theta_K \lor y\theta_K \star x\theta_K \neq 1\theta_K$ .

Although this Hilbert algebra with apartness  $[H : \theta]$ , constructed by the cocongruence  $\theta_K$ , does not have its counterpart in the classical theory of Hilbert algebras, it nevertheless appears naturally. Its existence is due to the specificity of Bishop's constructive framework. However, there is a strong link between Hilbert algebras with apartness  $H/(\theta_K^{\triangleleft}, \theta_K)$  and  $[H : \theta_K]$ .

**Theorem 3.9.** Let  $f: H \longrightarrow M$  be a se-homomorphism between Hilbert algebras with apartness. If we denote co-ideal Coker(f) by K, then there are the unique seepimorphisms  $\pi: H \longrightarrow H/(\theta_K^{\triangleleft}, \theta_K)$ , defined by  $\pi(x) = [x]$  and  $\vartheta: H \longrightarrow [H:\theta_K]$ , defined by  $\vartheta(x) = x\theta_K$ , the unique se-monomorphisms  $g: H/(\theta_K^{\triangleleft}, \theta_K) \longrightarrow M$ , defined by g([x]) = f(x) and  $h: [H:\theta_K] \longrightarrow M$ , defined by  $h(x\theta_K, and$  the unique injective, embedding and surjective se-homomorphism  $\varphi: H/(\theta_K^{\triangleleft}, \theta_K) \longrightarrow [H:\theta_K]$ , defined by  $\varphi([x]) = x\theta_K$ , such that

$$f = g \circ \pi, \ f = h \circ \vartheta \ and \ \vartheta = \pi \circ \varphi, \ g = h \circ \varphi \ and \ f = h \circ \varphi \circ \pi$$

are valid.

*Proof.* Since the proof of this theorem consists of a direct verification of the enumerated statements, we will omit their evidences.  $\Box$ 

### 4 Conclusion

In the paper we have introduced and analyze Hilbert algebras with apartness in the Bishop's constructive principled-philosophical-logic framework. The idea of Hilbert algebra built on a set with apartness  $(H, =, \neq)$  as a carrier of the algebraic structure is introduced in the Introductory Section of this article. In Section 3, which is a major part of this research, the properties of Hilbert with apartness are presented. This section is divided into seven subsections. While the concept of co-subalgebra was introduced and analyzed in subsection 3.1, the concept of co-order relations in

a Hilbert algebra with apartness was the focus of subsection 3.2. In subsections 3.3 and 3.4 some important properties of the concepts of co-ideals and co-filters in such algebras are demonstrated. Subsection 3.5 deals with the se-homomorphisms of Hilbert algebras with apartness. In the last subsection, we deal with the co-congruence generated by a co-idea and, through it, construct a tightly interconnected two different types of Hilbert algebras, one of which has no counterpart in the classical theory of Hilbert algebras.

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# A PARACONSISTENT ASP-LIKE LANGUAGE WITH TRACTABLE MODEL GENERATION

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#### Abstract

Answer Set Programming (ASP) is nowadays a dominant rule-based knowledge representation tool. Though existing ASP variants enjoy efficient implementations, generating an answer set remains intractable. The goal of this research is to define a new ASP-like rule language, 4SP, with tractable model generation. The language combines ideas of ASP and a paraconsistent rule language 4QL. Though 4SP shares the syntax of ASP and for each program all its answer sets are among 4SP models, the new language differs from ASP in its logical foundations, the intended methodology of its use and complexity of computing models.

As we show in the paper, 4QL can be seen as a paraconsistent counterpart of ASP programs stratified with respect to default negation. Although model generation for 4QL programs is tractable, dropping stratification makes it intractable for both 4QL and ASP. To retain tractability while allowing non-stratified programs, in 4SP we introduce trial expressions interlacing programs with hypotheses as to the truth values of default negations. This allows us to develop a model generation algorithm with deterministic polynomial time complexity.

We also show relationships among 4SP, ASP and 4QL.

### **1** Introduction and Motivations

Answer Set Programming (ASP) [5, 7, 17, 39, 40, 41, 44, 47, 48, 62] is a knowledge representation framework based on the logic programming and nonmonotonic reasoning

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paradigms that uses an answer set/stable model semantics for logic programs. Generating answer sets is intractable which is both an ASP strength and a weakness. The strength arises from concise representations of NP-complete problems and the use of efficient ASP solvers to conquer these problems. The weakness stems from potential lack of scalability: one can hardly expect efficient performance over large datasets. Even generating the first answer set may require time longer than could be allocated.

Another research line is represented by 4QL [49, 50, 51, 63], a four-valued paraconsistent rule language with tractable model generation and query answering. The language allows for disambiguating inconsistencies and reacting on ignorance in a nonmonotonic manner. For that purpose inspection operators for accessing truth values of literals have been introduced. However, tractability comes at a price of stratification over inspection operators. While the ASP semantics is basically three-valued with truth values t (true), f (false) and u (unknown) [25, 56], 4QL uses the fourth truth value, i, representing inconsistency.

Paraconsistent and paracoherent versions of logic programs and ASP have been investigated in the literature [4, 16, 21, 29, 33, 34]. However, to our best knowledge, no version of ASP enjoys tractable model generation. Many approaches use the logic  $B_4$  [8] as the base formalism. However,  $B_4$  may be problematic when used in the contexts we consider. Therefore, in 4QL and 4SP we use the  $L_4^+$  logic not sharing less intuitive features of  $B_4$ . The superscript '+' indicates that original logics are extended by introducing additional connectives.<sup>1</sup>

In order to motivate the use of a paraconsistent approach and the choice of  $L_4^+$  rather than  $B_4^+$ , consider sample rules of an imaginary rescue scenario listed as Program 1, where *resc* abbreviates "rescuer" and one is primarily interested in checking who is going to be saved by the rescuer, as specified in Lines 1–2 of the program where, as standard in rule-based languages, 'H := B.' denotes that the conjunction of formulas in B implies H, and 'H.' abbreviates 'H := true.'.

Program 1: Sample rules of the rescue scenario.
1 willSave(resc, P) := $\neg$ willSave(P, P), evacuable(P).
2 $\neg$ willSave(resc, P):- willSave(P, P).
3 willSave(eve, eve). evacuable(eve).
4 ¬willSave(jack, jack). evacuable(jack).
5 ¬willSave(resc, resc). evacuable(resc).

Program 1, derived from the barber paradox, has no consistent models. Indeed, the least set of its conclusions contains, among others,

<sup>&</sup>lt;sup>1</sup>'B' is used to indicate that we basically deal with the Belnap-Dunn logic. 'L' indicates that a linear ordering replaces the truth ordering of  $B_4$ . For logics used in this paper see Table 2.

$$willSave(resc, resc), \quad \neg willSave(resc, resc), \tag{1}$$

 $willSave(eve, eve), \quad \neg willSave(resc, eve), \quad (2)$ 

$$willSave(jack, jack), willSave(resc, jack).$$

$$(3)$$

Despite the inconsistency in (1), conclusions in (2)–(3) provide useful information about *eve* and *jack*. Of course, there may be more victims for whom conclusions are consistent. In fact, given that *evacuable*(P) is consistent, *willSave*(P) is consistent for all P other than *resc*. Importantly, inconsistent conclusions may be useful as well. First, they may indicate problematic situations calling for further attention. Second, when a generated plan makes a given goal inconsistent, executing the plan may be a better choice than doing nothing. E.g., if the goal is important, like helping victims, a plan leading to an inconsistent goal may be better than having no plan: though during planning there are arguments against achieving the goal as the plan's effect, there are also arguments that the goal will actually be accomplished. For many further arguments towards paraconsistency see [1, 3, 6, 11, 12, 13, 14, 15, 22, 23, 28, 36, 37, 64, 69] and numerous references there.

