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ARTICLES

Two Normalizations for Natural Deductions in Sequent Style

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Abstract

By using sequent derivations, we will show the role of -forms of derivations, -substitution and -the nature and forms of natural deduction rules, in defining reduction steps of normalization procedures for natural deduction systems.

Keywords: natural deduction, sequent systems, normalization, cut elimination

1 Introduction

The results of correspondences between derivations of natural deduction and sequent systems and connections normalization with cut elimination were presented in many papers (see for example [3, 8, 9, 12, 15, 20, 23, 26]). (A summary of these results was given in the Introduction of [8].) We mention only the well-known results about unsymmetrical correspondences of rules (an elimination from natural deduction corresponds to a left sequent rule together with a cut), and the problems with $\vee$ and $\exists$ for connections between the reduction steps of normalization and cut elimination (see [26]).

In this paper we will study correspondences of rules and connections between normalization and cut elimination, by using two natural deduction systems in sequent style with explicit substitution, the systems **NI** and **NE**, which cover predicate intuitionistic logic. Namely, we want to investigate these correspondences and connections for systems which have characteristics of natural deduction and sequent systems, under the following uniform conditions: **(1)** their derivations are of the

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same form (in our case, they are sequent derivations); **(2)** their rules are two versions of right sequent rules; and **(3)** substitution is an explicit rule of both systems. The first system is the system **NI** from [6], which is a standard natural deduction system from [9] (the calculus NJ), [16] or [22] in sequent style (the system **Ni** from 2.1.1 in [19] in sequent style (see 2.1.4 in [19])) with explicit substitution. The second system is the system **NE** (i.e. $\mathbf{NE}^z$ from [6] without indices), which is the extended natural deduction system $\mathcal{NE}$ from [1] and [2] (i.e. natural deduction with general elimination rules from [21]) in sequent style with explicit substitution. The system **NE** is similar to the system NG+Subst from [14] (i.e natural deduction in sequent calculus style+Comp from [24], i.e. the system NLI+Comp from [25]) (see the Note 2.1 below).

Some solutions of the problems with $\vee$ and $\exists$ of connections normalization with cut elimination mentioned above were presented in [8] and [3]. Moreover, in [8] the role of substitution in these connections was shown. The systems from [3] are a standard natural deduction system and a standard sequent system and they do not have characteristics from **(1)**-**(3)**. In [8] a standard natural deduction system and a sequent system for propositional intuitionistic logic were studied, and derivations of both systems were presented as terms. Thus, the systems from [8] have the characteristics from **(1)** and **(3)** (where substitution is considered as their rule). In this paper we demand the systems to have the characteristic from **(2)**, because we want to study the role of nature of inference rules in connections of natural deduction and sequent systems without the differences between right and left-right sequent rules. We will define the maps e and n, which connect the derivations of **NI** and **NE**, and the result will be the following: the role of explicit substitution in connections between their two versions of right rules are the same as the role of cut in connections between natural deduction rules and standard (i.e. left-right) rules of sequent systems. However, by explicit substitution, a derivation Π from **NI** and its ne-image $\mathsf{ne}\Pi$ (a derivation π from **NE** and its en-image $\mathsf{en}\pi$) will be different derivations in **NI** (**NE**), and they will be connected by some new reductions, reductions which add substitutions (reductions which simplify elimination rules and add substitutions).

Explicit substitution will make more complex maximum segments than the standard ones (see the Note 2.2 below). In the system **NI** all reductions of derivations will be usual reductions of normalization for standard natural deduction (see for example [16] or [22]) with explicit substitutions presented in sequent style. In the system **NE** the basic of reductions of derivations will be detour conversions of normalization for NLI from [25] in NLI+Comp, whose basics are detour conversions from natural deduction with general elimination rules from [21] (which are also $\mathcal{E}$max-conversions from the extended natural deduction system $\mathcal{NE}$ from [2], [5]) and conversions of sequent derivations from [3] (see the Note 5.1, the Note 7.3 and the Note 8.1 below).

We will show that each reduction step from **NI** has the corresponding reduction step in **NE**, and vice versa, but some reductions of derivations from **NE** concerning maximum segments, correspond to reductions of derivations from **NI** concerning substitutions only. Moreover, we will show that the normalization procedure for **NE** does not make the complete normalization procedure for **NI**: if in **NE** there is a sequence of reductions, by which a derivation π is reduced into a normal derivation π_N, then in **NI** that sequence makes corresponding sequence of reductions by which the n-image of π, the derivation $\mathsf{n}\pi$, is reduced into the n-image of π_N, the derivation $\mathsf{n}\pi_N$, without maximum segments, but it can have some substitutions. We note that these characteristics are similar to characteristics of the connection of normalization of standard natural deduction and cut elimination (see for example [8], [20] and [26]). On the other hand, we can leave out the sequent systems and we want to study two natural deduction systems **NI** and **NE** and the role of the nature of their rules and substitution in connections between their normalizations. We think that the answer to the question: "Is it important to compare these systems?" can be Kreisel's sentence "...the distinction between formal systems with the same set of theorems, in terms of the proofs expressed by their derivations, is meaningful" from [11] (p. 243).

2 The systems NI and NE

Our language will be the language of the first order predicate logic, i.e. it will have the logical connectives $\wedge$, $\vee$ and $\supset$ (i.e. $\Rightarrow$), the quantifiers $\forall$ and $\exists$, and the propositional constant $\bot$ (for absurdity). Bound variables will be denoted by $x, y, z, ...$, free variables by $a, b, c, ...$ and individual terms by r, s, t,.... $P, Q, R, ...$ will denote atomic formulae and $A, B, C, ..., A_1, ...$ will denote arbitrary formulae. $\Gamma, \Delta, \Lambda, ..., \Gamma_1$,... will denote finite multisets of formulae. Γ, Δ will denote the multiset, which is the union of Γ and Δ. $[A]^n$ will denote the finite multiset of n formulae of the same form A, $n{\geq}0$ (in some cases $n{>}0$); $[A]^n$, Γ will denote the union of the multiset $[A]^n$ and the multiset Γ, where Γ can contain other formulae of the form A, which are not from $[A]^n$. If Γ is the union of $[A_1]^{k_1}, ..., [A_m]^{k_m}$, $m>0$, $k_i>0$, $1{\leq}i{\leq}m$, where $A_1, ..., A_m$ are formulae of different forms, then the multiset which consists of $k_i{\cdot}n$ formulae of the form A_i, $1{\leq}i{\leq}m$, $n{\geq}1$, will be denoted by Γ^n.

A sequent of derivations from **NI** and **NE** is of the form $\Gamma{\to}C$, where Γ consists of its *left formulae* and C is its *right formula*. The axioms of **NI** and **NE** are *general axioms* (see, for example, 3.1.6 in [19], p. 55): i-axioms (i.e. initial axioms) are of the form $B, A{\to}A$ and $\bot$-axioms are of the form $B, \bot{\to}P$, where B may not exist.

2.1 The system NI

Postulates for the system **NI**.
Axioms: i-axioms (i.e. initial axioms): $B, A \to A$

$\bot$-*axioms:* $B, \bot \to P$, where P is an atomic formula different from $\bot$

Inference rules

$$\text{(sb)}\ \frac{\Gamma \to A \qquad [A]^n, \Delta \to C}{\Gamma^n, \Delta \to C},\ n>0$$

operational rules:

eliminations / *introductions*

$$(\supset\text{E})\ \frac{\Gamma \to A \supset B \qquad \Delta \to A}{\Gamma, \Delta \to B} \qquad\qquad (\supset\text{I})\ \frac{[A]^n, \Gamma \to B}{\Gamma \to A \supset B},\ n\geq 0$$

$$(\wedge\text{E}_1)\ \frac{\Gamma \to A\wedge B}{\Gamma \to A} \quad (\wedge\text{E}_2)\ \frac{\Gamma \to A\wedge B}{\Gamma \to B} \qquad\qquad (\wedge\text{I})\ \frac{\Gamma \to A \qquad \Delta \to B}{\Gamma, \Delta \to A\wedge B}$$

$$(\vee\text{E})\ \frac{\Gamma \to A\vee B \quad [A]^n, \Delta \to C \quad [B]^m, \Lambda \to C}{\Gamma, \Delta, \Lambda \to C}, n, m\geq 0 \quad (\vee\text{I}_1)\ \frac{\Gamma \to A}{\Gamma \to A\vee B} \quad (\vee\text{I}_2)\ \frac{\Gamma \to B}{\Gamma \to A\vee B}$$

$$(\forall\text{E})\ \frac{\Gamma \to \forall x A x}{\Gamma \to A\mathrm{t}} \qquad\qquad (\forall\text{I})\ \frac{\Gamma \to Aa}{\Gamma \to \forall x A x}$$

$$(\exists\text{E})\ \frac{\Gamma \to \exists x A x \quad [Aa]^n, \Delta \to C}{\Gamma, \Delta \to C}, n\geq 0 \qquad\qquad (\exists\text{I})\ \frac{\Gamma \to A\mathrm{t}}{\Gamma \to \exists x A x}$$

In ($\forall$I) and ($\exists$E) the variable a is the *proper variable* of these rules, and it has to satisfy the *restrictions on variables*: in ($\forall$I) (($\exists$E)) a does not appear in $\Gamma, \forall x A x$ ($\Gamma, \Delta, \exists x A x, C$). In the rule (sb) i.e. the *substitution*: the formula A is its *left sb-formula* (i.e. *substitution formula*), the formulae from $[A]^n$ are its *right sb-formulae*, and the right formula of its right upper sequent is its *premiss*. In an operational rule: the right formulae of its upper sequents are *premisses of* that *rule*, where the premiss, whose connective or quantifier is explicitly shown, is the *major premiss of* that *rule*, while the premisses which are not major are its *minor premisses*. In each rule: the right formula of its lower sequent is its *consequence*, and formulae which are not premisses, consequences or sb-formulae will be called *assumptions* of that rule. (We note that the assumptions of rules are left formulae of their sequents.) In ($\vee$E), ($\supset$I) and ($\exists$E) the assumptions from [ ] are their *discharged assumptions*.
$\Pi, \Pi', \overline{\overline{\Pi}}$, Π_1...will denote derivations in **NI**. A derivation Π with the end sequent $\Gamma{\to}A$ will be denoted by $\begin{array}{c}\Pi\\ \Gamma{\to}A\end{array}$. $\dfrac{\begin{array}{c}\Pi_1\\ \Gamma_1{\to}A_1\end{array}}{\Gamma{\to}A}\text{R}$, $\dfrac{\begin{array}{c}\Pi_1\\ \Gamma_1{\to}A_1\end{array}\ \begin{array}{c}\Pi_2\\ \Gamma_2{\to}A_2\end{array}}{\Gamma{\to}A}\text{R}$ or $\dfrac{\begin{array}{c}\Pi_1\\ \Gamma_1{\to}A_1\end{array}\ \begin{array}{c}\Pi_2\\ \Gamma_2{\to}A_2\end{array}\ \begin{array}{c}\Pi_3\\ \Gamma_3{\to}A_3\end{array}}{\Gamma{\to}A}\text{R}$ will denote a derivation with the last rule R and the end sequent $\Gamma{\to}A$.

2.2 The system NE

We define the natural deduction system **NE**, which is the system $\mathcal{NE}$ from [2] (or [1]) in the sequent style with explicit substitution (see also $\mathbf{NE}^z$ from [6]).
Postulates for the system **NE**.
Axioms: i-axioms (i.e. initial axioms): $B, A \rightarrow A$

$\perp$-*axioms:* $B, \perp\rightarrow P$, where P is an atomic formula different from $\perp$

Inference rules

$$\text{(sb)}\ \frac{\Gamma \rightarrow A \qquad [A]^n, \Delta \rightarrow C}{\Gamma^n, \Delta \rightarrow C},\ \ n>0$$

operational rules:

eliminations — *introductions*

$$(\supset\mathrm{E}\mathcal{E})\ \frac{\Gamma\rightarrow A\supset B \qquad \Delta\rightarrow A \qquad [B]^n, \Lambda\rightarrow C}{\Gamma, \Delta, \Lambda\rightarrow C}, n>0 \qquad (\supset\mathrm{I}\mathcal{E})\ \frac{[A]^n, \Gamma\rightarrow B}{\Gamma\rightarrow A\supset B}, n\geq 0$$

$$(\wedge\mathrm{E}\mathcal{E}_1)\ \frac{\Gamma\rightarrow A\wedge B\ \ [A]^n, \Delta\rightarrow C}{\Gamma, \Delta\rightarrow C}\ \ (\wedge\mathrm{E}\mathcal{E}_2)\frac{\Gamma\rightarrow A\wedge B\ \ [B]^m, \Lambda\rightarrow C}{\Gamma, \Lambda\rightarrow C}, n, m>0\ \ (\wedge\mathrm{I}\mathcal{E})\ \frac{\Gamma\rightarrow A \quad \Delta\rightarrow B}{\Gamma, \Delta\rightarrow A\wedge B}$$

$$(\vee\mathrm{E}\mathcal{E})\ \frac{\Gamma\rightarrow A\vee B\ \ [A]^n, \Delta\rightarrow C\ \ [B]^m, \Lambda\rightarrow C}{\Gamma, \Delta, \Lambda\rightarrow C}, n, m\geq 0 \qquad (\vee\mathrm{I}\mathcal{E}_1)\ \frac{\Gamma\rightarrow A}{\Gamma\rightarrow A\vee B}\ \ (\vee\mathrm{I}\mathcal{E}_2)\ \frac{\Gamma\rightarrow B}{\Gamma\rightarrow A\vee B}$$

$$(\forall\mathrm{E}\mathcal{E})\ \frac{\Gamma\rightarrow\forall xAx \qquad [At]^n, \Delta\rightarrow C}{\Gamma, \Delta\rightarrow C}, n>0 \qquad (\forall\mathrm{I}\mathcal{E})\ \frac{\Gamma\rightarrow Aa}{\Gamma\rightarrow\forall xAx}$$

$$(\exists\mathrm{E}\mathcal{E})\ \frac{\Gamma\rightarrow\exists xAx \qquad [Aa]^n, \Delta\rightarrow C}{\Gamma, \Delta\rightarrow C}, n\geq 0 \qquad (\exists\mathrm{I}\mathcal{E})\ \frac{\Gamma\rightarrow A\mathrm{t}}{\Gamma\rightarrow\exists xAx}$$

In $(\forall\mathrm{I}\mathcal{E})$ and $(\exists\mathrm{E}\mathcal{E})$ a is the *proper variable*, which has to satisfy the *restrictions on variables* as in **NI**. In the rules of **NE** *sb-formulae*, *premisses*, *major premisses*, *minor premisses*, *consequences* and *assumptions* are defined as in **NI**; $(\supset\mathrm{I}\mathcal{E})$ and all eliminations have *discharged assumptions* in [ ].
$\pi, \pi', \overline{\pi},\ \pi_1....$ will denote derivations in the system **NE**, and the derivations with the end sequent $\Gamma\rightarrow A$ and the last rule R will be presented as in **NI**.

Note 2.1. *The system* ***NE*** *(i.e. the system* $\boldsymbol{NE}^z$ *from [6] without indices) is similar to natural deduction in sequent calculus style+Comp from [23] (i.e the system NG+Subst from [14], i.e. the system NLI+Comp from [25]). Namely, the system* ***NE*** *is the extended natural deduction system* $\mathcal{NE}$ *from [1] and [2] (whose elimination rules for* $\wedge$, $\supset$ *(i.e.* $\Rightarrow$*) and* $\forall$ *were introduced in [18]) in sequent style with explicit substitution, the other systems are von Plato's natural deduction with general elimination rules from [21] in sequent style with explicit composition, and the system* $\mathcal{NE}$ *and von Plato's system from [21] are similar. However, there are the following differences between their inference rules: in* $\mathcal{NE}$ *there are two elimination* *rules for* $\wedge$ *which correspond to the elimination rule for* $\wedge$ *from von Plato's system from [21]; in* $\mathcal{NE}$ *discharged assumptions of the eliminations for* $\wedge$, $\supset$ *and* $\forall$ *must exist, i.e. their numbers are greater than 0 (i.e. in* ($\wedge$E$\mathcal{E}_1$), ($\wedge$E$\mathcal{E}_2$), ($\supset$E$\mathcal{E}$), ($\forall$E$\mathcal{E}$) *of* ***NE****:* $n, m > 0$*), while in von Plato's system from [21] their numbers are greater or equal 0. Moreover, in* $\mathcal{NE}$ *formulae with indices can be used. In [1] and [2] the system* $\mathcal{NE}$ *was defined as a system between standard natural deduction and a standard sequent system, so elimination rules of* $\mathcal{NE}$ *must have forms such that there are connections of that system with both, standard natural deduction and the sequent system too. There is also the following characteristic of the system* ***NE****: its substitution rule corresponds to several successive rules Comp (Subst) from [23] and [25] ([14]). The reason for the form of substitution in the system* ***NE*** *is the form of implicit substitution in natural deduction: one derivation of a formula A can substitute several assumptions A of the other derivation.*

2.3 Substitutions in derivations from NI and NE

First we will derive a rule in **NI** and **NE**, the *0-substitution rule*, which will be used in the connections of their derivations. The *0-substitution rule*, sb^0, is $\dfrac{\Delta \to C}{\Gamma, \Delta \to C}$, where Γ is a non-empty multiset of formulae, and that *0-substitution rule* will be denoted by sb^0_Γ because Γ is added by that rule sb^0. The i-axioms ($\perp$-axioms) of the form $B, A \to A$ ($B, \perp \to P$) can be derived by i-axioms $A \to A$ ($\perp$-axioms $\perp \to P$) and the rule *0-substitution*: $\dfrac{\Delta \to C}{\Gamma, \Delta \to C}\mathrm{sb}^0_\Gamma$, and vice versa. Namely,

(1) if pi is a derivation of **NI** or **NE**, $\begin{array}{c} pi \\ \Delta \to C \end{array}$, we derive: $\dfrac{\dfrac{\dfrac{\begin{array}{c} pi \\ \Delta \to C \end{array} \quad B_1, C \to C}{B_1, \Delta \to C}\mathrm{sb}}{\dots}}{\Gamma, \Delta \to C}$, $B_1, ..., B_m$ is Γ, $m \geq 1$, and **...** is the sequence of the rules: $\dfrac{B_1, ..., B_{j-1}, \Delta \to C \;\; B_j, C \to C}{B_1, ..., B_{j-1}, B_j, \Delta \to C}\mathrm{sb}$, $2 \leq j \leq m$;

(2) by $A \to A$, $\perp \to P$ and the rule sb^0 we derive: $\dfrac{A \to A}{B, A \to A}\mathrm{sb}^0_B$ and $\dfrac{\perp \to P}{B, \perp \to P}\mathrm{sb}^0_B$.

In [6] $\mathbf{NI}^z$ and $\mathbf{NE}^z$ (i.e. **NI** and **NE** with indices) were connected with sequent systems, and the axioms $B, A \to A$ and $B, \perp \to P$ were used for these connections. It is well-known that $B, A \to A$ and $B, \perp \to P$ as well as the rule sb^0 corresponds to the substructural rule thinning (i.e. weakening) of sequent systems. However, sb^0 shows, better than these axioms, the characteristic of the natural deduction rules that we always have thinning (several or one) when we need it. Thus, in the rest part of this paper we will study **NI** and **NE** with axioms $A{\to}A$, $\bot{\to}P$ and $(\mathrm{sb}^0)\dfrac{\Delta \to C}{\Gamma, \Delta \to C}\mathrm{sb}^0_\Gamma$, $\Gamma{\neq}\emptyset$. The premiss, the consequence and assumptions of (sb^0) are as of (sb).

In an operational rule (a rule) of **NI** or **NE**: each assumption from the multiset Γ or Λ (Δ) of its lower sequent is the *copy of* the corresponding assumption from the same multiset Γ or Λ (Δ) of an upper sequent of that rule; in (sb) for each assumption from Γ there are n assumptions in Γ^n, which are its *copies*; in (sb) and (sb^0) the consequence C is the *copy of* the premiss C. In ($\vee$E) and ($\exists$E) of **NI** (all eliminations of **NE**) the consequence C is the *copy of* each premiss C.

In a derivation from **NI** or **NE**: there is an *edge* from each assumption D of an upper sequent of a rule, which is not discharged, to its copy D in the lower sequent of that rule (it is one kind of edges from [7]); if $A_1, ..., A_n$ is a sequence of formulae of that derivation such that: (1) A_1 is either the left formula of an axiom, or a left formula in Γ from the lower sequent of a rule sb^0_Γ, i.e. it is not a copy of any formula, (2) for each $i{<}n$ there is an edge from A_i to A_{i+1}, and (3) there is no edge from A_n, then $A_1, ..., A_n$ will be called the *a-edge of* A_1.

2.4 Segments in derivations from NI and NE

The maximum segments of derivations from **NI** and **NE** will be defined as well-known maximum segments of derivations from natural deduction: a sequence of formulae of the same form, whose first formula is the consequence of an introduction of a connective (a quantifier) and its last formula is the major premiss of an elimination of that connective (that quantifier). However, these maximum segments will be possible with the premisses and the consequences of substitutions and 0-substitutions (see the Note 2.2 below).

In a derivation from **NI** or **NE**, the *segment* is a sequence of right formulae of its sequents, $A_1, ..., A_n$, $n{\geq}1$, where (1) all A_i, $1{\leq}i{\leq}n$, are of the same form, the form A; (2) for each $i<n$: A_i and A_{i+1} are a premiss and the consequence of a rule, where A_{i+1} is the copy of A_i; (3) if A_1 (A_n) is the consequence (a premiss) of a rule, then each of its premisses (its consequence) is not of the form A. If *sg* is a segment, then the number of operational rules whose premisses belong to *sg* will be the *length of sg*, $l(sg)$. If the first formula of sg:$A_1, ..., A_n$, A_1, is the consequence of an introduction of a connective (a quantifier) and its last formula, A_n, is the major premiss of an elimination of that connective (that quantifier), then *sg* is a *maximum segment.* A maximum segment *sg* whose length is 1 will be a *maximum formula.*

Note 2.2. *By the definitions of rules and segments in derivations from **NI** and **NE**,*

if $sg{:}A_1, ..., A_n$ *is a segment of a derivation*
(1) Π *from* ***NI*** *and* $l(sg){>}1$*, then each* A_i*,* $1{\leq}i{\leq}n{-}1$*, is* $-$*the minor premiss of* $\vee E$ *or* $\exists E$*, or* $-$*the premiss or the consequence of a substitution or a 0-substitution;*
(2) π *from* ***NE*** *and* $l(sg){>}1$*, then each* A_i*,* $1{\leq}i{\leq}n{-}1$*, is* $-$*the minor premiss of an elimination, or* $-$*the premiss or the consequence of a substitution or a 0-substitution.*

3 The reductions of derivations from NI and NE

Note 3.1. *By the substitution as an explicit rule in* ***NI*** *and* ***NE*** *we want to show the complexity of reduction steps of the normalization procedures for these systems, which cannot be seen in the normalization procedure for standard natural deduction (see the Sections 4 and 5 below). On the other hand, our normal derivations in* ***NI*** *and* ***NE*** *will be normal derivations as in standard natural deduction, i.e. normal derivations in* ***NI*** *and* ***NE*** *will be without maximum segments and substitutions (see 4.3 and 5.3 below). The 0-substitutions, as a characteristic of natural deduction, can be rules of normal derivations in* ***NI*** *and* ***NE****.*

To define standard normal derivations, we need some reductions by which a derivation can be transformed into a derivation without any substitution. In 3.1 (3.2) we will define sb-reductions ($\mathcal{E}$sb-reductions) for derivations from **NI** (**NE**) which will be needed for the elimination of substitutions (see the Lemma 3.1 and the Lemma 3.2 below). In the systems **NI** and **NE**: a derivation which does not have substitutions will be called the *sb-free derivation.*

In each reduction of reductions of derivations from **NI** (**NE**) in 3.1, 3.3. 4.1 and 4.2 (3.2, 3.3, 5.1 and 5.2) below, its *redex* will be Π (π), its *contractum* will be $\overline{\overline{\Pi}}$ ($\overline{\pi}$) and they will be in one line, when it is possible, on the left side and on the right side, respectively. By *pi* will be denoted a derivation which can be from **NI** or **NE**, i.e. it is a derivation Π in **NI** and a derivation π in **NE**.
If $\dfrac{pi}{\Sigma \to C}$ and $\dfrac{\overline{pi}}{\overline{\Sigma} \to C}$ are the redex and the contractum of a reduction in **NI** or **NE**, D is the left formula of a sequent in *pi* whose a-edge ends in Σ, D is from (1) an axiom of *pi*, then in $\overline{pi}$ that axiom has either (1.1) the corresponding axioms (either one (but, it can be above a new sb in $\overline{\overline{\Pi}}$ (see ($\mathrm{mf}_{\vee_1}$) from 4.1 below) or two (see Π_4 of Π and $\overline{\overline{\Pi}}$ in ($\mathrm{ms}^{\vee}_{\supset}$) from 4.1 below)), (1.2) one sb^0_{Φ} (see Π_2 of Π and $\mathrm{sb}^0_{\Delta'}$ of $\overline{\overline{\Pi}}$ in ($\mathrm{mf}_{\wedge_1}$) from 4.1 below), or (1.3) the corresponding axiom and one sb^0_{Φ} (see Π_1 of Π and $\overline{\overline{\Pi}}$ of (sb_0) in 3.1 below); (2) Θ in the lower sequent of an sb^0_{Θ} from *pi*, then in $\overline{pi}$ that rule has the corresponding sb^0_{Θ}. So, if we consider Σ and $\overline{\Sigma}$ as sets, then they are the same, but they can be different miltisets. In each reduction below if Σ is $\Gamma^k, \Delta^m, \Theta$, $\overline{\Sigma}$ is $\Gamma^k, \Delta^{m \cdot n}, \Theta, \Theta$ for some $k, m{\geq}1$, $n{>}1$, and a formula D^{Σ} from Σ

belongs to (i) Γ^k, then the *r-number of* D^Σ is 1; (ii) Δ^m, then the *r-number of* D^Σ is n; (iii) Θ, then the *r-number of* D^Σ is 2. Moreover, in definitions where we will write "pi'' is obtained from pi' by replacing its subderivation pi, which is the redex of one of reductions, with the contractum $\overline{pi}$ of that reduction" we will assume that the part of pi'' below the last sequent of $\overline{pi}$ is the part of pi' below the last sequent of pi, whose each multiset $[B]^n$ is replaced by the multiset which is obtained in the following way: each formula of $[B]^n$ from an a-edge, which contains a formula B of the last sequent of pi, is replaced by m formulae B, where m is the r-number of that formula B from the last sequent of pi.

$$\dfrac{\begin{array}{c} pi \\ \Gamma \to A \end{array}}{\Delta \to A}\, sb...$$

will denote a derivation from **NI** or **NE** which consists of pi, pi' (which is not written) and the sequence of substitutions and 0-substitutions, such that the formulae A from $\Gamma{\to}A$ and $\Delta{\to}A$ belong to a segment sg and the premiss and the consequence of each substitution and each 0-substitution from that sequence belongs to sg (for each substitution its premiss is A from sg, where its left upper sequent is arbitrary and it is the last sequent of a suderivation of the derivation above).

An example of such derivation is:

$$\dfrac{\begin{array}{c} pi_2 \\ \Phi{\to}E \end{array} \quad \dfrac{\dfrac{\begin{array}{c} pi_1 \\ \Theta{\to}D \end{array} \quad \dfrac{\begin{array}{c} pi' \\ \Lambda{\to}C \end{array} \quad \begin{array}{c} pi \\ \Gamma{\to}A \end{array}}{\Gamma_1{\to}A}\,\mathrm{sb}}{\Gamma_2{\to}A}\,\mathrm{sb}}{\Gamma_3{\to}A}\,\mathrm{sb}^0_\Sigma}{\Delta{\to}A}\,\mathrm{sb}$$

where Γ_1 is the union of Γ without $[C]^n$ ($n{>}0$) and Λ^n; Γ_2 is the union of Γ_1 without $[D]^m$ ($m{>}0$) and Θ^m; Γ_3 is Σ, Γ_2; and Δ is the union of Γ_3 without $[E]^l$ ($l{>}0$) and Φ^l.

3.1 The sb-reductions of derivations from NI

(sb$_{-r}$)

$$\dfrac{\begin{array}{c} \Pi_1 \\ \Gamma \to A \end{array} \quad A \to A}{\Gamma \to A}\,\mathrm{sb} \qquad\qquad \begin{array}{c} \Pi_1 \\ \Gamma \to A \end{array}$$

(sb$_{-l}$) is similar to (sb$_{-r}$), its redex ends with an sb with left upper sequent $A{\to}A$.

(sb$_0$)

$$\dfrac{\begin{array}{c} \Pi_1 \\ \Gamma \to A \end{array} \quad \dfrac{\begin{array}{c} \Pi_2 \\ [A]^l, \Delta \to B \end{array}}{[A]^{l+k}, \Lambda, \Delta \to B}\,\mathrm{sb}^0_{[A]^k,\Lambda}}{\Gamma^{l+k}, \Lambda, \Delta \to B}\,\mathrm{sb} \qquad\qquad \dfrac{\dfrac{\begin{array}{c} \Pi_1 \\ \Gamma \to A \end{array} \quad \begin{array}{c} \Pi_2 \\ [A]^l, \Delta \to B \end{array}}{\Gamma^l, \Delta \to B}\,\mathrm{sb}}{\Gamma^{l+k}, \Lambda, \Delta \to B}\,\mathrm{sb}^0_{\Gamma^k,\Lambda}$$

where $l+k>0$, $l>0$. If l is 0, then the sb with $[A]^l$ does not exist in $\overline{\Pi}$.

($\mathrm{sb}_{\wedge I}$)

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma\to A\end{array}\quad \dfrac{\begin{array}{c}\Pi_2'\\ [A]^l,\Delta\to B\end{array}\quad \begin{array}{c}\Pi_2''\\ [A]^k,\Lambda\to C\end{array}}{[A]^n,\Delta,\Lambda\to B\wedge C}\wedge I}{\Gamma^n,\Delta,\Lambda\to B\wedge C}\mathrm{sb} \qquad \dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \Gamma\to A\end{array}\quad \begin{array}{c}\Pi_2'\\ [A]^l,\Delta\to B\end{array}}{\Gamma^l,\Delta\to B}\mathrm{sb}\quad \dfrac{\begin{array}{c}\Pi_1\\ \Gamma\to A\end{array}\quad \begin{array}{c}\Pi_2''\\ [A]^k,\Lambda\to C\end{array}}{\Gamma^k,\Lambda\to C}\mathrm{sb}}{\Gamma^n,\Delta,\Lambda\to B\wedge C}\wedge I$$

where $n=l+k>0$. If l (k) is 0, then the sb with $[A]^l$ ($[A]^k$) does not exist in $\overline{\Pi}$.

($\mathrm{sb}_{\vee E}$), ($\mathrm{sb}_{\supset E}$) and ($\mathrm{sb}_{\exists E}$) are similar to ($\mathrm{sb}_{\wedge I}$).

($\mathrm{sb}_{\vee I_1}$)

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma\to A\end{array}\quad \dfrac{\begin{array}{c}\Pi_2\\ [A]^n,\Delta\to B\end{array}}{[A]^n,\Delta\to B\vee C}\vee I_1}{\Gamma^n,\Delta\to B\vee C}\mathrm{sb} \qquad \dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \Gamma\to A\end{array}\quad \begin{array}{c}\Pi_2\\ [A]^n,\Delta\to B\end{array}}{\Gamma^n,\Delta\to B}\mathrm{sb}}{\Gamma^n,\Delta\to B\vee C}\vee I_1$$

($\mathrm{sb}_{\vee I_2}$), ($\mathrm{sb}_{\supset I}$), ($\mathrm{sb}_{\forall I}$), ($\mathrm{sb}_{\exists I}$), ($\mathrm{sb}_{\wedge E_1}$), ($\mathrm{sb}_{\wedge E_2}$), ($\mathrm{sb}_{\forall E}$) are similar to ($\mathrm{sb}_{\vee I_1}$).

$\Pi' \to_{sb} \Pi''$ iff Π'' is obtained from Π' by replacing its subderivation Π, which is the redex of an sb-reduction, with the contractum $\overline{\Pi}$ of that sb-reduction.
Π' $sb{>}_n$ Π'' i.e. Π' *is sb-reduced into* Π'' iff there are $\Pi_0, ..., \Pi_n$, $n\geq 0$, such that Π_0 is Π', Π_n is Π'', and for all $i<n$ (when $n>0$): $\Pi_i \to_{sb} \Pi_{i+1}$.

Lemma 3.1. *In **NI** each* Π *can be sb-reduced into an sb-free derivation* Π^{sf}.

Proof. By using the property that each derivation Π': $\dfrac{\begin{array}{c}\Pi_1\\ \Gamma\to A\end{array}\quad \begin{array}{c}\Pi_2\\ [A]^n,\Delta\to B\end{array}}{\Gamma^n,\Delta\to B}\mathrm{sb}$, where Π_1 and Π_2 are sb-free, can be sb-reduced into an sb-free derivation. It can be proved by an induction on the pair $\langle$the degree of Π', the rank of $\Pi'\rangle$, where the degree of Π' and the rank of Π' are defined as in [9]. □

3.2 The $\mathcal{E}$sb-reductions of derivations from NE

($\mathcal{E}\mathrm{sb}_*$) is (sb_*), when $*$ is $-r$, $-l$ or 0.
($\mathcal{E}\mathrm{sb}_*$) is (sb_*), when $*$ is $\wedge I$, $\vee I_1$, $\vee I_2$, $\supset I$, $\forall I$ or $\exists I$.

($\mathcal{E}\mathrm{sb}_{\wedge E_1}$)

$$\dfrac{\begin{array}{c}\pi_1\\ \Gamma\to A\end{array}\quad \dfrac{\begin{array}{c}\pi_2'\\ [A]^l; \Delta\to B\wedge C\end{array}\quad \begin{array}{c}\pi_2''\\ [A]^k; [B]^m; \Lambda\to D\end{array}}{[A]^n; \Delta,\Lambda\to D}\wedge E\mathcal{E}_1}{\Gamma^n; \Delta,\Lambda\to D}\mathrm{sb} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_1\\ \Gamma\to A\end{array}\quad \begin{array}{c}\pi_2'\\ [A]^l; \Delta\to B\wedge C\end{array}}{\Gamma^l; \Delta\to B\wedge C}\mathrm{sb}\quad \dfrac{\begin{array}{c}\pi_1\\ \Gamma\to A\end{array}\quad \begin{array}{c}\pi_2''\\ [A]^k; [B]^m; \Lambda\to D\end{array}}{\Gamma^k; [B]^l; \Lambda\to D}\mathrm{sb}}{\Gamma^n; \Delta,\Lambda\to D}\wedge E\mathcal{E}_1$$

where $n=l+k>0$. If l (k) is 0, then the sb with $[A]^l$ ($[A]^k$) does not exist in $\overline{\pi}$.

$(\mathcal{E}\mathrm{sb}_{\wedge E_2})$, $(\mathcal{E}\mathrm{sb}_{\vee E})$, $(\mathcal{E}\mathrm{sb}_{\supset E})$, $(\mathcal{E}\mathrm{sb}_{\forall E})$, $(\mathcal{E}\mathrm{sb}_{\exists E})$ are similar to $(\mathcal{E}\mathrm{sb}_{\wedge E_1})$.

$\pi' \to_{\mathcal{E}sb} \pi''$ iff π'' is obtained from π' by replacing its subderivation π, which is the redex of an $\mathcal{E}$sb-reduction, with the contractum $\overline{\pi}$ of that $\mathcal{E}$sb-reduction.

$\pi'\ \mathcal{E}sb{>}_n\ \pi''$ i.e. π' *is* $\mathcal{E}$*sb-reduced into* π'' iff there are $\pi_0, ..., \pi_n$, $n{\geq}0$, such that π_0 is π', π_n is π'', and for all $i{<}n$ (when $n{>}0$): $\pi_i \to_{\mathcal{E}sb} \pi_{i+1}$.

Lemma 3.2. *In* ***NE*** *each* π *can be* $\mathcal{E}$*sb-reduced into an sb-free derivation* π^{sf}.