To illustrate the questionable features of  $B_4$ , assume that willSave(chris, chris) is unknown and willSave(resc, resc) is inconsistent. In such a case, in  $B_4$  we have:

$$(willSave(resc, resc) \lor willSave(chris, chris))$$
 is true; (4)

$$(willSave(resc, resc) \land willSave(chris, chris)) \text{ is false.}$$
(5)

The results (4)–(5) may be misleading to users sharing the classical understanding of  $\lor$  and  $\land$ , where one expects disjunction to be true (respectively, conjunction to be false) only when at least one of its arguments is true (respectively, false). In  $L_4^+$ , the disjunction in (4) is inconsistent and the conjunction in (5) is unknown. Consequently, we chose  $L_4^+$  and 4QL, adjusting the related algorithms to our needs.<sup>2</sup>

The original contributions of the paper include:

- a synthesis of ASP and 4QL: to design a new language, 4SP, with tractable model generation and capturing all queries computable in deterministic polynomial time;
- a generalized concept of stratification: to achieve the uniformity of presentation and comparability of ASP and 4QL programs;

<sup>&</sup>lt;sup>2</sup>Note, however, that in some other contexts it may be intuitive to assume a disjunction to be true even when none of its arguments is true. For examples see [19, 20] where informational semantics is considered, a defense of  $B_4$  in [67] where the canonical account of truth values is reminded as a tool for assessing consequences of what the computer is being told, or the discussion of  $B_4$  in [35].

- a concept of trial expressions allowing for setting hypotheses: to accomplish tractability of 4SP model generation;
- Algorithm 11: for generating well-supported 4SP models;
- Theorems 5.11, 6.11, 8.1 8.4: to show relationships among ASP, 4QL and 4SP as well as properties of 4SP.

The paper is structured as follows. In Section 2 we outline the methodology behind 4SP and discuss selected use cases of 4SP. In Sections 3 - 5 we recall the three- and four-valued logics considered in the paper as well as the ASP and 4QL languages. In Section 6 we introduce trial expressions and the 4SP language, and present an algorithm for 4SP model generation. In Section 7 we illustrate the use of 4SP using a simple case study. Section 8 is devoted to properties of 4SP and its relations to ASP and 4QL. Finally, Section 9 discusses related work and concludes the paper.

# 2 The Intended Methodology and Selected Use-Cases

Many application examples of paraconsistent reasoning are discussed, e.g., in [1, 3, 11, 12, 23, 64, 69]. 4SP can serve as a pragmatic tool for querying paraconsistent belief bases in most application domains and scenarios addressed there. Note that in this paper "reasoning" is based on querying rather than entailment. Below we outline the intended methodology of its use and present some further selected use cases.

## 2.1 The Intended Methodology

As we will show in Section 8, every 4SP program may have an exponential number of models. However, computing each model is tractable. We therefore replace the ASP "generate-and-test" methodology, particularly suitable for solving NP-complete problems, by the "generate-choose-and-use" methodology, where one:

- generates as many 4SP models as possible given particular time restrictions;
- selects the best models with respect to some externally defined criteria.

Model generation may be "blind" if no further information is available. With additional external knowledge it may be better directed. For example, generating literals in models may be directed by suitable probability distributions when available. Given a nonmonotonic reasoning support, one may tend to avoid *abnormal* literals, use defaults or results obtained from other nonmonotonic techniques.<sup>3</sup>

<sup>&</sup>lt;sup>3</sup>For a review of tractable versions of such forms of reasoning, compatible with our approach, see [49].

The criteria of selecting "the best" models are dependent on the particular goals to be achieved. E.g., one may choose models:

- minimizing the number of inconsistent literals r(.) for specific r's;
- minimizing the number of unknown literals r(.) for specific r's;
- minimizing the resource consumption, cost, risks involved, etc.;
- maximizing the probability of success or preferences' satisfaction;
- etc.

Note that 4SP does not provide specific means for expressing such criteria. It is meant to generate models specified by programs and then to supply them for evaluation, choice and use to other systems' components.

**Remark 2.1.** Let us emphasize that 4SP model generation is tractable. This contrasts with ASP, where generating each model is intractable. In 4SP one first generates an arbitrary set of hypotheses (being tractable) and then uses the hypotheses to generate a model. For each set of hypotheses there is a unique 4SP model and computing this model is tractable. Even though iterating through all sets of hypotheses is infeasible (requiring an exponential time), the intended methodology assumes generation of as many models as possible with the guarantee that the assumed (feasible) number of models will be generated. This is not the case in ASP, where one may generate candidates for answer sets and, provided that  $P \neq NP$ , finding even the first answer set may require a superpolynomial number of iterations.

## 2.2 Selected Use-Cases

## 2.2.1 Big Data Analytics

When big data is involved, e.g., collected from sensor networks, cyber-physical systems, IoT, health care systems, social media, smart cities, agriculture, finance, education, etc., uncertainty involving inconsistencies, noise, ambiguities and missing information, is inevitable [42]:

"most of the attribute values relating to the timing of big data [...] are missing due to noise and incompleteness. Furthermore, the number of missing links between data points in social networks is approximately 80% to 90% and the number of missing attribute values within patient reports transcribed from doctor diagnoses are more than 90%."

The aim of big data analytics is to discover hidden knowledge, e.g., leading to early detection of destructive diseases or simulating risky business decisions. When rule languages are used as (a support of) analytic tools, they typically use big data aggregates where lack of knowledge and inconsistencies may or have to be inherited. Indeed, the strategy of enforcing consistency may lead to loss of perhaps valuable information. For example [43],

"in health care systems, inconsistent information may be required to provide a full clinical perspective where no information loss is desirable".

In [37], among many others, the following example is appealing:

"in a government tax database inconsistencies in a taxpayer records are used to invoke queries into that taxpayer."

In the 4SP language truth values representing informational incompleteness and inconsistencies are first-class citizens: information gaps and inconsistencies are built into the 4SP's semantics. Though ASP is ready for handling missing information, when the involved facts or conclusions of rules are contradictory, its consistency requirement filters out all potentially useful models.

#### 2.2.2 Ontology Fusion

Fusing ontologies or belief bases may result in inconsistencies difficult or undesirable to recover [43, 45, 59]. Program 2 reflects the scenario discussed in [43, 59], where two ontologies are fused. Here *b*, *cns*, *ns*, *bp* stand for *brain*, *central nervous system*, *nervous system* and *body part*, respectively.

Program 2: Sample rules resulting from fusing ontologies.			
1 $cns(X) := b(X)$ .	/* shared by the 1st and the 2nd ontology */		
2 bp(X) := b(X).	/* shared by the 1st and the 2nd ontology */		
s ns(X) := cns(X).	/* from the 1st ontology */		
$4 \neg ns(X) := bp(X).$	/* from the 2nd ontology */		
5 b(o1).			

Conclusions of Program 2 are gathered in the set:

$$\{b(o1), cns(o1), bp(o1), ns(o1), \neg ns(o1)\},$$
(6)

with an obvious inconsistency manifesting itself in the presence of both ns(o1) and  $\neg ns(o1)$ . Though not an answer set, (6) is a 4SP model which can be used for further reasoning. While the ontology may be huge, the inconsistency affects only literals involving ns(.).

#### 2.2.3 Some Further Use-Cases

Let us still indicate some further use cases being directly relevant to the current paper:<sup>4</sup>

- actions in potentially inconsistent/incomplete environments [15];
- belief fusion and shadowing [14, 26];
- argumentation [27, 28];
- approximate reasoning [66].

## **3** Many-valued Logics Used in the Paper

In the rest of the paper we will focus on propositional rule languages. Of course, first-order variables are valuable as means to concisely express rule schemata. As we consider finite domains only, our results can be lifted to the case where first-order variables are present. Namely, first-order variables can be eliminated by *grounding* [39]. For example, the rule in Line 1 of Program 1 represents three rules, one for each constant, *resc, eve, jack*, occurring in the program:

willSave(resc, resc) := ¬willSave(resc, resc), evacuable(resc). willSave(resc, eve) := ¬willSave(eve, eve), evacuable(eve). willSave(resc, jack) := ¬willSave(jack, jack), evacuable(jack).

**Remark 3.1.** Assuming a fixed number of rules and relations occurring in a program, grounding only polynomially increases its size with respect to the number of constants involved. Since we will consider data complexity [2], this feature is sufficient (though not necessary: for many rule-based languages more efficient model generation algorithms exist).

We will use three- and four-valued logics  $K_3^+$  [46],  $P_3^+$  [55],  $L_4^+$  [52],  $B_4^+$  [8] using (suitable subsets of) truth values: f (false), u (unknown), i (inconsistent), t (true).<sup>5</sup> To keep the presentation uniform, the syntax of all considered logics is the same: for each considered logic we use a set of *propositional variables* and define the *set of formulas* to contain propositional variables and being closed under unary connectives  $\neg, \sim$  (classical and default negation), and binary ones  $\land, \lor, \rightarrow$  (conjunction, disjunction and implication).

For the purpose of this paper it suffices to assume that a *logic* is given by the set of formulas and interpretations assigning truth values to formulas. Let  $\mathcal{L} \in \{K_3^+, P_3^+, L_4^+, B_4^+\}$ 

<sup>&</sup>lt;sup>4</sup>Please consult also references in the quoted papers.

<sup>&</sup>lt;sup>5</sup>According to the convention we do not distinguish between truth values and constants denoting them.

	_	$\sim$	$\rightarrow$	f	u	í	t
f	t	t	f	t	t	t	t
u	u	t	u	t	t	t	t
í	í	í	í	f	f	t	f
t	f	f	t	f	f	t	t

Table 1: The semantics of negations and implication. For  $K_3^+$  (respectively,  $P_3^+$ ), the row and column labeled by  $\mathfrak{i}$  (respectively,  $\mathfrak{u}$ ) is to be removed.

be a logic. The set of its propositional variables is denoted by  $\mathcal{P}_{\mathcal{L}}$  and the set of truth values is denoted by  $\mathfrak{n}_{\mathcal{L}}$  where assume that:

$$\{\mathfrak{f},\mathfrak{t}\}\subseteq\mathfrak{n}_\mathcal{L}\subseteq\{\mathfrak{f},\mathfrak{u},\mathfrak{i},\mathfrak{t}\}$$

For the purpose of defining models, we have to designate a set of truth values to act as being true [57, 65]. The set of *designated truth values* is denoted by  $\delta_{\mathcal{L}}$  ( $\delta_{\mathcal{L}} \subseteq \mathfrak{n}_{\mathcal{L}}$ ).

To define the semantics of  $\mathcal{L}$  we first need to provide the semantics of connectives applied to truth values. Truth tables of negations and implication are shown in Table 1.

#### Remark 3.2.

- When we deal with less than four truth values, Table 1 needs to be reduced by removing rows and columns labeled by "irrelevant" truth values. For example, for  $K_3^+$  the row and column containing *i* is redundant, for  $P_3^+$  one removes those labeled by  $\mathfrak{u}$ .
- Intuitively, the default negation ~ p stands for "p is not true". However, while its traditional ASP meaning is "the truth value of p is f or u in the finally computed interpretation", when considered as a four-valued connective of 4SP, its meaning is "p's truth value is f, u or i in the interpretation computed up to now".
- The implication on {f, u, t} as well as on {f, i, t} is the implication of [61]. It has been generalized to {f, u, i, t} in [52].

The semantics of  $\land$  and  $\lor$  is standard:

$$\tau_1 \wedge \tau_2 \stackrel{\text{def}}{=} \text{glb}_{\leq} \{\tau_1, \tau_2\}; \tag{7}$$

$$\tau_1 \vee \tau_2 \stackrel{\text{def}}{=} \text{lub}_{\leq} \{\tau_1, \tau_2\},\tag{8}$$

where  $\tau_1, \tau_2 \in \mathfrak{n}_{\mathcal{L}}$  and lub, glb are respectively the least upper and the greatest lower bound with respect to ordering  $\leq$  chosen from Figure 1.



Figure 1: Truth orderings used in the paper where K stands for Kleene, P - Priest, L - Linearity, and B - for Belnap.

Logic	Extends	Truth values	Ordering
$K_3^+$	$K_3$	f, u, <u>t</u>	$\leq_K$ in Figure 1(a)
$P_{3}^{+}$	$P_3$	f, <u>í, t</u>	$\leq_P$ in Figure 1(b)
$L_4^+$	$L_4$	f, u, <u>í, t</u>	$\leq_L$ in Figure 1(c)
$B_4^+$	$B_4$	f, u, <u>í, t</u>	$\leq_B$ in Figure 1(d)

Table 2: Logics used in the paper with underlined designated truth values.

To define the semantics of formulas of  $\mathcal{L} \in \{K_3^+, P_3^+, L_4^+, B_4^+\}$  we assume a mapping w assigning truth values to propositional variables:

$$w: \mathcal{P}_{\mathcal{L}} \longrightarrow \mathfrak{n}_{\mathcal{L}}.$$
 (9)

Assignments (9) are extended to all formulas:

$$w(\neg A) \stackrel{\text{def}}{=} \neg w(A); \quad w(\sim A) \stackrel{\text{def}}{=} \sim w(A); \tag{10}$$

$$w(A \circ B) \stackrel{\text{def}}{=} w(A) \circ w(B), \text{ where } o \in \{\land, \lor, \rightarrow\}.$$
(11)

Logics used in the paper are listed in Table 2, where  $K_3$ ,  $P_3$  are three-valued logics of [46] and [55],  $B_4$ ,  $L_4$  are four-valued logics of [8] and [52], respectively.

**Definition 3.3** (Literals). Let  $p \in \mathcal{P}_{\mathcal{L}}$  be a propositional variable. By a classical literal (literal, for short) we mean an expression of the form p (positive literal) or  $\neg p$  (negative literal). The set of classical literals is denoted by  $\mathbb{L}$ . When  $\ell \in \mathbb{L}$ ,  $\neg \neg \ell$  is identified with  $\ell$ . By a default literal we understand an expression of the form  $\sim \ell$ , where  $\ell \in \mathbb{L}$ . The set of default literals is denoted by  $\mathbb{D}$ .

**Definition 3.4** (Interpretations, consistency). By an interpretation we mean a finite set  $\mathcal{I} \subseteq \mathbb{L}$ . An interpretation  $\mathcal{I}$  is consistent if, for every  $p \in \mathcal{P}_{\mathcal{L}}$ ,  $p \notin \mathcal{I}$  or  $\neg p \notin \mathcal{I}$ .

In what follows, the considered set of propositional variables,  $\mathcal{P}_{\mathcal{L}}$ , will always be finite. In such cases there is a one-to-one mapping between assignments (9) and interpretations allowing us to freely switch between them. Namely, given an interpretation  $\mathcal{I}$ , the corresponding assignment  $w^{\mathcal{I}}$  is:

$$w^{\mathcal{I}}(p) \stackrel{\text{def}}{=} \begin{cases} \mathbf{t} & \text{when } p \in \mathcal{I} \text{ and } \neg p \notin \mathcal{I}; \\ \mathbf{i} & \text{when } p \in \mathcal{I} \text{ and } \neg p \in \mathcal{I}; \\ \mathbf{u} & \text{when } p \notin \mathcal{I} \text{ and } \neg p \notin \mathcal{I}; \\ \mathbf{f} & \text{when } p \notin \mathcal{I} \text{ and } \neg p \in \mathcal{I}. \end{cases}$$

Of course,  $w^{\mathcal{I}}$  can be extended to all formulas using Equations (10) and (11). To simplify notation we will write  $\mathcal{I}(A)$  to stand for  $w^{\mathcal{I}}(A)$ . Conversely, given w, the corresponding interpretation  $\mathcal{I}^w$  is defined by:

$$\mathcal{I}^{w} \stackrel{\text{def}}{=} \{p \mid w(p) = \mathbf{t}\} \cup \{\neg p \mid w(p) = \mathbf{f}\} \cup \{p, \neg p \mid w(p) = \mathbf{i}\}.$$

**Definition 3.5** (Models). *Given a logic*  $\mathcal{L}$  *with the set of designated values*  $\mathfrak{d}_{\mathcal{L}} \subseteq \mathfrak{n}_{\mathcal{L}}$ *, we say that an interpretation*  $\mathcal{I}$  is a model *of a formula* A,  $\mathcal{I} \models_{\mathcal{L}} A$ *, when*  $\mathcal{I}(A) \in \mathfrak{d}_{\mathcal{L}}$ .