Proof. The proof is similar to the proof of the Lemma 3.1. □

3.3 The ss-reductions of derivations from NI and NE

The reductions (ss_n) and (ss_0) below are reductions of derivations from both systems, where pi_k are Π_k in **NI** and π_k in **NE**, $1{\leq}k{\leq}3$.

(ss_n)

$$\dfrac{\begin{array}{c}pi_1\\ \Gamma\to A\end{array} \quad \dfrac{\begin{array}{c}pi_2\\ [A]^l,\Delta\to B\end{array}\quad \begin{array}{c}pi_3\\ [A]^k,[B]^m,\Lambda\to C\end{array}}{[A]^{l\cdot m},\Delta^m,[A]^k,\Lambda\to C}\,\mathrm{sb}}{\Gamma^{(l\cdot m)+k},\Delta^m,\Lambda\to C}\,\mathrm{sb} \qquad \dfrac{\dfrac{\begin{array}{c}pi_1\\ \Gamma\to A\end{array}\quad\begin{array}{c}pi_2\\ [A]^l,\Delta\to B\end{array}}{\Gamma^l,\Delta\to B}\,\mathrm{sb}\quad \dfrac{\begin{array}{c}pi_1\\ \Gamma\to A\end{array}\quad \begin{array}{c}pi_3\\ [A]^k,[B]^m,\Lambda\to C\end{array}}{\Gamma^k,[B]^m,\Lambda\to C}\,\mathrm{sb}}{\Gamma^{(l\cdot m)+k},\Delta^m,\Lambda\to C}\,\mathrm{sb}$$

$n{=}(l\cdot m){+}k, m{>}0$. If l (k) is 0, then the sb with $[A]^l$ ($[A]^k$) does not exist in $\overline{pi}$.

(ss_0)

$$\dfrac{\dfrac{\begin{array}{c}pi_1\\ \Delta\to B\end{array}\quad\begin{array}{c}pi_2\\ [B]^m,\Lambda\to C\end{array}}{\Delta^m,\Lambda\to C}\,\mathrm{sb}}{\Gamma,\Delta^m,\Lambda\to C}\,\mathrm{sb}^0_\Gamma \qquad\qquad \dfrac{\begin{array}{c}pi_1\\ \Delta\to B\end{array}\quad \dfrac{\begin{array}{c}pi_2\\ [B]^m,\Lambda\to C\end{array}}{\Gamma,[B]^m,\Lambda\to C}\,\mathrm{sb}^0_\Gamma}{\Gamma,\Delta^m,\Lambda\to C}\,\mathrm{sb}$$

$\Pi' \to_s \Pi''$ in **NI** ($\pi' \to_s \pi''$ in **NE**), iff Π'' is obtained from Π' (π'' is obtained from π') by replacing its subderivation, which is the redex of an ss-reduction, with the contractum of that ss-reduction.

4 The reductions of derivations from NI

There are three kinds of reductions of derivations from the system **NI**:
(1) the *mf-reductions* and the *ms-reductions*, which correspond to reductions of derivations of the normalization procedure for standard natural deduction (see [22], [16] or [17]): the mf-reductions correspond to the reductions from II §2 of [16] (by which a maximum formula can be eliminated), while the ms-reductions correspond to the reductions from the proof of Theorem 1 in IV §1 of [16] (by which a maximum segment sg, $l(sg){>}1$, is replaced by a maximum segment sg', where $l(sg'){<}l(sg)$);

(2) the *sb_E-reductions*, which permute one rule sb or sb^0 below one elimination whose major premiss is the consequence of that rule; (3) the *sb^+-reductions*, which "add a new substitution" to some derivations. The sb_E-reductions and the sb^+-reductions will be necessary in the connections between **NI** and **NE** (see the Section 6.2 and the Section 7 below).

4.1 The mf-reductions and the ms-reductions

In the mf-reduction ($\mathrm{mf}_{\wedge_1}$) below, if *sb...* from its redex Π consists of k ($k\geq 0$) rules and for each $j\leq k$, jth rule of *sb...* in Π is: $-$ a substitution with the premiss $A\wedge B$, then jth rule of *sb...* in $\overline{\Pi}$ is the substitution with the same left upper sequent, the same multiset of assumptions in the right upper sequent, the same multiset of right sb-formulae, but its premiss is A; $-$ an sb^0_Θ with the premiss $A\wedge B$, then jth rule of *sb...* in $\overline{\Pi}$ is sb^0_Θ with the same multiset of assumptions in the upper sequent, but its premiss is A. In all other mf-reductions and ms-reductions the connection between *sb...* from Π and *sb...* from $\overline{\Pi}$ is the same.

$\diamondsuit$ ***mf-reductions***

($\mathrm{mf}_{\wedge_1}$)

$$\dfrac{\dfrac{\dfrac{\begin{array}{cc}\Pi_1 & \Pi_2\\ \Gamma'\to A & \Delta'\to B\end{array}}{\Gamma',\Delta'\to A\wedge B}\wedge I}{\Gamma,\Delta\to A\wedge B}\,sb...}{\Gamma,\Delta\to A}\wedge E_1 \qquad \dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \Gamma'\to A\end{array}}{\Gamma',\Delta'\to A}\,\mathrm{sb}^0_{\Delta'}}{\Gamma,\Delta\to A}\,sb...$$

($\mathrm{mf}_{\vee_1}$)

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \Gamma'\to A\end{array}}{\Gamma'\to A\vee B}\vee I_1}{\Gamma\to A\vee B}\,sb... \quad \begin{array}{c}\Pi_2\\ [A]^n,\Delta\to C\end{array} \quad \begin{array}{c}\Pi_3\\ [B]^m,\Lambda\to C\end{array}}{\Gamma,\Delta,\Lambda\to C}\vee E \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \Gamma'\to A\end{array}}{\Gamma\to A}\,sb... \quad \begin{array}{c}\Pi_2\\ [A]^n,\Delta\to C\end{array}}{\Gamma^n,\Delta\to C}\,\mathrm{sb}}{\Gamma^n,\Delta,\Lambda\to C}\,\mathrm{sb}^0_\Lambda$$

where $n>0$. If n is 0, then $\overline{\Pi}$ is $\overline{\Pi}$ above, where $\dfrac{\begin{array}{c}\Pi_2\\ \Delta\to C\end{array}}{A,\Delta\to C}\,\mathrm{sb}^0_A$ is instead of Π_2.

($\mathrm{mf}_\supset$)

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ [A]^n,\Gamma'\to B\end{array}}{\Gamma'\to A\supset B}\supset I}{\Gamma\to A\supset B}\,sb... \quad \begin{array}{c}\Pi_2\\ \Delta\to A\end{array}}{\Gamma,\Delta\to B}\supset E \qquad \dfrac{\begin{array}{c}\Pi_2\\ \Delta\to A\end{array} \quad \dfrac{\begin{array}{c}\Pi_1\\ [A]^n,\Gamma'\to B\end{array}}{[A]^n,\Gamma\to B}\,sb...}{\Delta^n,\Gamma\to B}\,\mathrm{sb}$$

where $n>0$. If n is 0, then $\overline{\Pi}$ is
$$\dfrac{\begin{array}{c}\Pi_2\\ \Delta\to A\end{array}\qquad \dfrac{\begin{array}{c}\Pi_1\\ \dfrac{\Gamma'\to B}{\Gamma\to B}\,sb...\end{array}}{A,\Gamma\to B}\,\mathrm{sb}^0_A}{\Delta,\Gamma\to B}\,\mathrm{sb}.$$

($\mathrm{mf}_{\wedge_2}$) and ($\mathrm{mf}_{\forall}$) are similar to ($\mathrm{mf}_{\wedge_1}$); ($\mathrm{mf}_{\vee_2}$) and ($\mathrm{mf}_{\exists}$) are similar to ($\mathrm{mf}_{\vee_1}$).

$\begin{array}{c}\Pi\\ \dfrac{\Gamma\to A}{\Delta\to A}\vee\exists Es...\end{array}$ will denote a derivation which consists of Π and Π' (or Π, Π' and Π'') (Π' (Π' and Π'') is (are) not written) and the sequence of $\vee E$, $\exists E$, substitutions and 0-substitutions, such that: the formulae A from $\Gamma\to A$ and $\Delta\to A$ belong to a segment sg, and the premiss and the consequence of each substitution and each 0-substitution and one minor premiss of each $\vee E$, $\exists E$ from that sequence belong to sg (for each $\vee E$, $\exists E$: the subderivation, whose end sequent is its upper sequent with its major premiss, is arbitrary; for each sb and each sb^0: as in $sb...$). The example:

$$\dfrac{\begin{array}{c}\Pi_3\\ \Phi\to B\end{array}\quad \dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \Theta_1\to C\vee D\end{array}\quad \begin{array}{c}\Pi_2\\ [C]^k,\Theta_2\to A\end{array}\quad \dfrac{\begin{array}{c}\Pi'\\ \Lambda\to E\end{array}\quad\begin{array}{c}\Pi\\ \Gamma\to A\end{array}}{[D]^l,\Gamma_1\to A}\,\mathrm{sb}}{\Gamma_2\to A}\,\vee\mathrm{E}}{\Gamma_3\to A}\,\mathrm{sb}^0_\Sigma}{\Delta\to A}\,\mathrm{sb},$$

where $[D]^l,\Gamma_1$ is the union of Γ without $[E]^n$ ($n>0$) and Λ^n; Γ_2 is $\Theta_1,\Theta_2,\Gamma_1$; Γ_3 is Σ,Γ_2; and Δ is the union of Γ_3 without $[B]^m$ ($m>0$) and Φ^m.

◇ ***ms-reductions***

In each ms-reduction, $\vee\exists Es...$ from its redex Π and its contractum $\overline{\Pi}$ are the same.

($\mathrm{ms}^{\vee}_{\wedge_1}$) The redex Π is

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \Gamma'\to A\vee B\end{array}\quad \dfrac{\dfrac{\begin{array}{c}\Pi_2'\\ \Delta''\to C\end{array}\quad\begin{array}{c}\Pi_2''\\ \Delta'''\to D\end{array}}{\Delta'',\Delta'''\to C\wedge D}\,\wedge I}{[A]^n,\Delta'\to C\wedge D}\,\vee\exists Es...\quad \begin{array}{c}\Pi_3\\ [B]^m,\Lambda'\to C\wedge D\end{array}}{\Gamma',\Delta',\Lambda'\to C\wedge D}\,\vee E}{\Gamma,\Delta,\Lambda\to C\wedge D}\,sb...}{\Gamma,\Delta,\Lambda\to C}\,\wedge E_1$$

where Π_3 can have $\wedge I$ as the subderivation of Π which ends with $[A]^n,\Delta'\to C\wedge D$ and the contractum $\overline{\Pi}$ is

$$
\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \Gamma'\to A\vee B\end{matrix} \quad \dfrac{\dfrac{\dfrac{\begin{matrix}\Pi_2'\\ \Delta''\to C\end{matrix} \quad \begin{matrix}\Pi_2''\\ \Delta'''\to D\end{matrix}}{\Delta'',\Delta'''\to C\wedge D}\wedge I}{[A]^n,\Delta'\to C\wedge D}\vee\exists Es...}{[A]^n,\Delta'\to C}\wedge E_1 \quad \dfrac{\begin{matrix}\Pi_3\\ [B]^m,\Lambda'\to C\wedge D\end{matrix}}{[B]^m,\Lambda'\to C}\wedge E_1}{\Gamma',\Delta',\Lambda'\to C}\vee E}{\Gamma,\Delta,\Lambda\to C}sb...
$$

($\mathrm{ms}^{\vee}_{\wedge_2}$), ($\mathrm{ms}^{\vee}_{\forall}$), ($\mathrm{ms}^{\exists}_{\wedge_1}$), ($\mathrm{ms}^{\exists}_{\wedge_2}$) and ($\mathrm{ms}^{\exists}_{\forall}$) are similar to ($\mathrm{ms}^{\vee}_{\wedge_1}$).

($\mathrm{ms}^{\vee}_{\supset}$) The redex Π is

$$
\dfrac{\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \Gamma'\to A\vee B\end{matrix} \quad \dfrac{\dfrac{\begin{matrix}\Pi_2\\ [C]^l,\Delta''\to D\end{matrix}}{\Delta''\to C\supset D}\supset I}{[A]^n,\Delta'\to C\supset D}\vee\exists Es... \quad \begin{matrix}\Pi_3\\ [B]^m,\Lambda'\to C\supset D\end{matrix}}{\Gamma',\Delta',\Lambda'\to C\supset D}\vee E}{\Gamma,\Delta,\Lambda\to C\supset D}sb... \quad \begin{matrix}\Pi_4\\ \Theta\to C\end{matrix}}{\Gamma,\Delta,\Lambda,\Theta\to D}\supset E
$$

where Π_3 can have $\supset I$ as the subderivation of Π which ends with $[A]^n, \Delta'\to C\supset D$ and the contractum $\overline{\Pi}$ is

$$
\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \Gamma'\to A\vee B\end{matrix} \quad \dfrac{\dfrac{\dfrac{\begin{matrix}\Pi_2\\ [C]^l,\Delta''\to D\end{matrix}}{\Delta''\to C\supset D}\supset I}{[A]^n,\Delta'\to C\supset D}\vee\exists Es... \quad \begin{matrix}\Pi_4\\ \Theta\to C\end{matrix}}{[A]^n,\Delta',\Theta\to D}\supset E \quad \dfrac{\begin{matrix}\Pi_3\\ [B]^m,\Lambda'\to C\supset D\end{matrix} \quad \begin{matrix}\Pi_4\\ \Theta\to C\end{matrix}}{[B]^m,\Lambda',\Theta\to D}\supset E}{\Gamma',\Delta',\Lambda',\Theta,\Theta\to D}\vee E}{\Gamma,\Delta,\Lambda,\Theta,\Theta\to D}sb...
$$

($\mathrm{ms}^{\vee}_{\vee_1}$), ($\mathrm{ms}^{\vee}_{\vee_2}$), ($\mathrm{ms}^{\vee}_{\exists}$), ($\mathrm{ms}^{\exists}_{\vee_1}$), ($\mathrm{ms}^{\exists}_{\vee_2}$), ($\mathrm{ms}^{\exists}_{\supset}$) and ($\mathrm{ms}^{\exists}_{\exists}$) are similar to ($\mathrm{ms}^{\vee}_{\supset}$).

The mf-reductions and the ms-reductions make the set of the *m-reductions*.

$\Pi' \to_{mf_*} \Pi''$ ($\Pi' \to_{ms^{\bullet}_{*}} \Pi''$), where $*\in\{\wedge_1,\wedge_2,\vee_1,\vee_2,\supset,\forall,\exists\}$, $\bullet\in\{\vee,\exists\}$ and they can be omitted, iff Π'' is obtained from Π' by replacing its subderivation Π, which is the redex of an mf-reduction (mf_*) (an ms-reduction ($\mathrm{ms}^{\bullet}_{*}$)), with the contractum $\overline{\Pi}$ of that mf-reduction (ms-reduction).

$\Pi' \to_m \Pi''$ iff either $\Pi' \to_{mf} \Pi''$ or $\Pi' \to_{ms} \Pi''$.

4.2 The sb_E-reductions and the sb^+-reductions

♢ ***sb_E-reductions*** The contractum $\overline{\overline{\Pi}}$ of an sb_E-reduction is obtained from its redex Π by permuting one rule sb or sb^0 below one elimination whose major premiss is the consequence of that rule. For example, the redex and the contractum of (sb-$E_\exists$) are

$$\dfrac{\dfrac{\begin{matrix}\Pi_1'\\ \Theta\to C\end{matrix}\quad \begin{matrix}\Pi_1''\\ [C]^k, \Gamma\to\exists xAx\end{matrix}}{\Theta^k, \Gamma\to\exists xAx}\,\mathrm{sb}\qquad \begin{matrix}\Pi_2\\ [Aa]^n, \Delta\to D\end{matrix}}{\Theta^k, \Gamma, \Delta\to D}\,\exists E \quad\text{and}\quad \dfrac{\begin{matrix}\Pi_1'\\ \Theta\to C\end{matrix}\qquad \dfrac{\begin{matrix}\Pi_1''\\ [C]^k, \Gamma\to\exists xAx\end{matrix}\quad \begin{matrix}\Pi_2\\ [Aa]^n, \Delta\to D\end{matrix}}{[C]^k, \Gamma, \Delta\to D}\,\exists E}{\Theta^k, \Gamma, \Delta\to D}\,\mathrm{sb}$$

while the redex and the contractum of (sb^0-$E_\vee$) are

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \Gamma\to A\vee B\end{matrix}}{\Theta, \Gamma\to A\vee B}\,\mathrm{sb}^0_\Theta\quad \begin{matrix}\Pi_2\\ [A]^n, \Delta\to D\end{matrix}\quad \begin{matrix}\Pi_3\\ [B]^m, \Lambda\to D\end{matrix}}{\Theta, \Gamma, \Delta, \Lambda\to D}\,\vee E \text{ and } \dfrac{\dfrac{\begin{matrix}\Pi_1\\ \Gamma\to A\vee B\end{matrix}\ \begin{matrix}\Pi_2\\ [A]^n, \Delta\to D\end{matrix}\ \begin{matrix}\Pi_3\\ [B]^m, \Lambda\to D\end{matrix}}{\Gamma, \Delta, \Lambda\to D}\,\vee E}{\Theta, \Gamma, \Delta, \Lambda\to D}\,\mathrm{sb}^0_\Theta$$

$\Pi' \to_{E_*} \Pi''$, where $* \in \{\wedge_1, \wedge_2, \vee, \supset, \forall, \exists\}$ and it can be omitted iff Π'' is obtained from Π' by replacing its subderivation Π, which is the redex of either an (sb-E_*) or an (sb^0-E_*) reduction, with the contractum $\overline{\overline{\Pi}}$ of that reduction.

♢ ***sb^+-reductions*** The redex Π of an sb^+-reduction ends with the elimination of $\wedge, \supset$ or $\forall$ and its contractum $\overline{\overline{\Pi}}$ is Π with one substitution whose right upper sequent is an i-axiom.

$$(+_{\wedge_1})\quad \dfrac{\begin{matrix}\Pi_1\\ \Gamma\to A\wedge B\end{matrix}}{\Gamma\to A}\,\wedge E_1 \qquad\qquad \dfrac{\dfrac{\begin{matrix}\Pi_1\\ \Gamma\to A\wedge B\end{matrix}}{\Gamma\to A}\,\wedge E_1\quad A\to A}{\Gamma\to A}\,\mathrm{sb}$$

$(+_{\wedge_2})$ similar to $(+_{\wedge_1})$

$$(+_{\supset})\quad \dfrac{\begin{matrix}\Pi_1\\ \Gamma\to A\supset B\end{matrix}\quad \begin{matrix}\Pi_2\\ \Delta\to A\end{matrix}}{\Gamma, \Delta\to B}\,\supset E \qquad\qquad \dfrac{\dfrac{\begin{matrix}\Pi_1\\ \Gamma\to A\supset B\end{matrix}\quad \begin{matrix}\Pi_2\\ \Delta\to A\end{matrix}}{\Gamma, \Delta\to B}\,\supset E\quad B\to B}{\Gamma, \Delta\to B}\,\mathrm{sb}$$

$$(+_{\forall})\quad \dfrac{\begin{matrix}\Pi_1\\ \Gamma\to\forall xAx\end{matrix}}{\Gamma\to At}\,\forall E \qquad\qquad \dfrac{\dfrac{\begin{matrix}\Pi_1\\ \Gamma\to\forall xAx\end{matrix}}{\Gamma\to At}\,\forall E\quad At\to At}{\Gamma\to At}\,\mathrm{sb}$$

$\Pi' \to_{+_*} \Pi''$, where $* \in \{\wedge_1, \wedge_2, \supset, \forall\}$ and it can be omitted iff Π'' is obtained from Π' by replacing its subderivation Π, which is the redex of a $(+_*)$ reduction, with the contractum $\overline{\overline{\Pi}}$ of that reduction.

4.3 The normal derivations in the system NI

If Π is sb-free and it does not have any subderivation which is the redex of an m-reduction, then Π will be called a *normal derivation* in **NI**.

If there is a sequence $\Pi_0, ..., \Pi_n$, $n{>}0$, such that Π_0 is Π', Π_n is Π'', and for all $i < n$:
(1) either Π_i sb$>_{k_i}\Pi_{i+1}$, $k_i{>}0$, or $\Pi_i \to_m \Pi_{i+1}$, then Π' *s-max*$>_n$ Π'', i.e. the derivation Π' *is s-max-reduced into* the derivation Π'';
(2) $\Pi_i \to_\circ \Pi_{i+1}$, where $\circ \in \{s, E, +\}$, then $\Pi' >^\circ_n \Pi''$;
(3) either Π_i *s-max*$>_1$ Π_{i+1} or $\Pi_i \to_\circ \Pi_{i+1}$, where $\circ \in \{s, E, +\}$, then Π'*s-max*$>^\circ_n\Pi''$.

5 The reductions of derivations from NE

There are three kinds of reductions of derivations from the system **NE**:
(1) the *$\mathcal{E}$mf-reductions* and the *$\mathcal{E}$ms-reductions*, where the $\mathcal{E}$mf-reductions eliminate maximum formulae, while the $\mathcal{E}$ms-reductions decrease the lengths of maximum segments (for details see the Note 5.1 below); (2) the *gz-reductions* (Gentzen-Zucker's reductions), which replace one elimination by its "simple form" and one substitution; (3) the *sb$^-$-reductions*, which "delete one substitution" in some derivations. The gz-reductions and the sb$^-$-reductions will be necessary in the connections between **NI** and **NE** (see the Section 6.2 and the Section 7 below).

Note 5.1. *The basics of the $\mathcal{E}$mf-reductions are the $\mathcal{E}$maxf-conversions of normalization for the system $\mathcal{NE}$ from [2] and [5] (which are the same as detour conversions of normalization for Plato's system from [21]). The basics of the $\mathcal{E}$ms-reductions ($\mathcal{E}ms^\bullet_*$), where $\bullet \in \{\wedge_1, \wedge_2, \vee, \supset, \forall, \exists\}$ and $* \in \{\vee_1, \vee_2, \exists\}$, are the $\mathcal{E}$maxs-conversions of normalization for $\mathcal{NE}$ (i.e permutation conversions of normalization for the system from [21]). Furthermore these $\mathcal{E}$ms-reductions and the $\mathcal{E}$mf-reductions of sb-free derivations from **NE** are detour conversions and permutation conversions, respectively, of normalization for NLI from [25]. The other $\mathcal{E}$ms-reductions, the reductions ($\mathcal{E}ms^\bullet_*$), where $\bullet \in \{\wedge_1, \wedge_2, \vee, \supset, \forall, \exists\}$ and $* \in \{\wedge_1, \wedge_2, \supset, \forall\}$, are defined by using $\diamondsuit$**IX**$\diamondsuit$-conversions for derivations of the sequent system $\mathcal{S}$ from [3] (whose sources are Zucker's reductions from 7.8.2 in [26]). Namely, the redex of a $\diamondsuit$**IX**$\diamondsuit$-conversion*

$$\text{is}\quad \dfrac{\dfrac{\overset{\mathcal{D}_1}{\Delta' \to C}}{\Delta \to C}\,\mathrm{R1} \qquad \dfrac{\overset{\mathcal{D}_2}{\Lambda' \to E}}{C, \Lambda \to E}\,\mathrm{R2}}{\Delta, \Lambda \to E}\,\mathrm{cut},$$

R1 *is the left rule for $\vee$ or $\exists$,* R2 *is the left rule for $\wedge$, $\supset$ or $\forall$ (so, in the redex above* R1 *(*R2*) has two upper sequents when it is the left rule for $\vee$ ($\supset$)), and the contractum is the result of conversions like Zucker's conversions from*

7.8.2 in [26] (it is not the result of the permutation of the cut above R1*, which is a standard reduction step). In **NE** these conversions will be used as follows: the redex of an $\mathcal{E}$ms-reduction ends with two eliminations (the upper elimination corresponds to* R1*, but it can be an elimination of $\wedge, \vee, \supset, \forall$ or $\exists$ not only of $\vee$ or $\exists$, and the last elimination, which can be an elimination of $\wedge, \supset$ or $\forall$, corresponds to* R2*) and its contractum will be obtained from that redex similarly as in $\diamondsuit$**IX**$\diamondsuit$-conversions.*

5.1 The $\mathcal{E}$mf-reductions and the $\mathcal{E}$ms-reductions

In all reductions below *sb*... from Π and $\overline{\Pi}$ are connected as in m-reductions in **NI**.
$\diamondsuit$ ***$\mathcal{E}$mf-reductions***

($\mathcal{E}\mathrm{mf}_{\wedge_1}$)

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\Gamma' \to A} \quad \overset{\pi_2}{\Delta' \to B}}{\Gamma', \Delta' \to A \wedge B}\wedge I\mathcal{E}}{\Gamma, \Delta \to A \wedge B}sb... \qquad \overset{\pi_3}{[A]^n, \Lambda \to C}}{\Gamma, \Delta, \Lambda \to C}\wedge E\mathcal{E}_1 \qquad \dfrac{\dfrac{\dfrac{\overset{\pi_1}{\Gamma' \to A}}{\Gamma', \Delta' \to A}\mathrm{sb}^0_{\Delta'}}{\Gamma, \Delta \to A}sb... \qquad \overset{\pi_3}{[A]^n, \Lambda \to C}}{\Gamma^n, \Delta^n, \Lambda \to C}\mathrm{sb}$$

($\mathcal{E}\mathrm{mf}_{\vee_1}$)

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\Gamma' \to A}}{\Gamma' \to A \vee B}\vee I\mathcal{E}_1}{\Gamma \to A \vee B}sb... \quad \overset{\pi_2}{[A]^n, \Delta \to C} \quad \overset{\pi_3}{[B]^m, \Lambda \to C}}{\Gamma, \Delta, \Lambda \to C}\vee E\mathcal{E} \quad \dfrac{\dfrac{\dfrac{\overset{\pi_1}{\Gamma' \to A}}{\Gamma \to A}sb... \quad \overset{\pi_2}{[A]^n, \Delta \to C}}{\Gamma^n, \Delta \to C}\mathrm{sb}}{\Gamma^n, \Delta, \Lambda \to C}\mathrm{sb}^0_{\Lambda}$$

where $n>0$. If n is 0, then $\overline{\pi}$ is $\overline{\pi}$ above, where $\dfrac{\overset{\pi_2}{\Delta \to C}}{A, \Delta \to C}\mathrm{sb}^0_A$ is instead of π_2.

($\mathcal{E}\mathrm{mf}_{\supset}$)

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{[A]^n, \Gamma' \to B}}{\Gamma' \to A \supset B}\supset I\mathcal{E}}{\Gamma \to A \supset B}sb... \quad \overset{\pi_2}{\Delta \to A} \quad \overset{\pi_3}{[B]^m, \Lambda \to C}}{\Gamma, \Delta, \Lambda \to C}\supset E\mathcal{E} \quad \dfrac{\dfrac{\overset{\pi_2}{\Delta \to A} \quad \dfrac{\overset{\pi_1}{[A]^n, \Gamma' \to B}}{[A]^n, \Gamma \to B}sb...}{\Delta^n, \Gamma \to B}\mathrm{sb} \quad \overset{\pi_3}{[B]^m, \Lambda \to C}}{\Delta^{n \cdot m}, \Gamma^m, \Lambda \to C}\mathrm{sb}$$

when $n>0$.The subderivation of $\overline{\pi}$ with end $[A]^n, \Gamma \to B$ is $\dfrac{\dfrac{\overset{\pi_1}{\Gamma' \to B}}{\Gamma \to B}sb...}{A, \Gamma \to B}\mathrm{sb}^0_A$, when $n=0$.

$(\mathcal{E}\mathrm{mf}_{\wedge_2})$, $(\mathcal{E}\mathrm{mf}_{\forall})$ and $(\mathcal{E}\mathrm{mf}_{\exists})$ are similar to $(\mathcal{E}\mathrm{mf}_{\wedge_1})$. $(\mathcal{E}\mathrm{mf}_{\vee_2})$ is similar to $(\mathcal{E}\mathrm{mf}_{\vee_1})$.

$\dfrac{\overset{\pi}{\Gamma\to A}}{\Delta\to A}Es...$ will denote a derivation which consists of π and π' (or π, π' and π'') (π' (π' and π'') is (are) not written) and the sequence of eliminations, substitutions and 0-substitutions, such that the formulae A from $\Gamma\to A$ and $\Delta\to A$ belong to a segment sg, and the premisses and the consequences of these substitutions and 0-substitutions and one minor premiss of each elimination from that sequence belong to sg (see derivations with $\vee\exists Es...$ from 4.1 above).

◇ ***$\mathcal{E}$ms-reductions***

In each $\mathcal{E}$ms-reduction, $Es...$ from its redex π and its contractum $\overline{\pi}$ are the same.

$(\mathcal{E}ms^{\wedge_1}_{\wedge_1})$ The redex π is

$$\cfrac{\cfrac{\cfrac{\overset{\pi_1}{\Gamma'\to A\wedge B}\qquad \cfrac{\cfrac{\overset{\pi_2'}{\Delta''\to C}\quad \overset{\pi_2''}{\Delta'''\to D}}{\Delta'',\Delta'''\to C\wedge D}\wedge I\mathcal{E}}{[A]^n,\Delta'\to C\wedge D}Es...}{\Gamma',\Delta'\to C\wedge D}\wedge E\mathcal{E}_1}{\Gamma,\Delta\to C\wedge D}sb...\qquad \overset{\pi_3}{[C]^k,\Lambda\to E}}{\Gamma,\Delta,\Lambda\to E}\wedge E\mathcal{E}_1$$

the contractum $\overline{\pi}$ is

$$\cfrac{\cfrac{\cfrac{\overset{\pi_1}{\Gamma'\to A\wedge B}\qquad \cfrac{\cfrac{\cfrac{\overset{\pi_2'}{\Delta''\to C}\quad \overset{\pi_2''}{\Delta'''\to D}}{\Delta'',\Delta'''\to C\wedge D}\wedge I\mathcal{E}}{[A]^n,\Delta'\to C\wedge D}Es...\quad C\to C}{[A]^n,\Delta'\to C}\wedge E\mathcal{E}_1}{\Gamma',\Delta'\to C}\wedge E\mathcal{E}_1}{\Gamma,\Delta\to C}sb...\qquad \overset{\pi_3}{[C]^k,\Lambda\to E}}{\Gamma^k,\Delta^k,\Lambda\to E}\mathrm{sb}$$

$(\mathcal{E}ms^{\wedge_1}_{\supset})$ The redex π is

$$\cfrac{\cfrac{\cfrac{\overset{\pi_1}{\Gamma'\to A\wedge B}\qquad \cfrac{\cfrac{\overset{\pi_2}{[C]^k,\Delta''\to D}}{\Delta''\to C\supset D}\supset I\mathcal{E}}{[A]^n,\Delta'\to C\supset D}Es...}{\Gamma',\Delta'\to C\supset D}\wedge E\mathcal{E}_1}{\Gamma,\Delta\to C\supset D}sb...\qquad \overset{\pi_3}{\Lambda\to C}\quad \overset{\pi_4}{[D]^m,\Theta\to E}}{\Gamma,\Delta,\Lambda,\Theta\to E}\supset E\mathcal{E}$$

and the contractum $\overline{\pi}$ is

$$
\begin{prooftree}
\AxiomC{π_1}
\noLine
\UnaryInfC{$\Gamma'\to A\wedge B$}
\AxiomC{π_2}
\noLine
\UnaryInfC{$[C]^k, \Delta''\to D$}
\RightLabel{$\supset I\mathcal{E}$}
\UnaryInfC{$\Delta''\to C\supset D$}
\RightLabel{$Es\ldots$}
\UnaryInfC{$[A]^n, \Delta'\to C\supset D$}
\AxiomC{π_3}
\noLine
\UnaryInfC{$\Lambda\to C$}
\AxiomC{$D\to D$}
\RightLabel{$\supset E\mathcal{E}$}
\TrinaryInfC{$[A]^n, \Delta', \Lambda\to D$}
\RightLabel{$\wedge E\mathcal{E}_1$}
\BinaryInfC{$\Gamma', \Delta', \Lambda\to D$}
\RightLabel{$sb\ldots$}
\UnaryInfC{$\Gamma, \Delta, \Lambda\to D$}
\AxiomC{π_4}
\noLine
\UnaryInfC{$[D]^m, \Theta\to E$}
\RightLabel{sb}
\BinaryInfC{$\Gamma^m, \Delta^m, \Lambda^m, \Theta\to E$}
\end{prooftree}
$$

$(\mathcal{E}ms^{\wedge_1}_{\wedge_2})$, $(\mathcal{E}ms^{\wedge_1}_{\forall})$, $(\mathcal{E}ms^{\wedge_2}_{\wedge_1})$, $(\mathcal{E}ms^{\wedge_2}_{\wedge_2})$, $(\mathcal{E}ms^{\wedge_2}_{\forall})$, $(\mathcal{E}ms^{\supset}_{\wedge_1})$, $(\mathcal{E}ms^{\supset}_{\wedge_2})$, $(\mathcal{E}ms^{\supset}_{\forall})$, $(\mathcal{E}ms^{\forall}_{\wedge_1})$, $(\mathcal{E}ms^{\forall}_{\wedge_2})$, $(\mathcal{E}ms^{\forall}_{\forall})$ are similar to $(\mathcal{E}ms^{\wedge_1}_{\wedge_1})$. $(\mathcal{E}ms^{\wedge_2}_{\supset})$, $(\mathcal{E}ms^{\forall}_{\supset})$ are similar to $(\mathcal{E}ms^{\wedge_1}_{\supset})$.

$(\mathcal{E}ms^{\wedge_1}_{\vee_1})$ The redex π is

$$
\begin{prooftree}
\AxiomC{π_1}
\noLine
\UnaryInfC{$\Gamma'\to A\wedge B$}
\AxiomC{π_2}
\noLine
\UnaryInfC{$\Delta''\to C$}
\RightLabel{$\vee I\mathcal{E}_1$}
\UnaryInfC{$\Delta''\to C\vee D$}
\RightLabel{$Es\ldots$}
\UnaryInfC{$[A]^n, \Delta'\to C\vee D$}
\RightLabel{$\wedge E\mathcal{E}_1$}
\BinaryInfC{$\Gamma', \Delta'\to C\vee D$}
\RightLabel{$sb\ldots$}
\UnaryInfC{$\Gamma, \Delta\to C\vee D$}
\AxiomC{π_3}
\noLine
\UnaryInfC{$[C]^k, \Lambda\to E$}
\AxiomC{π_4}
\noLine
\UnaryInfC{$[D]^l, \Theta\to E$}
\RightLabel{$\vee E\mathcal{E}$}
\TrinaryInfC{$\Gamma, \Delta, \Lambda, \Theta\to E$}
\end{prooftree}
$$

the contractum $\overline{\pi}$ is

$$
\begin{prooftree}
\AxiomC{π_1}
\noLine
\UnaryInfC{$\Gamma'\to A\wedge B$}
\AxiomC{π_2}
\noLine
\UnaryInfC{$\Delta''\to C$}
\RightLabel{$\vee I\mathcal{E}$}
\UnaryInfC{$\Delta''\to C\vee D$}
\RightLabel{$Es\ldots$}
\UnaryInfC{$[A]^n, \Delta'\to C\vee D$}
\AxiomC{π_3}
\noLine
\UnaryInfC{$[C]^k, \Lambda\to E$}
\AxiomC{π_4}
\noLine
\UnaryInfC{$[D]^l, \Theta\to E$}
\RightLabel{$\vee E\mathcal{E}$}
\TrinaryInfC{$[A]^n, \Delta', \Lambda, \Theta\to E$}
\RightLabel{$\wedge E\mathcal{E}_1$}
\BinaryInfC{$\Gamma', \Delta', \Lambda, \Theta\to E$}
\RightLabel{$sb\ldots$}
\UnaryInfC{$\Gamma, \Delta, \Lambda, \Theta\to E$}
\end{prooftree}
$$

$(\mathcal{E}ms^{\wedge_1}_{\vee_2})$, $(\mathcal{E}ms^{\wedge_1}_{\exists})$, $(\mathcal{E}ms^{\wedge_2}_{\vee_1})$, $(\mathcal{E}ms^{\wedge_2}_{\vee_2})$, $(\mathcal{E}ms^{\wedge_2}_{\exists})$, $(\mathcal{E}ms^{\supset}_{\vee_1})$, $(\mathcal{E}ms^{\supset}_{\vee_2})$, $(\mathcal{E}ms^{\supset}_{\exists})$, $(\mathcal{E}ms^{\forall}_{\vee_1})$, $(\mathcal{E}ms^{\forall}_{\vee_2})$ and $(\mathcal{E}ms^{\forall}_{\exists})$ are similar to $(\mathcal{E}ms^{\wedge_1}_{\vee_1})$.