## 4 Answer Set Programming

We will focus on normal ASP programs.<sup>6</sup>

#### Definition 4.1 (Syntax of normal ASP programs).

Let  $\ell_1, \ldots, \ell_k, \ell_{k+1}, \ldots, \ell_m \in \mathbb{L} \cup \{\mathfrak{f}, \mathfrak{u}, \mathfrak{t}\}$  and  $\ell \in \mathbb{L}$ . A normal ASP rule (ASP rule, for short) is an expression of the form:

$$\ell := \ell_1, \dots, \ell_k, \sim \ell_{k+1}, \dots, \sim \ell_m.$$
(12)

It is further assumed that  $0 \le k \le m$  and when  $k + 1 \le m$  then  $k > 0.^7$  The literal  $\ell$  is called the conclusion (head) and the part after ':-' is called the premises (body) of rule (12). A rule with the empty premises is called a fact and is written as ' $\ell$ .'. A rule without default literals is called pure.

A normal ASP program (ASP program, for short) is a finite set of normal rules. A program is pure if it contains pure rules only.

Note also that the empty conjunction, thus the empty body, is assumed to be t.

<sup>&</sup>lt;sup>6</sup>The results can be extended to disjunctive programs where disjunctions correspond to choice rules. We leave this extension for future research.

<sup>&</sup>lt;sup>7</sup>Default literals have to be "guarded" by a classical literal.

The basic rule-based reasoning principle of ASP is:

- if premises of rule (12) evaluate to t, add  $\ell$  to the set of conclusions. (13)

The semantics of ASP programs is given by answer sets.

**Definition 4.2** (Models of ASP programs; Answer Sets). By a model of an ASP program  $\Pi$  we mean a consistent interpretation  $\mathcal{I}$  satisfying all rules of  $\Pi$  understood as implications:

$$\mathcal{I}\models_{K_3^+} (\ell_1 \wedge \ldots \wedge \ell_k \wedge \sim \ell_{k+1} \wedge \ldots \wedge \sim \ell_m) \to \ell.$$

If  $\Pi$  is pure then an answer set of  $\Pi$  is the least (with respect to  $\subseteq$ ) model of  $\Pi$ , if exists. If  $\Pi$  contains  $\sim$  then  $\mathcal{I}$  is an answer set of  $\Pi$  iff  $\mathcal{I}$  is the least model of  $\Pi^{\mathcal{I}}$ , where  $\Pi^{\mathcal{I}}$  is obtained from  $\Pi$  by substituting each default literal  $\sim \ell$  occurring in  $\Pi$  by its truth value  $\mathcal{I}(\sim \ell)$ .

Let us now provide a simple (naive) algorithm for generating an answer set for an ASP program. We first need Algorithm 3 generating minimal interpretations for pure programs. To keep presentation simple, it is based on the naive bottom up evaluation - see [2]. Each literal of the form  $\neg p$  is treated as a fresh propositional variable, say p', so pure ASP programs can be seen as DATALOG programs to which the original naive bottom up evaluation applies. Generating an answer set can be done nondeterministically, as in Algorithm 4.

#### Algorithm 3: function generateLeast( $\Pi$ );

1	/* returns the least model of a pure ASP program $\Pi$ */
2	set $\mathcal{I} = \emptyset$ ;
3	while there is a rule $\ell := \beta$ . $\in \Pi$ such that $\mathcal{I}(\beta) = t$ and $\ell \notin \mathcal{I}$ do
4	$  lset \mathcal{I} = \mathcal{I} \cup \{\ell\}; $
5	return $\mathcal{I}$ .

# Algorithm 4: function generateAnswerSet(Π); 1 /\* returns an answer set of an ASP program Π if exists \*/ 2 set I = a nondeterministically generated consistent interpretation; 3 set J = generateLeast(Π<sup>T</sup>); /\* Π<sup>T</sup> is defined in Def. 4.2 \*/ 4 if I = J then 5 return I.

The following theorem is well known (see, e.g., [7, 17, 53]).

**Theorem 4.3.** Given an ASP program  $\Pi$ , generating an answer set for  $\Pi$  is an NP-complete problem with respect to the number of propositional variables in  $\Pi$ .

# 5 The 4QL Language

The 4QL language has been introduced in [49, 50]. In this paper we shall use its extended version of [51].<sup>8</sup> 4QL allows for paraconsistent reasoning, using the  $L_4^+$  logic. Rather than default negation  $\sim$ , inspection operators are used as defined below.

**Definition 5.1** (Inspection operators). *By an* inspection operator we understand any expression of the form  $\ell \in T$ , where  $\ell \in \mathbb{L}$  and  $T \subseteq \{\mathfrak{f}, \mathfrak{u}, \mathfrak{i}, \mathfrak{t}\}$ . The meaning of inspection operators depends on the actual interpretation  $\mathcal{I}$ :

$$\mathcal{I}(\ell \in T) \stackrel{\text{def}}{=} \begin{cases} \mathsf{t} & \text{when } \mathcal{I}(\ell) \in T; \\ \mathsf{f} & \text{otherwise.} \end{cases}$$
(14)

The set of inspection operators is denoted by  $\mathbb{I}$ .

When truth values are restricted to three values of  $K_3^+$ , default negation of ASP can be defined by:

$$\sim \ell \stackrel{\text{def}}{\equiv} (\ell \in \{\mathfrak{f}, \mathfrak{u}\}).$$
 (15)

The original version of 4QL uses modules but, for the sake of uniformity, we skip them here. In order to compare 4QL, ASP, and 4SP as well as to achieve the full power of 4QL without using modules, let us introduce a general form of stratification with respect to a set of arbitrary expressions  $\mathbb{E}$ , e.g., consisting of default literals or expressions involving inspection operators (a definition similar in spirit but using modules has been provided in [63]).

**Definition 5.2** (Stratification). Given a finite set S of 4QL (or ASP) rules and a set of expressions  $\mathbb{E}$ , we say that S is stratifiable with respect to  $\mathbb{E}$  when  $S = S_1 \cup ... \cup S_r$  such that for  $1 \le i \ne j \le r$ ,  $S_i \cap S_j = \emptyset$  and:

- for every conclusion  $\ell$  of a rule in S, there is  $1 \le i \le r$  such that all rules with conclusions  $\ell, \neg \ell$  are in  $S_i$  ( $\ell$  is fully defined in  $S_i$ );
- whenever an expression  $e \notin \mathbb{E}$  appears in premises of a rule in  $S_i$ , for  $1 \le i \le r$ , classical literals appearing in e are fully defined in  $S_j$  for some  $1 \le j \le i$ ;
- whenever an expression  $e \in \mathbb{E}$  appears in premises of a rule in  $S_i$  for  $1 \le i \le r$ , classical literals occurring in e are fully defined in  $S_j$  for some  $1 \le j < i$ .

 $\triangleleft$ 

<sup>&</sup>lt;sup>8</sup>An open source interpreter of 4QL and its doxastic extensions (see [14] and references there) is available via 4ql.org.

**Remark 5.3.** Stratification of Definition 5.2 generalizes stratification used in DATALOG<sup>¬</sup> (see, e.g., [2]). To verify whether a set of 4QL rules is stratifiable, one can easily adapt the deterministic polynomial time algorithm checking stratification for DATALOG<sup>¬</sup> programs. Finding a stratification, if exists, is also tractable.

Definition 5.4 (Syntax of 4QL programs).

Let  $\ell \in \mathbb{L}$ ,  $\ell_{11}, \ldots, \ell_{1k_1}, \ldots, \ell_{m1}, \ldots, \ell_{mk_m} \in \mathbb{L} \cup \mathbb{I} \cup \{\mathfrak{f}, \mathfrak{u}, \mathfrak{i}, \mathfrak{t}\}$ . A 4QL rule is an expression of the following form, where semicolon ';' stands for disjunction, with conjunction ',' binding stronger:

$$\ell := \ell_{11}, \dots, \ell_{1k_1}; \dots; \ell_{m1}, \dots, \ell_{mk_m}.$$
<sup>(16)</sup>

A rule without occurrences of inspection operators is called pure. A 4QL program is a finite set of 4QL rules stratifiable with respect to  $\mathbb{I}$ . A program is pure if it contains pure rules only.

Rule (16) is interpreted as the following implication of  $L_4^+$ :

$$((\ell_{11} \wedge \ldots \wedge \ell_{1k_1}) \vee \ldots \vee (\ell_{m1} \wedge \ldots \wedge \ell_{mk_m})) \to \ell.$$
 (17)

The reasoning principles of 4QL are (13) together with:

- if premises of rule (16) evaluate to i, add l,  $\neg l$  to the set of conclusions. (18)

**Remark 5.5.** *Though Principles* (13) *and* (18) *are natural, they appear problematic when disjunction is concerned. As an example, consider Program 5.* 

Program 5: Program illustrating a disjunction issue.
$1 reachable(base, P) := can_reach(base, P, ground).$
2 $reachable(base, P) := can_reach(base, P, air).$

When one of premises of rules in Lines 1-2 of Program 5 evaluates to t with the other one being i, according to Principles (13), (18), reachable(base, P) becomes i. On the other hand, base is reachable from P either by ground or by air, so the conclusion reachable(base, P) should intuitively be true. In the classical logic the above rules are equivalent to:

 $reachable(base, P) := can_reach(base, P, ground) \lor can_reach(base, P, air).$  (19)

Therefore one needs a disjunction in rules' bodies. In fact, using rule (19) one indeed concludes that reachable(base, P) is t in each logic listed in Table 2.

Accordingly, disjunction is explicit in 4QL (and 4SP) rules. Due to its nonmonotonic behavior, it requires a nonstandard computation engine.<sup>9</sup> For details see [49, 51] and further parts of the current paper.

<sup>&</sup>lt;sup>9</sup>A DATALOG-like evaluation applied to reducts in ASP is far from being sufficient here.

The semantics of 4QL is defined by well-supported models, where *well-supportedness* guarantees that all conclusions are derived using reasoning starting from facts [49, 50, 51]. For a definition of well supported models see [50]. To put well-supportedness into perspective, below we provide a new, equivalent definition generalizing the concept of loops [48].

**Definition 5.6** (Dependency graph). Let  $\Pi$  be a pure 4QL program. By a dependency graph of  $\Pi$  we understand a directed graph with vertices labeled by classical literals occurring in  $\Pi$ . There is an arc from  $\ell$  to  $\ell'$  if there is a rule in  $\Pi$  whose head is  $\ell$  and  $\ell'$  appears in the rule's body.

**Definition 5.7** (Loop). A non-empty subset L of literals occurring in a pure 4QL program  $\Pi$  is called a loop of  $\Pi$  if for any  $\ell, \ell' \in L$ , there is a path of non-zero length from  $\ell$  to  $\ell'$  in the dependency graph of  $\Pi$ , such that all the vertices in the path are in L.

By  $R^{-}(L, \Pi)$  we understand the set of rules:

 $\{\ell := B_1; \ldots; B_m. \in \Pi \mid \ell \in L \text{ and there is } 1 \le i \le m \text{ such that}$  for all  $\ell'$  in  $B_i, \ell' \notin L\}.$  (20)

**Definition 5.8** (Well-supported model). An interpretation  $\mathcal{I}$  is a well-supported model of a pure 4QL program  $\Pi$  iff  $\mathcal{I}$  is the least (with respect to  $\subseteq$ ) model of  $\Pi$ , <sup>10</sup> and for every loop L of  $\Pi$ , if there is  $\ell \in L$  such that  $\mathcal{I}(\ell) \in {\mathbf{i}, \mathbf{t}}$  then:

- $\mathcal{I}(\ell) = \mathbf{t}$  iff there is a rule ' $\ell$ :- B.' in  $R^-(L, \Pi)$  such that  $\mathcal{I}(B) = \mathbf{t}$  and there are no rules ' $\ell$ :- C.', ' $\neg \ell$ :- D.' in  $R^-(L, \Pi)$  such that  $\mathcal{I}(C) = \mathbf{i}$  or  $\mathcal{I}(D) \in {\mathbf{i}, \mathbf{t}};$
- $\mathcal{I}(\ell) = \mathfrak{i}$  iff there is a rule ' $\ell := B$ .' or ' $\neg \ell := B$ .' in  $R^-(L, \Pi)$  such that  $\mathcal{I}(B) = \mathfrak{i}$  or there are rules ' $\ell := C$ .', ' $\neg \ell := D$ .' in  $R^-(L, \Pi)$  with  $\mathcal{I}(C) = \mathfrak{t}$  and  $\mathcal{I}(D) = \mathfrak{t}$ .

Algorithm 6 presents a high-level pseudocode for computing well-supported models for pure 4QL programs. It is further formalized as Algorithm 7 (implementing Line 4 of Algorithm 6), and Algorithm 8 (computing the well-supported model). Note that all conclusions inferred by Algorithm 6 are supported by facts and no conclusion is obtained using a proposition defeated later.

**Remark 5.9.** To generalize Algorithm 8 to non-pure programs, one can find its stratification (without losing tractability – see Remark 5.3) and eliminate inspection operators stratum by stratum, starting from the lowest one.<sup>11</sup> Let  $S_i$  be the lowest stratum where inspection operators occurs. Their truth values are determined in strata lower than  $S_i$ . Substituting all inspection operators occurring in  $S_i$  by the determined truth values makes  $S_1 \cup \ldots \cup S_i$ a pure program for which Algorithm 8 applies. This procedure is to be iterated until all strata have been dealt with. **Algorithm 6:** A pseudocode for computing the well supported model for a given pure 4QL program.

1	repeat
2	generate the least set of conclusions;
3	retract conclusions based on defeated premises, i.e., premises at some point being true
	but later becoming inconsistent;
4	correct (minimally) the obtained set of literals to make it a model (to satisfy rules
	with inconsistent premises and not inconsistent conclusions)
5	until no further retractions are needed

**5 until** no further retractions are needed.

Algorithm 7: function findCorrection( $\Pi$ ,  $\mathcal{I}$ );

/\* returns the correction of I with respect to a pure 4QL program Π \*/
 set J = Ø;
 set K = I;
 while there is a rule 'ℓ:- β.' ∈ Π such that K(β) = i and K(ℓ) ≠ i do
 set J = J ∪ {ℓ, ¬ℓ};
 set K = K ∪ {ℓ, ¬ℓ};
 return J.

The following theorem is proved in [50, 51].

#### Theorem 5.10.

- For every 4QL program Π there is exactly one well-supported model and it can be generated in deterministic polynomial time with respect to the number of propositional variables in Π.
- 4QL captures deterministic polynomial time over linearly ordered domains. That is, every query computable in deterministic polynomial time can be expressed in 4QL whenever a linear ordering over the domain is available in the 4QL vocabulary.

When stratification is not required, like in the case of ASP, generating well supported models for 4QL programs becomes an NP-complete problem.

The following theorem shows a close correspondence between (stratified) ASP and 4QL programs.

<sup>&</sup>lt;sup>10</sup>In the four-valued case, minimality substitutes the completion of [18] used together with loop formulas to characterize answer sets for ASP programs [48].

<sup>&</sup>lt;sup>11</sup>Here we understand stratification in the sense of Definition 5.2 with  $\mathbb{E} = \mathbb{I}$ .

#### Algorithm 8: function generateWsm4ql( $\Pi$ )

**Theorem 5.11.** Let  $\Pi$  be an ASP program stratifiable with respect to the set of default literals  $\mathbb{D}$  and  $\Pi'$  be a 4QL program obtained from  $\Pi$  by substituting default negations by inspection operators as in (15). Then the well supported model  $\mathcal{I}$  of  $\Pi'$  is the answer set of  $\Pi$  iff  $\mathcal{I}$  is consistent. If  $\mathcal{I}$  is inconsistent then  $\Pi$  has no answer set.

# 6 The 4SP Language

The syntax of 4SP programs is very similar to ASP programs. We extend the language by allowing disjunctions and the truth constant *i* to appear in rules' bodies.