$(\mathcal{E}ms^{\supset}_{\supset})$ The redex π is

$$
\begin{prooftree}
\AxiomC{π_1}
\noLine
\UnaryInfC{$\Gamma'\to A\supset B$}
\AxiomC{π_2}
\noLine
\UnaryInfC{$\Lambda'\to A$}
\AxiomC{π_3}
\noLine
\UnaryInfC{$[C]^m, \Delta''\to D$}
\RightLabel{$\supset I\mathcal{E}$}
\UnaryInfC{$\Delta''\to C\supset D$}
\RightLabel{$Es\ldots$}
\UnaryInfC{$[B]^n, \Delta'\to C\supset D$}
\RightLabel{$\supset E\mathcal{E}$}
\TrinaryInfC{$\Gamma', \Lambda', \Delta'\to C\supset D$}
\RightLabel{$sb\ldots$}
\UnaryInfC{$\Gamma, \Lambda, \Delta\to C\supset D$}
\AxiomC{π_4}
\noLine
\UnaryInfC{$\Theta\to C$}
\AxiomC{π_5}
\noLine
\UnaryInfC{$[D]^l, \Phi\to E$}
\RightLabel{$\supset E\mathcal{E}$}
\TrinaryInfC{$\Gamma, \Lambda, \Delta, \Theta, \Phi\to E$}
\end{prooftree}
$$

the contractum $\overline{\pi}$ is

$$
\begin{prooftree}
\AxiomC{π_1}\noLine
\UnaryInfC{$\Gamma'\to A\supset B$}
\AxiomC{π_2}\noLine
\UnaryInfC{$\Lambda'\to A$}
\AxiomC{π_3}\noLine
\UnaryInfC{$[C]^m,\Delta''\to D$}
\RightLabel{$\supset I\mathcal{E}$}
\UnaryInfC{$\Delta''\to C\supset D$}
\RightLabel{$Es\ldots$}
\UnaryInfC{$[B]^n,\Delta'\to C\supset D$}
\AxiomC{π_4}\noLine
\UnaryInfC{$\Theta\to C$}
\AxiomC{$D\to D$}
\RightLabel{$\supset E\mathcal{E}$}
\TrinaryInfC{$[B]^n,\Delta',\Theta\to D$}
\RightLabel{$\supset E\mathcal{E}$}
\TrinaryInfC{$\Gamma',\Lambda',\Delta',\Theta\to D$}
\RightLabel{$sb\ldots$}
\UnaryInfC{$\Gamma,\Lambda,\Delta,\Theta\to D$}
\AxiomC{π_5}\noLine
\UnaryInfC{$[D]^l,\Phi\to E$}
\RightLabel{sb}
\BinaryInfC{$\Gamma^l,\Lambda^l,\Delta^l,\Theta^l,\Phi\to E$}
\end{prooftree}
$$

$(\mathcal{E}ms^{\exists}_{\supset})$ and $(\mathcal{E}ms^{\vee}_{\supset})$ are similar to $(\mathcal{E}ms^{\supset}_{\supset})$.

$(\mathcal{E}ms^{\vee}_{\wedge_1})$ The redex π and the contractum $\overline{\pi}$ are

$$
\begin{prooftree}
\AxiomC{π_1}\noLine
\UnaryInfC{$\Gamma'\to A\vee B$}
\AxiomC{π_2'}\noLine
\UnaryInfC{$\Delta''\to C$}
\AxiomC{π_2''}\noLine
\UnaryInfC{$\Delta'''\to D$}
\RightLabel{$\wedge I\mathcal{E}$}
\BinaryInfC{$\Delta'',\Delta'''\to C\wedge D$}
\RightLabel{$Es\ldots$}
\UnaryInfC{$[A]^n,\Delta'\to C\wedge D$}
\AxiomC{π_3}\noLine
\UnaryInfC{$[B]^m,\Lambda'\to C\wedge D$}
\RightLabel{$\vee E\mathcal{E}$}
\TrinaryInfC{$\Gamma',\Delta',\Lambda'\to C\wedge D$}
\RightLabel{$sb\ldots$}
\UnaryInfC{$\Gamma,\Delta,\Lambda\to C\wedge D$}
\AxiomC{π_4}\noLine
\UnaryInfC{$[C]^k,\Theta\to E$}
\RightLabel{$\wedge E\mathcal{E}_1$}
\BinaryInfC{$\Gamma,\Delta,\Lambda,\Theta\to E$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{π_1}\noLine
\UnaryInfC{$\Gamma'\to A\vee B$}
\AxiomC{π_2'}\noLine
\UnaryInfC{$\Delta''\to C$}
\AxiomC{π_2''}\noLine
\UnaryInfC{$\Delta'''\to D$}
\RightLabel{$\wedge I\mathcal{E}$}
\BinaryInfC{$\Delta'',\Delta'''\to C\wedge D$}
\RightLabel{$Es\ldots$}
\UnaryInfC{$[A]^n,\Delta'\to C\wedge D$}
\AxiomC{$C\to C$}
\RightLabel{$\wedge E\mathcal{E}_1$}
\BinaryInfC{$[A]^n,\Delta'\to C$}
\AxiomC{π_3}\noLine
\UnaryInfC{$[B]^m,\Lambda'\to C\wedge D$}
\AxiomC{$C\to C$}
\RightLabel{$\wedge E\mathcal{E}_1$}
\BinaryInfC{$[B]^m,\Lambda'\to C$}
\RightLabel{$\vee E\mathcal{E}$}
\TrinaryInfC{$\Gamma',\Delta',\Lambda'\to C$}
\RightLabel{$sb\ldots$}
\UnaryInfC{$\Gamma,\Delta,\Lambda\to C$}
\AxiomC{π_4}\noLine
\UnaryInfC{$[C]^k,\Theta\to E$}
\RightLabel{sb}
\BinaryInfC{$\Gamma^k,\Delta^k,\Lambda^k,\Theta\to E$}
\end{prooftree}
$$

where π_3 can have $\wedge I\mathcal{E}$ as the subderivation of π which ends with $[A]^n,\Delta'\to C\wedge D$

$(\mathcal{E}ms^{\vee}_{\wedge_2})$, $(\mathcal{E}ms^{\vee}_{\forall})$, $(\mathcal{E}ms^{\exists}_{\wedge_1})$, $(\mathcal{E}ms^{\exists}_{\wedge_2})$ and $(\mathcal{E}ms^{\exists}_{\forall})$ are similar to $(\mathcal{E}ms^{\vee}_{\wedge_1})$.

$(\mathcal{E}ms^{\exists}_{\vee_1})$ The redex π is

$$
\begin{prooftree}
\AxiomC{π_1}\noLine
\UnaryInfC{$\Gamma'\to\exists xAx$}
\AxiomC{π_2}\noLine
\UnaryInfC{$\Delta''\to C$}
\RightLabel{$\vee I\mathcal{E}_1$}
\UnaryInfC{$\Delta''\to C\vee D$}
\RightLabel{$Es\ldots$}
\UnaryInfC{$[Aa]^k,\Delta'\to C\vee D$}
\RightLabel{$\exists E\mathcal{E}$}
\BinaryInfC{$\Gamma',\Delta'\to C\vee D$}
\RightLabel{$sb\ldots$}
\UnaryInfC{$\Gamma,\Delta\to C\vee D$}
\AxiomC{π_3}\noLine
\UnaryInfC{$[C]^n,\Lambda\to E$}
\AxiomC{π_4}\noLine
\UnaryInfC{$[D]^m,\Theta\to E$}
\RightLabel{$\vee E\mathcal{E}$}
\TrinaryInfC{$\Gamma,\Delta,\Lambda,\Theta\to E$}
\end{prooftree}
$$

the contractum $\overline{\pi}$ is

$$\dfrac{\begin{array}{c}\pi_1\\ \Gamma'\to\exists xAx\end{array} \quad \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_2\\ \Delta''\to C\end{array}}{\Delta''\to C\vee D}\vee I\mathcal{E}_1}{[Aa]^k,\Delta'\to C\vee D}Es... \quad \begin{array}{c}\pi_3\\ [C]^n,\Lambda\to E\end{array} \quad \begin{array}{c}\pi_4\\ [D]^m,\Theta\to E\end{array}}{[Aa]^k,\Delta',\Lambda,\Theta\to E}\vee E\mathcal{E}}{\dfrac{\Gamma',\Delta',\Lambda,\Theta\to E}{\Gamma,\Delta,\Lambda,\Theta\to E}sb...}\exists E\mathcal{E}$$

$(\mathcal{E}ms^{\exists}_{\vee_2})$, $(\mathcal{E}ms^{\exists}_{\exists})$, $(\mathcal{E}ms^{\vee}_{\vee_1})$, $(\mathcal{E}ms^{\vee}_{\vee_2})$ and $(\mathcal{E}ms^{\vee}_{\exists})$ are similar to $(\mathcal{E}ms^{\exists}_{\vee_1})$.
The $\mathcal{E}$mf-reductions and the $\mathcal{E}$ms-reductions make the set of the *$\mathcal{E}$m-reductions*.

$\pi'\to_{\mathcal{E}mf_*}\pi''$ $(\pi'\to_{\mathcal{E}ms^{\bullet}_{*}}\pi'')$, $*\in\{\wedge_1,\wedge_2,\vee_1,\vee_2,\supset,\forall,\exists\}$, $\bullet\in\{\wedge_1,\wedge_2,\vee,\supset,\forall,\exists\}$ and they can be omitted iff π'' is obtained from π' by replacing its subderivation π, which is the redex of an $\mathcal{E}$mf-reduction $(\mathcal{E}\mathrm{mf}_*)$ (an $\mathcal{E}$ms-reduction $(\mathcal{E}ms^{\bullet}_{*})$), with the contractum $\overline{\pi}$ of that $\mathcal{E}$mf-reduction ($\mathcal{E}$ms-reduction).

$\pi'\to_{\mathcal{E}m}\pi''$ iff either $\pi'\to_{\mathcal{E}mf}\pi''$ or $\pi'\to_{\mathcal{E}ms}\pi''$.

5.2 The gz-reductions and the sb$^-$-reductions

Note 5.2. *The connections which were introduced by Gentzen in Section V§5 in [9] (and they were used in Zucker's reductions from 7.8.2 in [26]), will be our gz-reductions. Their forms are the result of the connection between natural deduction rules and sequent rules mentioned in **(2)** from the Introduction, more precisely, the connection between elimination rules of natural deduction and left rules of sequent systems. These reductions will connect each derivation from **NE** with its image of the composition of the maps* n *and* e*,* e∘n*, which connects the set of derivations of **NE** with itself (see the Section 6.2 below).*

◊ ***gz-reductions***

$(\mathrm{gz}_{\wedge_1})$

$$\dfrac{\begin{array}{c}\pi_1\\ \Gamma\to A\wedge B\end{array} \quad \begin{array}{c}\pi_2\\ [A]^n,\Delta\to C\end{array}}{\Gamma,\Delta\to C}\wedge E\mathcal{E}_1 \qquad \dfrac{\dfrac{\begin{array}{c}\pi_1\\ \Gamma\to A\wedge B\end{array} \quad A\to A}{\Gamma\to A}\wedge E\mathcal{E}_1 \quad \begin{array}{c}\pi_2\\ [A]^n,\Delta\to C\end{array}}{\Gamma^n,\Delta\to C}\mathrm{sb}$$

$(\mathrm{gz}_{\wedge_2})$ are similar to $(\mathrm{gz}_{\wedge_1})$

$(\mathrm{gz}_{\supset})$

$$\dfrac{\begin{array}{c}\pi_1\\ \Gamma\to A\supset B\end{array} \quad \begin{array}{c}\pi_2\\ \Lambda\to A\end{array} \quad \begin{array}{c}\pi_3\\ [B]^n,\Delta\to C\end{array}}{\Gamma,\Lambda,\Delta\to C}\supset E\mathcal{E} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_1\\ \Gamma\to A\supset B\end{array} \quad \begin{array}{c}\pi_2\\ \Lambda\to A\end{array} \quad B\to B}{\Gamma,\Lambda\to B}\supset E\mathcal{E} \quad \begin{array}{c}\pi_3\\ [B]^n,\Delta\to C\end{array}}{\Gamma^n,\Lambda^n,\Delta\to C}\mathrm{sb}$$

(gz$_\forall$)

$$\dfrac{\begin{matrix}\pi_1 & \pi_2\\ \Gamma\to\forall xAx & [At]^n,\Delta\to C\end{matrix}}{\Gamma,\Delta\to C}\forall E\mathcal{E} \qquad \dfrac{\dfrac{\begin{matrix}\pi_1 & \\ \Gamma\to\forall xAx & At\to At\end{matrix}}{\Gamma\to At}\forall E\mathcal{E} \quad \begin{matrix}\pi_2\\ [At]^n,\Delta\to C\end{matrix}}{\Gamma^n,\Delta\to C}\,\mathrm{sb}$$

$\pi' \to_{gz_*} \pi''$, where $* \in \{\wedge_1, \wedge_2, \supset, \forall\}$ and it can be omitted iff π'' is obtained from π' by replacing its subderivation π, which is the redex of a gz-reduction, with the contractum $\overline{\pi}$ of that gz-reduction.

♢ ***sb$^-$-reductions*** In connections of reductions from **NI** and **NE** ($\mathcal{E}$sb$_{-r}$) from the Section 3.2 will be called the sb$^-$-reduction, i.e. π is $\dfrac{\begin{matrix}\pi_1 & \\ \Gamma\to A & A\to A\end{matrix}}{\Gamma\to A}\mathrm{sb}$, and $\overline{\pi}$ is $\begin{matrix}\pi_1\\ \Gamma\to A\end{matrix}$.

$\pi'\to_-\pi''$ iff π'' is obtained from π' by replacing its subderivation, which is the redex of an sb$^-$-reduction, with the contractum of that sb$^-$-reduction.

5.3 The normal derivations in the system NE

If π is sb-free and it does not have any subderivation which is the redex of an $\mathcal{E}$m-reduction, then π will be a *normal derivation* in **NE**.

If there is a sequence $\pi_0, ..., \pi_n$, $n>0$, such that π_0 is π', π_n is π'', and for all $i < n$:
(1) either π_i $\mathcal{E}$sb$>_{k_i}\pi_{i+1}$, $k_i>0$, or $\pi_i\to_{\mathcal{E}m}\pi_{i+1}$, then the derivation π' $\mathcal{E}$*s-max*$>_n$ π'', i.e. the derivation π' *is* $\mathcal{E}$*s-max-reduced into* the derivation π'';
(2) $\pi_i \to_\circ \pi_{i+1}$, where $\circ \in \{s, gz, -\}$, then $\pi' >^\circ_n \pi''$;
(3) either $\pi_i\mathcal{E}$*s-max*$>_1\pi_{i+1}$ or $\pi_i\to_\circ \pi_{i+1}$, where $\circ\in\{s, gz, -\}$, then π' $\mathcal{E}$*s-max*$>^\circ_n$ π''.

6 The maps e, n, n ∘ e and e ∘ n

6.1 The maps e and n

We define the maps e:Der(**NI**) $\longrightarrow$ Der(**NE**) and n:Der(**NE**) $\longrightarrow$ Der(**NI**), where Der(**NI**) and Der(**NE**) are the sets of derivations from **NI** and **NE**, respectively. First we note that a derivation *pi* from **NI** or **NE** has *proper variable property (PVP)* if in *pi* for each elimination of ∃ and each introduction of ∀ the following holds: its proper variable occurs only in sequents above the lower sequent of that rule and it does not occur as a proper variable in any other rule. A derivation from **NI** and **NE** can be effectively transformed into a derivation with PVP (see [9, III, 3.10] for details), so we assume that our derivations in **NI** and **NE** have PVP. The *length of* a *derivation pi* from **NI** or **NE** is the number of its inferences rules.

The e-image of a derivation Π with the end sequent $\Theta \to C$ is the derivation $\mathsf{e}\Pi$ with the same end sequent. The map e will be defined by the induction of the length of the derivation Π.

(1) $$\Pi: \quad C \to C \qquad\qquad \mathsf{e}\Pi: \quad C \to C$$

(2) $$\bot \to P \qquad\qquad \bot \to P$$

(3) $$\dfrac{\overset{\Pi_1}{\Gamma \to A} \qquad \overset{\Pi_2}{[A]^n, \Delta \to C}}{\Gamma^n, \Delta \to C}\,\mathrm{sb} \qquad\qquad \dfrac{\overset{\mathsf{e}\Pi_1}{\Gamma \to A} \qquad \overset{\mathsf{e}\Pi_2}{[A]^n, \Delta \to C}}{\Gamma^n, \Delta \to C}\,\mathrm{sb}$$

(4) $$\dfrac{\overset{\Pi_1}{\Delta \to C}}{\Gamma, \Delta \to C}\,\mathrm{sb}^0_\Gamma \qquad\qquad \dfrac{\overset{\mathsf{e}\Pi_1}{\Delta \to C}}{\Gamma, \Delta \to C}\,\mathrm{sb}^0_\Gamma$$

(5) $$\dfrac{\overset{\Pi_1}{[A]^n, \Gamma \to B}}{\Gamma \to A \supset B}{\supset}I \qquad\qquad \dfrac{\overset{\mathsf{e}\Pi_1}{[A]^n, \Gamma \to B}}{\Gamma \to A \supset B}{\supset}I\mathcal{E}$$

(6) $$\dfrac{\overset{\Pi_1}{\Gamma \to A \supset B} \qquad \overset{\Pi_2}{\Delta \to A}}{\Gamma, \Delta \to B}{\supset}E \qquad\qquad \dfrac{\overset{\mathsf{e}\Pi_1}{\Gamma \to A \supset B} \qquad \overset{\mathsf{e}\Pi_2}{\Delta \to A} \qquad B \to B}{\Gamma, \Delta \to B}{\supset}E\mathcal{E}$$

(7) $$\dfrac{\overset{\Pi_1}{\Gamma \to A} \qquad \overset{\Pi_2}{\Delta \to B}}{\Gamma, \Delta \to A \wedge B}{\wedge}I \qquad\qquad \dfrac{\overset{\mathsf{e}\Pi_1}{\Gamma \to A} \qquad \overset{\mathsf{e}\Pi_2}{\Delta \to B}}{\Gamma, \Delta \to A \wedge B}{\wedge}I\mathcal{E}$$

(8) $$\dfrac{\overset{\Pi_1}{\Gamma \to A \wedge B}}{\Gamma \to A}{\wedge}E_1 \qquad\qquad \dfrac{\overset{\mathsf{e}\Pi_1}{\Gamma \to A \wedge B} \qquad A \to A}{\Gamma \to A}{\wedge}E\mathcal{E}_1$$

(9) the last rule of Π is $\wedge E_2$, then $\mathsf{e}\Pi$ is defined as in (8).

(10) $$\dfrac{\overset{\Pi_1}{\Gamma \to A}}{\Gamma \to A \vee B}{\vee}I_1 \qquad\qquad \dfrac{\overset{\mathsf{e}\Pi_1}{\Gamma \to A}}{\Gamma \to A \vee B}{\vee}I\mathcal{E}_1$$

(11) the last rule of Π is $\vee I_2$, then $\mathsf{e}\Pi$ is defined as in (10).

(12) $$\dfrac{\overset{\Pi_1}{\Gamma \to A \vee B} \quad \overset{\Pi_2}{[A]^n, \Delta \to C} \quad \overset{\Pi_3}{[B]^m, \Lambda \to C}}{\Gamma, \Delta, \Lambda \to C}{\vee}E \qquad \dfrac{\overset{\mathsf{e}\Pi_1}{\Gamma \to A \vee B} \quad \overset{\mathsf{e}\Pi_2}{[A]^n, \Delta \to C} \quad \overset{\mathsf{e}\Pi_3}{[B]^m, \Lambda \to C}}{\Gamma, \Delta, \Lambda \to C}{\vee}E\mathcal{E}$$

$$(13)\quad \dfrac{\begin{array}{c}\Pi_1\\ \Gamma\to Aa\end{array}}{\Gamma\to\forall xAx}\forall I \qquad\qquad \dfrac{\begin{array}{c}\mathsf{e}\Pi_1\\ \Gamma\to Aa\end{array}}{\Gamma\to\forall xAx}\forall I\mathcal{E}$$

$$(14)\quad \dfrac{\begin{array}{c}\Pi_1\\ \Gamma\to\forall xAx\end{array}}{\Gamma\to At}\forall E \qquad\qquad \dfrac{\begin{array}{c}\mathsf{e}\Pi_1\\ \Gamma\to\forall xAx\end{array}\qquad At\to At}{\Gamma\to At}\forall E\mathcal{E}$$

$$(15)\quad \dfrac{\begin{array}{c}\Pi_1\\ \Gamma\to At\end{array}}{\Gamma\to\exists xAx}\exists I \qquad\qquad \dfrac{\begin{array}{c}\mathsf{e}\Pi_1\\ \Gamma\to At\end{array}}{\Gamma\to\exists xAx}\exists I\mathcal{E}$$

$$(16)\quad \dfrac{\begin{array}{c}\Pi_1\\ \Gamma\to\exists xAx\end{array}\qquad \begin{array}{c}\Pi_2\\ [Aa]^n,\Delta\to C\end{array}}{\Gamma,\Delta\to C}\exists E \qquad\qquad \dfrac{\begin{array}{c}\mathsf{e}\Pi_1\\ \Gamma\to\exists xAx\end{array}\qquad \begin{array}{c}\mathsf{e}\Pi_2\\ [Aa]^n,\Delta\to C\end{array}}{\Gamma,\Delta\to C}\exists E\mathcal{E}$$

The n-image of a derivation π which ends with $\Theta\to C$ is the derivation $\mathsf{n}\pi$ whose end sequent is $\widetilde{\Theta}\to C$, where for each formula D^{Θ} from Θ there is the multiset $[D]^m$ in $\widetilde{\Theta}$, $m\geq 1$, such that: if D^{Θ} belongs to an a-edge of the formula from an axiom of π (the lower sequent of an sb^0), then the multiset $[D]^m$ consists of all formulae from $\widetilde{\Theta}$ of the form D, which belong to an a-edge of the formula D from the n-image of that axiom, an axiom of $\mathsf{n}\pi$ (the lower sequent of the n-image of that sb^0). Thus, if $\Theta\to C$ is $[A]^n,\Delta\to C$, then for each assumption A^i from $[A]^n$ (i denotes members of $[A]^n$) there is the multiset $[A^i]^{m_i}$, $1\leq i\leq n$, in $\widetilde{\Theta}$, so the multiset $[A]^n$ from Θ has the corresponding multiset $[A]^{n1}$ in $\widetilde{\Theta}$, which consists of formulae from $[A^i]^{m_i}$, $1\leq i\leq n$, i.e. $n1=m_1+\ldots+m_n$. The map n will be defined by the induction of the length of π.

$$(1)\quad \begin{array}{c}\pi\\ C\to C\end{array} \qquad\qquad \begin{array}{c}\mathsf{n}\pi\\ C\to C\end{array}$$

$$(2)\quad \bot\to P \qquad\qquad \bot\to P$$

$$(3)\quad \dfrac{\begin{array}{c}\pi_1\\ \Gamma\to A\end{array}\qquad \begin{array}{c}\pi_2\\ [A]^n,\Delta\to C\end{array}}{\Gamma^n,\Delta\to C}\,\mathrm{sb} \qquad\qquad \dfrac{\begin{array}{c}\mathsf{n}\pi_1\\ \widetilde{\Gamma}\to A\end{array}\qquad \begin{array}{c}\mathsf{n}\pi_2\\ [A]^{n1},\widetilde{\Delta}\to C\end{array}}{\widetilde{\Gamma}^{n1},\widetilde{\Delta}\to C}\,\mathrm{sb}$$

$$(4)\quad \dfrac{\begin{array}{c}\pi_1\\ \Delta\to C\end{array}}{\Gamma,\Delta\to C}\,\mathrm{sb}^0_{\Gamma} \qquad\qquad \dfrac{\begin{array}{c}\mathsf{n}\pi_1\\ \widetilde{\Delta}\to C\end{array}}{\Gamma,\widetilde{\Delta}\to C}\,\mathrm{sb}^0_{\Gamma}$$

$$(5)\quad \dfrac{\begin{array}{c}\pi_1\\ [A]^n,\Gamma\to B\end{array}}{\Gamma\to A\supset B}\supset I\mathcal{E} \qquad\qquad \dfrac{\begin{array}{c}\mathsf{n}\pi_1\\ [A]^{n1},\widetilde{\Gamma}\to B\end{array}}{\widetilde{\Gamma}\to A\supset B}\supset I$$

$$(6)\ \dfrac{\overset{\pi_1}{\Gamma \to A\supset B} \quad \overset{\pi_2}{\Delta \to A} \quad \overset{\pi_3}{[B]^n, \Lambda \to C}}{\Gamma, \Delta, \Lambda \to C}\supset E\mathcal{E} \qquad \dfrac{\dfrac{\overset{\mathsf{n}\pi_1}{\widetilde{\Gamma}\to A\supset B} \quad \overset{\mathsf{n}\pi_2}{\widetilde{\Delta}\to A}}{\widetilde{\Gamma}, \widetilde{\Delta}\to B}\supset E \qquad \overset{\mathsf{n}\pi_3}{[B]^{n1}, \widetilde{\Lambda} \to C}}{\widetilde{\Gamma}^{n1}, \widetilde{\Delta}^{n1}, \widetilde{\Lambda}\to C}\ \mathrm{sb}$$

$$(7)\ \dfrac{\overset{\pi_1}{\Gamma \to A} \qquad \overset{\pi_2}{\Delta \to B}}{\Gamma, \Delta \to A\wedge B}\wedge I\mathcal{E} \qquad \dfrac{\overset{\mathsf{n}\pi_1}{\widetilde{\Gamma} \to A} \qquad \overset{\mathsf{n}\pi_2}{\widetilde{\Delta} \to B}}{\widetilde{\Gamma}, \widetilde{\Delta} \to A\wedge B}\wedge I$$

$$(8)\ \dfrac{\overset{\pi_1}{\Gamma \to A\wedge B} \quad \overset{\pi_2}{[A]^n, \Delta \to C}}{\Gamma, \Delta \to C}\wedge E\mathcal{E}_1 \qquad \dfrac{\dfrac{\overset{\mathsf{n}\pi_1}{\widetilde{\Gamma} \to A\wedge B}}{\widetilde{\Gamma} \to A}\wedge E_1 \qquad \overset{\mathsf{n}\pi_2}{[A]^{n1}, \widetilde{\Delta} \to C}}{\widetilde{\Gamma}^{n1}, \widetilde{\Delta} \to C}\ \mathrm{sb}$$

(9) the last rule of π is $\wedge E\mathcal{E}_2$, then $\mathsf{n}\pi$ is defined as in (8).

$$(10)\ \dfrac{\overset{\pi_1}{\Gamma \to A}}{\Gamma \to A\vee B}\vee I\mathcal{E}_1 \qquad \dfrac{\overset{\mathsf{n}\pi_1}{\widetilde{\Gamma} \to A}}{\widetilde{\Gamma} \to A\vee B}\vee I_1$$

(11) the last rule of π is $\vee I\mathcal{E}_2$, then $\mathsf{n}\pi$ is defined as in (10).

$$(12)\ \dfrac{\overset{\pi_1}{\Gamma\to A\vee B} \quad \overset{\pi_2}{[A]^n, \Delta\to C} \quad \overset{\pi_3}{[B]^m, \Lambda\to C}}{\Gamma, \Delta, \Lambda \to C}\vee E\mathcal{E} \qquad \dfrac{\overset{\mathsf{n}\pi_1}{\widetilde{\Gamma}\to A\vee B} \quad \overset{\mathsf{n}\pi_2}{[A]^{n1}, \widetilde{\Delta}\to C} \quad \overset{\mathsf{n}\pi_3}{[B]^{m1}, \widetilde{\Lambda}\to C}}{\widetilde{\Gamma}, \widetilde{\Delta}, \widetilde{\Lambda} \to C}\vee E$$

$$(13)\ \dfrac{\overset{\pi_1}{\Gamma \to Aa}}{\Gamma \to \forall x Ax}\forall I\mathcal{E} \qquad \dfrac{\overset{\mathsf{n}\pi_1}{\widetilde{\Gamma} \to Aa}}{\widetilde{\Gamma} \to \forall x Ax}\forall I$$

$$(14)\ \dfrac{\overset{\pi_1}{\Gamma \to \forall x Ax} \quad \overset{\pi_2}{[At]^n, \Delta \to C}}{\Gamma, \Delta \to C}\forall E\mathcal{E} \qquad \dfrac{\dfrac{\overset{\mathsf{n}\pi_1}{\widetilde{\Gamma} \to \forall x Ax}}{\widetilde{\Gamma} \to At}\forall E \qquad \overset{\mathsf{n}\pi_2}{[At]^{n1}, \widetilde{\Delta} \to C}}{\widetilde{\Gamma}^{n1}, \widetilde{\Delta} \to C}\ \mathrm{sb}$$

$$(15)\ \dfrac{\overset{\pi_1}{\Gamma \to At}}{\Gamma \to \exists x Ax}\exists I\mathcal{E} \qquad \dfrac{\overset{\mathsf{n}\pi_1}{\widetilde{\Gamma} \to At}}{\widetilde{\Gamma} \to \exists x Ax}\exists I$$

$$(16)\ \dfrac{\overset{\pi_1}{\Gamma \to \exists x Ax} \quad \overset{\pi_2}{[Aa]^n, \Delta \to C}}{\Gamma, \Delta \to C}\exists E\mathcal{E} \qquad \dfrac{\overset{\mathsf{n}\pi_1}{\widetilde{\Gamma} \to \exists x Ax} \quad \overset{\mathsf{n}\pi_2}{[Aa]^{n1}, \widetilde{\Delta} \to C}}{\widetilde{\Gamma}, \widetilde{\Delta} \to C}\exists E$$

6.2 The derivations Π_{NE} in NI and the derivations π_{EN} in NE

We will present the forms of derivations which are results of the compositions $\mathsf{n} \circ \mathsf{e}$ and $\mathsf{e} \circ \mathsf{n}$. It will be shown that: the $\mathsf{n} \circ \mathsf{e}$-image of a derivation Π from **NI**, $\mathsf{ne}\Pi$, is obtained from Π by several (may be 0) sb$^+$-reductions (the Lemma 6.2(2)(3)); and the $\mathsf{e} \circ \mathsf{n}$-image of a derivation π from **NE**, $\mathsf{en}\pi$, is obtained from π by several (may be 0) gz-reductions (the Lemma 6.4(2)(3)). We note that $\mathsf{ne}\Pi$ and $\mathsf{en}\pi$ will be used in connections between reductions of derivations from **NI** and **NE** in the Section 7.

If $\Pi >_n^+ \overline{\Pi}$ and $\overline{\Pi}$ does not have any subderivation which is the redex of an sb$^+$-reduction, then $\overline{\Pi}$ will be Π_{NE} for Π; while Π_{NE} is Π, when Π does not have any subderivation which is the redex of an sb$^+$-reduction.

Note 6.1. *The* e*-image of an elimination* R *of* $*$, $*\in\{\wedge, \vee, \supset, \forall, \exists\}$, *from* Π *is the corresponding elimination* R^{e} *in* $\mathsf{e}\Pi$, *and the* n*-image of* R^{e} *is: -the elimination* R, *when* $*$ *is* $\vee$ *or* $\exists$; *-the elimination* R *with an sb whose upper sequents are its consequence and an i-axiom, when* $*$ *is* $\wedge$, $\supset$ *or* $\forall$. *So,* $\mathsf{ne}\Pi$ *is* Π *with added one sb after each elimination of* $\wedge$, $\supset$ *and* $\forall$, *i.e. the derivation* Π_{NE} *(the Lemma 6.2(2)).*

Lemma 6.1. *For each derivation* Π *from* ***NI*** *there is the derivation* Π_{NE}.

Proof. By an induction on the length of the derivation Π. ☐

Lemma 6.2. *In the system* ***NI*** *for each derivation* Π:
(1) Π *has a maximum segment iff* Π_{NE} *has a maximum segment;*
(2) the derivation $\mathsf{ne}\Pi$ *is* Π_{NE};
(3) if Π *does not have any elimination of* $\wedge, \supset$ *and* $\forall$, *then* $\mathsf{ne}\Pi$ *is* Π.

Proof. (1) By the definition of the derivation Π_{NE} for a derivation Π.
(2) and (3) By the definitions of the maps n and e. ☐

If $\pi >_n^{gz} \overline{\pi}$ and $\overline{\pi}$ does not have any subderivation which is the redex of a gz-reduction, then $\overline{\pi}$ will be π_{EN} for π; while π_{EN} is π, when π does not have any subderivation which is the redex of a gz-reduction.

Note 6.2. *The* n*-image of an elimination* R *of* $\wedge, \supset$ *or* $\forall$ *from* π *is the corresponding elimination* R^{n} *with a substitution* sb^{R} *below it in* $\mathsf{n}\pi$, *and by the map* e, R^{n} *goes to the corresponding elimination whose last upper sequent is an i-axiom and* sb^{R} *goes to a substitution. So,* π *and* $\mathsf{en}\pi$ *are connected by gz-reductions, i.e.* $\mathsf{en}\pi$ *is the derivation* π_{EN} *(the Lemma 6.4(2)).*

Lemma 6.3. *For each derivation* π *from* ***NE*** *there is the derivation* π_{EN}.

Proof. By one induction on the length of the derivation π. ☐

Lemma 6.4. *In the system* $\boldsymbol{NE}$ *for each derivation* π*:*
(1) if π_{EN} *has a maximum segment, then* π *has a maximum segment;*
(2) the derivation $\mathsf{en}\pi$ *is* π_{EN}*;*
(3) if π *does not have any elimination of* $\wedge, \supset$ *and* $\forall$*, then* $\mathsf{en}\pi$ *is* π*.*

Proof. (1) By the definition of the derivation π_{EN} for a derivation π.
(2) and (3) By the definitions of the maps n and e. □

By the definitions of sb-reductions, $\mathcal{E}$sb-reductions, ss-reductions and the maps e and n, we have the following two lemmata.

Lemma 6.5. *For each derivation* Π *from the system* $\boldsymbol{NI}$*:*
(1) if $\Pi \to_{sb} \overline{\Pi}$*, then* $\mathsf{e}\Pi \to_{\mathcal{E}sb} \mathsf{e}\overline{\Pi}$*;*
(2) if Π *is sb-free, then* $\mathsf{e}\Pi$ *is sb-free.*

Lemma 6.6. *For each derivation* π *from the system* $\boldsymbol{NE}$*:*
(1) if $\pi \to_{\mathcal{E}sb} \overline{\pi}$*, then either there is* Π *from* $\boldsymbol{NI}$ *and* $\mathsf{n}\pi \to_{s} \Pi \to_{sb} \mathsf{n}\overline{\pi}$ *or* $\mathsf{n}\pi \to_{sb} \mathsf{n}\overline{\pi}$*;*
(2) if π *is sb-free, then (2.1)* $\mathsf{n}\pi$ *is sb-free, when* π *does not have any elimination of* $\wedge, \supset$ *and* $\forall$*; (2.2)* $\mathsf{n}\pi$ *has substitutions, otherwise.*

7 The connections between reductions

Note 7.1. *We will show that the* e*-image of an m-reduction (i.e. the* e*-images of its redex and its contractum) from* $\boldsymbol{NI}$ *depends on the main sign of the formula of the maximum segment of its redex. If the main sign of that formula is: – either* $\vee$ *or* $\exists$*, then in* $\boldsymbol{NE}$ *the* e*-image of that reduction is one reduction of the same kind (the Lemma 7.1(2) and the Lemma 7.2(2)); – either* $\wedge$*,* $\supset$ *or* $\forall$*, then in* $\boldsymbol{NE}$ *the* e*-image of that reduction is one reduction of the same kind with one* sb^{-}*-reduction (the Lemma 7.1(1) and the Lemma 7.2(1)).*

In the Lemma 7.1 and the Lemma 7.2 below for each $\Pi \to_m \overline{\Pi}$ it is sufficient to prove the case when the subderivation of Π which is the redex of the reduction is Π.