**Definition 6.1** (Syntax of 4SP programs).

Let  $\ell \in \mathbb{L}$  and  $\ell_{11}, \ldots, \ell_{1k_1}, \ldots, \ell_{m1}, \ldots, \ell_{mk_m} \in \mathbb{L} \cup \mathbb{D} \cup \{\mathfrak{f}, \mathfrak{u}, \mathfrak{i}, \mathfrak{t}\}$ . A 4SP rule is an expression of the following form, where semicolon ';' stands for disjunction, with conjunction ',' binding stronger:

$$\ell := \ell_{11}, \dots, \ell_{1k_1}; \dots; \ell_{m1}, \dots, \ell_{mk_m}.$$
<sup>(21)</sup>

A 4SP rule is pure if it does not involve default negation. A 4SP program is a finite set of 4SP rules. A 4SP program is pure if it contains pure rules only.

**Remark 6.2.** Note that we require a nonempty conclusion, so ASP-like constraints are excluded. With ASP-like constraints, tractability could be lost. Indeed, iterating through the set of hypotheses (see Definition 6.5) could take superpolynomial time when models could be rejected by constraints.

The key step towards tractable model generation is to use the four valued default negation, as defined in Table 1. To illustrate the idea, let us start with supportedness losing in the non-stratified case. **Example 6.3.** When 4QL programs may be non-stratified, supportedness may be lost. To illustrate the issue, consider non-stratifiable 4QL rules in Program 9.<sup>12</sup> The program has no well-supported models: the rule in Line 1 makes p true so the second rule makes q true, too. That way p loses its support so cannot be derived. Consequently, q cannot be derived, so its value becomes u and so on.

Program 9: Supportedness losing.

1  $p := q \in \{\mathbf{u}, \mathbf{f}\}$ . /\*  $p := \sim q$ . – when restricted to ASP \*/ 2 q := p.

When Line 1 is replaced by ' $p:- \sim q$ .', with the four-valued  $\sim$ , after applying the rules it would be natural to consider  $\sim q$  inconsistent. Now, Principle (18) together with the rule in Line 1 make p inconsistent and the rule in Line 2 makes q inconsistent. So the well-supported model would become  $\{p, \neg p, q, \neg q\}$ .

In 4SP, rather than using two-valued inspection operators, we will use the four-valued default negation  $\sim$ . However, we encounter the next issue, illustrated in Example 6.4.

**Example 6.4.** Consider Program 10. It consists of two non-stratifiable rules. Both  $\{p\}$  and  $\{q\}$  are its well-supported models. Their generation depends on the order of rules: when the rule in Line 1 is applied first, the result is  $\{p\}$ , being  $\{q\}$  otherwise.

Program 10: A further non-stratifiability effect.	
1 $p:- \sim q$ .	
$2 q := \sim p.$	

Of course, the semantics should not depend on the order of rules' application. We therefore consider programs in the context of *trial expressions* allowing one to select truth values of default negations.

**Definition 6.5** (Trial expression). *By a* trial expression *we understand an expression of the form:* 

$$\Pi \operatorname{with} \mathcal{H}, \tag{22}$$

where  $\Pi$  is a program with ASP syntax in the sense of Definition 4.1 and  $\mathcal{H}$ , called a set of hypotheses for  $\Pi$ , is a finite set of expressions of the form  $\sim \ell \ll t$  or  $\sim \ell \ll f$  such that for every and only literal  $\ell'$  occurring in  $\Pi$  within an expression  $\sim \ell'$ , the literal  $\ell'$  occurs (as a subexpression) in  $\mathcal{H}$ .

<sup>&</sup>lt;sup>12</sup>Program 9 is a set of rules being neither a 4QL nor a 4SP program.

The set of trial expressions is denoted by  $\mathbb{T}$ .

Intuitively,  $\Pi$  with  $\mathcal{H}$  amounts to assuming the truth values of default negations in  $\mathcal{H}$  and verifying whether the hypotheses have been consistent with the results of  $\Pi$ . For a particular default negation occurring in  $\Pi$ ,

- first assume that the truth value  $\sim \ell$  is t (respectively, f);
- if, during generating an answer set for Π, the value of ~l appears not to be t (respectively, not to be f), correct the truth value of ~l assigning it the value i.

Note that we do not allow expressions of the form  $\sim \ell \ll i$ . First, the role of trial expressions is to try to "guess" consistent solutions with inconsistency being an undesirable effect. Second,  $\sim \ell \ll i$  can be expressed by  $\{\sim \ell \ll f, \sim \neg \ell \ll f\}$ . This follows from the fact that  $\sim \ell$  is f only when  $\ell$  is t.

**Example 6.6.** Let  $\Pi$  denote Program 10. Then:

$$\Pi \operatorname{with} \{ \sim q \twoheadleftarrow \mathfrak{t}, \sim p \twoheadleftarrow \mathfrak{f} \}$$
(23)

is intended to mean that the value of  $\sim q$  is hypothesized to be true and the value of  $\sim p$  is hypothesized to be false. Hence, (23) results in  $\{p\}$ , confirming the hypotheses. On the other hand, consider:

 $\Pi \operatorname{with} \{ \sim p \twoheadleftarrow \mathbf{t}, \sim q \twoheadleftarrow \mathbf{t} \}.$ (24)

The rules of Program 10 generate  $\{p,q\}$  as conclusions, violating both hypotheses expressed in (24). Therefore, both  $\sim p$  and  $\sim q$  become i and the answer set for (24) becomes  $\{p, \neg p, q, \neg q\}$ .

**Remark 6.7.** Note that hypotheses of (22) may mutually affect knowledge bases represented by programs. For example,  $\neg p$  is inconsistent with the hypothesis that  $\sim p \ll \mathfrak{f}$ . If  $\neg p$  is a generated conclusion,  $\sim p \ll \mathfrak{f}$  together with  $\neg p$  makes  $\sim p$  inconsistent, what only happens when p is  $\mathfrak{i}$ . This calls for placing p in the generated model, too. Such side-effects have to be reflected in the model generation algorithm.

In importing truth values from an interpretation  $\mathcal{I}$  to the set of hypotheses  $\mathcal{H}$  we deal with the situation when the imported truth values are not u. Therefore, we are on the grounds of  $P_3^+$  and can use the following rules:<sup>13</sup>

if  $\ell \in \mathcal{I}$  then add  $\sim \ell \ll \mathfrak{f}$  to  $\mathcal{H}$ . (25)

<sup>&</sup>lt;sup>13</sup>The rule (25) applies to  $\ell$  being either positive or negative.

Of course, when  $\ell$  is  $\mathfrak{i}$  in  $\mathcal{I}, \ell, \neg \ell \in \mathcal{I}$  so the rule (25), applied twice, results in adding to  $\mathcal{H}$  both  $\sim \ell \leftarrow \mathfrak{f}$  and  $\sim \neg \ell \leftarrow \mathfrak{f}$ .

Conversely,

if 
$$\{\sim \ell \leftarrow f, \sim \ell \leftarrow t\} \subseteq \mathcal{H}$$
 then add  $\ell, \neg \ell$  to  $\mathcal{I}$ ; (26)

else if 
$$\sim \ell \ll f \in \mathcal{H}$$
 then add  $\ell$  to  $\mathcal{I}$ . (27)

**Definition 6.8** (Semantics of 4SP). *Given an* 4SP *program*  $\Pi$ , *by a well-supported model of*  $\Pi$  *we mean any well-supported model for* ' $\Pi$  with  $\mathcal{H}$ ' where  $\mathcal{H}$  *is an arbitrary set of hypotheses for*  $\Pi$ .

For computing well-supported models of 4SP programs we still need the following definition.