Lemma 7.1. *If* $\Pi \to_{mf_*} \overline{\Pi}$ *in the system* $\boldsymbol{NI}$*, then*
(1) there is π *from* $\boldsymbol{NE}$ *such that* $\mathsf{e}\Pi \to_{\mathcal{E}mf_*} \pi >_1^- \mathsf{e}\overline{\Pi}$*, when* $*$ *is* $\wedge_1$*,* $\wedge_2$*,* $\forall$ *or* $\supset$*;*
(2) $\mathsf{e}\Pi \to_{\mathcal{E}mf_*} \mathsf{e}\overline{\Pi}$*, when* $*$ *is* $\vee_1$*,* $\vee_2$ *or* $\exists$*.*

Proof. (1) We consider the case $\Pi \to_{mf_{\wedge_1}} \overline{\Pi}$ from 4.1. By the definition of e and $\mathcal{E}$mf-reductions, we have that the e-image of Π, $\mathsf{e}\Pi$, contains a redex of the reduction $\mathcal{E}\mathrm{mf}_{\wedge_1}$, whose contractum without the last substitution is $\mathsf{e}\overline{\Pi}$, i.e. $\mathsf{e}\Pi$ is

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathsf{e}\Pi_1 & \mathsf{e}\Pi_2\\ \Gamma'\to A & \Delta'\to B\end{matrix}}{\Gamma',\Delta'\to A\wedge B}\wedge I\mathcal{E}}{\Gamma,\Delta\to A\wedge B}sb\ldots \quad A\to A}{\Gamma,\Delta\to A}\wedge E\mathcal{E}_1$$ and in **NE**: $\mathsf{e}\Pi\to_{\mathcal{E}mf_{\wedge_1}}\pi$, for π: $$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathsf{e}\Pi_1\\ \Gamma'\to A\end{matrix}}{\Gamma',\Delta'\to A}\mathrm{sb}^0_{\Delta'}}{\Gamma,\Delta\to A}sb\ldots \quad A\to A}{\Gamma,\Delta\to A}\mathrm{sb}$$

and $\pi\to_{-}\mathsf{e}\overline{\Pi}$.Thus, $\mathsf{e}\Pi\to_{\mathcal{E}mf_{\wedge_1}}\pi>_1^-\mathsf{e}\overline{\Pi}$. The other cases are similar to the case above.
(2) By the definitions of mf-reductions, $\mathcal{E}$mf-reductions and the map e. $\square$

Lemma 7.2. *If $\Pi\to_{ms^\bullet_*}\overline{\Pi}$ in the system **NI**, then*
*(1) there is π in **NE** such that $\mathsf{e}\Pi\to_{\mathcal{E}ms^\bullet_*}\pi>_1^-\mathsf{e}\overline{\Pi}$, $\bullet$ is $\vee$ or $\exists$, $*$ is $\wedge_1$, $\wedge_2$, $\forall$ or $\supset$;*
(2) $\mathsf{e}\Pi\to_{\mathcal{E}ms^\bullet_*}\mathsf{e}\overline{\Pi}$, $\bullet$ is $\vee$ or $\exists$, $*$ is $\vee_1$, $\vee_2$ or $\exists$.

Proof. (1) We consider the case $\Pi\to_{ms^\vee_{\wedge_1}}\overline{\Pi}$ from 4.1. By the definition of e, $\mathsf{e}\Pi$ is

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathsf{e}\Pi_1\\ \Gamma'\to A\vee B\end{matrix} \quad \dfrac{\dfrac{\begin{matrix}\mathsf{e}\Pi_2' & \mathsf{e}\Pi_2''\\ \Delta''\to C & \Delta'''\to D\end{matrix}}{\Delta'',\Delta'''\to C\wedge D}\wedge I\mathcal{E}}{[A]^n,\Delta'\to C\wedge D}\vee\exists Es\ldots \quad \begin{matrix}\mathsf{e}\Pi_3\\ [B]^m,\Lambda'\to C\wedge D\end{matrix}}{\Gamma',\Delta',\Lambda'\to C\wedge D}\vee E\mathcal{E}}{\Gamma,\Delta,\Lambda\to C\wedge D}sb\ldots \quad C\to C}{\Gamma,\Delta,\Lambda\to C}\wedge E\mathcal{E}_1$$

in **NE**, where $\vee\exists Es\ldots$ is the sequence of $\vee E\mathcal{E}$, $\exists E\mathcal{E}$, sb and sb^0 which are e-images of the rules of $\vee\exists Es\ldots$ from Π. By $\mathcal{E}$ms-reductions: $\mathsf{e}\Pi\to_{\mathcal{E}ms^\vee_{\wedge_1}}\pi$, where π is

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathsf{e}\Pi_1\\ \Gamma'\to A\vee B\end{matrix} \quad \dfrac{\dfrac{\dfrac{\begin{matrix}e\Pi_2' & \mathsf{e}\Pi_2''\\ \Delta''\to C & \Delta'''\to D\end{matrix}}{\Delta'',\Delta'''\to C\wedge D}\wedge I\mathcal{E}}{[A]^n,\Delta'\to C\wedge D}\vee\exists Es\ldots \quad C\to C}{[A]^n,\Delta'\to C}\wedge E\mathcal{E}_1 \quad \dfrac{\begin{matrix}\mathsf{e}\Pi_3\\ [B]^m,\Lambda'\to C\wedge D\end{matrix} \quad C\to C}{[B]^m,\Lambda'\to C}\wedge E\mathcal{E}_1}{\Gamma',\Delta',\Lambda'\to C}\vee E\mathcal{E}}{\Gamma,\Delta,\Lambda\to C}sb\ldots \quad C\to C}{\Gamma,\Delta,\Lambda\to C}\mathrm{sb}$$

By the definition of e, $\mathsf{e}\overline{\Pi}$ is π without the last sb, i.e. $\pi\to_{-}\mathsf{e}\overline{\Pi}$. Thus, we have $\mathsf{e}\Pi\to_{\mathcal{E}ms^\vee_{\wedge_1}}\pi>_1^-\mathsf{e}\overline{\Pi}$. The other cases are similar to the case above.
(2) By the definitions of ms-reductions, $\mathcal{E}$ms-reductions and the map e. $\square$

Note 7.2. *We will show that the n-image of an $\mathcal{E}$mf-reduction (i.e. the n-images of its redex and its contractum) from **NE** is the corresponding mf-reduction (its redex* *and its contractum) in **NI** (the Lemma 7.3).However, the* n*-image of an* $\mathcal{E}$*ms-reduction from **NE**, which has the corresponding ms-reduction in **NI**, is that ms-reduction ((1)-(2) of the Lemma 7.4); while the* n*-images of all other* $\mathcal{E}$*ms-reductions from **NE** are* sb_E*-reductions or* sb^+*-reductions in **NI** ((3)-(4) of the Lemma 7.4).*

In the Lemma 7.3 and the Lemma 7.4 below for each $\pi \to_{\mathcal{E}m} \overline{\pi}$ it is sufficient to prove the case when the subderivation of π which is the redex of the reduction is π.

Lemma 7.3. *If* $\pi \to_{\mathcal{E}mf_*} \overline{\pi}$ *in the system **NE**, then*
(1) $\mathsf{n}\pi \to_{mf_*} \mathsf{n}\overline{\pi}$*, when* $*$ *is* $\wedge_1$*,* $\wedge_2$*,* $\forall$ *or* $\supset$*;*
(2) $\mathsf{n}\pi \to_{mf_*} \mathsf{n}\overline{\pi}$*, when* $*$ *is* $\vee_1$*,* $\vee_2$ *or* $\exists$*.*

Proof. (1) We consider the case $\pi \to_{\mathcal{E}mf_\supset} \overline{\pi}$, $n>0$, from 5.1. By the definition of n, $\mathsf{n}\pi$ and $\mathsf{n}\overline{\pi}$ are:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\mathsf{n}\pi_1}{[A]^{n1},\widetilde{\Gamma}'\to B}}{\widetilde{\Gamma}'\to A\supset B}\supset I}{\widetilde{\Gamma}\to A\supset B}sb\ldots \quad \overset{\mathsf{n}\pi_2}{\widetilde{\Delta}\to A}}{\widetilde{\Gamma},\widetilde{\Delta}\to B}\supset E \quad \overset{\mathsf{n}\pi_3}{[B]^{m1},\widetilde{\Lambda}\to C}}{\widetilde{\Gamma}^{m1},\widetilde{\Delta}^{m1},\widetilde{\Lambda}\to C}\mathrm{sb} \text{ and } \dfrac{\dfrac{\overset{\mathsf{n}\pi_2}{\widetilde{\Delta}\to A}\quad \dfrac{\overset{\mathsf{n}\pi_1}{[A]^{n1},\widetilde{\Gamma}'\to B}}{[A]^{n1},\widetilde{\Gamma}\to B}sb\ldots}{\widetilde{\Delta}^{n1},\widetilde{\Gamma}\to B}\mathrm{sb}\quad \overset{\mathsf{n}\pi_3}{[B]^{m1},\widetilde{\Lambda}\to C}}{\widetilde{\Delta}^{n1\cdot m1},\widetilde{\Gamma}^{m1},\widetilde{\Lambda}\to C}\mathrm{sb},$$

respectively. So, $\mathsf{n}\pi \to_{mf_\supset} \mathsf{n}\overline{\pi}$. The other cases are similar to the case above.
(2) The cases are similar to the cases from (1). □

The n-image of $\overset{\pi}{\dfrac{\Gamma\to A}{\Delta\to A}Es\ldots}$, whose formulae A are from a segment sg, is $\overset{\mathsf{n}\pi}{\dfrac{\widetilde{\Gamma}\to A}{\widetilde{\Delta}\to A}\vee\exists Es\ldots}$ where the formulae A are from a segment sg' because, by the definition of n, for the rules of the sequence $Es\ldots$:−the n-image of an elimination of $\vee$ ($\exists$) is an elimination of $\vee$ ($\exists$); −the n-image of a substitution (a 0-substitution) is a substitution (a 0-substitution); −the n-image of an elimination of $\wedge$, $\supset$ or $\forall$ is the corresponding elimination together with a substitution, whose left upper sequent is the lower sequent of that elimination, and its premiss and its consequence are from sg'.

Lemma 7.4. *If* $\pi \to_{\mathcal{E}ms^\bullet_*} \overline{\pi}$ *in the system **NE**, then*
(1)(1.1) there is Π *in **NI** such that* $\mathsf{n}\pi \to_{ms^\bullet_*} \Pi >^+_2 \mathsf{n}\overline{\pi}$*,* $\bullet$ *is* $\vee$ *and* $*$ *is* $\wedge_1,\wedge_2,\forall$ *or* $\supset$*;*
(1.2) there is Π *in **NI** such that* $\mathsf{n}\pi \to_{ms^\bullet_*} \Pi >^+_1 \mathsf{n}\overline{\pi}$*,* $\bullet$ *is* $\exists$ *and* $*$ *is* $\wedge_1,\wedge_2,\forall$ *or* $\supset$*;*
(2) $\mathsf{n}\pi \to_{ms^\bullet_*} \mathsf{n}\overline{\pi}$*,* $\bullet$ *is* $\vee$ *or* $\exists$*,* $*$ *is* $\vee_1,\vee_2$ *or* $\exists$*;*
(3) there is Π *in **NI** such that* $\mathsf{n}\pi >^E_k \Pi >^+_1 \mathsf{n}\overline{\pi}$*,* $k\geq 1$*,* $\bullet$ *and* $*$ *are* $\wedge_1$*,* $\wedge_2$*,* $\forall$ *or* $\supset$*;*
(4) $\mathsf{n}\pi >^E_k \mathsf{n}\overline{\pi}$*,* $k\geq 1$*,* $\bullet$ *is* $\wedge_1,\wedge_2,\forall$*,* $\supset$*,* $*$ *is* $\vee_1,\vee_2$*,* $\exists$*.*
In (3) and (4) $\mathsf{n}\pi$ *and* $\mathsf{n}\overline{\pi}$ *have the maximum segments which correspond to the maximum segments of* π *and* $\overline{\pi}$*, respectively.*

Proof. (1) We consider the case $\pi \to_{\mathcal{E}ms^{\vee}_{\wedge_1}} \overline{\pi}$ from 5.1. By the definition of n, $\mathsf{n}\pi$ is

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\mathsf{n}\pi_1}{\widetilde{\Gamma'}\to A\vee B}\quad \dfrac{\dfrac{\overset{\mathsf{n}\pi_2'}{\widetilde{\Delta''}\to C}\quad \overset{\mathsf{n}\pi_2''}{\widetilde{\Delta'''}\to D}}{\widetilde{\Delta''},\widetilde{\Delta'''}\to C\wedge D}\wedge I}{[A]^{n1},\widetilde{\Delta'}\to C\wedge D}\vee\exists Es... \quad \overset{\mathsf{n}\pi_3}{[B]^{m1},\widetilde{\Lambda'}\to C\wedge D}}{\widetilde{\Gamma'},\widetilde{\Delta'},\widetilde{\Lambda'}\to C\wedge D}\vee E}{\widetilde{\Gamma},\widetilde{\Delta},\widetilde{\Lambda}\to C\wedge D}sb...}{\widetilde{\Gamma},\widetilde{\Delta},\widetilde{\Lambda}\to C}\wedge E_1 \quad \overset{\mathsf{n}\pi_4}{[C]^{k1},\widetilde{\Theta}\to E}}{\widetilde{\Gamma}^{k1},\widetilde{\Delta}^{k1},\widetilde{\Lambda}^{k1},\widetilde{\Theta}\to E}\,\text{sb}$$

$\vee\exists Es...$ is the sequence of $\vee E$, $\exists E$, sb and sb^0 which belong to the n-images of the rules of $Es...$ from π. By the definition of ms-reductions, $\mathsf{n}\pi \to_{ms^{\vee}_{\wedge_1}} \Pi$, where Π is

$$\dfrac{\dfrac{\dfrac{\overset{\mathsf{n}\pi_1}{\widetilde{\Gamma'}\to A\vee B}\quad \dfrac{\dfrac{\dfrac{\overset{\mathsf{n}\pi_2'}{\widetilde{\Delta''}\to C}\quad \overset{\mathsf{n}\pi_2''}{\widetilde{\Delta'''}\to D}}{\widetilde{\Delta''},\widetilde{\Delta'''}\to C\wedge D}\wedge I}{[A]^{n1},\widetilde{\Delta'}\to C\wedge D}\vee\exists Es...}{[A]^{n1},\widetilde{\Delta'}\to C}\wedge E_1 \quad \dfrac{\overset{\mathsf{n}\pi_3}{[B]^{m1},\widetilde{\Lambda'}\to C\wedge D}}{[B]^{m1},\widetilde{\Lambda'}\to C}\wedge E_1}{\widetilde{\Gamma'},\widetilde{\Delta'},\widetilde{\Lambda'}\to C}\vee E}{\widetilde{\Gamma},\widetilde{\Delta},\widetilde{\Lambda}\to C}sb... \quad \overset{\mathsf{n}\pi_4}{[C]^{k1},\widetilde{\Theta}\to E}}{\widetilde{\Gamma}^{k1},\widetilde{\Delta}^{k1},\widetilde{\Lambda}^{k1},\widetilde{\Theta}\to E}\,\text{sb},$$

$\mathsf{n}\overline{\pi}$ is

$$\dfrac{\dfrac{\dfrac{\overset{\mathsf{n}\pi_1}{\widetilde{\Gamma'}\to A\vee B}\quad \dfrac{\dfrac{\dfrac{\dfrac{\overset{\mathsf{n}\pi_2'}{\widetilde{\Delta''}\to C}\quad \overset{\mathsf{n}\pi_2''}{\widetilde{\Delta'''}\to D}}{\widetilde{\Delta''},\widetilde{\Delta'''}\to C\wedge D}\wedge I}{[A]^{n1},\widetilde{\Delta'}\to C\wedge D}\vee\exists Es...}{[A]^{n1},\widetilde{\Delta'}\to C}\wedge E_1 \quad C\to C}{[A]^{n1},\widetilde{\Delta'}\to C}\text{sb} \quad \dfrac{\dfrac{\overset{\mathsf{n}\pi_3}{[B]^{m1},\widetilde{\Lambda'}\to C\wedge D}}{[B]^{m1},\widetilde{\Lambda'}\to C}\wedge E_1 \quad C\to C}{[B]^{m1},\widetilde{\Lambda'}\to C}\text{sb}}{\widetilde{\Gamma'},\widetilde{\Delta'},\widetilde{\Lambda'}\to C}\vee E}{\widetilde{\Gamma},\widetilde{\Delta},\widetilde{\Lambda}\to C}sb... \quad \overset{\mathsf{n}\pi_4}{[C]^{k1},\widetilde{\Theta}\to E}}{\widetilde{\Gamma}^{k1},\widetilde{\Delta}^{k1},\widetilde{\Lambda}^{k1},\widetilde{\Theta}\to E}\,\text{sb}.$$

Thus, $\mathsf{n}\overline{\pi}$ is Π with two substitutions whose right upper sequents are $C\to C$, i.e. $\Pi >_2^+ \mathsf{n}\overline{\pi}$. The other cases are similar to this case.

(2) The cases are similar to the cases from (1).

(3) The cases are similar to the cases from (4).

(4) We consider the case $\pi \to_{\mathcal{E}ms^{\wedge_1}_{\vee_1}} \overline{\pi}$ from 5.1. By the definition of n, $\mathsf{n}\pi$ and $\mathsf{n}\overline{\pi}$ are

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\mathsf{n}\pi_1}{\widetilde{\Gamma'}\to A\wedge B}}{\widetilde{\Gamma'}\to A}\wedge E_1 \quad \dfrac{\dfrac{\overset{\mathsf{n}\pi_2}{\widetilde{\Delta''}\to C}}{\widetilde{\Delta''}\to C\vee D}\vee I_1}{[A]^{n1},\widetilde{\Delta'}\to C\vee D}\vee\exists Es...}{\widetilde{\Gamma'}^{n1},\widetilde{\Delta'}\to C\vee D}\,\mathrm{sb}}{\widetilde{\Gamma}^{n1},\widetilde{\Delta},\to C\vee D}\,sb... \quad \overset{\mathsf{n}\pi_3}{[C]^{k1},\widetilde{\Lambda}\to E} \quad \overset{\mathsf{n}\pi_4}{[D]^{l1},\widetilde{\Theta}\to E}}{\widetilde{\Gamma}^{n1},\widetilde{\Delta},\widetilde{\Lambda},\widetilde{\Theta}\to E}\vee E \quad \text{and}$$

$$\dfrac{\dfrac{\dfrac{\overset{\mathsf{n}\pi_1}{\widetilde{\Gamma'}\to A\wedge B}}{\widetilde{\Gamma'}\to A}\wedge E_1 \quad \dfrac{\dfrac{\dfrac{\overset{\mathsf{n}\pi_2}{\widetilde{\Delta''}\to C}}{\widetilde{\Delta''}\to C\vee D}\vee I_1}{[A]^{n1},\widetilde{\Delta'}\to C\vee D}\vee\exists Es... \quad \overset{\mathsf{n}\pi_3}{[C]^{k1},\widetilde{\Lambda}\to E} \quad \overset{\mathsf{n}\pi_4}{[D]^{l1},\widetilde{\Theta}\to E}}{[A]^{n1},\widetilde{\Delta'},\widetilde{\Lambda},\widetilde{\Theta}\to E}\vee E}{\widetilde{\Gamma'}^{n1},\widetilde{\Delta'},\widetilde{\Lambda},\widetilde{\Theta}\to E}\,\mathrm{sb}}{\widetilde{\Gamma}^{n1},\widetilde{\Delta},\widetilde{\Lambda},\widetilde{\Theta}\to E}\,\mathrm{sb}...$$

where $\vee\exists Es...$ is the sequence of $\vee E$, $\exists E$, sb and sb^0 which belong to the n-images of the rules from $Es...$ in π. So, $\mathsf{n}\pi{>}^{E}_{k}\mathsf{n}\overline{\pi}$ ($k{=}l{+}1$, l is the number of rules in $sb...$) and $\mathsf{n}\pi$ and $\mathsf{n}\overline{\pi}$ have the maximum segments which correspond to the maximum segments of π and $\overline{\pi}$, respectively. For all other reductions the proofs are similar. □

Note 7.3. ***NI*** *and* ***NE*** *are the sequent forms of* $\mathcal{N}$ *and* $\mathcal{NE}$ *from [2], respectively. However, the forms of some reductions (i.e. conversions) for maximum segments in derivations from* ***NE*** *and* $\mathcal{NE}$ *are different (see the Note 5.1 above). The connections between reductions for maximum segments from* $\mathcal{NE}$ *and* ***NE*** *and their images in* $\mathcal{N}$ *and* ***NI***, *respectively, are the following: if* $\pi_{\mathcal{NE}}\to_{\mathcal{E}maxs^{\bullet}_{*}}\overline{\pi}_{\mathcal{NE}}$ *is the conversion from* $\mathcal{NE}$ *corresponding to* $\pi\to_{\mathcal{E}ms^{\bullet}_{*}}\overline{\pi}$ *from the Lemma 7.4 above, and* $\pi\to_{\mathcal{E}ms^{\bullet}_{*}}\overline{\pi}$ *is from*
(i) (1) or (3), then $\pi\to_{\mathcal{E}ms^{\bullet}_{*}}\overline{\pi}$ *and* $\pi_{\mathcal{NE}}\to_{\mathcal{E}maxs^{\bullet}_{*}}\overline{\pi}_{\mathcal{NE}}$ *are different and in the case – (1):* $\mathsf{n}\pi$ *and* $\mathsf{n}\overline{\pi}$ *are connected by the corresponding ms-reduction and* sb^{+}*-reductions in* ***NI***, *but in* $\mathcal{N}$ *there is not any conversion which connects the images of* $\pi_{\mathcal{NE}}$ *and* $\overline{\pi}_{\mathcal{NE}}$ *(see the Theorem 7.6(II) in [2]); – (3):* $\mathsf{n}\pi$ *and* $\mathsf{n}\overline{\pi}$ *are connected by some reductions, which are not ms-reductions in* ***NI***, *while in* $\mathcal{N}$ *the images of the derivations* $\pi_{\mathcal{NE}}$ *and* $\overline{\pi}_{\mathcal{NE}}$ *are the same (see the Theorem 7.5 in [2]).*
(ii) (2) or (4), then $\pi\to_{\mathcal{E}ms^{\bullet}_{*}}\overline{\pi}$ *is the same as* $\pi_{\mathcal{NE}}\to_{\mathcal{E}maxs^{\bullet}_{*}}\overline{\pi}_{\mathcal{NE}}$ *from* $\mathcal{NE}$, *and in* ***NI*** *and* $\mathcal{N}$, *respectively, their images are reductions: – of the same kind, in the case (2) (see the Theorem 7.6(I) from [2]); –with the same property as the images of the reductions from the case (3), in the case (4) (see the Theorem 7.5 from [2]).*

Lemma 7.5. *In the systems* ***NI*** *and* ***NE***:

(1) if Π *is normal in* ***NI****, then* $\mathsf{e}\Pi$ *is normal in* ***NE****;*
(2) if π *is normal in* ***NE****, then* $\mathsf{n}\pi$ *does not have any maximum segment.*
Proof. (1) By Lemma 6.5(2), Lemma 6.2(2), Lemma 6.2(1), Lemma 7.3, Lemma 7.4.
(2) By Lemma 6.6(2), Lemma 6.4(2), Lemma 6.4(1), Lemma 7.1, Lemma 7.2. □

Theorem 7.1. *If* Π *s-max*$>_n$ $\overline{\Pi}$ *in the system* ***NI****,*
then $\mathsf{e}\Pi$ $\mathcal{E}$*s-max*$>_l^-$ $\mathsf{e}\overline{\Pi}$*,* $l \geq n$*, in the system* ***NE****.*
Proof. By Lemma 6.5(1), Lemma 7.1 and Lemma 7.2. □

Theorem 7.2. *If* π $\mathcal{E}$*s-max*$>_n$ $\overline{\pi}$ *in the system* ***NE****,*
then $\mathsf{n}\pi$ *s-max*$>_l^{sE+}$ $\mathsf{n}\overline{\pi}$*,* $l \geq n$*, in the system* ***NI****.*
Proof. By Lemma 6.6(1), Lemma 7.3 and Lemma 7.4. □

Note 7.4. *We give the sketch of the proof of the normalization theorem for the system* ***NE*** *i.e. the proof of the property: each derivation* $\widetilde{\pi}$ *can be* $\mathcal{E}$*s-max-reduced into a normal derivation* $\widetilde{\pi}_N$*. Namely, it can be proved that the derivations of the system* ***NE*** *have the following properties: (1) if* π' *is* $\dfrac{\begin{array}{cc}\pi_1 & \pi_2\\ \Gamma\to A & [A]^n, \Delta\to C\end{array}}{\Gamma, \Delta\to C}\,\mathrm{sb}$*, where* π_1 *and* π_2 *are without maximum segments and* π' *is* $\mathcal{E}$*sb-reduced into a derivation* π''*, then* π'' *is without maximum segments (by an induction on the length of the derivation* π'*); (2) if* π *is the redex of an* $\mathcal{E}$*m-reduction and it does not contain the redex of any other* $\mathcal{E}$*m-reduction, then* π *can be* $\mathcal{E}$*s-max-reduced into a normal derivation* $\overline{\pi}$ *(by an induction on the length of the maximum segment, where we use the Lemma 3.2 and the property (1)). Thus, the normalization theorem for the system* ***NE*** *can be proved by an induction on the number of maximum segments in a derivation* $\widetilde{\pi}$ *such that we consider its subderivation as* π *from (2) and we use the property (2). (Similarly for the system* ***NI****.)*

Now we will prove the theorem about the connection between normalization procedures for **NI** and **NE**, where: either $\Pi >_k^+ \Pi_{NE}$ or Π is Π_{NE} (either $\pi >_j^{gz} \pi_{EN}$ or π is π_{EN}) will be denoted by $\Pi \geq_k^+ \Pi_{NE}$ ($\pi \geq_j^{gz} \pi_{EN}$).

Theorem 7.3. *For each derivation* Π *from* ***NI*** *and each derivation* π *from* ***NE****:*
(1) (1.1) if $\mathsf{e}\Pi$ $\mathcal{E}$*s-max*$>_n$ $\overline{\pi}$ *and* $\overline{\pi}$ *is normal, then*
$\Pi \geq_k^+ \Pi_{NE}$ *s-max*$>_l^{sE+}$ $\mathsf{n}\overline{\pi}$*,* $l \geq n$*, and* $\mathsf{n}\overline{\pi}$ *does not have any maximum segment;*
(1.2) if $\mathsf{e}\Pi$ *is normal, then* Π *does not have any maximum segment;*
(2) (2.1) if $\mathsf{n}\pi$ *s-max*$>_n$ $\overline{\Pi}$ *and* $\overline{\Pi}$ *is normal, then*
$\pi \geq_j^{gz} \pi_{EN}$ $\mathcal{E}$*s-max*$>_l^-$ $\mathsf{e}\overline{\Pi}$*, where* $l \geq n$ *and* $\mathsf{e}\overline{\Pi}$ *is a normal derivation;*
(2.2) if $\mathsf{n}\pi$ *is normal, then* π *is normal.*

Proof. (1) (1.1) By Lemma 6.1, Lemma 6.2(2), Theorem 7.2 and Lemma 7.5(2).
(1.2) By Lemma 7.5(2), Lemma 6.1, Lemma 6.2(2) and Theorem 6.2(1).
(2) (2.1) By Lemma 6.3, Lemma 6.4(2), Theorem 7.1 and Lemma 7.5(1).
(2.2) By Lemma 6.4(3) and Lemma 7.5(1). □

Note 7.5. *We note that the normalization procedure from **NI** makes the normalization procedure in **NE**, i.e. the normalization theorem for **NE** is a consequence of the normalization theorem for **NI** (the Theorem 7.3(2)). However, because of the definition of its normal derivations (see the Note 2.2), the normalization procedure for **NE** makes only the sequence of reduction steps which ends with one derivation without maximum segments in **NI** (the Theorem 7.3(1)).*

8 Conclusions and remarks

(I) By using explicit substitution it was shown that the connections between two versions of right sequent rules, the rules of **NI** and **NE**, are the same as the connections between natural deduction rules and left-right sequent rules, i.e. standard rules of sequent systems (see the Notes 6.1 and 6.2 above). Moreover, by reductions which make Π_{NE} for a derivation Π from **NI** (π_{EN} for a derivation π from **NE**), sb^{+}-reductions (gz-reductions), these connections are precisely described.

(II) Explicit substitution gives the more complex notion of maximum segments than the standard ones (see the Note 3.2) and new reductions of derivations: sb-reductions and ss-reductions in **NI** and $\mathcal{E}$sb-reductions and ss-reductions in **NE**. The standard reduction steps concerning maximum segments of the normalization procedures for **NI** and **NE** are well connected (see the Lemmata 7.1-7.4). But, these connections need the reductions concerning substitutions: sb_{E}-reductions and sb^{+}-reductions in **NI** and sb^{-}-reductions in **NE**. Namely, each m-reduction of the normalization procedure for **NI** has the corresponding $\mathcal{E}$m-reduction of the normalization procedure for **NE** with an sb^{-}-reduction (in some cases) (the Lemmata 7.1 and 7.2); and some $\mathcal{E}$m-reductions of the normalization procedure for **NE** have the corresponding m-reductions of the normalization procedure for **NI** with sb^{+}-reductions and sb_{E}-reductions (in some cases) (the Lemmata 7.3 and 7.4). However, it is important to note that some $\mathcal{E}$m-reductions of derivations from **NE** concerning maximum segments, correspond to reductions of derivations from **NI** concerning substitutions only (the Lemma 7.4(3)(4)).

Note 8.1. *We recall that standard permutation conversions from von Plato's system from [21] (i.e. $\mathcal{E}$maxs-conversions from $\mathcal{NE}$ from [2] and [5]) are not sources of all $\mathcal{E}$ms-reductions in **NE** (see the Note 5.1). Now we present the connections between* *the reduction ($\mathcal{E}ms^{\wedge_1}_{\wedge_1}$) from $\boldsymbol{NE}$ with the corresponding standard conversion from $\mathcal{NE}$ (i.e. von Plato's system from [21]) and the corresponding reduction from a sequent system. If the redex of the reduction ($\mathcal{E}ms^{\wedge_1}_{\wedge_1}$) from the Section 5.1 is reduced into the derivation $\overline{\overline{\pi}}$:*

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \Gamma'\to A\wedge B\end{matrix} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\pi'_2\\ \Delta''\to C\end{matrix} \qquad \begin{matrix}\pi''_2\\ \Delta'''\to D\end{matrix}}{\Delta'',\Delta'''\to C\wedge D}\wedge I\mathcal{E}}{[A]^n,\Delta'\to C\wedge D}Es... \qquad \begin{matrix}\pi_3\\ [C]^k,\Lambda\to E\end{matrix}}{[A]^n,\Delta',\Lambda\to E}\wedge E\mathcal{E}_1}{\Gamma',\Delta',\Lambda\to E}\wedge E\mathcal{E}_1}{\Gamma,\Delta,\Lambda\to E}sb...$$

where the source of that reduction is the conversion ($\mathcal{E}maxs^{\wedge_1}_{\wedge_1}$) of standard $\mathcal{E}$maxs-conversions from $\mathcal{NE}$ (i.e. a permutation conversion from von Plato's system from [21]), then the n-images of π and $\overline{\overline{\pi}}$, $\mathsf{n}\pi$ and $\mathsf{n}\overline{\overline{\pi}}$, are

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\mathsf{n}\pi_1\\ \widetilde{\Gamma'}\to A\wedge B\end{matrix}}{\widetilde{\Gamma'}\to A}\wedge E_1 \qquad \dfrac{\dfrac{\begin{matrix}\mathsf{n}\pi'_2\\ \widetilde{\Delta''}\to C\end{matrix} \qquad \begin{matrix}\mathsf{n}\pi''_2\\ \widetilde{\Delta'''}\to D\end{matrix}}{\widetilde{\Delta''},\widetilde{\Delta'''}\to C\wedge D}\wedge I\mathcal{E}}{[A]^{n1},\widetilde{\Delta'}\to C\wedge D}\vee\exists Es...}{\widetilde{\Gamma'}^{n1},\widetilde{\Delta'}\to C\wedge D}\text{sb}}{\widetilde{\Gamma'}^{n1},\widetilde{\Delta'}\to C\wedge D}sb...}{\widetilde{\Gamma}^{n1},\widetilde{\Delta}\to C}\wedge E_1 \qquad \begin{matrix}\mathsf{n}\pi_3\\ [C]^{k1},\widetilde{\Lambda}\to E\end{matrix}}{\widetilde{\Gamma}^{n1\cdot k1},\widetilde{\Delta}^{k1},\widetilde{\Lambda}\to E}\text{sb}$$

and

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathsf{n}\pi_1\\ \widetilde{\Gamma'}\to A\wedge B\end{matrix}}{\widetilde{\Gamma'}\to A}\wedge E_1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\mathsf{n}\pi'_2\\ \widetilde{\Delta''}\to C\end{matrix} \qquad \begin{matrix}\mathsf{n}\pi''_2\\ \widetilde{\Delta'''}\to D\end{matrix}}{\widetilde{\Delta''},\widetilde{\Delta'''}\to C\wedge D}\wedge I\mathcal{E}}{[A]^{n1},\widetilde{\Delta'}\to C\wedge D}\vee\exists Es...}{[A]^{n1},\widetilde{\Delta'}\to C}\wedge E_1 \qquad \begin{matrix}\mathsf{n}\pi_3\\ [C]^{k1},\widetilde{\Lambda}\to E\end{matrix}}{[A]^{n1\cdot k1},\widetilde{\Delta'}^{k1},\widetilde{\Lambda}\to E}\text{sb}}{\widetilde{\Gamma'}^{n1\cdot k1},\widetilde{\Delta'}^{k1},\widetilde{\Lambda}\to E}\text{sb}}{\widetilde{\Gamma}^{n1\cdot k1},\widetilde{\Delta}^{k1},\widetilde{\Lambda}\to E}sb...$$

respectively, and $\mathsf{n}\pi$ and $\mathsf{n}\overline{\overline{\pi}}$ are not redex and contractum of any m-reduction in $\boldsymbol{NI}$ (the same property of the systems from [2] was mentioned in the part (i) of the Note 7.3). But, if we make a map from a sequent system into $\boldsymbol{NE}$ (by using, for example,

ψ from [2]), then the reduction: π is reduced to $\overline{\overline{\pi}}$ is one step in an image of the following reduction from standard cut elimination for that sequent system:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_2\\ A,\widetilde{\Delta'}\to C\wedge D\end{array}}{A\wedge B,\widetilde{\Delta'}\to C\wedge D}\wedge L_1 \quad \dfrac{\begin{array}{c}\mathcal{D}_3\\ C,\widetilde{\Lambda}\to E\end{array}}{C\wedge D,\widetilde{\Lambda}\to E}\wedge L_1}{A\wedge B,\widetilde{\Delta'},\widetilde{\Lambda}\to E}\,\text{cut} \quad \textit{is reduced to} \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}_2\\ A,\widetilde{\Delta'}\to C\wedge D\end{array} \quad \dfrac{\begin{array}{c}\mathcal{D}_3\\ C,\widetilde{\Lambda}\to E\end{array}}{C\wedge D,\widetilde{\Lambda}\to E}\wedge L_1}{A,\widetilde{\Delta'},\widetilde{\Lambda}\to E}\,\text{cut}}{A\wedge B,\widetilde{\Delta'},\widetilde{\Lambda}\to E}\wedge L_1$$

*It is well known that the sequent systems have different sets of reductions for cut elimination. The reductions of one of them agree with reductions of normalization for standard natural deduction (see [8] and [3]), and the reductions of the other set agree with reductions of normalization for natural deduction with general elimination rules (see [21], [23], [2] and [5]). Now we can say that **NE** can have two different sets of reductions for normalization, where one of them agrees with normalization, while the other agrees with cut elimination.*

(III) The normalization procedure for **NE** does not make the complete normalization procedure for **NI** (see the Note 7.5 and the Theorem 7.3(1)). Thus, in **NI** and **NE** there are the same conditions: derivations are in the sequent style, both systems have explicit substitution and their rules are right rules; but, the different forms of these rules (we can say their different natures) and the complexity of the meaning of the notion substitution make the main role in the connections of normalization procedure for **NI** and **NE**. Namely, the connection between normalizations for **NI** and **NE** is like the connection between standard normalization and cut elimination.

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Tsien's Power-of-two Law in a Neuromorphic Network Model Suitable for Artificial Intelligence

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Abstract

We show that Tsien's "power-of-two" law concerning the basic wiring and computational logic of brain circuits, which has been confirmed in various animal studies, may be derived in a recently developed model of neuronal networks.

Keywords: Tsien's power-of-two law, theory of connectivity, wiring logic, cell assemblies, sequent calculus, neuromorphic networks.

1 Introduction

A neuromorphic network model was derived via abstractions of a combination of physical, chemical and computational principles in [1, 2, 3]. This model differs in several respects from the standard notion of a neural net. (Please see below). Neurons compile ultra local chemical computations—which suffice to regulate unicellular life—into distributed multiplexed electrical computational forms. The approach we took in building our model was to hide most of the immense complexity of the intracellular chemical computational structure by lowering the resolution of observable states in a formalized fashion [1]. As noted in [1], when variables are hidden in this way, a structure emerges which is similar to that of quantum theory. (Although actual quantum physics is surely not a theory of hidden variables, a theory of hidden variables will have quantum-like aspects: ours is one of those.) Thus, we begin at the

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mesoscale, taking our computational unit to be a model of the neuron. Our neuron model is *bicameral*, being resolved into an input "chamber" or node corresponding to that part of the soma to which the dendrites are attached, while the hillock (or trigger zone)-*cum*-axon initial segment, or indeed the entire axon, is regarded as a separate output chamber or node. In this way different values of membrane potential across the cell may be accommodated [2, 3].

The computational logic of a single network of such bicameral neurons (henceforth *b-neurons*) is identified with the logic of its space of states. In contrast to the hidden internal cellular microscale chemical computational structure, this may be thought of as the logic of the mesoscale. Computational logicians have discovered a way of extrapolating this mesoscale logic to the macroscale: namely, by embedding the logic of a single space (i.e. in our case, a single network) into a logic of many spaces (i.e. in our case, many networks). This latter logic is expressed in the form of a Gentzen sequent calculus: these matters are discussed in full in [3] and sketched below in section 3. The advantages of doing this are at least three-fold. Firstly, it provides a means of naturally synthesizing larger structures from smaller units, which, if done judiciously, may mirror the corresponding biological processes, and secondly, computations in the logic, if found, would be amenable to analyses along the lines of standard computability theory. Thirdly, the logical formulation can reveal macroscale structural properties that lie beyond the scope of the mathematically intractable physical models, which resemble the most complex models of many body theory, but are even more complex, since those circumstances that ameliorate the analogous physics, such as symmetries and known force laws, are entirely absent in brain networks.

An immediate upshot of these considerations is that our formalism imposes a wiring constraint upon the networks considered that precisely implements Tsien's power-of-two law which describes the basic wiring and computational logic of the brain [4]. Tsien put forth the Theory of Connectivity that cell assemblies across many brain regions are organized via a power-of-two permutation logic to generate a variety of principal cell cliques encoding specific-to-general features, respectively, covering all mathematical possibilities [4, 5, 6]. Tsien's power-of-two law has been recently validated across multiple cortical, subcortical and limbic circuits processing a range of cognitive functions such as social recognition of faces and body parts and classifications of food experiences and fear episodic memories [7]. It has also been found serendipitously (for the case of 3 inputs) in a recent study of macaque brains [8]. Our model pre-supposes the "standard" model of a multipolar neuron, with any number of inputs and/or outputs, whose soma can be approximately resolved as described above. Owing to the staggering diversity of neuron types, not all neurons conform to this standard. For instance, amacrine retinal cells, which can have axons emanating directly from their dendrites, likely cannot be so resolved [9]. (This is true also for certain dopaminergic neurons.) On the other hand, we conjecture that principal excitatory cells, such as pyramidal cells, do conform to this standard and this will be the cell type to keep in mind in what follows.

Our model differs in several respects from most neural network models as used for instance in AI applications, being closer to the biological paradigm. Our bicameral neuron model necessarily fulfills the biological rubric which contends that *computation is done in the synapses*, which correspond to our output chambers. This would not be possible with an irreducible monadic neuron model. In addition, the use of the canonical multilinear connectives of tensor and exterior product enables an effective simulation of some of the effects of substrate connectivity, implemented in biology independently of the principal axonal connections by direct connectivity via interneurons, direct or gap electrical junctions, or chemical media such as neurotransmitters, hormones, etc. In artificial systems this would be tantamount to an independent means of processing synaptic weights, perhaps by means of dedicated memristor-like circuitry. Finally, and most pertinently for the purposes of this paper, the model does not impose *a priori* wiring patterns or constraints. Rather, these may arise spontaneously, as does Tsien's power-of-two law, the subject of the present paper. It would appear that artificial implementations of such a model may be feasible, and this possibility will be taken up elsewhere.

It is remarkable to note that a single mathematical construct, known as the exterior algebra of a vector space, kept coming up in the course of these studies [2, 3], first in the quasi-physical model itself, and then again and surprisingly, in the computational logic of the model. This object was discovered before the middle of the nineteenth century by H. Grassmann, who seemed to appreciate its beautiful symmetric structure but whose writings on it were were famously misunderstood by his contemporaries and their successors into the twentieth century, probably because the mathematical vocabulary and notations of the time were not up to the complexities of the many operations this object allows. Cf. [10]. It can be described in modern terms roughly as follows. Consider any vector space, V say. Then there exists a certain algebra $E(V)$ and a linear map of V into $E(V)$ with the properties that the square in $E(V)$ of the image under this map of every vector in V is zero, and $E(V)$ is the "smallest" algebra with this property (see Appendix A.3 for details). Then $E(V)$ is unique up to algebra isomorphism. It is called the *exterior algbera* of V. It is not obvious from this definition that $E(V)$ has a *graded* structure but the mathematics compels it. This grading is the origin of Tsien's power-of-two law, which emerges as a mathematical necessity in our model. (This object is an example, in more modern terminology, of a *graded Hopf algebra.* The algebraic aspect of it was rediscovered by early workers in quantum theory as the state space of a collection of fermionic quanta. To physicists it is known as the *Fermi-Dirac*, or *Fermi-Dirac-Fock* space of V.)

We have attempted to make this paper as self contained as possible. All that is required mathematically is a knowledge of elementary linear algebra and the notion of an algebra. An appendix (Appendix A) summarizing what is needed from multilinear algebra is supplied.

The layout of the paper is as follows. Section 2 contains a brief review of our network model in quasi-physical terms; section 3 briefly describes the system of computational logic inherent in the model and its possible relation to memory functions; section 4 applies the formalism to simulate a set of large-scale *in vivo* neural recording experiments Tsien and his group performed to confirm his postulated power-of-two law, and to derive the law itself. There are two appendices: the mathematical one mentioned above, and another containing the fragment of the sequent logic needed in section 4.

2 A brief sketch of the model and its associated spaces

We start with networks of basic nodes (modelling parts of neuronal somata) which can contain real values (*à la* local membrane potentials). Arbitrary numbers of edges are directed outward from a node to other nodes (*à la* axonal terminals) and an arbitrary number of edges are directed inward to a node from other nodes (*à la* dendritic inputs). Note that this definition includes the degenerate case of no edges at all, namely just a collections of nodes with no links between them.

Suppose the nodes of such a network $\mathscr{N}_A$ with N nodes are labelled in a certain order $n_1^A, \ldots, n_N^A$. Then the real N-dimensional vector space of the corresponding (real) vector of nodal values $(\lambda_1, \ldots, \lambda_N)$ is spanned by the basis of vectors e_i^A which is the vector with 1 in the i-th position and zeroes elsewhere. Thus a possible vector of values may be expressed as

$$\mathbf{v} = \sum_{i=1}^{N} \lambda_i e_i^A \tag{2.1}$$

where λ_i is the value *in* the node n_i^A. In general these values will depend on time. We shall for the time being denote this "state" space by $\mathfrak{H}_A$ and regard it as a real Hilbert space with the usual Euclidean inner product. (The extent to which the linearity of this structure is relevant is discussed in [1, 2].) We emphasize that this is the space of all possible vectors the network may hold.

2.1 Network combinations

There are various ways of combining networks. By way of illustration, one way is to merely consider the network formed by juxtaposing, or putting together, networks without positing any connections between them. If this is done with networks $\mathscr{N}_A$ and $\mathscr{N}_B$, say, then the state space of the combined system is easily seen to be the direct sum of the individual spaces:

$$\mathfrak{H}_A \oplus \mathfrak{H}_B. \tag{2.2}$$

We shall not require this construct for the logical purposes of this paper. There is another, more subtle, combination that will be of greater concern to us here. Namely, an association among nodes that is by means of a *substrate*, or other type of connection, different and in addition to the network of axon-like connections itself. In nature there are various mechanisms whereby ensembles of cells become associated co-operatively with each other, either by spatial proximity or by sharing the same neuromodulatory chemical environment, or because they are jointly orchestrated or tuned, for instance by a shared substrate of interneurons or other cells. In this kind of combination (or pairing in the case of two networks) a node n_i^A of a network $\mathscr{N}_A$ may be associated with a node n_j^B of a network $\mathscr{N}_B$ in such a way that the pair (n_i^A, n_j^B) may act as a unit, which contains a single value. The state of such a unitized pair is well modeled by the tensor product $e_i^A \otimes e_j^B$ of the corresponding states (cf. Appendix A.1 if necessary). A value assigned to this paired state cannot be attributed to any one constituent state since $\lambda e_i^A \otimes e_j^B = e_i^A \otimes \lambda e_j^B = \lambda(e_i^A \otimes e_j^B)$, $\lambda e_i^A \otimes \mu e_j^B = \mu e_i^A \otimes \lambda e_j^B = \lambda\mu(e_i^A \otimes e_j^B)$ and similarly for any number of tensorial tuples. That is to say, in keeping with the doctrine of hidden variables, such a combined tuple of nodes *hides* the location of a value λ: for instance the state $\lambda(e_i^A \otimes e_j^B) = \lambda e_i^A \otimes e_j^B = e_i^A \otimes \lambda e_j^B$ cannot distinguish between the circumstance that the value λ is in the i-node and the circumstance that is it in the j-node. Thus the space of all possible such pairs is the tensor product of the individual spaces,

$$\mathfrak{H}_A \otimes \mathfrak{H}_B \tag{2.3}$$

in this case, and similarly for any number of networks. In this way the details of the microscopic computations proceeding inside each node and among an ensemble of co-acting nodes, are hidden, in keeping with our doctrine of hidden variables, while mesoscale effects are taken into account. This closely mimics the way in which interneurons among other mechanisms orchestrate the activity of principal neurons. Moreover, should one of these values momentarily become zero, then the probability of the combined state being available (or firing) falls to zero. This is both synchronizing and inhibiting. It simulates in a vastly simplified but effective manner the kind of inhibitory synchronization effected by biological interneurons, a hugely diversified group including basket cells and chandelier cells, which connect local principal neurons and induce very rapid inhibitory signalling between networks of them, having synapses of both chemical and electrical types (bidirectional gap junctions). (We note that these cells have their own panoply of neurotransmitters, such as GABA in addition to the electrical connections.) Of course, the joint value could go up, and in that case there are problems for the neural applications we have in mind. Cf. section 2.3 for further discussion. (The ability of this model to take such substrate or interstitial liaisons into account is one way in which the model differs from the standard neural net model.)

2.2 The bicameral neuron

As noted, our computational unit will be taken to be the *b-neuron* which comprises a pair of our primitive nodes, regarded as two "chambers," modelling the "standard" neuron. Namely, one node is regarded as the *input* chamber corresponding to that part of a neuronal soma to which the dendritic inputs are attached, while the second is the *output* chamber corresponding to the hillock-*cum*-axon initial segment along which the action potential is conducted upon the firing of the neuron. In this way we can accommodate the values of the membrane potential at the two extreme ends of the neuron, which will generally be different upon its firing. Once the graded input potential achieves the threshold value, assuming it does, the action potential is triggered and occupies the output chamber. The value, or amplitude, of this action potential will in general be different from, and independent of, the incoming somatic potential subsequent to the firing. Since the basic neuron now comprises two nodes, its state space is a two dimensional state space isomorphic to $\mathbb{R}^2 \cong \mathbb{R} \oplus \mathbb{R}$, where $\mathbb{R}$ denotes the field of real numbers and where the first term accommodates the state space of the input node and the second term accommodates the state space for the output node, and is interpreted as a two node system. As such we have something very like a *qubit*, which is the quantum two state analogue of the classical *bit* and is taken as a basic computational unit in the hopeful discipline known as quantum computing. Denoting the generic vector in the extended state space $\mathbb{R}^2$ of such a two node system by (λ, μ) we found a pair of operators

$$\mathfrak{a}^T(\lambda, \mu) := (0, \lambda), \tag{2.4}$$

$$\mathfrak{a}(\lambda, \mu) := (\mu, 0). \tag{2.5}$$

where $(\ )^T$ denotes operator transpose or adjoint relative to the Euclidean inner product. These operators satisfy the following *anticommutation relations*:

$$\mathfrak{a}\mathfrak{a}^T + \mathfrak{a}^T\mathfrak{a} = 1, \tag{2.6}$$

$$\mathfrak{a}^2 = 0, \tag{2.7}$$

$$(\mathfrak{a}^T)^2 = 0. \tag{2.8}$$

It is quickly checked that these operators are indeed mutually adjoint. These relations identify the system as a fermion-like one, since it can be shown that they represent respectively the annihilation and creation of a fermion-like quantum of one type.

The operator $\mathfrak{a}^T$ represents the internal transfer of the current value in the input node from that node to the output node and thus represents a *preparation* to fire the neuron: if λ has reached the threshold value then, at that moment, it becomes a value of the action potential and moves into the trigger zone or hillock, replacing any value μ that may be there (equation (2.4)). At the same time there is still a graded potential registering upon the input node. That is to say, the action potential state $(0, \lambda)$ must at this moment be added to the current state of the input node which is of the form $(\nu, 0)$ for some ν. We note that the operator $\mathfrak{a}^T$ is not an observable of the system, but rather an internal unobservable operation that in a sense stands for the electrochemical ionic flows, etc., that physically transfer the potential across the cell body, while hiding the hugely complex details.

As a space of *states* of the underlying b-neuron this space has additional algebraic structure which captures an important aspect of actual neurons and reveals an improved mathematical model of the association connective in the presence of a co-activating substrate that overcomes a problem with the $\otimes$-product. To see this, first note that insofar as the operator $\mathfrak{a}^T$ represents a preparation to fire the b-neuron, that fact that $(\mathfrak{a}^T)^2 = 0$ means that this preparation cannot be executed twice at the same time: a neuron cannot fire while it is already firing. Now consider the subspace of the algebra of operators on $\mathbb{R}^2$, namely $\operatorname{End}\mathbb{R}^2$, generated by the identity operator I and $\mathfrak{a}^T$, which may be written

$$\mathcal{A} := \mathbb{R}I \oplus \mathbb{R}\mathfrak{a}^T \subset \operatorname{End}\mathbb{R}^2. \tag{2.9}$$

Here we have explicitly written in the two generators I and $\mathfrak{a}^T$. It is immediately seen that $\mathcal{A}$ is in fact a (commutative) *subalgebra* of $\operatorname{End}\mathbb{R}^2$ in view of equation (2.8). Now note that the inclusion map $\iota : \mathbb{R}\mathfrak{a}^T \hookrightarrow \mathcal{A}$ is easily seen to satisfy the UMP of Appendix A.3 so that $\mathcal{A} \cong E(\mathbb{R}\mathfrak{a}^T)$ as an algebra. If we make explicit the state representing the b-neuron's output node, e say, then we may express the state space of the b-neuron in the form

$$E(\mathbb{R}e) = \mathbb{R} \oplus \mathbb{R}e \tag{2.10}$$

where $e^2 = e \wedge e$, which will be seen to express the fact that the neuron cannot fire twice simultaneously and at the same time realizes the exterior product $\wedge$ as essentially the unique combinator modeling this property mathematically.

2.3 Network state spaces and the significance of the exterior product

For a network $\mathscr{N}_A$ of b-neurons, where the fanned in (dendritic) edges to a b-neuron enter the input node, and the fanned out (axonal) edges exit the output node, let us label the output nodes $(n_1^A, \ldots, n_N^A)$ so that the corresponding basis states are $\{e_1^A, \ldots, e_N^A\}$. Then, taking into account the possible presence of substrate connectivity of the type discussed in section 2.1, the state space for the whole network is the tensor product of the individual b-neuron state spaces, namely:

$$\begin{aligned} E(\mathbb{R}e_1^A) \otimes \cdots \otimes E(\mathbb{R}e_N^A) &\cong E(\mathbb{R}e_1^A \oplus \cdots \oplus \mathbb{R}e_N^A) && (2.11) \\ &= E(\mathfrak{H}_A) && (2.12) \\ &= \mathbb{R} \oplus \mathfrak{H}_A \oplus \bigwedge^2 \mathfrak{H}_A \oplus \cdots \oplus \bigwedge^N \mathfrak{H}_A && (2.13) \end{aligned}$$

by the fundamental identity equation (A.7). This is an algebra isomorphism when the exterior product is used on the right and the graded product is used on the left.

In [2] we derived a dynamics for such a network which turns out to be analogous to the dynamics of physical systems of many fermionic quanta of the *quasispin* type [11], except that it is over the real numbers. In this it also resembles the Ising-like models of Hopfield et al. [12].

In this formulation a basic homogeneous element in $E(\mathfrak{H}_A)$ of the form $e_{i_1} \wedge \ldots \wedge e_{i_p}$ represents the state in which the (output) nodes $n_{i_1}^A, \ldots, n_{i_p}^A$, are *occupied*, which means that the corresponding b-neurons are *firing* (or are ON) while all the others are not (or are OFF). Consequently, they were described in [2] as *firing patterns*, and this terminology was extended to all the states in $E(\mathfrak{H}_A)$ (since in that formulation superpositional states were allowed).

Now we note a problem with the $\otimes$-product mentioned earlier. We have interpreted a tensor of the form $\lambda e_i \otimes e_j = e_i \otimes \lambda e_j$ as a state of a pair of corresponding nodes associated or combined by a substrate of some kind so that the pair acts as a unit and registers the single value $\lambda \in \mathbb{R}$. If this value decreases to zero, the combined state contributes nothing to any firing pattern: the probability of firing falls to zero [2]. Thus the substrate connection admits *inhibitory* input into nodes, and for this reason we identified it with the substrate provided in actual brains by the network of interneurons. But this kind of connection would also admit excitatory inputs since the algebra admits any real value for λ so it may increase as well as decrease. However, in this case, a problem would arise for self connection of nodes via this substrate, whose basic state would be of the form $\lambda e_i \otimes e_i$. If a node were connected to itself in this manner and admitted excitatory input, then it would run the risk, in the application to b-neurons, of precipitating the forbidden state of double firing of the b-neuron.

It is therefore significant that the exterior product, $\wedge$, which, we note, arose spontaneously out of our adoption of the operator $\mathfrak{a}$ in equation (2.4) or equation (2.5), expressly excludes this possibility, since $\lambda e_i \wedge e_i = 0$, while at the same time also admitting inhibitory substrate connections, since it has the same multilinear property of sharing scalar values as the $\otimes$-product.

The delineaments of Tsien's law for networks of our type can already be discerned in the structure of the space of firing patterns as displayed on the right hand side of equation (2.13). Namely, that it is a *graded* structure, the grading reflecting all possible firing subpatterns. Thus, the 0-th grade $\mathbb{R}$ represents the "vacuum" state of zero occupancy or no firing, the first grade $\bigwedge^1 \mathfrak{H}_A = \mathfrak{H}_A$ the space of states of all possible single firings, ... , the k-th grade $\bigwedge^k \mathfrak{H}_A$ the space of states of all possible k-tuple firing subpatterns, etc. Since $\dim \bigwedge^k \mathfrak{H}_A = \binom{\dim \mathfrak{H}_A}{k} = \binom{N}{k}$ there are

$$\sum_{k=1}^{N} \binom{N}{k} = 2^N - 1 \tag{2.14}$$

independent such firing subpatterns. This is Tsien's power-of-two formula. How it fits into a computational scheme is the subject of the rest of this article.

3 Computation and memory

The computational scheme adopted in [3], which externalizes and multiplexes the computational logic of a single network, turns out to be identical with a fragment of a self-dual version of system of deductive logic introduced by Girard in the late 1980s called Linear Logic [13, 14, 15, 16]. It is applied in [3] to give an algorithmic account of the memory function known as *cued pattern completion.* Here we shall give a very brief account of this system, dubbed **GN** (for Gentzen Neurologic).

A *Gentzen sequent calculus* is a metacalculus for dealing wholesale with deductions in a possibly notional underlying deductive system. It is appropriate to our endeavor here since its modus operandi is to hide the details of a possibly notional underlying deduction. Thus, the basic term, a *sequent*, is written

$$\Gamma \vdash \Delta, \tag{3.1}$$

where Γ and Δ are sets of formulas, which has the informal reading as "$\bigwedge \Gamma \Rightarrow \bigvee \Delta$," where here the symbol $\bigwedge$ means logical conjunction, not exterior product. The details of the deduction "$\Rightarrow$" are hidden in "$\vdash$". As metacalculi for natural deduction systems these sequent calculi delineate certain symmetries and structural aspects of the underlying deductive system which are hidden if one remains at the lower level of the underlying system. The organizing power of the style has had a major impact on the proof theoretic aspects of deductive logic.

The sequent calculus idea lends itself to other interpretations. A revolution occurred when Girard realized that it could be used to effect an extremely refined computational theory of resource management: here, a sequent of the form (3.1) is read, roughly speaking, as the depiction of a process in which the resource Γ is consumed to produce the resource Δ. (These resources may be evanescent, being possibly changed or used up in the process of "passing through" the turnstile.) The concomitant logical rules and connectives then acquire entirely new and more general interpretations and obey new laws. Thus arose the system known as Linear Logic (**LL**).

The inferences in a sequent calculus are expressed in the form of a fraction with the denominator sequent being *inferred* from the numerator sequent above it. Only a small subset of the axiomatic inferences for our system, dubbed **GN**, is needed for this paper and it is given in Appendix B.

In applications, one adds "non-logical" axioms to the **GN** rules, for instance to depict a brain as a family of linked networks (see paragraph below), and other sorts of given sequents. The resulting family of proofs or deductions is known by logicians as a **GN**-*theory.*

We shall cut through the rigorous formal procedures usually imposed by logicians and apply this system informally. We are already half-way there by using the same symbol $\otimes$ for the logical connective in corresponding rules for **GN** shown in Appendix B as is used in the category of vector spaces. In a properly formal approach the atomic formulas of the logic are mapped to (or *interpreted as*) certain objects in a category, and this association is extended in an obvious way to all well formed formulas. This must be done in such a way that sequents are mapped to morphisms in the category so that equivalent proofs are mapped to the same morphism in the category. This can be done for **LL** and this fragment of it, but we shall short circuit this process, or, rather, assume it has already been done, and just regard the formulas (also known as *types*) $A, B, \ldots$, as finite dimensional real

Hilbert spaces (formerly written $\mathfrak{H}_A, \mathfrak{H}_B, \ldots$) read as representing the state spaces of networks $\mathscr{N}_A, \mathscr{N}_B, \ldots$ (This is no restriction on finite dimensional vector spaces since any such space is the state space of at least one network, namely the degenerate one with the same number of nodes as the dimension of the space). Sequents $A \vdash B$ will then represent linear maps of A into B. There are special ones that arise by linking of $\mathscr{N}_A$ to $\mathscr{N}_B$ via the fanning out nature of the edges emanating from our nodes. Namely, a network $\mathscr{N}_A$ may be connected ("downstream") to a network $\mathscr{N}_B$ by specifying an immediate (single-link) axon-like connection of node n_i^A to certain nodes n_j^B, or none, for each i. At the state space level this determines an assignment in which $e_i \mapsto \sum_j \alpha_{ij} e_j^B$ and some real values α_{ij}. In a biological network the real values α_{ij} will be determined by the graded potential induced upon the receiving nodes n_j^B by an action potential discharged by n_i^A and will factor in also the adjacency matrix of the connection of $\mathscr{N}_A$ to $\mathscr{N}_B$. Thus, although the map *determines* a linear map $\mathfrak{H}_A \to \mathfrak{H}_B$ it is itself not linear in the relevant variables (not shown) upon which the α_{ij} depend (cf. [2, 3]). Such linking sequents may be composed via the Cut rule to produce connections of arbitrary length. This type of map is like a *wiring diagram* of a connection of $\mathscr{N}_A$ to $\mathscr{N}_B$ describing a potential (i.e. *possible*) situation. In general, the coefficients will depend on time which we have not yet taken into account. Note that since the adjacency matrix of the connection is factored into these coefficients, and may depend on time, the possible dynamic plasticity of axons is inherent in the structure. Moreover, we have not labeled individual sequents. This is because they may be considered to be classes of maps between the interpreted types involved, whose details are, again, hidden. (Cf. [2, 3] for more detail.) It is important to note that other sorts of maps of state spaces must also be allowed, and are treated as sequents on the same logical footing as these linking sequents. (Thus a sequent resembles, but is more general than, a wiring connection between one layer of a standard neural network and the next.)

In this interpretation of formulas or types as vector spaces and sequents as maps thereof, we interpret the empty set of formulas as the vector space $\mathbb{R}$, and sequences of types (appearing to the left of the turnstile) as the tensor product of their members, which is consistent with the rule L$\otimes$ (equation (B.6)). Note that **GN** allows only single formulas, or none, to the right of the turnstile. This is of some computational import: namely it renders the system *intuitionistic* or *constructive* [3].

3.1 Storage and the rule of Contraction

In ordinary logic the rule of Contraction, which takes the sequent form

$$\frac{A, A, \Gamma \vdash D}{A, \Gamma \vdash D}, \tag{3.2}$$

is interpreted as expressing the fact that the $A \wedge A$ conjunct (i.e. A *and* A) implicit in the top line can be replaced by the single instance of A in the bottom line. In the Girardian resource paradigm it has a different connotation: if A is used twice in a production of D then it may be used only once. The implication here is that the resource A is *storable* since then only one instance of it is needed, other instances being retrievable at will from its putative store. Clearly this may not happen if A is evanescent: it may not be usable more than once if it is damaged or used up. For this reason Girard introduced the modal operator $!(\ )$, called *of course*, which produces a storable version of A when applied to it, and so restores the rule of Contraction (LC, equation (B.3)). We have dubbed formulas that satisfy the rule LC *storage capable*. The other rules of **GN** involving ! are concomitant.

Now it turns out—and this is the crucial point—that the exterior algebra of a vector space exactly models the ! operator in the category of finite dimensional vector spaces. That is to say, if A represents the state space of a network ($\mathfrak{H}_A$ in our old notation), then we can take $!A$ (or $E(\mathfrak{H}_A)$ in our old notation) to represent the storage capable version of A. (A proof is given in [3, 17] among other places.) But at the same time, $!A$ represents the state space of the same network with b-neurons as nodes, and again at the same time, this space *also* represents the space of *firing patterns* of the latter network. Thus firing patterns are storable commodities. (This does not exhaust the convergence of interpretations meeting in the exterior algebra. By the Plücker embedding (Appendix A.4) each firing pattern determines a unique subspace of A itself and therefore these patterns reflect the logic of the space A. Tsien's law is ultimately a manifestation of this observation.)

This property of the exterior algebra of a vector space comes about because its algebraic structure is only half the story. As noted in Appendix A.2 it has another, dual, *coalgebra* structure, with coproduct $\psi : !A \to !A \otimes !A$, which is used to implement the rule LC.

Sequents of the form $!A \vdash B$ now have the interpretation as maps of the *firing patterns* (of the A-network) which is to say, insofar as the sequent represents a wiring diagram, it concerns those wirings relevant to all possible subsets of *firing* b-neurons. Again, this will ultimately manifest Tsien's law since, reverting to equations (2.11) to (2.13), we have, from the interpretation of the last sequent, the linear maps (in the new notation), $A \hookrightarrow !A \to B$, $\bigwedge^2 A \hookrightarrow !A \to B$, $\bigwedge^3 A \hookrightarrow !A \to B$, $\ldots$, implying an independent wiring of each possible co-acting subnetwork of firing b-neurons into whatever network the A-network fires into.

3.2 The timing of sequents

In [3] we introduced a minimal timing scheme for sequents, such that only sequents which hold at the same time may be deemed to be amenable to the rules of **GN**. Thus, a basic sequent $A \vdash B$, of the type considered above, if valid at time t, which we shall write $A \vdash_t B$ where t is a real number residing in some subinterval of the non-negative real numbers, represents in a sense a wiring *diagram* at time t. It represents a certain state of affairs relating to how the network $\mathscr{N}_A$ is connected to the network $\mathscr{N}_B$ at time t, which also includes information concerning the possible firing activity of $\mathscr{N}_A$'s nodes relative to N_B's nodes at time t. As such it represents an *eventuality*, or possible outcome at time t, rather than an event.

As noted, only sequents holding at the same time may be combined according to these rules. An analysis of the rules then showed that only one of them, namely the storage rule LC, could be deemed to take a discernible time to physically implement and then only when the network involved is *synchronized* at the time involved. The other rules are either changes to the representing Hilbert spaces reflecting unphysical processes or book-keeping devices. We say that a network of identical b-neurons is *co-activated* or *synchronized* at time t if all of its b-neurons that are firing at the time t are at the same point in the progress of their action potentials.

Writing $A \vdash_t B$ if the sequent $A \vdash B$ holds at time t, the LC rule is now written

$$\frac{!A, !A, \Gamma \vdash_t D}{!A, \Gamma \vdash_{t+\tau_c^A} D.}\mathrm{LC}_t \tag{3.3}$$

That is to say, if $!A, !A, \Gamma \vdash_t D$ represents a firing-cum-wiring diagram at time t (and only the subsets of firing b-neurons are involved in this diagram, as noted above), and all the A b-neurons that are firing at that moment are at the same point of their action potentials, then this diagram may be replaced by the diagram $!A, \Gamma \vdash_{t+\tau_c^A} D$ at time $t + \tau_c^A$. This is the time at which all the A b-neurons would reach their rest states after t. (This value τ_c^A will itself depend on t so we take it to be the average such time, pending a finer analysis. In humans the duration of an action potential is approximately 1 ms to 1.5 ms after initiation, so the average interval between this and the attainment of the rest state is $\tau_c^A \sim 2$ ms.)

The following are proved in [3]:

Theorem 1
Suppose $!A \vdash_{t_0} !B$ for some $t_0 \geqslant 0$ and that the A network is synchronized at t for all $t \geqslant t_0$. Then $!A \vdash_{t_0+n\tau_c^A} !B$ for $n = 1, 2, 3, \ldots$

Suppose there exists a time $t_0 \geqslant 0$ such that, if $!A \vdash_{t_0} !B$, then $!A \vdash_t !B$ for all $t \in [t_0, t_0 + \tau_c^A]$. Then we shall say that the sequent $!A \vdash_{t_0} !B$ (or $!A \vdash !B$) is *persistent* at t_0.

Theorem 2
Suppose $!A \vdash !B$ is persistent at t_0, $!A \vdash_{t_0} !B$, and the A network is synchronized for all $t \geqslant t_0$. Then $!A \vdash_t !B$ for all $t \geqslant t_0$.

The first of these shows that sequents involving storage capable networks such as those presumably involved in memory functions, will, if the upstream network is synchronized, refresh at discrete intervals, until the synchronization is vitiated. (Of course each refreshment is interpreted by a different map, whose details can be exposed if necessary.)

The second result shows that it does not take much latency time in the upstream (or antecedent) network for this refreshment process to become continuous.

These results are useful in discussions of possible memory functions [3].

3.3 Retrieval

In [2] we derived the infinitesimal generator of time translation of the states of bicameral networks of our type. This operator upon the real Hilbert space of states or firing patterns, of the from $!A$ for instance, is the analogue of the Hamiltonian of an ordinary quantum system, except that in our case it acts upon a real space, and is not orthogonal, which is the real analogue of hermiticity, a condition demanded of physical quantum observables, but not demanded in our case. However, the eigenstates of this operator, as in the physical quantum case, are stationary, or stable, at least over short periods of time, and moreover are local attractors. (Cf. [2] where we analyzed the eigenstates of the thirteen three node motifs, and [3].) For this reason, as in the case of ordinary attractor networks, we attribute possible cognitive significance to such eigenstates. For instance, we assume that they may represent memory traces or *memoranda*: the experience of a memory *is* the firing of the neurons involved in the relevant firing pattern.

In [3] we consider the issue of retrieval in our setup in some detail. Here we give a brief account of that discussion. In ordinary computing parlance, this operation is dual to the operation of storage. In our setup it has a somewhat different connotation. Thus, in ordinary quantum mechanics “retrieval” would correspond to a measurement operation, and in the standard Copenhagen interpretation this amounts to a projection upon an eigenstate or eigenspace of the Hamiltonian. If we regard a retrieval operation in our context as being similarly implemented by a projection upon an eigenspace of our Hamiltonian analogue, then such projections, being linear maps, may be included as non-logical axioms in our basic logical setup. In this sense the operations of storage and retrieval, both being implemented via sequents in the calculus, are *unified*. In our context, this process of projection, onto an eigenstate say, reflects the collapse of all the possible choices of firing patterns at a particular moment onto one that is of cognitive significance, such as a memory trace.

We noted in [3] that:

1. The image under a linear map of an eigenstate inherits its stability against decoherence and other dynamical effects, regardless of the dynamics of the target network.

2. Any subspace of the state space of a network, such as an eigenspace, may be embedded into the state space of a subnetwork of the network.

In this way, we can render the operation of retrieval in terms of sequents. Thus an incoming stimulus at time t_0, say, corresponding to a "stored" or already established A-network, causes a collapse, or projection, of $!A$ onto an eigenspace. (Since eigenvectors are local attractors [2] this may at first be a collapse onto a "nearby" subspace.) As noted above this eigenspace may be embedded into the state space of a subnetwork, $!D$ say. Then the projection of $!A$ onto the eigenspace composed with the embedding of the latter into $!D$ interprets a sequent $!A \vdash_{t_0} !D$ which instantiates the act of retrieval. We shall regard $!D$ as a network that registers, like a memory, the result of the retrieval. The network $!D$ may also feed the result out, via projection axons modeled by further sequents, not necessarily shown. (The use of the word "projection" in "projection axon" has of course nothing to do with its mathematical use earlier: it is another case of unfortunate duplication of jargon across disciplines.) In this case we may regard this sequent as implementing the wiring diagram for an *activation* of the A network.

We showed further in [3] that if the A-network is co-activated (or synchronized) and τ_c^A is small enough, we would have persistence at t_0 and therefore, by Theorem 2, we would have $!A \vdash_t !D$ for all $t \geqslant t_0$ until the breakdown, for instance, of synchronicity. This possibly lengthy lifetime of the connection seems appropriate to a working memory. In general, the lifetime of the retrieved memorandum may be modulated in various ways.

4 Successive activations and Tsien's law

We shall in this section finally apply our formalism (rather informally) to model the experimental setup used by Tsien et al. to confirm his power-of-two law in organizing cell assemblies and show how some of these findings follow from our assumptions. In this series of careful experiments rodents were exposed to various classes of new memorable experiences (fearful episodic events such as earthquakes or appetitive food experiences such as eating candy) and the activation of certain cortical areas were recorded and analyzed (cf. for instance [7]).

We note that in the context of this section the term *clique* will be taken to mean a group of neurons exhibiting similar tuning properties [4, 7]. Thus our basic homogenous firing patterns may be taken to represent the states of such cliques.