**Definition 6.9** (Interlace). Let  $\mathcal{I}$  be an interpretation and  $\mathcal{H}$  be a set of hypotheses for a 4SP program. By an interlace of  $\mathcal{I}$  and  $\mathcal{H}$ , denoted by  $interlace(\mathcal{I}, \mathcal{H})$ , we mean a minimal interpretation  $\mathcal{J}$  such that  $\mathcal{I} \subseteq \mathcal{J}$  and  $\mathcal{J}$  is closed under application of rules (25)–(27).  $\triangleleft$ 

Algorithm 11 computes the well-supported model for a 4SP program  $\Pi$  when a set of hypotheses  $\mathcal{H}$  is given. The function 'generateWsmAux' is obtained from Algorithm 8 by adding  $\mathcal{H}$  as a parameter and replacing Line 5 by:

5' set 
$$\mathcal{I} = interlace(Inc \cup \{\ell \mid \ell \in PTrue\}, \mathcal{H}).$$
 (28)

Algorithm 11: function generateWsm4sp( $\Pi, \mathcal{H}$ )
1 /* returns the well-supported model $\mathcal I$ for 'II with $\mathcal H$ ' */
2 <b>return</b> generateWsmAux( $\Pi, \mathcal{H}$ ).

In the light of Theorem 8.2, the intended use of Algorithm 11 instantiates the methodology outlined in Section 2.1, where model generation depends on assigning truth values to hypotheses.

**Remark 6.10.** Algorithm 11 substantially differs from the algorithms for generating wellsupported models for 4QL programs due to the use of interlace() in (28).

It is also important to notice that whenever we deal with 4SP programs stratifiable with respect to  $\mathbb{D}$ , the well supported model (in the sense of Theorem 5.11) contains a minimal (wrt  $\subseteq$ ) set of inconsistent literals. In such cases it is then worth to first generate this well-supported model rather than start from arbitrary sets of hypotheses (se also Section 7).

**Theorem 6.11.** If a 4SP program is stratifiable with respect to  $\mathbb{D}$  then its well supported model  $\mathcal{I}$  (in the sense of Theorem 5.11) is minimally inconsistent, that is for every set  $\mathcal{H}$ , the set of inconsistent literals in  $\mathcal{I}$  is included in (or equal to) the set of inconsistent literals in the model for  $\Pi$  with  $\mathcal{H}$ .

PROOF To compute  $\mathcal{I}$  one can apply Algorithm 8 stratum by stratum. Now minimality follows from the fact that, in each iteration, Algorithm 8 uses DATALOG-based computation (Algorithm 3) generating the least models, thus makes inconsistent those and only literals that have to be inconsistent.

# 7 A Case Study

To illustrate the use of 4SP, consider Program 12, modifying Program 1 where loc(P, L) stands for "person P is in location L" and path(X, L) stands for "there is a path between locations X and L".

Program 12: Sample rules of the modified rescue scenario.
1 willSave(resc, P) :- $\sim$ willSave(P, P), evacuable(P).
$2 \neg willSave(resc, P) := willSave(P, P).$
3 $evacuable(P)$ :- $\sim \neg reachable(base, P)$ .
4 reachable $(X, P)$ :- path $(X, L)$ , loc $(P, L)$ .
5 $path(X,L)$ :- $path(X,Y), path(Y,L).$
6 $path(base, l_1)$ . $\neg path(base, l_1)$ . $path(base, l_2)$ . $path(l_1, l_3)$ . $path(l_2, l_3)$ .
7 $loc(resc, base)$ . $loc(eve, l_1)$ . $loc(jack, l_3)$ .
8 willSave(resc, resc).

First observe that Program 12 has no answer sets. Indeed the rule in Line 2 together with the fact in Line 8 make the literal *willSave(resc, resc)* inconsistent. Therefore, in the rest of this section we deal with 4SP.

The graph specified by *path*() in Program 12 is shown in Figure 2.



Figure 2: The graph specified by *path()* in Program 12.

According to the 4QL convention, variables occurring in premises of a rule and not occurring in its conclusion are existentially quantified in premises [51]. Therefore, the rule in Line 5 is to be understood as implicitly quantified with  $\exists Y$ :

$$path(X,L) := "\exists Y \ path(X,Y), path(Y,L)".$$
(29)

Since we deal with finite domains, existential quantifiers abbreviate disjunctions:

$$\exists Y(A(Y)) \stackrel{\text{def}}{=} (A(a_1) \lor \ldots \lor A(a_n)), \tag{30}$$

where  $a_1, \ldots, a_n$  are the domain elements. For example, when  $X = base, L = l_3$ , (29) is equivalent to:<sup>14</sup>

$$path(base, l_3) := \underbrace{path(base, l_1)}_{t}, \underbrace{path(l_1, l_3)}_{t}; \underbrace{path(base, l_2)}_{t}, \underbrace{path(l_2, l_3)}_{t}.$$
(31)

Thus, the truth value of  $path(base, l_3)$  is  $((\mathbf{i} \wedge \mathbf{t}) \vee (\mathbf{t} \wedge \mathbf{t})) = (\mathbf{i} \vee \mathbf{t}) = \mathbf{t}$ .

To eliminate first-order variables in rules other than in Line 3 we use grounding. E.g., Line 1 of Program 12 represents three rules:

$$willSave(resc, eve) := \sim willSave(eve, eve), evacuable(eve).$$
 (32)

$$willSave(resc, jack) := \sim willSave(jack, jack), evacuable(jack).$$
 (33)

$$willSave(resc, resc) := \sim willSave(resc, resc), evacuable(resc).$$
 (34)

Program 12 is stratifiable with respect to  $\sim$  so, according to Theorem 6.11, let us first analyze its outcomes without using hypotheses and trial expressions.

According to the rule in Line 4, reachable(base, eve) is  $\mathfrak{i}$ , reachable(base, jack) is  $\mathfrak{t}$  and reachable(base, resc) is  $\mathfrak{u}$ . Thus the rule in Line 3 results in evacuable(eve) being  $\mathfrak{i}$ , evacuable(jack) being  $\mathfrak{t}$  and evacuable(resc) being  $\mathfrak{t}$ .<sup>15</sup>. As noticed earlier, the truth value of willSave(resc, resc) is  $\mathfrak{i}$ . Using (32) and (33), we infer that willSave(resc, eve) is  $\mathfrak{i}$  and willSave(resc, jack) is  $\mathfrak{t}$ .

According to our definition of semantics, the model sketched above is obtained using the set of hypotheses compatible with assumptions made in nonmonotonic rules (using  $\sim$ ). For example,  $\mathcal{H}$  would in this case contain:

$$\begin{array}{ll} \sim \neg \textit{reachable}(\textit{base},\textit{eve}) \twoheadleftarrow t, & \sim \neg \textit{willSave}(\textit{eve},\textit{eve}) \twoheadleftarrow t, \\ \sim \neg \textit{reachable}(\textit{base},\textit{jack}) \twoheadleftarrow t, & \sim \neg \textit{willSave}(\textit{jack},\textit{jack}) \twoheadleftarrow t, \\ \sim \neg \textit{reachable}(\textit{base},\textit{resc}) \twoheadleftarrow t, & \sim \neg \textit{willSave}(\textit{resc},\textit{resc}) \twoheadleftarrow t. \end{array}$$

<sup>&</sup>lt;sup>14</sup>Without violating the result, we only list literals with truth value different from **u**.

<sup>&</sup>lt;sup>15</sup>Notice the nonmonotonic nature of rule in Line 3: due to the use of  $\sim$  we deduced the truth of *evacuable(resc)* on the basis of the unknown value of literal  $\neg$ *reachable(base, resc)* in the rule's premise.

Of course, other sets of hypotheses are also allowed and can result in different outcomes. For example, replacing  $\sim \neg reachable(base, jack) \ll t$  by  $\sim \neg reachable(base, jack) \ll f$  would make the conclusions *evacuable(jack)* and *willSave(resc, jack)* inconsistent.

## 8 Properties of 4SP

Let us now focus on the most important properties of 4SP.

**Theorem 8.1.** For every 4SP program  $\Pi$  and every set of hypotheses  $\mathcal{H}$  for  $\Pi$  provided as an input to Algorithm 11, the unique well-supported model is computed in deterministic polynomial time in the number of propositional variables in  $\Pi$  and the size of  $\mathcal{H}$ .

PROOF Note that interlace, in total, can add at most a linear (w.r.t. the number of propositional variables in  $\Pi$ ) number of new literals to the well-supported model. It now suffices to use the corresponding complexity result of [51] where computing well-supported models for 4QL is proved tractable.