We shall assume that our networks satisfy the conditions laid out above wherever necessary. Thus the modeled neuron's soma should be resolvable into two chambers, namely an input node and an output node as above, the model to keep in mind being pyramidal or other excitatory principal cells, residing in certain areas (such as the hippocampal CA1 region, amygdala or in cortical layers L2–L6).

Here we shall attempt to simulate the general experimental setup. We shall consider first the simplest case in which single cells are assumed to be activated by each stimulus. We shall retain our earlier convention that state spaces of single b-neurons be denoted by a lower case Roman character, such as $!a$, say. Here a is interpreted as the one dimensional state space of the output node of a b-neuron, $!a$ denoting the two dimensional exterior algebra of a, the state space of the b-neuron. We assume, as noted, that some b-neuron, n_1 say, will be activated by the first stimulus in the chosen class (fearful event-1, i.e. earthquake; taste-1, i.e. candy, etc.) at time t_0 say. This is described as in section 3.3 by a planned "activation" sequent of the form:

$$!a_1 \vdash_{t_0} !D_1 \tag{4.1}$$

(where a single b-neuron is trivially synchronized, so we may apply Theorem 1). We suppose also that there is some long term memory store (or LTM) $!M$ that the D network may feed out to, at some later time t_1, via a sequent $!D \vdash_{t_1} !M$. Then we have

$$\frac{!a_1 \vdash_{t_1} !D_1 \quad !D_1 \vdash_{t_1} !M}{!a_1 \vdash_{t_1} !M}\textsc{Cut} \tag{4.2}$$

from which follows

$$\cfrac{\cfrac{!a_1 \vdash_{t_1} !M \qquad !a_1 \vdash_{t_1} !D_1}{!a_1, !a_1 \vdash_{t_1} !M \otimes !D_1}\mathrm{R}\otimes}{!a_1 \vdash_{t_1+\tau_c^1} !M \otimes !D_1.}\mathrm{LC}_{t_1} \tag{4.3}$$

Now it follows for all s from Ax that

$$\dfrac{\dfrac{!M \vdash_s !M}{!M, !D_1 \vdash_s !M}\,\mathrm{LW}}{!M \otimes !D_1 \vdash_s !M}\,\mathrm{L}\otimes \tag{4.4}$$

so that CUTting the last two conclusions we obtain

$$!a_1 \vdash_{t_1+\tau_c^1} !M \tag{4.5}$$

(and similarly $!a_1 \vdash_{t_1+\tau_c^1} !D_1$) and at later times in view of Theorem 1. In other words, some time after first activation, $!a_1$ is mapped into LTM.

Now suppose that the second stimulus (startle-2 or taste-2, etc.) is experienced by a second neuron n_2:

$$!a_2 \vdash_{t_2} !D_2 \tag{4.6}$$

where t_2 is some later time. Then, with (4.5) we have

$$\dfrac{\dfrac{!a_1 \vdash_{t_2} !M \qquad !a_2 \vdash_{t_2} !D_2}{!a_1, !a_2 \vdash_{t_2} !M \otimes !D_2}\,\mathrm{R}\otimes}{!a_1 \otimes !a_2 \vdash_{t_2} !M \otimes !D_2.}\,\mathrm{L}\otimes \tag{4.7}$$

As above we then have

$$!a_1 \otimes !a_2 \vdash_{t_2} !M \tag{4.8}$$

and

$$!a_1 \otimes !a_2 \vdash_{t_2} !D_2 \tag{4.9}$$

and at later times in view of Theorem 1. Equation (4.9) shows that the planned activation of $!a_2$ entails the planned *reactivation* of $!a_1$ (which will fire into D_2 but that is still an activation). If we had repeated the first stimulus instead of inducing the second, we would have had instead of (4.7):

$$\dfrac{\dfrac{!a_1 \vdash_{t_2} !M \qquad !a_1 \vdash_{t_2} !D_1}{!a_1, !a_1 \vdash_{t_2} !M \otimes !D_2}\,\mathrm{R}\otimes}{!a_1 \vdash_{t_2+\tau_c^1} !M \otimes !D_2}\,\mathrm{LC}_{t_2} \tag{4.10}$$

from which follows as usual that $!a_1 \vdash !M$ and $!a_1 \vdash !D_1$ at a later time. That is to say, we return to the initial state of affairs.

Now we expand the antecedent in the sequents in (4.7) through (4.9) using the identity (A.7). Here we denote by e_i the generator of the space a_i:

$$
\begin{aligned}
!a_1 \otimes !a_2 &= !(a_1 \oplus a_2) && (4.11)\\
&= \mathbb{R} \oplus (\mathbb{R}e_1 \oplus \mathbb{R}e_2) \oplus \bigwedge^2 (\mathbb{R}e_1 \oplus \mathbb{R}e_2) && (4.12)\\
&= \mathbb{R} \oplus (\mathbb{R}e_1 \oplus \mathbb{R}e_2) \oplus \mathbb{R}(e_1 \wedge e_2) && (4.13)\\
&= \mathbb{R} \oplus (a_1 \oplus a_2) \oplus (a_1 \wedge a_2). && (4.14)
\end{aligned}
$$

Each of these summand spaces is mapped into the LTM state space as in section 3.1 and the last expression depicts the possible firing patterns after the second stimulus has been applied. It shows the reactivation of the firing patterns associated with the first stimulus. (The first factor $\mathbb{R}$ denotes the "vacuum" or state of no firing patterns.)

If either a_1 or a_2 fires then the co-activated pair (or *clique*) of cells represented by $a_1 \wedge a_2$ must fire as a unit, presumably axonally projecting as a pair of cells into some other brain region ($!D_1$ or $!D_2$ or $!M$). This then is the "general" response to any of the repertoire of stimuli (startles, or tastes, etc.), which so far consists only of the first two experiences. An obvious barcode-like diagram (as in [7] for instance) which depicts the firing activity of the two neurons after the second stimulus would look like this:

$$
\begin{array}{rl}
a_1 & \blacksquare\square \\
a_2 & \square\blacksquare \\
a_1 \wedge a_2 & \blacksquare\blacksquare
\end{array}
\qquad (4.15)
$$

where the columns correspond to the neurons so far activated listed in temporal order from left to right, and the rows represent the firing patterns of the set of cells, two in this instance, yielding $2^2 - 1 = 3$ possible co-firings (or firing patterns) of the constituent subsets (or cliques) which is the number of rows. To reiterate, the first column shows possible activity because the second stimulus reinstates the possible activation of the first associated neuron.

Adding a third stimulus and attendant neuron yields

$$
\begin{aligned}
!a_1 \otimes !a_2 \otimes !a_3 &= !(a_1 \oplus a_2 \oplus a_3) && (4.16)\\
&= \mathbb{R} \oplus (a_1 \oplus a_2 \oplus a_3) \oplus \bigwedge^2 (a_1 \oplus a_2 \oplus a_3) \oplus \bigwedge^3 (a_1 \oplus a_2 \oplus a_3) && (4.17)\\
&= \mathbb{R} \oplus (a_1 \oplus a_2 \oplus a_3) \oplus (a_1 \wedge a_2 \oplus a_2 \wedge a_3 \oplus a_1 \wedge a_3) \oplus \\
&\qquad \oplus (a_1 \wedge a_2 \wedge a_3). && (4.18)
\end{aligned}
$$

The same phenomenon occurs with the new stimulus leading to reactivations of the earlier firing patterns. We note again the emergence of the power-of two law as in section 3.2: there are $2^3 - 1 = 7$ cliques or firing patterns. Again every possible combination of such patterns is accommodated and mapped into LTM, with $a_1 \wedge a_2 \wedge a_3$ covering the general case of this class of stimulus, and the others becoming more specialized as we lower the grade of firing pattern (i.e. as we go up the rows of the barcode). Again the barcode is of the expected form after the third stimulus:

$$\begin{array}{rl}
a_1 & \blacksquare\square\square \\
a_2 & \square\blacksquare\square \\
a_3 & \square\square\blacksquare \\
a_1 \wedge a_2 & \blacksquare\blacksquare\square \\
a_2 \wedge a_3 & \square\blacksquare\blacksquare \\
a_1 \wedge a_3 & \blacksquare\square\blacksquare \\
a_1 \wedge a_2 \wedge a_3 & \blacksquare\blacksquare\blacksquare
\end{array} \tag{4.19}$$

In the case of four inputs, there are $2^4 - 1 = 15$ cliques (or number of rows):

$$\begin{array}{rl}
a_1 & \blacksquare\square\square\square \\
a_2 & \square\blacksquare\square\square \\
a_3 & \square\square\blacksquare\square \\
a_4 & \square\square\square\blacksquare \\
a_1 \wedge a_2 & \blacksquare\blacksquare\square\square \\
a_1 \wedge a_3 & \blacksquare\square\blacksquare\square \\
a_1 \wedge a_4 & \blacksquare\square\square\blacksquare \\
a_2 \wedge a_3 & \square\blacksquare\blacksquare\square \\
a_2 \wedge a_4 & \square\blacksquare\square\blacksquare \\
a_3 \wedge a_4 & \square\square\blacksquare\blacksquare \\
a_1 \wedge a_2 \wedge a_3 & \blacksquare\blacksquare\blacksquare\square \\
a_1 \wedge a_2 \wedge a_4 & \blacksquare\blacksquare\square\blacksquare \\
a_1 \wedge a_3 \wedge a_4 & \blacksquare\square\blacksquare\blacksquare \\
a_2 \wedge a_3 \wedge a_4 & \square\blacksquare\blacksquare\blacksquare \\
a_1 \wedge a_2 \wedge a_3 \wedge a_4 & \blacksquare\blacksquare\blacksquare\blacksquare
\end{array} \tag{4.20}$$

which reproduces Figure 1B of [7].

In this simplified case of one cell per stimulus each barcode is contained as a sub-barcode of the next one. As each new single cell is added the reactivations of the previous ones maintain the same pattern within the new barcode.

4.1 Functional connectivity motifs

As noted by Tsien, it is generally not just a single cell that responds to a stimulus but rather a (large) collection or network of tuned and/or co-spiking cells, which he terms

Functional Connectivity Motifs, or FCMs. Such collections would be selectively advantageous since redundancy is proof against the failure of individual cells. If these FCMs were modeled by networks of our type, then they obey Tsien's power-of-two law by mathematical necessity since their state spaces are exterior algebras. The origin of the law resides in the structure of this algebra and its operation within a particular FCM is just the local manifestation of this structure. In fact it manifests globally also, by means of the identity (A.7), and determines how FCMs interact. This explains other results of the experiments.

We shall keep in mind that our FCMs will be thought to model assemblies or networks of pyramidal cells typically residing in the hippocampal CA1 region and amygdala or principal excitatory neurons in cortical layers L2–L6, with axonal projections into other areas. Thus, we now replace the single b-neurons n_i of the last paragraph by FCM networks $\mathscr{N}_{A_i}$, each responding to members of a class of stimuli (startles, tastes, etc.) The only difference from the cases treated algorithmically in the last section is that the A_i, replacing the a_i used there, are of unknown (but probably very large) dimension, and the only parts of the computations given there that are different are the expansions corresponding to the ones in equation (4.14) and (4.18) which now go on much longer since the dimensions of the spaces involved are very much higher. This will have repercussions on the reactivation of previous firing patterns which will vitiate the simple barcodes of the last paragraph.

The fact that the power-of-two law that obtains for the case of FCMs consisting of single cells extends to the case of arbitrary FCM networks follows from the fundamental identity (A.7) as follows. Let us denote the exterior algebra $!A$ stripped of its zero grade vacuum sector by $(!A)_+ = A \oplus \bigwedge^2 A \oplus \cdots$. Then from (A.7) we have

$$!(A_1 \oplus A_2) \cong\, !A_1 \otimes !A_2 \tag{4.21}$$

where this is an algebra isomorphism with the graded product on the right. Whence:

$$\begin{aligned} \mathbb{R} \oplus (!(A_1 \oplus A_2))_+ &\cong (\mathbb{R} \oplus (!A_1)_+) \otimes (\mathbb{R} \oplus (!A_2)_+) && (4.22)\\ &\cong \mathbb{R} \oplus (!A_1)_+ \oplus (!A_2)_+ \oplus [(!A_1)_+ \otimes (!A_2)_+]\,. && (4.23) \end{aligned}$$

Now since the vacuum sectors are generated by the respective identities of the algebras on both sides and the isomorphism is one of algebras, the identities must correspond. Consequently we have:

$$(!(A_1 \oplus A_2))_+ \cong (!A_1)_+ \oplus (!A_2)_+ \oplus [(!A_1)_+ \otimes (!A_2)_+] \tag{4.24}$$

or

$$(!A_1 \otimes !A_2)_+ \cong [(!A_1)_+ \otimes (!A_2)_+] \oplus (!A_1)_+ \oplus (!A_2)_+. \tag{4.25}$$

This relation may be checked by a brute force computation of both sides.

We claim that this relation now generates barcodes of the type depicted in the last section, but with $(!A_i)_+$ replacing a_i and $\otimes$ replacing $\wedge$. That is to say, the new barcodes repeat the power-of-two structure but the solid blocks corresponding to the a_i are now replaced by the *entire* barcodes for the A_i-networks. To see this we consider the formal equation

$$(xy)_+ = x_+y_+ + x_+ + y_+ \tag{4.26}$$

noting its similarity to equation (4.25). Now substitute yz for y in the above equation and expand accordingly. We obtain, after some easy work:

$$\begin{aligned}
(xyz)_+ &= x_+(yz)_+ + x_+ + (yz)_+ &(4.27)\\
&= x_+[y_+z_+ + y_+ + z_+] + x_+ + [y_+z_+ + y_+ + z_+] &(4.28)\\
&= x_+y_+z_+ + x_+y_+ + y_+z_+ + x_+z_+ + x_+ + y_+ + z_+. &(4.29)
\end{aligned}$$

Note that with three symbols on the left, there are the correct $2^3 - 1 = 7$ terms on the right. It is easy to prove that with n symbols on the left we get the correct $2^n - 1$ terms on the right. This applies *mutatis mutandis* to equation (4.25) and thus emerges the same barcode structure as in the unicellular case but with the solid blocks now containing the entire barcodes of the A_i-networks themselves, namely the spaces $(!A_i)_+$, which can be very complicated. As more stimuli are experienced, this pattern repeats, with the prior barcodes being incorporated into the current one, as in the unicellular case. This seems to be borne out by the barcodes depicted for instance in [7]. Since we are allowing superpositions of states, current contributions may interfere with prior ones and complicate the picture as the experiment proceeds. This topic and others will be taken up elsewhere.

5 Conclusions

We have argued that Tsien's power-of-two law (Theory of Connectivity), reported as the basic wiring and computational logic of neural circuits underlying intelligent cognitions, is inherent as a mathematical necessity in a recently introduced network model whose nodes simulate neurons with somata resolvable into two I/O subunits, a class we conjecture to include pyramidal cells and other principal cells (excitatory or inhibitory projection cells). We applied the model, through its inherent logical structure, to computationally deduce and simulate the large-scale neural recording experiments performed by Tsien to confirm his reported power-of-two law. Other aspects of the observed results could thereby also be explained.

6 Appendices

A Some multilinear algebra

References for this appendix include [18, 19, 20, 21, 22].

A.1 Tensor products

We take vector spaces over a field (or modules over a ring) k which we may take to be $\mathbb{R}$, the field we will be using here. For n of them, $V_1, V_2, \ldots, V_n$ and another one W (they do not need to be finite dimensional here) one may have *multilinear* functions $f\colon V_1 \times V_2 \times \ldots \times V_n \to W$ meaning that f is linear in each variable separately. Linear means it preserves vector addition and scalar multiplication. One might envisage a complicated theory of such multilinear functions generalizing the theory of linear functions of a single variable. However such a theory is not necessary, since for any such collection of vector spaces $V_1, V_2, \ldots, V_n$ there exists a single vector space $T(V_1, V_2, \ldots, V_n)$ satisfying the following Universal Mapping Property (UMP):

There exists a multilinear map $\iota\colon V_1 \times V_2 \times \ldots \times V_n \to T(V_1, V_2, \ldots, V_n)$ such that for any multilinear map $f\colon V_1 \times V_2 \times \ldots \times V_n \to W$ there exists a unique linear map $\tilde{f}\colon T(V_1, V_2, \ldots, V_n) \to W$ such that $\tilde{f} \circ \iota = f$.

It is easy to prove that such a $T(V_1, V_2, \ldots, V_n)$ must be unique up to isomorphism of vector spaces. It is called the *tensor product* of the vector spaces involved and written $V_1 \otimes V_2 \otimes \cdots \otimes V_n = \bigotimes_{i=1}^n V_i$. Usually the base field or ring is appended to the tensor sign as in $\otimes_k$ since often algebraists have many fields and/or rings to deal with simultaneously. If the field is not in doubt, as here, it is omitted. In this way multilinear maps are turned into linear ones and there is no need for a separate theory. One may think of the tensor product as the vector space generated by basis vectors of the form $a_1 \otimes a_2 \otimes \cdots \otimes a_n, a_i \in V_i$ subject to linear additivity in each variable and the scalar multiplication property $\lambda a_1 \otimes a_2 \otimes \cdots \otimes a_n = a_1 \otimes \lambda a_2 \otimes \cdots \otimes a_n = \ldots = a_1 \otimes a_2 \otimes \cdots \otimes \lambda a_n,\ \lambda \in k$. If V_i has dimension d_i then $\dim(\bigotimes_{i=1}^n V_i) = d_1 d_2 \ldots d_n$.

Properties of tensor products may be derived entirely through the use of the UMP stated above. For instance, suppose linear maps $f_i : V_i \to W_i$, $i = 1, \ldots, n$ are given. Then $f_1 \times \ldots \times f_n : V_1 \times \ldots \times V_n \to W_1 \times \ldots \times W_n$ is multilinear so that its composition with the linear ι -map of $W_1 \times \ldots \times W_n$ into $W_1 \otimes \cdots \otimes W_n$ is also multlinear so that it may be lifted to produce a linear map $V_1 \otimes \cdots \otimes V_n \to W_1 \otimes \cdots \otimes W_n$ as in the diagram above. This map is denoted $\bigotimes_{i=1}^n f_i = f_1 \otimes \cdots \otimes f_n$.

A.2 Exterior products

With notation as in the last section, let $V^p := \overbrace{V \times \ldots \times V}^{p}$ for $p \geqslant 2$. Then, a multilinear function $f : V^p \to W$ is said to be *alternating* if

$$f(v_1, \ldots, v_i, v_i, \ldots, v_p) = 0. \tag{A.1}$$

It follows from multilinearity that interchanging any pair of adjacent variables changes the sign of the value of f and from this that the same holds for the interchange of any pair of variables. Then it follows that the repetition of any pair of variables causes the value of f to vanish. There is a UMP for alternating maps similar to the one for general multilinear maps. The unique vector space that plays the rôle of the tensor product $\bigotimes^p V$ in this case is written $\bigwedge^p V$, and called the *exterior* product. It is generated by elements of the form $v_1 \wedge v_2 \wedge \ldots \wedge v_p$, $v_i \in V$. This element is multilinear in its arguments and alternating in the sense described above for f. Thus for instance $v \wedge v = 0$ and if $v \wedge w = 0$ then v and w generate the same one dimensional subspace, i.e. are *colinear*. (For, if v and w were not linearly dependent they could be included in a basis for V in which case $v \wedge w$ would be a basis element of $\bigwedge^2 V$ which could not be the zero vector.) The map corresponding to ι in the last section sends $(v_1, \ldots, v_p)$ to $v_1 \wedge v_2 \wedge \ldots \wedge v_p$. It is not hard to show that, if the dimension of V is n, then $\dim \bigwedge^p V = \binom{n}{p}$. Note that $\dim \bigwedge^n V = 1$ and that $\bigwedge^p V = \{0\}$ if $p > n$. A useful intuition is that the exterior product $v_1 \wedge v_2 \wedge \ldots \wedge v_p$ is a vector representing the volume of the polytope bounded by the vectors $v_1, \ldots, v_p$ normal to the surface of this polytope.

A.3 Exterior algebras

These exterior products may be assembled into a unital associative algebra (i.e. an associative algebra containing a unit) having a certain universal mapping property (UMP) with respect to linear maps $f : V \to A$ into such an algebra A having the property that $f(v)^2 = 0$ for all $v \in V$. Namely, there exists an associative unital algebra $E(V)$ for any vector space V, and a linear map $\kappa : V \to E(V)$ having the property mentioned, such that if $f : V \to A$ is any linear map into any associative unital algebra A having that "square free" property, then there exists a unique algebra morphism $\tilde{f} : E(V) \to A$ such that $\tilde{f} \circ \kappa = f$. $E(V)$ is then necessarily unique up to algebra isomorphism. One may take this algebra to be the *exterior algebra* of V defined by

$$E(V) := \bigoplus_{k \geqslant 0} \bigwedge^k V \tag{A.2}$$

which is easily seen to satisfy the UMP. In case V is finite dimensional, of dimension n, say, this direct sum terminates to give

$$E(V) = \bigwedge^{0} V \oplus \bigwedge^{1} V \oplus \bigwedge^{2} V \oplus \cdots \oplus \bigwedge^{n} V \tag{A.3}$$

$$= \mathbb{R} \oplus V \oplus \bigwedge^{2} V \oplus \cdots \oplus \bigwedge^{n} V. \tag{A.4}$$

Here we take $\bigwedge^0 V = \mathbb{R}$ and $\bigwedge^1 V = V$. Note that $\dim E(V) = 2^n$. The algebra multiplication is given by wedging together two elements of the summands—called *homogenous elements*—in the order given, and extending by linearity to the whole space, with elements in $\bigwedge^0 V = \mathbb{R}$ just acting as scalars in the usual way. The map $\kappa : V \to E(V)$ is given by $\kappa(v) = v$ considered to lie in the summand $\bigwedge^1 V = V$. This algebra has many interesting properties and symmetries which were understood by H. Grassmann in the middle of the 19th century but whose published treatment of it was famously misunderstood by his contemporaries, probably because of limitations in the notations of the time. We shall rehearse a few of these properties here. First we note that for two finite dimensional vector spaces V and W the map

$$\bigwedge^{m} V \otimes \bigwedge^{n} W \longrightarrow \bigwedge^{m+n} (V \oplus W) \tag{A.5}$$

given by $(v_1 \wedge \ldots \wedge v_m) \otimes (w_1 \wedge \ldots \wedge w_n) \mapsto v_1 \wedge \ldots \wedge v_m \wedge w_1 \wedge \ldots \wedge w_n$ induces an isomorphism of vector spaces

$$\bigoplus_{k=0}^{p} (\bigwedge^{k} V \otimes \bigwedge^{p-k} W) \cong \bigwedge^{p} (V \oplus W) \tag{A.6}$$

from which we obtain an isomorphism of vector spaces

$$E(V \oplus W) \cong E(V) \otimes E(W) \tag{A.7}$$

which is *not* an isomorphism of algebras when the ordinary tensor product algebra multiplication is used on the right hand side of equation (A.7). There is, however, an algebra product structure on the right hand side that does render that isomorphism above an isomorphism of algebras: it is called the *graded* product and it is described as follows. For homogeneous elements $a, c \in E(V)$, $b, d \in E(W)$ the *graded product* on the algebra $E(V) \otimes E(W)$ is determined by the definition:

$$(a \otimes b)(c \otimes d) := (-1)^{\deg(b)\deg(c)}(ac \otimes bd), \tag{A.8}$$

where the degree $\deg(f)$ of a homogeneous element f is the power of the exterior product to which it belongs, also called the *grade* of f. The graded multiplication above may be canonically extended to any number of tensor products of algebras. (The notion of grading for algebraic objects was codified in the 1950s by some of the algebraists of Paris (Bourbaki and Chevalley [18])).

Let us consider the case when V is one dimensional, with basis element e, say. Then clearly $E(V) = E(\mathbb{R}e) = \mathbb{R} \oplus V \cong \mathbb{R} \oplus \mathbb{R}e$ in our earlier notation, where, as an element in the algebra of $E(\mathbb{R}e)$, $e^2 = e \wedge e = 0$. It is immediate that $E(\mathbb{R}e)$ is commutative as an algebra. Now, for any finite dimensional vector space V with basis $\{e_1, \ldots, e_n\}$ we have

$$E(V) \cong E(\mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_n) \cong E(\mathbb{R}e_1) \otimes \cdots \otimes E(\mathbb{R}e_n) \tag{A.9}$$

as vector spaces. As noted above, this is not an isomorphism of algebras with the ordinary tensor product multiplication on the right hand side, since this would be commutative as all of the $E(\mathbb{R}e_i)$ are, while the left hand side is not. However, as mentioned, the right hand side with graded product is isomorphic with the exterior product on the left hand side. Thus, the exterior algebra may be described in terms of graded tensor products of algebras isomorphic with $E(\mathbb{R}e)$ (cf. [22]). The reader may note the similarity of such tensor products to the notion of qubit registers in the parlance of quantum computation.

If $f : V \to W$ is a linear map of vector spaces, there is a unique map of algebras $E(f) : E(V) \to E(W)$ that extends $f : V \to W$. This may be proved using the UMP for exterior algebra. It is easy to see that it is given by the linear extension of the assignments $E(f)(1) = 1, E(f)(v_1 \wedge \ldots \wedge v_k) = f(v_1) \wedge \ldots \wedge f(v_k)$.

The exterior algebra of a finite dimensional k vector space V has additional elements of structure which are not usually invoked in the physical context of the Fermi-Dirac space of a many fermion system, but will be of interest to us here. First consider the vector space maps given by $V \to V \oplus V$, $v \mapsto (v, v)$ and $V \to \{0\}$, $v \mapsto 0$. As above these respectively induce algebra maps

- $\psi_V : E(V) \to E(V \oplus V) \cong E(V) \otimes E(V)$ called the *coproduct* which is an algebra map given on elements $v \in V \subset E(V)$ by

  $$\psi_V(v) = 1 \otimes v + v \otimes 1. \tag{A.10}$$

  This coproduct makes $E(V)$ a *coalgebra*;

- $c_V : E(V) \to E(\{0\}) = k$, called the *counit*, given by the projection of $E(V)$ onto its first component.

Together with the algebra structure these maps give $E(V)$ the structure of a graded *Hopf algebra* in which the product, coproduct, unit, and counit intertwine in certain ways that need not concern us here.

A.4 The Plücker embedding

For a (real) vector space V, let $\mathbf{Gr}(p, V)$ denote the family of subspaces of V of dimension p, the notation $\mathbf{Gr}$ being in honor of Grassmann. The special case of $p = 1$ is called the *projective space* of V, and is denoted by $\mathbf{P}(V)$. Exterior products may be used to obtain an explicit representation of $\mathbf{Gr}(p, V)$, namely the map

$$\psi : \mathbf{Gr}(p, V) \longrightarrow \mathbf{P}(\bigwedge^p V) \tag{A.11}$$

given, for a p-dimensional subspace $W \subseteq V$, with basis $\{w_1, \ldots, w_p\}$, by

$$\psi(W) := \mathbb{R} w_1 \wedge \ldots \wedge w_p \tag{A.12}$$

is well-defined. For, another basis of W is related to this basis by a matrix with a non-vanishing determinant and the corresponding exterior product is the previous one, namely $w_1 \wedge \ldots \wedge w_p$, multiplied by this determinant and so specifies the same element in $\mathbf{P}(\bigwedge^p V)$. Moreover, it is not hard to see that $w \in W$ if and only if $w \wedge \psi(W) = 0$, showing that ψ is one-to-one, or *injective*. It is called the *Plücker embedding*.

Intuitively this last result can be interpreted as follows: if the volume of the $(p+1)$-dimensional polytope obtained by adding w as another side to the p-dimensional polytope with sides $w_1, \ldots, w_p$ is zero, then w must lie in the polytope, and conversely. This is easily seen when $p = 2$: if adding a third vector to the two dimensional polytope with sides w_1, w_2 produces a (3-dimensional) polytope of zero volume, then w must lie in the plane determined by w_1, w_2, and conversely.

B A fragment of the Gentzen sequent calculus GN

This system was introduced and discussed in [3].

Here capital Greeks stand for finite sequences of formulas including possibly the empty one, and D stands for either a single formula or no formula, i.e. the empty sequence, and when it appears in the form $\otimes D$, the $\otimes$ symbol is presumed to be absent when D is empty. If Γ denotes the sequence $A_1, A_2, \ldots, A_n$ then $!\Gamma$ will denote the sequence $!A_1, !A_2, \ldots, !A_n$. The Girard *of course* exponential operator ! is sometimes pronounced "bang."

The rules or axioms are expressed as a set of inferences written as fractions as mentioned in the text, with labels appended to the right of the inference line. Deductions, which are called *proofs* in this context, are written as trees of these fractions.

GN
(fragment)
Structural Rules
EXCHANGE

$$\frac{\Gamma, A, B, \Gamma' \vdash D}{\Gamma, B, A, \Gamma' \vdash D}\text{LE} \tag{B.1}$$

WEAKENING

$$\frac{\Gamma \vdash D}{\Gamma, !A \vdash D}\text{LW} \tag{B.2}$$

CONTRACTION

$$\frac{!A, !A, \Gamma \vdash D}{!A, \Gamma \vdash D}\text{LC} \tag{B.3}$$

The Identity Group
AXIOM

$$A \vdash A \quad \text{Ax} \tag{B.4}$$

CUT

$$\frac{\Gamma \vdash A \quad A, \Gamma' \vdash D}{\Gamma, \Gamma' \vdash D}\text{CUT} \tag{B.5}$$

Multiplicative Logical Rules
CONJUNCTIVE (MULTIPLICATIVE) CONNECTIVE

$$\frac{\Gamma, A, B \vdash D}{\Gamma, A \otimes B \vdash D}\text{L}\otimes \tag{B.6}$$

$$\frac{\Gamma \vdash A \quad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \otimes B}\text{R}\otimes \tag{B.7}$$

We remark again that the rule of CONTRACTION is paradigmatic of storability of the contracted type. If a storable resource is needed twice to produce some other resource, then it is needed only once, since it can be retrieved from the putative store for use again. We shall call such a type *storage capable*. In this calculus, types are generally not storage capable, owing to their evanescence, but (following Girard) they become so when operated upon by the *of course* operator !. Please see section 3 for a more detailed discussion.

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Infinity in Computable Probability

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Abstract

Does combining a finite collection of objects infinitely many times guarantee the construction of a particular object? Here we use recursive function theory to examine the popular scenario of an infinite collection of typing monkeys reproducing the works of Shakespeare. Our main result is to show that it is possible to assign typing probabilities in such a way that while it is impossible that no monkey reproduces Shakespeare's works, the probability of any finite collection of monkeys doing so is arbitrarily small. We extend our results to target-free writing, and end with a broad discussion and pointers to future work.

1 Introduction

Since at least the time of Aristotle [1], the concept of combining a finite number of objects infinitely many times has been taken to imply certainty of construction of a particular object. In a frequently-encountered modern example of this argument, at least one of infinitely many monkeys, producing a character string equal in length to the collected works of Shakespeare by striking typewriter keys in a uniformly random manner, will with probability one reproduce the collected works. In the following, the term "monkey" can (naturally) refer to some (abstract) device capable of producing sequences of letters of arbitrary (fixed) length at a reasonable speed.

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Recursive function theory is one possible model for computation; Russian recursive mathematics is a reasonable formalization of this theory [4].[1] Here we show that, surprisingly, within recursive mathematics it is possible to assign to an infinite number of monkeys probabilities of reproducing Shakespeare's collected works in such a way that while it is impossible that no monkey reproduces the collected works, the probability of *any* finite number of monkeys reproducing the works of Shakespeare is *arbitrarily* small. The method of assigning probabilities depends only on the desired probability of success and not on the size of any finite subset of monkeys.

Moreover, the result extends to reproducing all possible texts of any finite given length. However, in the context of implementing an experiment or simulation computationally (such as the small-scale example in [10]; see also [7]), the fraction among all possible probability distributions of such *pathological* distributions is vanishingly small provided sufficiently large samples are taken.

2 The Classical Experiment

The classical infinite monkey theorem [2, 6] can be stated as follows: given an infinite amount of time, a monkey hitting keys on a typewriter with uniformly random probability will almost certainly type the collected works of William Shakespeare [11]. We use a slightly altered (but equivalent) theorem involving an infinite collection of monkeys, and give an intuitive direct proof.

Let a string of characters of length $w \in \mathbb{N}^+$ over a given alphabet A (of size $|A|$, including punctuation) be called a *w-string*. For example, "*banana*" is a 6-string over the alphabet $\{a, b, n\}$. Suppose each monkey is given a computer keyboard with $|A|$ keys, each corresponding to a different character. Suppose also that the experiment is so contrived that each monkey will type its w-string in finite time.

Theorem 1. *At least one of infinitely many monkeys typing w-strings, as described in the previous paragraph, will almost certainly produce a perfect copy of a target w-string in finite time.*

Proof. Recall that for this theorem, the probability that any given monkey strikes any particular key is uniformly distributed. Let the target w-string be T_w. The chance of a given monkey producing T_w is simply the probability of him typing each

[1]The history of the relationship between classical logic and computation is long and complex, and beyond the scope of this paper. These results do indicate that mathematics using constructive logics, such as the Russian recursive mathematics used here, seems to be more suited to the simulation of the work of a computer than classical logic.

character of the target text in the correct position, or

$$\underbrace{\frac{1}{|A|} \times \frac{1}{|A|} \times \cdots \times \frac{1}{|A|}}_{w \text{ terms}} = \left(\frac{1}{|A|}\right)^w .$$

Therefore, the probability that a given monkey *fails* to produce T_w is

$$1 - \left(\frac{1}{|A|}\right)^w .$$

Now, if we examine the output of m monkeys, then the probability that *none* of these monkeys produces T_w is

$$T_w(m) = \left(1 - \left(\frac{1}{|A|}\right)^w\right)^m .$$

Therefore, the probability that at least one monkey of m produces T_w is

$$P(m) = 1 - T_w(m) .$$

Now

$$\lim_{m \to \infty} P(m) = 1.$$

□

In other words, as the number of monkeys tends to infinity, at least one will almost certainly produce the required string.

However, any real-world experiment that attempts to show this will, unless the target w-string and $|A|$ are quite small, be very likely to fail, since the probabilities involved are tiny. For example, taking the English alphabet (together with punctuation and capitalization) to have 64 characters, a simple computation shows that, if the monkeys are typing 6-strings, the chances of a monkey typing "banana" correctly are

$$\left(\frac{1}{64}\right)^6 = \frac{1}{68719476736} \approx 1.5 \times 10^{-11}. \tag{1}$$

If it takes one second to check a single monkey's output, then the number of seconds that will elapse before we have a 50% chance of finding a monkey that has typed "banana" correctly is outside the precision of typical computing software. Of course, if some monkeys have a preference for typing a certain letter more often than others — say 'a' — then this probability can be much larger. Indeed, it is non-uniformity among monkeys that we exploit to derive our main result in §4.

Results such as (1) have been interpreted [8, p.53] as saying that "The probability of [reproducing the collected works of Shakespeare] is therefore zero in any operational sense…". In §4 we show that this probability can be made arbitrarily small in any sense, operational or otherwise.

3 A Simple, Classical Non-Uniform Version

What if the monkeys do not necessarily strike their keys in a uniformly distributed manner? In this case, we might prescribe a certain probability for a particular monkey to type a particular w-string (and this probability need not be the same from one monkey to the next). Before we reach our main result, we outline a non-uniform classical probability distribution such that for any $\varepsilon > 0$ the probability of success by monkey m is arbitrarily small, but with the probability of failure still zero. If we allow our probability distribution to be a function of m as well as ε then the following distribution will suffice:

$$1 - p_k(m, \varepsilon) = \delta(m - k)(\varepsilon - \sigma) + \delta(m + 1 - k)(1 - \varepsilon + \sigma) \ ,$$

where p_k is the probability of failing at monkey k, the Dirac delta function $\delta(s) = 1$ for $s = 0$ and is zero otherwise, and $0 < \sigma < \varepsilon$. Here, the probability of finding the target w-string at or before the m^{th} monkey, $P(m)$, is less than the prescribed ε, but success is still certain — we need merely look at $m + 1$ monkeys.

In the following section we show that, surprisingly, this can be achieved with a probability distribution dependent only on ε, and not on m. That is, it is possible to produce a computable distribution so that, while each monkey produces Shakespeare's works with nonzero probability, actually finding the culprit among *any* finite subcollection is very unlikely. To do so, we invoke a result from recursive mathematics.

4 The Successful Monkey is Arbitrarily Elusive

Within recursive mathematics, there is a theorem sometimes referred to as the *singular covering theorem*, originally proved by Tseitin and Zaslavsky (1956), and independently by Kreisel and Lacombe (1957) (see [9]): given a compact set K, for every positive ε, one can construct a computable open rational ε-bounded covering of K.[2] It can be restricted to the interval $[0, 1]$ as follows:

Theorem 2. *For each $\varepsilon > 0$ there exists a sequence $(I_k)_{k=1}^{\infty}$ of bounded open rational intervals in $\mathbb{R}$ such that*

(i) $[0, 1] \subset \bigcup_{k=1}^{\infty} I_k$, *and*

[2] Related theorems with detailed proofs and discussion were published by Tseitin and Zaslavsky in [12]. We hasten to add that while this may seem esoteric, the results really provide commentary on much more mainstream ideas such as computer simulations, since constructive logics are much more suited to theorizing about these.

(ii) $\sum_{k=1}^{n} |I_k| < \varepsilon$ *for each* $n \in \mathbb{N}^+$.

Our principal result, Theorem 3, follows from this theorem, and highlights the tension between classical probability theory and its constructive counterpart as outlined in [5].

To set up our principal theorem, we first define M to be an infinite, enumerable set of monkeys (the *monkeyverse*), and for any natural number m the *m-troop* of monkeys to be the first m monkeys in M. Note that for any given monkey it is decidable whether that monkey has produced a given finite target string.

Theorem 3. *Given a finite target w-string T_w and a positive real number ε, there exists a computable probability distribution on M of producing w-strings such that:*

(i) the classical probability that no monkey in M produces T_w is 0*; and*

(ii) the probability of a monkey in any m-troop producing T_w is less than ε.

Proof. Suppose that the hypotheses of the theorem are satisfied. As above, let $P(m)$ be the probability that a monkey in the m-troop has produced T_w, and let p_k be the probability that the k^{th} monkey has *not* produced T_w. Then

$$P(m) = 1 - \prod_{k=1}^{m} p_k.$$

Given $0 < \varepsilon < 1$, compute $\varepsilon_0 = -\log(1-\varepsilon)$. For this ε_0, construct the singular cover $(I_k)_{k=1}^{\infty}$ as per Theorem 2. Then set

$$p_k = \exp\left(-|I_k|\right).$$

To prove (i), observe that $0 < p_k < 1$ for each k. The monotone convergence theorem now ensures that the product $\prod_{k=1}^{m} p_k$ *classically* tends to 0, hence it is (classically) impossible that no monkey produces T_w.

On the other hand, we have (computably)

$$-\log(p_k) = |I_k|,$$

whence, by the singular covering theorem,

$$\sum_{k=1}^{m} -\log(p_k) = \sum_{k=1}^{m} |I_k| < \varepsilon_0 = -\log(1-\varepsilon)$$

for all $m \in \mathbb{N}^+$. Some rearranging shows that

$$\log\left(\prod_{k=1}^{m} p_k\right) = \sum_{k=1}^{m} \log(p_k) > \log(1-\varepsilon)$$

and hence

$$\prod_{k=1}^{m} p_k > 1 - \varepsilon.$$

Then the probability of any member of the m-troop producing T_w is

$$P(m) = 1 - \prod_{k=1}^{m} p_k < \varepsilon$$

for *any* positive natural number m. This proves (ii). □

Thus, the chances of us actually *finding* the monkey that produces the collected works of Shakespeare can be made arbitrarily small, and the classical intuition that, since we have an infinite number of monkeys, Shakespeare's works must be typed by *some* monkey is of no help in *locating* the successful monkey.

We emphasize that, in contrast to the case in §3, the pathological distribution in Theorem 3 does *not* depend on m, the size of the troop we search.[3]

One might argue that it is easy to assign probabilities in such a way that any finite search will almost certainly not yield the monkey that produced it — by letting each monkey produce the target w-string with probability zero. However, in this case, *no* monkey will produce it. Our theorem shows that, even in the case where it is (classically) impossible that no monkey produces the target, it is still possible to make the probability of finding the monkey that accomplishes the necessary task arbitrarily small.

5 Target-Free Writing

One criticism of the above line of reasoning is that the experimenter requires knowledge of the target. There, the output of each monkey was tested against the collected works of Shakespeare: only if every character matched would it pass the test. However, suppose now that we wish to recreate Shakespeare's work armed only with knowledge of the total character length in some alphabet. That is, we know that we require one of the $|A|^w$ possible w-strings. Can we guarantee to complete the

[3]Contrasting the classical with the computational view in the same proof may prove counterintuitive. We are hoping to shed light on why the *intuitive* result—that it is (in the classical abstract world) impossible that *no* monkey produces Shakespeare's works—clashes with the fact that it may be incredibly difficult (in the concrete computational world) to nail the cheeky monkey that did it. What sense to make of the product $\prod p_k$ of monkeys failing to produce Shakespeare classically tending to 0? The problem here is the *rate* at which it does so—this rate is computationally untractable.

list (without repetition) and therefore recreate the collected works of Shakespeare (somewhere)? We note that the list can be shortened by checking for grammar etc.[4]; here we consider the worst case of the complete list, without repetition, of w-strings.

Corollary 4. *Any list of finite strings is completed in finite time with arbitrarily small probability.*

The proof relies on applying Theorem 3 multiple times using standard calculations.

6 Pathological Distributions Are Arbitrarily Rare

At first sight, Theorem 3 might appear to destroy any hope of finding the successful monkey. However, we have the following:

Theorem 5. *The probability that the probability distribution on the monkeyverse is constructed in such a way as to make the constructive probability of finding the desired monkey arbitrarily small, is arbitrarily small.*

Proof. Given $0 < \varepsilon < 1$, in order for the probability distribution to be pathological, the probability of *any* monkey in the m-troop outputting T_w cannot exceed ε. Therefore the fraction of pathological distributions over an m-troop is at most ε^m, and

$$\lim_{m\to\infty} \varepsilon^m = 0.$$

□

In short, we can make the fraction of pathological distributions arbitrarily small if we search sufficiently large m-troops. Here, then, is an a priori justification for large sample sizes in the case of computational simulations.

7 Discussion and Further Work

Recall that, throughout this paper, we take the term "monkey" to refer to some device capable of producing arbitrary but finite sequences of letters — computers satisfy this criterion. The theorems presented in this paper therefore have implications for computer simulations. In particular, when performing simulations of a

[4]Truncating the list in this way may be desirable in order to avoid being overwhelmed by "meaningless cacophonies, verbal farragoes, and babblings" [3].

probabilistic nature, the experimenter needs to ensure that pathological distributions do not arise, or arise rarely enough to provide a measure of confidence in the conclusion.

It should also be noted that the classical non-uniform distribution presented above suggests that a pathological situation can never be ruled out with certainty, since if the experimenter tests just one more monkey, the result may be vastly different than observed earlier in the simulation. With practical considerations in mind, there will be some point at which costs (ethical and/or material) outweigh the benefit of testing further monkeys.

The proof of Theorem 3 required a result from constructive mathematics. We conjecture that such a result is classically impossible, since the singular covering theorem is classically not true.

A deeper fact here is that from the classical viewpoint, the computable reals have zero measure, and all finite texts produced by monkeys correspond to the rationals (or some other convenient subset of computable reals). The context of the results, then, would indicate that a careful constructive study of probability distributions provides *a priori* motivation for repetition of simulations for accuracy (to rule out accidental pathological distributions generated by computer programs), and has potentially more to say about issues involving computer simulations.

There is the further issue of what model of constructive mathematics provides a good framework for this sort of work. Philosophically there is tension between the intuitionistic free choice-sequence approach and the computable sequence approach, and within these approaches are further complications by sensitivity of the theory to the validity (or invalidity) of the various versions of König's Lemma. It is not the aim here to go deeply into these issues, which could lead to a lengthy series of papers. In the interest of brevity, we leave such explorations for future research.

It has not escaped our attention that science and mathematics have each been considered to be "games" of recombining a finite set of characters (even if we do not yet know what they all are). Even if we consider only finite strings which are syntactically sound, and not contradicted by empirical evidence, our result shows that completing such a list is not necessarily even *likely* to happen within *any* finite time, such as a human lifespan, the duration of a civilisation, or even the age of the universe.

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Formal Periodic Steady-State Analysis of Power Converters in Time-Domain

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Abstract

Time-domain based periodic steady-state analysis is an indispensable component to analyze switching functionality and design specifications of power electronics converters. Traditionally, paper-and-pencil proof methods and computer-based tools are used to conduct the time-domain based steady-state analysis of these converters. However, these techniques do not provide an accurate analysis due to their inability to model and analyze continuous behaviors exhibited by the power electronics converters. On the other hand, an accurate analysis is direly needed in many safety and cost-critical power electronics applications, such as biomedical, hybrid electric vehicles, and aerospace engineering. To alleviate the issues pertaining to the above-mentioned techniques, we propose a methodology, based on higher-order-logic theorem proving, to conduct the time-domain based steady-state analysis of power electronics converters in this paper. The proposed methodology is primarily based on a formalized switching function analysis technique, ordinary linear differential equations and steady-state conditions of the systems. To illustrate the usefulness of proposed formalization, we present the formal time-domain steady-state analysis of a commonly used DC-DC Buck converter.

1 Introduction

Power electronics converters are an integral part of, almost, every realizable electrical/electronics system, as a power processing stage, to meet their power requirements [10]. These systems are typically composed of semiconductor devices, like switches, energy storage and dissipative elements, i.e., inductors, capacitors, and resistors, and integrated circuits for control. Generally, periodic steady-state analysis is a mandatory preprocessing step for the small-signal analysis, which is used to evaluate the performance of the converter. Moreover, time-domain based analysis is necessary for the study of the switching functionality, which is central to the power conversion operation of the converters [10]. However, switching is a highly non-linear phenomenon and therefore leads to significant modeling, analysis and design challenges of these systems.

Traditionally, paper-and-pencil proof methods or computer-based numerical techniques are used to perform the time-domain based steady-state analysis of the power electronics systems. The paper-and-pencil proofs are usually based on many assumptions, such as small-ripple approximations, and averaging techniques to linearize the nonlinear behavior of the systems to analyze the systems in steady-state [10]. These linearized models, expressed as ordinary linear differential equations, are then simulated using a variety of computer based simulation tools, such as MATLAB Simulink, Saber, PSpice, to evaluate the performance of the power electronics systems. Generally, these computer based simulation tools use discretized time or frequency domain models of the systems and numerical integration methods [7] for solving the differential equations of the converters [8]. Therefore, the above-mentioned techniques cannot ascertain an accurate and reliable analysis of the power converters due to inherent approximation based nature of these techniques. For example, the accuracy of paper-and-pencil proof methods is usually limited by the underlying approximate linearized model. On the other hand, the nonlinear analysis is, mathematically, not tractable and due to human involvement is highly likely error prone. Similarly, the numerical methods employed in the simulation techniques, based upon the discretization of time or frequency, lead to truncation errors and also cannot accurately model the hybrid behavior, i.e., continuous behavior driven by discrete events, exhibited by power converters [22]. To address this issue, computer algebra systems, which are software programs for the symbolic processing of mathematical expressions, are also employed for the analysis of such systems [16]. However, the symbolic processing is based on the unverified program codes, and therefore prone to bugs [21]. Thus, given the aforementioned inaccuracies, these traditional techniques should not be relied upon for the analysis of power electronics systems, especially when they are used in safety-critical areas, such as implantable medical devices [3] and automotive industry [9], and mission-critical areas, such as aerospace engineering [13], where bugs may lead to heavy monetary or human life loss.

In recent years, formal methods have been extensively employed for the accurate analysis of a variety of hardware and software systems. The transfer function of DC-DC converters has been verified [6] in the frequency domain using higher-order-logic theorem proving based on the signal flow graph and Mason's gain formula. The transfer function is then used to reason about the efficiency, stability and resonance of pulse width modulation push-pull DC-DC converter and 1-boost cell DC-DC converter. However, the nature of formalization does not permit to reason about the interesting features of switch, which is a key element of power electronic converters. Model checking has also been used for the analysis of the DC-DC Buck circuit [18] [20] using a hybrid automaton equivalent model of circuit to verify the reachability and safety properties of the circuit. However, the state-based modeling of the circuit does not allow to describe the exact continuous behavior of power converters circuits. Moreover, the state-space explosion issues also limit the scope of model checking for the verification of continuous and hybrid systems. To the best of our knowledge, there is no formal approach in the literature that explicitly allows us to verify the nonlinear aspects pertaining to the modeling and time-domain based steady-state analysis of power electronics systems.

The main motivation of this paper is to develop a formal logical framework for the time-domain based steady-state analysis of power converters. The main challenge in this direction is to be able to model and analyze the continuous structural or topological changes under the switching action [5], which are usually modeled using the Heaviside step function [1], i.e.,

$$u(t) = \begin{cases} 1 & 0 < t \\ 1/2 & t = 0 \\ 0 & t < 0 \end{cases} \tag{1}$$

The topological changes deter the explicit use of conventional circuit analysis techniques, such as mesh and node analysis, for investigating the implementation of the circuit by using the behavior of its individual components and its overall behavior [17]. Another notable consequence is that the switching action introduces piecewise functions, which are also expressed in terms of the Heaviside step function, in the analysis that in turn cannot be analyzed using linear mathematical techniques based on the Riemann integral theory, such as differential chain rule and integration by part. To tackle the former issue, we propose to use the switching function technique [17], which is a commonly used circuit analysis technique that allows to incorporate the topological changes of the circuit in the analysis. We tackled the piecewise nature of the functions in our formal framework by using the Gauge or Henstock-Kurzweil integral [15]. The Gauge integral is characterized by the Gauge function for the tagged division of an interval over which the function is to be integrated. This simple, but novel, alteration allows us to integrate the functions with countable singularities or the functions that are continuous but not differentiable everywhere on the given interval. It, particularly, supported us in the formal verification of an interesting notion of the Heaviside step function as a generalized function [14] which is widely used to describe discontinuous phenomena in physics and engineering disciplines. As a generalized function, the Heaviside step function acts as an operator on a test function $f(x)$, which needs to be smooth everywhere, as:

$$\int_a^b h(x-c)f(x) = \int_c^b f(x) \qquad \forall\, a\, b\, c.\ a < c < b \tag{2}$$

The smoothness of test function also plays a pivotal role in the differentiation of the piecewise functions involving the Heaviside step function in the formal time-domain based periodic steady-state analysis of power converters.

Besides these foundations, the proposed formalization is based on the formalizations of linear ordinary differential equations and steady-state conditions. The homogeneous linear differential equations using real analysis have been formalized in HOL to model the cyber-physical systems [19]. In this paper, we have extended the logical framework, presented in [19], to the non-homogeneous linear differential equations using complex analysis to formally model the dynamic behavior of the power converters. We have used the multi-variable integral, differential, transcendental and topological theories to define the steady-state conditions due to the piecewise nature of the functions involved in the analysis.

The formalization in this paper is done using the HOL-Light theorem prover [11], which supports formal reasoning about higher-order logic. The main motivation behind this choice is the availability of reasoning support about multi-variable integral, differential, transcendental and topological theories [12], which are the foremost foundations required for the formalization of time-domain based steady-state analysis of power electronics systems.

The rest of the paper is organized as follows: We describe some preliminaries regarding the periodic steady-state analysis of power electronics converters in Section 2. In Section 3, we present the proposed methodology. The formalization of the switching function technique, ordinary differential equations and steady-state conditions in Section 4. We utilize this formalization to formally verify a Power converter circuit, i.e., DC-DC buck converter in Section 5. Finally, Section 6 concludes the paper.

2 Periodic Steady-state Analysis of Power Converters

Power converter circuits use continuous switching among different circuit configurations to achieve the desired power conversion, such as dc-dc, dc-ac, ac-dc and ac-ac. In each circuit configuration, also called mode or state of the converter, the behavior of the circuit variables can be expressed as differential equations with initial conditions from the previous mode at the switching instance. Therefore, the standard approach for the time-domain analysis of these converters consists of developing the differential equations for each mode of the circuit based on the Kirchoff's voltage or current laws to describe the dynamic behavior of these circuits.

Mathematically, the behavior of these systems can be described as:

$$\begin{aligned} H(t, y_1, y_1^1, ..., y_n^{m_n}) &= p(t) & t \in [t_{n-1}, t_n], n, m_n \in \mathbb{N} \\ y_n^k(t_n) &= y_{n-1}^k(t_{n-1}) & k \in \mathbb{N} \\ y_0^1(t_0) &= 0 \end{aligned} \tag{3}$$

Where, H and p are functions of an independent variable t, a dependent variable y_n and its m_n-th order derivative in the corresponding n-th mode, respectively. In power converters, the time is considered as an independent variable, whereas, the voltage or current of the energy storage components is considered as a dependent variable. The order, i.e., m_n, of an ordinary differential equation of the power converter, in the n-th mode, is determined by the number of energy storage elements constituting the mode. The function $p(t)$ is referred to as a non-homogeneous term, which can be zero or non-zero in the n-th mode, depending upon the presence of source in the n-th mode of a power converter. Initially, the value of dependent variable is considered zero, i.e., $y_0^1(t_0) = 0$, however, later on the value of the dependent variable in one mode becomes an initial value for the next mode, i.e., $y_n^k(t_n) = y_{n-1}^k(t_{n-1})$, when switching instance occurs. Whereas, k is the order of the derivative of the dependent variable evaluated at a specific time instance.

For the brevity of the notion, transient and steady-state time-domain behavior of a DC-DC power converter is presented in Fig. 1, base on the above-mentioned standard approach. DC-DC power converter circuits are designed to step-up or step down the dc voltage levels applied at their input. Fig. 1 shows the output behavior, y_t, of a DC-DC power converter under the switching action represented by a rectangular switch wave form, S_w.

In periodic steady-state, the dependent variables of a power converter circuit attain an equilibrium and repeat the behavior over a time period, T_p, constituting

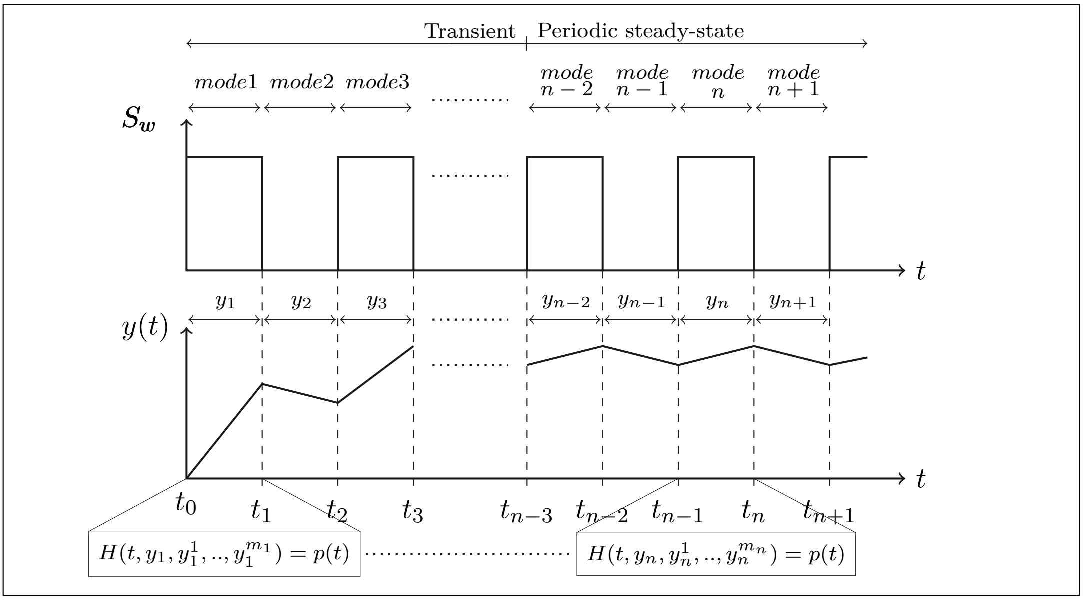


Figure 1: **Dynamic behavior of the output, $y(t)$, of a DC-DC power converter under switching action, represented by the switching wave form, S_w.**

l modes. Mathematically, the periodic steady-state behavior of a power converter over one time period, when $t \to \infty$, can be represented as:

$$\begin{gathered} H(t, y_n, y_n^1, ..., y_n^{m_n}) = p(t) \quad t \in T, T \in \bigcup_{i=1}^{l} \left[t'_{i-1}, t'_i\right], m_n, n, l \in \mathbb{N} \\ y^k(t'_0) = y^k(t'_0 + T_p) \quad T_p = t'_{max(i)} - t'_0, k \in \mathbb{N} \end{gathered} \tag{4}$$

Equation (4) reduces the problem to the identification of the modes in one time period, $T_p = t'_{max(i)} - t'_0$, of the circuit, which is the length of time over which the modes of a power circuit converter repeat themselves. The function y is a piecewise function defined over l modes. Whereas, $y^k(t'_0) = y^k(t'_0 + T_p)$ refers to the steady-state conditions of the system variable at reference switching time instances, t'_0, and T_p, and k represents the k-th order derivative of the variable.

Fig. 2 illustrates the behavior of the output of a DC-DC power converter in steady-state, which is mathematically modeled in Equation 4. The output, $y(t)$, of the converter exhibits a repetitive behavior over the time period T_p in l modes. In literature, waveforms of the dependent variable, y, are used for the periodic steady-sate analysis of the power converters by applying the principle of inductor

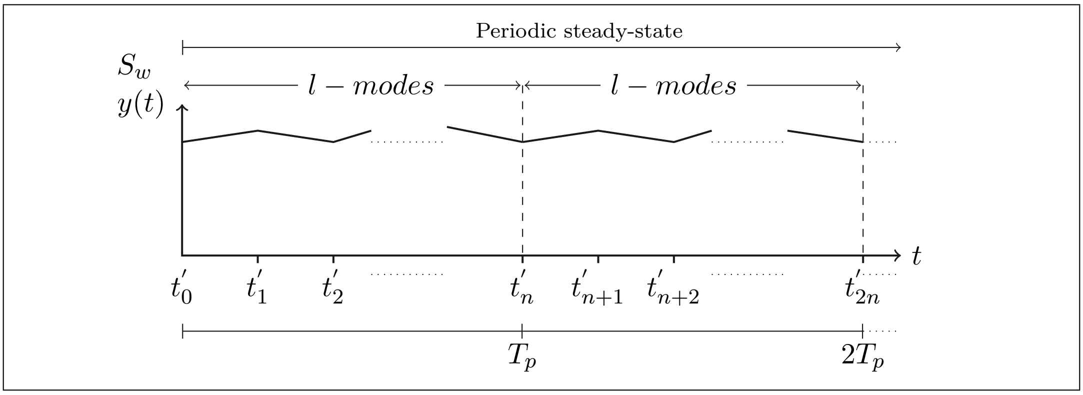


Figure 2: **Behavior of the output, $y(t)$, of a DC-DC power converter in Periodic steady-state.**

volt-second or capacitor-charge, along, with small-ripple approximations to reduce the complexity of the analysis by compromising the accuracy [10].

In this paper, we propose a logical framework for the formal verification of the periodic steady-state analysis of power converters in time domain, which are mathematically represented by Equation 4. The challenges to develop a logical framework for the formal verification of the aforementioned problem are two fold. Firstly, we intend to develop a higher-order logic formalization capable of incorporating the topological structural changes over the time period, i.e., $T \in \bigcup_{i=1}^{l} \left[t_{i-1}^{'}, t_i^{'}\right]$, thus, enabling us to formally model and reason about the implementation behavior of these circuits within the sound core of the HOL-Light theorem prover. Second we want to develop a formal library of foundations, including; differential equations, concepts from operational calculus described by Equation 2, to formally reason and verify the highly nonlinear behavior of the circuit variables involved in the formal periodic steady-state analysis of these circuits, in higher-order logic. The respective subsections of Section 4 address these challenges by presenting the formalization of switching function technique, differential equations and solution of these differential equations, respectively, to conduct the formal periodic steady-state analysis of power converters in the time-domain.

In the next section, we present the proposed methodology for the formal periodic steady-state analysis of the power converters, in a higher-order-logic theorem prover, i.e., HOL-Light.

3 Proposed Methodology

We propose to use higher-order-logic theorem proving, as shown in Fig. 3, in order to formally verify the power converters operating in the periodic steady-state. The first step in the proposed methodology is to build a formal model for the switching function technique and linear order differential equations to formally express the implementation and specification of power converter circuits, in higher-order logic. The proposed formal modeling of switching function technique is based on the formal definitions of an ideal semiconductor switch, energy storage and dissipative elements, and Kirchoff's current and voltage laws. Whereas, the formal modeling of the linear ordinary differential equation is used for the formal specification of the behavior of each mode of the power converter circuit. The aforementioned two formal models can then be used to formally assert and analyze the implementation of the circuits, as a theorem, using the sound core of HOL-Light. Moreover, the formal specification of ordinary linear differential equations is also used to formally

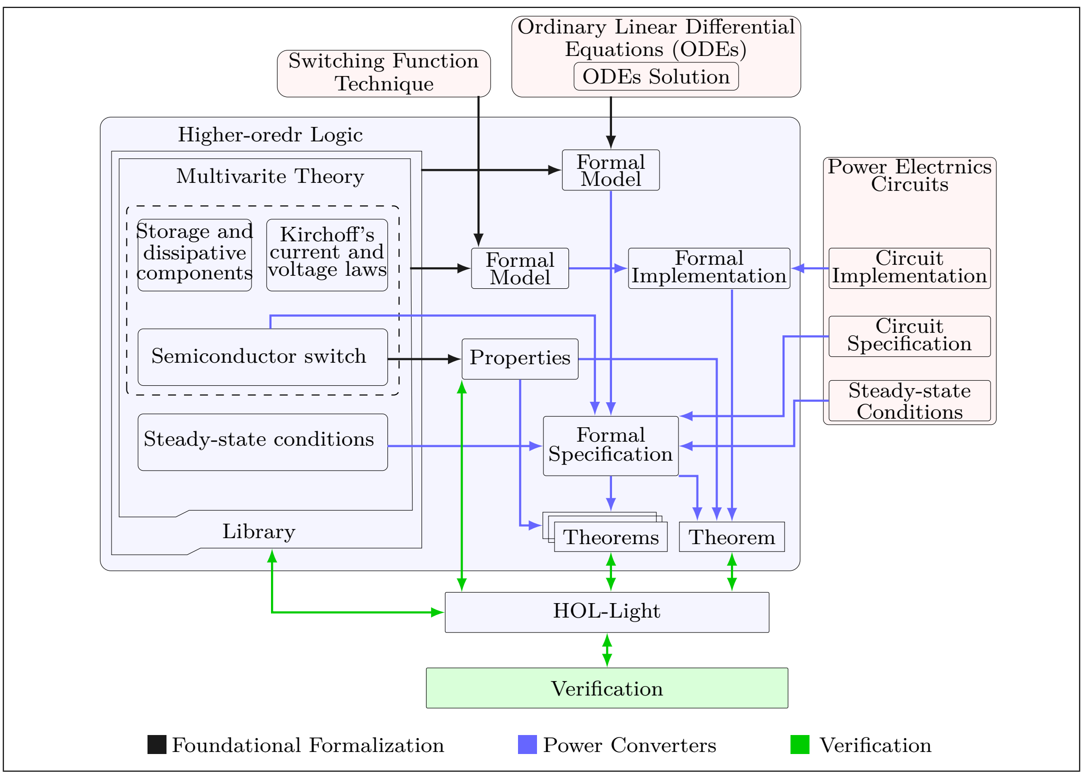


Figure 3: **Proposed Methodology**

verify the correctness of the solutions of these equations. As the steady-state analysis is based upon the formal modeling of the linear ordinary differential equations and their solutions, therefore, in the next step, we propose to formally define the steady-state conditions to conduct the formal analysis of power converters, as shown in Fig 3. These formal definitions, along with multi-variable theories of HOL-Light, are used to formally verify the theorems that are required to conduct the formal steady-state analysis of power converters. Finally, the switch is formalized using the Heaviside step function, and its related properties, such as integration and derivation of piecewise functions involving Heaviside step function, are formally verified. As the switching functionality plays the most vital role in characterizing the nonlinear behavior of the power converters therefore these formally verified properties are used in, almost, every aspect of the formalization and verification.

4 Foundational Formalizations

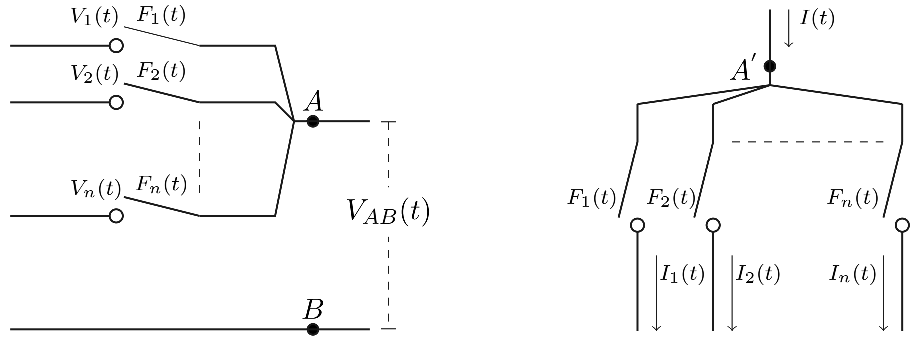
4.1 Formal Model of the Switching Function Technique

In power converter circuits, semiconductor devices such as, diodes, BJTs (bipolar junction transistors), MOSFETs (metal oxide semiconductor field effect transistors), IGBTs (insulated gate bipolar transistors) etc, are used for performing the switching operation. These semiconductor devices play a vital role in the development of reliable, cost-effective and highly efficient converters [4]. Although, these devices differ in their physics and physical properties, however, as a switch, their function is to connect or disconnect a path or subcircuit, in a converter circuit, to achieve the desired conversion. Therefore, the functionality of an ideal semiconductor device as a switch can be modeled using the Heaviside function, i.e., Equation (1), in HOL-Light:

Definition 1: ⊢ ∀ t.
```
semi_switch t = if t < &0 then &0 else
                (if t = &0 then &1 / &2 else &1)
```

Definition 1 models the functionality of a semiconductor switch as a real value `1`, for connected status, and `0`, for disconnected status, in higher-order logic. Whereas, at the switching instance `t`, it has value `1/2`. The `&` is a typecasting operator in HOL-Light that maps a number to a real number. In our formalization, we use switch status or switching function to refer connected or disconnected switch.

The switching operation is central to the power converters functionality, however, it hinders the straightforward usage of the conventional circuit theory techniques, such as Kirchoff's voltage and current laws. The switching function technique relies



(a) Voltage at switching junction (b) Current at switching junction

Figure 4: **Switching function technique**

on the superposition theorem of the voltage or current to express the behavior of these quantities in the presence of a switch in the circuit. It is based on the conceptualization of the switch as a modulating function for the input and output power. Based on this notion, the voltages and the currents in the presence of a switch component can be expressed as [1];

$$V_{AB}(t) = \sum_{i=1}^{n} V_i(t)F_i(t) \qquad n \in \mathbb{N} \tag{5a}$$

$$I_i(t) = I(t)\sum_{i=1}^{n} F_i(t) \qquad n \in \mathbb{N} \tag{5b}$$

Equation 5(a), describes voltage at the switch junction, in a mesh, in terms of switching functions. Fig. 4(a) is a pictorial representation of the concept, where n voltage sources are connected to a point, A, through n switches. The voltage, V_{AB}, is then the superposition of the input voltages, however, the contribution of each voltage is dependent upon the associated switching function. Similarly, Equation 5(b), describes the current at a node, $A^{'}$, which has n switches. Fig. 4(b) describes the situation where current, $I(t)$, is supplied to n paths of the circuit through n switches. Each path receives the fraction of total current depending upon its switch status, $F_n(t)$.

Voltages and currents at the switching junction in higher-order logic are defined, as:

Definition 2: $\vdash$ $\forall$ `mod_lst volt_lst t.`

```
switch_volt mod_lst volt_lst t =
vsum (0..LENGTH mod_lst - 1) (λ n.  EL n volt_lst t * Cx (EL n mod_lst))
```

The function `switch_volt` describes the voltage at the switch junction using Equation 5(a). It accepts a list, `volt_lst`, which contains all the possible voltage drops at the switching junction, a list of modes, `mod_lst`, which contains the switch status or switching function for each mode, and `t` is the time, which indicates that this function is time dependent. Whereas, `Cx` is a HOL-Light function, which is used to map a real number, representing the switching function, to a complex number.

Definition 3: ⊢ ∀ `mod_lst curr t.  switch_current mod_lst curr t =`
```
curr t * vsum (0..LENGTH mod_lst - 1) (λ n.  Cx (EL n mod_lst))
```

Definition 3 formally models the current at the switching junction using Equation 5(b). It accepts an argument `curr`, which represents the total supplied current to the switch junction, a list of modes, `mod_lst`, which contains the switch status or switching function for each mode, and `t`, which represents time.

To accomplish the formal modeling of the switching function technique, we also formalize the Kirchoff's voltage and current laws:

Definition 4: ⊢ ∀ `vol_lst t.  kvl vol_lst t =`
```
vsum (0..LENGTH vol_lst - 1) (λn.  EL n vol_lst t) = Cx (&0)
```

Definition 5: ⊢ ∀ `cur_lst t.  kcl cur_lst t =`
```
vsum (0..LENGTH cur_lst - 1) (λn.  EL n cur_lst t) = Cx (&0)
```

The `kvl` and `kcl` functions accept lists of type $(\mathbb{R} \rightarrow \mathbb{C})$, to express the behavior of the time dependent voltages and currents in the given power converter circuit and a time variable `t`. They return the predicates that guarantee that the sum of the voltages in a loop or sum of the currents at a node are zero for all the time instants.

The voltages and currents in Definitions 2 and 3 are piecewise functions due to switching action. We formally verified the result of Equation (2) to conduct the formal analysis involving such functions:

Theorem 1: ⊢ ∀ `f a b c x.`
```
A1:(∀t.  (λx.  f (x)) differentiable_on s) ∧
A2:~(real_interval [a,b] = {}) ∧
A3:c ∈ [a, b]
    ⇒ ∫_a^b (λx.  semi_switch x c ) * f (x)) = ∫_c^b (λx.  f (x))
```

The Assumption `A1` ensures the differentiability of a test function, `f`, over `s`. Whereas, `s`:$(\mathbb{R} \rightarrow \mathbb{B})$ is a set-theoretic definition of the intervals in higher-order logic, over real numbers. For a given real interval `[a,b]`, it represents all possible real intervals, which are subsets of the given real interval. Therefore, Assumption `A1` ensures the differentiability of a test function over all subsets of the given real interval `[a,b]`. Assumptions `A2` and `A3` ensure that the interval is non-empty and point `c` lies within the interval `[a, b]`. The conclusion of the Theorem 1 formally verifies the affect of applying the Heaviside step function on a test function, i.e., changes the limit of integral. Theorem 1 is formally verified using the formal definition of Gauge integral and its properties, available in HOL-Light theorem prover. This formally verified result plays a very key role in the formal reasoning of the systems which exhibit nonlinear behavior, such as power converters circuits.

The above formalization enables us to formally model and analyze the nonlinear behavior exhibited by the power converters, due to switching action, in higher-order logic.