To generate all well-supported models for a given 4SP program  $\Pi$  it suffices to iterate Algorithm 11 with sets  $\mathcal{H}$  reflecting different choices of truth values t, f assigned to default literals in  $\Pi$ . Therefore we have the following proposition.

**Proposition 8.2.** For every 4SP program there is at most an exponential number of wellsupported models with respect to the number of default literals occurring in  $\Pi$ .

The following theorem shows that classical answer sets are preserved.

**Theorem 8.3.** For every ASP program  $\Pi$ , the set of all well-supported models of  $\Pi$  (being a 4SP program) contains all answer sets of  $\Pi$ .

PROOF If there are no answer sets for  $\Pi$  then the conclusion is obvious. Otherwise, let  $\mathcal{I}$  be an answer set for  $\Pi$ . To obtain  $\mathcal{I}$  as a 4SP model it suffices to assume the set of hypotheses reflecting the contents of  $\mathcal{I}$ , i.e., to consider  $\Pi$  with  $\mathcal{H}$ , where:<sup>16</sup>

$$\mathcal{H} \stackrel{\text{def}}{=} \{ \sim \ell \twoheadleftarrow \mathfrak{f} \mid \ell \in \mathcal{I}, \sim \ell \text{ occurs in } \Pi \} \cup \{ \sim \ell \twoheadleftarrow \mathfrak{t} \mid \ell, \neg \ell \notin \mathcal{I}, \sim \ell \text{ occurs in } \Pi \}.$$
(35)

Indeed:

- when  $\ell \in \mathcal{I}$ , we have that  $\mathcal{I}(\sim \ell) = \mathfrak{f}$ ;
- when  $\ell, \neg \ell \notin \mathcal{I}$ , we have that  $\mathcal{I}(\sim \ell) = \mathbf{t}$ .

<sup>&</sup>lt;sup>16</sup>Note that  $\ell$  in (35) can be a positive or a negative literal.

Thus,  $\mathcal{H}$  defined in (35) forces the correct truth values of expressions involving  $\sim$ .

Let  $4SP^{I}$  denotes an extension of 4SP obtained by allowing inspection operators I to occur in premises of rules. We have the following theorem.

### Theorem 8.4.

- 1. Model generation for  $4SP^{\mathbb{I}}$  programs stratifiable with respect to  $\mathbb{I}$  is tractable.
- 2. Model generation for 4SP<sup>I</sup> programs without the stratifiability requirement is NPcomplete.

#### Proof

1. When a  $4SP^{\mathbb{I}}$  program is stratifiable with respect to  $\mathbb{I}$  it suffices to compute the wellsupported model stratum by stratum, starting from the stratum not containing inspection operator. In this case, when an inspection operator  $\ell \in T$  is being evaluated, the truth value of  $\ell$  is computed in an earlier stratum so verifying  $\ell \in T$  becomes trivial.

2. In order to show NP-completeness of model generation for  $4\text{SP}^{II}$  it suffices to show a  $4\text{SP}^{II}$  implementation of three-colorability of a graph (a well-known NP-complete problem). Let e(X,Y) denotes edges, v(X) denotes vertices and c(X,Z) stands for "the vertex X has color Z". Then Program 13 contains a  $4\text{SP}^{II}$  program whose models, if exist, are all colorings of the graph using the colors r, g, b.

**Program 13:** An implementation of three-colorability of a graph in  $4SP^{I}$ .

First, note that Program 13 has only consistent models (contains neither negation  $\neg$  nor  $\sim$ ). Second, we used first-order variables X, Y, Z as means to concisely represent propositional rules. Grounding of the program contains at most quadratic number of rules with respect to the number of constants occurring in facts  $e(\ldots)$ . Therefore, the conclusion of the theorem is true.

Let  $4SP^{FO}$  denotes 4SP where one is allowed to use first-order representation of rules Then we have the following theorem. **Theorem 8.5.**  $4SP^{FO}$  captures deterministic polynomial time over linearly ordered domains.

PROOF Stratified DATALOG<sup>¬</sup> captures deterministic polynomial time over linearly ordered domains (see, e.g., [2]). Stratified DATALOG<sup>¬</sup> programs can be reduced to 4SP programs by grounding. Well-supported 4SP models of such programs are just the standard models of stratified DATALOG<sup>¬</sup> programs.

Given a problem  $\pi$  computable in deterministic polynomial time and a linear ordering on the domain, there is a DATALOG<sup>¬</sup> program  $\Pi$  capturing  $\pi$ . The rules in the program are fixed, so the size of grounding of  $\Pi$  is at most polynomially larger than  $\Pi$ . Therefore, the grounding of  $\Pi$ , being a 4SP program, captures  $\pi$ , too. The set of hypotheses is empty so, according to Theorem 8.1, computing models of grounding of  $\Pi$  is tractable with respect to the number of propositional variables (grounded literals) which proves the result.

We then have the following obvious corollary.

**Corollary 8.6.** 4SP captures deterministic polynomial time over linearly ordered domains.

 $\triangleleft$ 

# 9 Related Work and Conclusions

The paper combines two threads: ASP [5, 7, 17, 39, 40, 41, 47, 48, 62] and 4QL [49, 50, 51, 63]. A detailed comparison of the selected features of ASP, 4QL and 4SP as well as models used in these languages is provided in Tables 3 and 4.

Language	Number of models	Finding a model	Stratification	Consistency
ASP	$\leq Exp$	NP	No	Yes
4ql	= 1	Р	Yes	No
4sp	$\geq$ 1, $\leq$ Exp	Р	No	No

Table 3: A comparison of ASP, 4QL and 4SP features.

Models	Truth	Consistency	Minimality	Supportedness
Stable models	t, f	Yes	Yes	Yes
Answer Sets	t, f, u	Yes	Yes	Yes
Well-supported	t, f, u, í	No	No	Yes

Table 4: A comparison of the discussed models.

In [54] the logic of here-and-there (HT) is used to define a direct declarative semantics

for ASP. HT can be defined by means of a five-valued logic. However, none of these values represents inconsistency.

Paraconsistent logic programming has been studied in [16] where Kripke-Kleene semantics investigated in [33] has been extended to a four-valued framework founded on Belnap logic  $B_4$ . The first paraconsistent approach to ASP has been proposed in [58], where a  $B_4^+$ -based framework is used and extended to six- and nine-valued frameworks for reasoning with inconsistency. Unlike [58], we use  $L_4^+$  together with trial/inspection operators as uniform means for disambiguating inconsistencies and completing missing knowledge in a nonmonotonic manner. For a survey of paraconsistent approaches to logic programming see also [21].

Paracoherent ASP [29] aims at reasoning from ASP programs lacking answer sets due to cyclic dependencies of atoms and their default negations. Program 9 with Line 1 substituted by the equivalent (with respect to  $K_3^+$ ) ASP rule ' $p:-\sim q$ .' is an example of such a dependence. For paracoherent reasoning [29] consider semi-stable models of [58] and semi-equilibrium models. In 4SP, Program 9 has a single model with both p and q inconsistent, { $p,\neg p, q,\neg q$ }, the same no matter whether  $\sim q$  is assumed t or f. Both, for semi-stable and semi-equilibrium semantics, model generation is proved intractable.

A hierarchy of tractable classes of stable models (over  $K_3^+$ ) has been reported in [9]. It reflects programs distance from their stratifiability. However, one of complexity factors considered there is, in the worst case, exponential with respect to number of propositional variables, what makes it intractable in the framework we consider.

Sufficient conditions for ASP guaranteeing tractability of answer set generation have been identified in the literature [31, 32]. Also, tractable default reasoning subsystems that can be translated into ASP have been considered in many sources, including [10, 30, 38, 60]. However, these approaches cover substantial subclasses of the general problem for which we have achieved tractability.

In summary, we have defined the 4SP language combining the ASP and 4QL ideas. We have gained tractability of model generation by relaxing the consistency requirement. That way, a prevalent use of paraconsistency allowed us to achieve tractability of model generation for an ASP-like language which, to our best knowledge, has not been achieved before. 4SP is intended to serve as a tool complementary/parallel to ASP, being useful for querying potentially inconsistent knowledge bases and providing models when ASP model generation fails due to complexity reasons or inconsistencies involved.

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