4.2 Ordinary Linear Differential Equation

An n^{th}-order ordinary linear differential equation can be represented as:

$$a_n(t)\frac{d^n y(t)}{dx} + a_{n-1}(t)\frac{d^{n-1} y(t)}{dx} + ... + a_0(t)y(t) = p(t) \tag{6}$$

We formalized the n^{th}-order derivative function in higher-order logic as follows:

Definition 6:
```
⊢ ∀ n f t.  (n_vec_deri 0 f t = f t )   ∧
  (∀ n.  n_vec_deri (SUC n) f t =
             n_vec_deri n (λ t.  vector_derivative f at t) t)
```

The function `n_vec_deri` accepts a positive integer `n` that represents the order of the derivative, the function `f`:($\mathbb{R} \rightarrow \mathbb{C}$) that represents the complex-valued function that needs to be differentiated, and the variable `t`:($\mathbb{R}$) that is the variable with respect to which we want to differentiate the function `f`. It returns the n^{th}-order derivative of `f` with respect to `t`. Now, based on this definition, we can formalize the left-hand side (LHS) and right-hand side (RHS) of Equation (6) in HOL-Light as the following definitions:

Definition 7:
```
⊢ ∀ P y t.  diff_eq_lhs A f t =
     vsum (0..LENGTH A) (λ n.  Cx ( EL n A t) * n_vec_deri n f t)
```

Definition 8:
```
⊢ ∀ L y t.  diff_eq_rhs L p t =
     vsum (0..LENGTH L) (λ n.  Cx (EL n L) * EL n p t)
```

In the above definitions, `A` and `L` are the coefficient's lists, `f`:($\mathbb{R} \rightarrow \mathbb{C}$) and `p(t)`:($\mathbb{R} \rightarrow \mathbb{C}$) are complex-valued functions, and `t`:($\mathbb{R}$) is the time variable to formally model the linear ordinary differential equation. Definition 6 is also used to formally define the steady-state condition of the power converters as:

Definition 9: ⊢ ∀ n. (steady_state 0 f T_p =
(n_vec_deri 0 f (&0) = n_vec_deri 0 f T_p)) ∧
(steady_state (SUC n) f T_p =
(n_vec_deri (SUC n) f (&0) = n_vec_deri (SUC n) f T_p))

The above generic formalization allows to formally model the dynamic behavior of systems represented by differential equations. We have utilized this formalization to formally specify and reason the periodic steady-state behavior of power converters, described in Equation 4.

4.3 Solution of Linear Differential Equations

The general solution to non-homogeneous Equation (6) is expressed as

$$y(t) = y_h(t) + y_p(t) = \sum_{i=1}^{n} c_i y_i(t) + y_p(t) \tag{7}$$

Where, $y_h(t)$ is the linear combination of the fundamental solutions of Equation (6) when $p(t) = 0$, and y_p is the particular solution corresponding to Equation (6) when $p(t) \neq 0$.

The formal verification of the correctness of the solution of linear differential equation, i.e., Equation (6), is based on the linearity property of the derivatives, which we have formally verified for the complex-valued functions as:

Theorem 2: ⊢ ∀ n f h t.
A1: (λ m t. m ≤ n ⇒ (λ t. n_vec_deri m f t) differentiable at t) ∧
A2: (λ m t. m ≤ n ⇒ (λ. n_vec_deri m h t) differentiable at t)
⇒ n_vec_deri n (λt. Cx a * f t + Cx b * h t) t =
Cx a * n_vec_deri (λt. f t) t + Cx b * n_vec_deri (λt. g t) t

We formally verified the solution of a linear differential equation, represented by Equation (7), in the HOL-Light theorem prover as follows:

Theorem 3: ⊢ ∀ Y_h C Y_p A L p t.
A1: (n_differentiable_fn Y_h (LENGTH A)) ∧
A2: (n_differentiable_fn Y_p (LENGTH L)) ∧
A3: (n_homo_soln A Y_h t) ∧
A4: (n_nonhomo_soln A L Y_h Y_p t)
⇒ diff_eq_lhs A (λ t. linear_sol C Y_h t + Y_p t = diff_equ_rhs L p t

In Theorem 3, Assumptions `A1` and `A2` ensure the n^{th}-order differentiability of the fundamental solutions, given as a list `Yh`, and particular solution, provided as a list `Yp`, respectively. The predicate in the Assumption `A3`, i.e., `n_order_homo_eq_soln_list`, ensures that each element of the list `Yh` is a solution of the given differential equation, when $p(t) = 0$ in Equation (6), where `L` is the list of coefficients. Similarly, the predicate in Assumption `A4`, i.e., `n_order_nonhomo_eq_soln_list`, ensures that the particular solution, `Yp`, satisfies the differential Equation (6). The function `linear_sol`, used in the conclusion of Theorem 2, models the linear solution combination of fundamental solutions, i.e., $\sum_{i=1}^{n} c_i y_i(t)$, using the lists of solution functions `Yh` and arbitrary constants `C`. The formal verification of Theorem 3 is based on Theorem 1 and the formally verified lemma about solution of homogeneous differential equation, i.e., when $p(t) = 0$ in Equation (6). More details about the modeling and verification steps can be found in our proof script [2]. The formalization, presented in this section, is generic and provides sufficient support to formally model and reason about different aspects of a power converters' circuits including; implementation and behavior, specification, correctness of the solution of differential equations representing the behavior of circuits, and also the steady-state behavior of quantities of interests, such as voltages and currents. The corresponding proof script, which is available for download at [2], has 3000 lines of HOL-Light code and requires about 350 man hours of development time.

5 DC-DC Buck Converter

The DC-DC buck converter is a commonly used power converter that steps down a given input to a desired output level. In a DC-DC Buck converter, operating in a continuous conduction mode, a switch controls the flow of energy from the raw source, Vs, to the output by periodically switching between Positions 1 and 2, as shown in Fig 5. The energy is stored in the inductor when the switch is at Position 1, and is dissipated to the output circuitry, when the switch is at Position 2.

The circuit has two modes, i.e., $n = 2$, defined by the switching instances, t_0, t_{on}, and t_{off}. In periodic steady-state the circuit will repeat its behavior periodically over the time period T_p. Moreover, due to periodic steady-state the dependence on t_0 can be dropped and therefore have assigned $t_0 = 0$ in our analysis. Applying Kirchoff's current and voltage laws in switch Positions 1 and 2, gives the following differential equations for the respective modes:

$$
\begin{aligned}
i_L &= i_C + i_R \\
\frac{d^2}{dt^2}V_{out}^1(t) + \frac{1}{RC}\frac{d}{dt}V_{out}^1(t) + \frac{1}{LC}V_{out}^1(t) &= \frac{V_s}{LC} \\
V_{out}^1(t) &= c_1 e^{s_1 t} + c_2 e^{s_2 t} + V_s
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
i_L &= -i_c - i_R \\
\frac{d^2}{dt^2}V_{out}^2(t) + \frac{1}{RC}\frac{d}{dt}V_{out}^2(t) + \frac{1}{LC}V_{out}^2(t) &= 0 \\
V_{out}^2(t) &= c_3 e^{s_3 t} + c_4 e^{s_4 t}
\end{aligned} \tag{9}
$$

Where, V_{out} is the output voltage of the converter, as shown in the Fig. 5, and s_1, s_2, s_3 and s_4 are the roots of the characteristic equation of the converter in two modes. Moreover, $s_1 = s_3$ and $s_2 = s_4$ due to the identical characteristic equations. The solution of Equations (8-9), over the time period T_c, can be written using the Heaviside step function as

$$V_{out}(t) = u(t - t_{on})V_{out}^1(t) + (1 - u(t - t_{on}))V_{out}^2(t) \tag{10}$$

In the periodic steady-state, the voltage of the DC-DC buck converter satisfies the following conditions

$$V_{out}(0) = V_{out}(T)\,,\ \ \frac{d}{dt}V_{out}(0) \quad = \frac{d}{dt}V_{out}(T) \tag{11}$$

The steady-state conditions provide two algebraic equations, however, there are four constants involved in the solution. Two more algebraic equations can be obtained from the continuity of the voltage, i.e., V_{out}, due to continuous conduction mode of

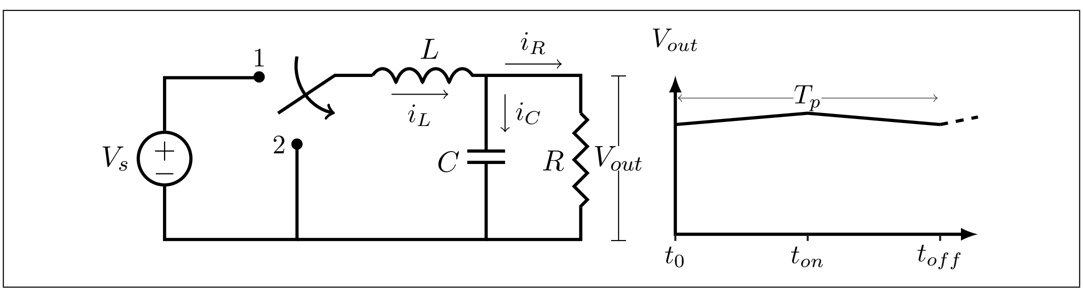


Figure 5: **DC-DC buck Converter**

Component	Current Relationship
Resistor	$\mathtt{I_R(t)} = \frac{\mathtt{V(t)}}{\mathtt{R}}$
Capacitor	$\mathtt{I_C(t)} = \mathtt{C}\frac{\mathtt{dV(t)}}{\mathtt{d(t)}}$
Inductor	$\mathtt{I_L} = \mathtt{i_0} + \frac{1}{\mathtt{R}}\int_0^{\mathtt{t}} \mathtt{V(t)}$

Table 1: **Basic quantities in DC-DC converter**

the circuit, i.e.,

$$V^1_{out}(t_{on}) = V^2_{out}(t_{on}) \, , \; \frac{d}{dt}V^1_{out}(t_{on}) \;\; = \frac{d}{dt}V^2_{out}(t_{on}) \tag{12}$$

Equations (11-12) are used to specify the periodic steady-state voltage that allows finding the minimum and peak conduction currents in steady-state. These currents can then be used to determine ripple currents, which are essentially crucial in specifying the components in the design of the converters.

The first step, in the formalization of the DC-DC Buck converter consists of using the switching function technique to write the switch junction voltages, which in turn requires to formally define the currents of inductor, capacitor and resistor elements. The mathematical expressions for these elements are presented in Table 1, which are formally defined as,

Definition 10: ⊢ ∀ $\mathtt{i_o}$ L v. ind_curr v L $\mathtt{i_o}$ =
    (λ t. $\mathtt{i_o}$ + Cx (&1 / L) * integral (interval [&0, t]) v)

Definition 11: ⊢ ∀ C v. cap_curr C v =
    (λ t. Cx C * vector_derivative v (at t))

Definition 12: ⊢ ∀ v R. res_curr R v = (λ t. v t * Cx (&1 / R))

Where, `R` , `C` and `L` represent the resistance, capacitance and inductances of the resistor, capacitor and inductor of the circuit. $\mathtt{i_o}$ is the initial value of the inductor current, whereas, `v` represents the voltage drop across the circuit elements, at any time `t`. Now, using Definitions 2, 4, 5, 10, 11, and 12, we can formalize the implementation of DC-DC Buck converter as:

Definition 13: ⊢ ∀ $\mathtt{i_o}$ L C R $\mathtt{V_s}$ $\mathtt{V_{out}}$ $\mathtt{V_L}$ $\mathtt{t_{on}}$ t.
  buck_ckt_impl $\mathtt{i_o}$ L C R Vs $\mathtt{V_{out}}$ $\mathtt{V_L}$ $\mathtt{t_{on}}$ t =
  (Vl = switch_volt [λt. Cx $\mathtt{V_s}$ - $\mathtt{V_{out}}$ t; (λt. -$\mathtt{V_{out}}$ t)]
     [&1 - semi_switch (t - t_on); semi_switch (t - $\mathtt{t_{on}}$] t)

```
∧ (∀t.  ∼(t = t_on) ⇒
      kcl [ind_curr (λt.  V_L t) L i_o; cap_curr C (λt.  -V_out t);
          res_curr R (λt.  -V_out t)] t )
```

In the above definition, V_s is the supply voltage, V_{out} is the voltage drop at the junction of all these components, with respect to the ground, and V_L is the voltage drop across the inductor. However, due to the the presence of the switching junction, we model the inductor voltage, in the first conjunct, using the `switch_volt` function, which is provided with two lists; one for all the possible voltage drops, and the other with all the corresponding switching functions for every mode, and an independent variable `t`. Where, t_{on}, is the exact switching instant. This voltage is then used to apply the conventional Kirchoff's current law, using the function `kcl`, which accepts a list of currents, and an independent variable, i.e., `t`.

This implementation model results in the ordinary linear differential equations of the system, which can be described using Definitions 7 and 8 as:

Definition 14: $\vdash \forall\ i_o\ V_s\ V_{out}\ L\ C\ R\ t_{on}\ t.$
$\texttt{buck\_diff\_equ}\ i_o\ V_s\ V_{out}\ L\ C\ R\ t_{on}\ t =$
$\texttt{if}\ (t < t_{on})\ \texttt{then diff\_eq\_lhs}\ [\frac{1}{LC}; \frac{1}{RC};\ 1]\ (V_{out}(t))\ t =$
$\quad \texttt{diff\_eq\_rhs}\ [\frac{V_s}{LC}]\ [1]\ t$
$\texttt{else diff\_eq\_lhs}\ [\frac{1}{LC}; \frac{1}{RC};\ 1]\ (V_{out}(t))\ t = \texttt{diff\_eq\_rhs}\ [0]\ [0]\ t$

According to the proposed methodology, as a first step, we formally verify the implementation and behavior of the Buck converter using the formal model of switching function technique and linear order differential equations as:

Theorem 4: $\vdash \forall\ i_0\ V_s\ V_L\ V_{out}\ L\ C\ R\ t_{on}\ T_p\ t\ .$
A1: $(\forall\ t.\ V_L\ \texttt{continuous\_on}\ [0, t]\ \wedge$
A2: $\sim (C = 0)\ \wedge$
A3: $(t \in (0,\ T_p))\ \wedge$
A4: $\sim(t = t_{on})\ \wedge$ **A5:** $(t_{on} \in (0,\ T_p))\ \wedge$
A6: $(\forall\ t.\ \texttt{differentiable\_n\_vec\_deri}\ 1\ V_{out}\ t)\ \wedge$
A7: $\texttt{buck\_ckt\_impl}\ i_0\ L\ C\ R\ V_s\ V_{out}\ V_L\ t_{on}\ t$
$\quad \Rightarrow \texttt{buck\_diff\_equ}\ i_0\ V_s\ V_{out}\ L\ C\ R\ t_{on}\ t$

Assumption `A1` ensures that the converter is operating in the continuous conduction mode. Assumption `A2` prevents a division by zero case in the formal analysis. Assumptions `A3`-`A4` ensure that the time is over one time period of the system and does not include the singularities, at $t_0 = 0$, $t = t_{on}$ and $t = T_p$, due to switching action. Whereas, Assumptions `A5` specifies that the switching time, $t = t_{on}$, lies within the open interval defined by the single time period of the circuit. Assumption

A6 formally specifies the differentiability of the function, V_{out}, and its first derivative. The predicate `differentiable_n_vec_deri` accepts a number, `n`, and function, `f`, and specifies the differentiability of the function upto its n^{th}-derivative. Finally, Assumption A7 specifies the formal implementation of the power converter circuit using Definition 13. The formal proof of Theorem 4 involves taking derivative of Assumption A7, which consists of piecewise functions, by employing Theorem 1.

Following the proposed methodology, the next task is to formally verify the correctness of the solution of the ordinary linear differential equations of the Buck converter in HOL-Light. Therefore, we define the piecewise solution, i.e., Equation (10), of the Buck converter in higher-order logic as:

Definition 15: $\vdash \forall$ V_s c_1 c_2 c_3 c_4 s_1 s_2 t_{on} t.
solution V_s c_1 c_2 c_3 c_4 s_1 s_2 t_{on} t =
linear_sol [c_1; c_2] (cexp_list [s_1; s_2]) t *
Cx (semi_switch (t - t_{on})) +
linear_sol [c_3; c_4] (cexp_list [s_1; s_2]) t *
Cx (&1 - semi_switch (t - t_{on})

Where V_s is the supply voltage, c_1, c_2, c_3 and c_4 are arbitrary constants, s_1 and s_2 are the roots of homogeneous differential equations corresponding to Equations (7) and (8), respectively. Whereas, the `cexp_list` function is a higher-order-logic function to express the exponential form of the solution for real and distinct roots, i.e., s_1 and s_2, of the circuit. It is defined as:

Definition 16: $\vdash \forall$ x. (cexp_list [] = []) $\wedge$
cexp_list (CONS s t) = CONS (λx. cexp (s * Cx (x))) (cexp_list t)

Next, using Definition 15, we formally verify the correctness of the solution of the differential equations, in each mode of the converter, in HOL-Light as:

Theorem 5: $\vdash \forall$ i_0 V_s V_{out} L C R c_1 c_2 c_3 c_4 s_1 s_2 t_{on} T_p t .
A1: $(\forall$ t. $\sim$(t = t_{on}) $\Rightarrow$ V_{out} = solution V_s c_1 c_2 c_3 c_4 s_1 s_2 t_{on} t) $\wedge$
A2: $(s_1 = -\frac{1}{2RC} + \frac{1}{2}\sqrt{\frac{1}{(RC)^2} - \frac{4}{LC}}\ ) \wedge$
A3: $(s_2 = -\frac{1}{2RC} - \frac{1}{2}\sqrt{\frac{1}{(RC)^2} - \frac{4}{LC}}\ ) \wedge$
A4: $(4\ R^2\ C \leq L) \wedge$
A5: $(0 < L) \wedge$
A6: $(0 < R) \wedge$
A7: $(0 < C) \wedge$
A8: $(t \in (0,\ T_p)) \wedge$
A9: $\sim(t = t_{on}) \wedge$
A10: $(t_{on} \in (0,\ T_p))$
$\Rightarrow$ buck_diff_equ i_0 V_s V_{out} L C R t_{on} t

Assumption A1 formally defines the output voltage V_{out} as a piecewise function, over the time period, T_p, of the converter circuit. Assumptions A2-A3 formally specify the roots of the equation. Assumption A4 formally specifies the condition on the circuit parameters for real and distinct roots. Assumptions A5-A7, ensure the positive values of inductance, resistance and capacitance of the circuit. Assumptions A8-A9 ensure that the time is over one time period of the system and does not include the singularities, at $t_0 = 0$, $t = t_{on}$ and $t = T_p$, due to switching action. Whereas, Assumptions A10 specifies that the switching time, $t = t_{on}$, lies within the open interval defined by the single time period of the circuit.

The formal verification of Theorem 5 utilized the formally verified results of Theorems 1 and 3.

Finally, we present the formally verified results of periodic steady-state voltage of of the DC-DC Buck converter as:

Theorem 6: $\vdash \forall$ V_s V_{out} c_1 c_2 c_3 c_4 s_1 s_2 t_{on} t T_p.

A1: $(t \in (0,\ T_p))$ $\wedge$
A2: $\sim(t = t_{on})$ $\wedge$
A3: $(t_{on} \in (0,\ T_p))$ $\wedge$
A4: $(\forall$ t. $\sim$ $(t = t_{on}) \Rightarrow V_{out}$ = solution V_s c_1 c_2 c_3 c_4 s_1 s_2 t_{on} t$)$ $\wedge$
A5: $(\forall$ t. n_vec_deri 1 $(\lambda$ t. V_{out} t) continuous at t$)$ $\wedge$
A6: $\sim$ $(s_2 - s_1 = 0)$ $\wedge$
A7: steady_state 1 V_{out} t $\Rightarrow$

$$\Bigg(V_{out}(0) = \left(\frac{s_2}{s_2 - s_1}\right)\Big[\Big(V_{out}(0) + \tfrac{1}{s_2}\tfrac{d}{dt}V_{out}(0) - V_s\Big)\ e^{-t_{on}s_1} + V_s\Big]\ e^{-T_p s_1} + \left(\frac{s_1}{s_2 - s_1}\right)\Big[\Big(-V_{out}(0) - \tfrac{1}{s_1}\tfrac{d}{dt}V_{out}(0) + V_s\Big)\ e^{-t_{on}s_1} - V_s\Big]\ e^{-T_p s_2}\Bigg)\ \wedge$$

$$\Bigg(-\tfrac{d}{dt}V_{out}(0) = \left(\frac{s_1 s_2}{s_2 - s_1}\right)\Big[\Big(V_{out}(0) + \tfrac{1}{s_2}\tfrac{d}{dt}V_{out}(0) - V_s\Big)\ e^{-t_{on}s_1} + V_s\Big]\ e^{-T_p s_1} + \left(\frac{s_1 s_2}{s_2 - s_1}\right)\Big[\Big(-V_{out}(0) - \tfrac{1}{s_1}\tfrac{d}{dt}V_{out}(0) + V_s\Big)\ e^{-t_{on}s_1} - V_s\Big]\ e^{-T_p s_2}\Bigg)$$

Assumptions A1 and A2 formally specify the analysis over one time period with singularities, at $t = 0$, $t = t_{on}$ and $t = T_p$, excluded. Whereas, Assumptions A3 specifies that the switching time, $t = t_{on}$, lies within the open interval defined by the single time period of the circuit. Assumption A4 formally defines the output voltage V_{out} as a piecewise function, over the time period, T_p, of the converter circuit. Assumption A5 formally specifies the continuity of the function and its derivative, to ensure the continuous conduction mode. Assumption A6 prevents the division by zero case in the analysis, and finally, Assumption A7 defines the steady-state of the buck converter.

The formal proof of Theorem 6 essentially consists of finding the values of the function and its derivative at $t = 0$ and $t = T_p$, in limit sense, and the values of arbitrary constants c_1, c_2, c_3 and c_4 by utilizing the continuity assumption A5 and the one-sided limits concepts due to singularities at $t = 0$, $t = t_{on}$ and $t = T_p$, due to switching action. More details about the proof can be found at [2].

The proposed foundational formalization of switching function technique and linear differential equations allowed us to formally specify and verify the nonlinear behavior of the DC-DC Buck converters in a very straightforward manner. Theorem 4 verifies that the implementation and behavior of the Buck converter by explicitly specifying the conditions on the piecewise functions, e.g., voltages in the case of DC-DC Buck converter, in the continuous conduction operating mode of the converter. The formally verified result is very helpful in the topology selection of the converter, which is usually the first step in the design procedure and, in practice, consists of an intuitive selection of topology for a given design specification. Moreover, Theorem 5 formally verifies the correction of the solution of the linear order differential equations representing the power converter behavior. This result plays a vital role in the performance evaluation. Once the implementation and behavior (Theorem 4), and the solution (Theorem 5) of the DC-DC Buck converter is formally verified, then Theorem 6 formally verifies the relationship among different parameters of the circuit, such as voltage and circuit components, in periodic steady-state. This result is instrumental in formal verification of the design objectives, such as desired voltage levels and component values, of the circuit. However, unlike traditional techniques these formally verified results give exact conditions in terms of the parameters of the Buck converter as they have been formally verified using a sound theorem prover. Moreover, these results are generic in terms of universally quantified variables and contain an exhaustive set of assumptions required for the validity of the results.

6 Conclusion

In this paper, we presented a formal methodology to conduct the formal time-domain based periodic steady-state analysis of power converters. The power converters are characterized by the switching functionality, which imparts to the structural changes of the converter circuit and a nonlinear mathematical analysis. To model the structural changes in the circuit, we developed the formal model of the circuit analysis technique, called switching function technique, and also developed a formal model of linear differential equations to formally specify the behavior of the converters. To cater for the nonlinearities in the analysis, the integral property of the Heaviside step function as a generalized function is verified. This logical formalism is then applied to the DC-DC Buck converter to formally verify the implementation and behavior of the converter's circuit, solution of its linear ordinary differential equations in all modes of the converter's circuit and the steady-state voltage relationship of the DC-DC Buck converter.

The proposed formalization can be extended to incorporate the formal small-signal modeling analysis of the power converters. Moreover, the formalization is based upon the complex valued functions to formally analyze the periodic steady-state analysis of power converters, which are characterized by the discontinuity due to switching action, therefore, the formalization is also equally applicable to analyze many other discontinuous phenomenon ubiquitous in many fields of Physics and engineering.

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Pseudo EMV-algebras. I. Basic Properties

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Abstract

We introduce pseudo EMV-algebras which are a non-commutative generalization of both MV-algebras and generalized Boolean algebras. The existence of a top element is not assumed. The paper has two parts. In the present one we study basic properties of pseudo EMV-algebras as ideals and homomorphisms. The class of all pseudo EMV-algebras is not a variety and rather a more general class, called a q-variety, but similar to a variety. We study representable pseudo EMV-algebras, normal-valued ones, and pseudo EMV-algebras whose every maximal ideal is normal.

The second part shows that every pseudo EMV-algebra without top element can be embedded into a pseudo EMV-algebra with top element as a maximal and normal ideal of the latter one. We present a categorical equivalence of the category of pseudo EMV-algebras without top element with a special category of pseudo MV-algebras or with a special category of ℓ-groups. Finally, we study states as finitely additive mappings as well as state-morphisms on pseudo EMV-algebras and we present their representation as an integral over a regular Borel probability measure.

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1 Introduction

A deep study of many valued algebras started with seminal papers by Chang, [2, 3]. It was recognized that every interval in an Abelian unital ℓ-group gives an example of MV-algebras. Mundici [34] established that every MV-algebra is isomorphic to the MV-algebra of the interval in some unital Abelian ℓ-group, and more, he established a categorical equivalence between the category of MV-algebras and the category of Abelian unital ℓ-groups. A non-commutative generalization of MV-algebras was introduced independently in [26] as pseudo MV-algebras and in [37] as generalized MV-algebras. Also for them in [12], their representation by unital ℓ-groups not necessarily Abelian together with a categorical equivalence was established. Fuzzy logic connected with these non-commutative algebraic structures was studied in [29]. In the last years, there appeared a lot of non-commutative algebraic structures like pseudo BL-algebras [8, 9], pseudo hoops [27], pseudo effect algebras [16, 17], residuated lattices, see e.g. [25].

Recently in [18], the authors introduced EMV-algebras as a common generalization of MV-algebras and Boolean rings. These algebras do not assume the existence of a top element, similarly as do not generalized Boolean algebras. On the other hand they behave locally as MV-algebras on each interval $[0, a]$, where a is a Boolean element of the EMV-algebra, in addition, conjunction and disjunction are assumed, negation only as relative complements for comparable elements, but a total complement is not assumed. For them a version of the Loomis–Sikorski theorem was established in [19], states were investigated in [20], and free and week free EMV-algebras were described in [21].

The aim of the present paper is to generalize EMV-algebras to a non-commutative variant, called pseudo EMV-algebras, in a similar way how MV-algebras were generalized to pseudo MV-algebras.

The main aims of the paper, which is divided into two parts, are:

Part I.

(1) We investigate the basic properties of pseudo EMV-algebras.

(2) The existence of a maximal ideal in MV-algebras or pseudo MV-algebras is a trivial task. Since a pseudo EMV-algebra does not have a top element in general, the existence of a maximal ideal is not so evident. We show that each non-trivial pseudo EMV-algebra possesses at least one maximal ideal.

(3) We describe congruences of pseudo EMV-algebras.

(4) The class of pseudo EMV-algebras is not closed under forming subalgebras, so this class is not a variety and rather a more general structure, close to a variety, called a q-variety. We investigate q-subvarieties of pseudo EMV-algebras and we show that the lattice of q-subvarieties is uncountable. In particular, we study rep-

resentable pseudo EMV-algebras, normal-valued pseudo EMV-algebras, and pseudo EMV-algebras where each maximal ideal is normal.

Part II.

(5) We present the Basic Representation Theorem showing that every pseudo EMV-algebra M without top element can be embedded into a pseudo EMV-algebra N with top element as a maximal and normal ideal of N. This result generalizes an analogous result for generalized Boolean algebras from [5, Thm 2.2].

(6) We establish a categorical equivalence of the category of pseudo EMV-algebras without top element with a special category of pseudo MV-algebras and with a special category of ℓ-groups.

(7) We introduce a state as an additive functional s on M with values in the real interval [0,1] such that there is an element x_0 with $s(x_0) = 1$. We describe state-morphisms which are exactly extremal states. States on pseudo EMV-algebras generalize states on MV-algebras studied in [35]. In contrast to EMV-algebras, it can happen that a pseudo EMV-algebra is stateless. The existence of at least one maximal ideal that is also normal is a necessary and sufficient condition for existence of a state.

(8) We investigate topological properties of the weak topology of states. We show that every state is a weak limit of a net of convex combinations of state-morphisms.

(9) We represent every state on a pseudo EMV-algebra as an integral over a unique regular Borel probability measure on the Borel σ-algebra of the state space. It generalizes an analogous result for MV-algebras established in [31, 36].

The paper is organized as follows. Part I: Section 2 gathers basic facts of pseudo MV-algebras. Pseudo EMV-algebras are introduced in Section 3 where their basic properties are established. Section 4 deals with ideals, homomorphisms, congruences, and with q-varieties of pseudo EMV-algebras. Prime ideals and q-varieties of pseudo EMV-algebras are studied in Section 5.

Part II: Representation of proper pseudo EMV-algebras as a maximal and normal ideal of a pseudo EMV-algebra with top element is investigated in Section 6. Section 7 describes a categorical equivalence of the category of pseudo EMV-algebras without top element with a special category of ℓ-groups. States and state-morphisms on pseudo EMV-algebras are introduced in Section 8. We introduce the weak topology of states and exhibit the topological properties of the state space. In particular, if a pseudo EMV-algebra does not have a top element, then the space of state-morphisms is a locally compact Hausdorff space whose one-point compactification is affinely isomorphic to the space of state-morphisms of the representing pseudo EMV-algebra with top element. An integral representation of a state by a unique regular Borel probability measure on the Borel σ-algebra of the state-space is proved in Section 9.

2 Basic Notions a Results

Pseudo MV-algebras as a non-commutative generalization of MV-algebras were defined independently in [26] as pseudo MV-algebras and in [37] as generalized MV-algebras. In this section we gather basic properties of pseudo MV-algebras.

Definition 2.1. *A pseudo MV-algebra is an algebra* $(M; \oplus, ^-, ^\sim, 0, 1)$ *of type* $(2, 1, 1, 0, 0)$ *such that the following axioms hold for all* $x, y, z \in M$ *with an additional binary operation* $\odot$ *defined via*

$$y \odot x = (x^- \oplus y^-)^\sim$$

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(A2) $x \oplus 0 = 0 \oplus x = x$;

(A3) $x \oplus 1 = 1 \oplus x = 1$;

(A4) $1^\sim = 0$; $1^- = 0$;

(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;

(A6) $x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x$;

(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;

(A8) $(x^-)^\sim = x$.

As usually, we assume that $\odot$ has higher binding priority than $\wedge$ and $\oplus$, and $\oplus$ is higher than $\vee$. We denote by $\mathcal{PMV}$ the variety of pseudo MV-algebras.

If we define a partial order $\leq$ on M by $x \leq y$ iff $x^- \oplus y = 1$, then M is a distributive lattice with $x \vee y = x \oplus (x^\sim \odot y)$ and $x \wedge y = x \odot (x^- \oplus y)$. In addition, $0 \leq x \leq 1$ for each $x \in M$ and $x \leq y$ iff $y \oplus x^\sim = 1$.

A pseudo MV-algebra is an MV-algebra iff $\oplus$ is a commutative binary operation.

A non-empty subset I of M is an *ideal* of M if (i) $a \leq b \in I$ implies $a \in I$, and (ii) if $a, b \in I$, then $a \oplus b \in I$. The sets M and $\{0\}$ are ideals of M. An ideal $I \neq M$ of M is (i) *maximal* if it is not a proper subset of any proper ideal of M, and (ii) *normal* if $a \oplus I := \{a \oplus b \colon b \in I\} = \{c \oplus a \colon c \in I\} =: I \oplus a$ for any $a \in M$. Using Zorn's Lemma, a maximal ideal exists in every pseudo MV-algebra, albeit there are pseudo MV-algebras that have no maximal ideal that is normal, see [11]. On the other side, the class of pseudo MV-algebras M such that every maximal ideal of M is normal is a variety, see [13, Thm 4.1]. We say that an element a of a pseudo MV-algebra M is *Boolean* if $a \wedge a^- = 0$, equivalently, $a \wedge a^\sim = 0$, equivalently $a \oplus a = a$. Let $B(M)$ be the set of Boolean elements, then $B(M)$ is a Boolean algebra that is an

MV-algebra and a subalgebra of M, and $a^- = a^\sim$; we put $a' = a^-$. It is the biggest Boolean subalgebra of M, and it can be called also as the *Boolean skeleton* of M. For basic properties of pseudo MV-algebras see [26].

Pseudo MV-algebras are intimately connected with ℓ-groups. We remind that a *po-group* (= partially ordered group) is a group $(G;+,-,0)$ written additively endowed with a partial order $\le$ such that, for $g,h \in G$ with $g \le h$ we have $a+g+b \le a+h+b$ for all $a,b \in G$. If the partial order $\le$ is a lattice order, G is said to be an *ℓ-group*. The *positive cone* of a po-group G is the set $G^+ = \{g \in G\colon 0 \le g\}$. An element $u \in G^+$ is a *strong unit* of G if, given $g \in G$, there is an integer $n \ge 1$ such that $g \le nu$. A couple (G,u), where G is an ℓ-group and u is a fixed strong unit of G, is said to be a *unital ℓ-group*. For non-explained notions about ℓ-groups, please, consult e.g. [24, 28].

A prototypical example of pseudo MV-algebras is from ℓ-groups: If u is a strong unit of a (not necessarily Abelian) ℓ-group G, set

$$\Gamma(G,u) := [0,u]$$

and

$$\begin{aligned} x \oplus y &:= (x+y) \wedge u, \\ x^- &:= u - x, \\ x^\sim &:= -x + u, \\ x \odot y &:= (x - u + y) \vee 0, \end{aligned}$$

then $\Gamma(G,u) := ([0,u]; \oplus, ^-, ^\sim, 0, u)$ is a pseudo MV-algebra [26]. The converse statement is also true as it follows from the basic representation of pseudo MV-algebras by unital ℓ-groups, see [12]:

Theorem 2.2. *For any pseudo MV-algebra M, there exists a unique (up to isomorphism of unital ℓ-groups) unital ℓ-group (G,u) with a strong unit u such that $M \cong \Gamma(G,u)$. The functor Γ defines a categorical equivalence of the category of pseudo MV-algebras with the category of unital ℓ-groups.*

We note that if $x^- = x^\sim$ for each $x \in M$, this does not mean that M is an MV-algebra. Indeed, let G be a non-commutative ℓ-group. Let $\mathbb{Z}$ be the group of integers. Define the lexicographic product $\mathbb{Z} \overrightarrow{\times} G$ endowed with a partial order $(n,f) \le (m,g)$ iff $n < m$ or $n = m$ and $f \le g$. Then $(1,0)$ is a strong unit for $\mathbb{Z} \overrightarrow{\times} G$ and in the pseudo MV-algebra $M := \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1,0))$, we have $x^- = x^\sim$ for each $x \in M$ and M is not an MV-algebra.

According to [12], we can define a partial operation $+$ for elements of a pseudo MV-algebra M: $x+y$ is defined iff $y \odot x = 0$, then $x+y = x \oplus y$. Using Theorem 2.2, we see that $x + y$ is equal to the group addition of x and y. In addition, $y \odot x = 0$ iff $x \leq y^-$ iff $y \leq x^\sim$, and the operation $+$ is associative, [10, 15]. If $x \leq y$, there are two unique elements $z_1, z_2 \in M$ such that $x + z_1 = y = z_2 + x$. We write $z_1 = x \backslash y$ and $z_2 = y/x$. If we use the group representation, then $z_1 = -x + y$ and $z_2 = y - x$.

Hence, we can define for each $x \in M$

$$0x = 0, \quad 1x = x, \quad (n+1)x = nx + x,\ n \geq 1,$$

assuming nx and $nx + x$ are defined in the pseudo MV-algebra M.

Now let $x \in M$. Using properties of the partial order $\leq$ on M and the equality $x^{-\sim} = x = x^{\sim -}$, we have

$$x^- = \min\{z \in M \colon z \oplus x = 1\} \tag{2.1}$$

and

$$x^\sim = \min\{z \in M \colon x \oplus z = 1\}. \tag{2.2}$$

Let $a \in M$ be a Boolean element of a pseudo MV-algebra M. The interval $[0, a] := \{x \in M \colon 0 \leq x \leq a\}$ can be converted into a pseudo MV-algebra as follows: For $x \in [0, a]$, we set $x^{-_a} = a/x$ and $x^{\sim_a} = x \backslash a$, and since $a \oplus a = a$, we put $\oplus_a = \oplus|_{[0,a]\times[0,a]}$. Then $M_a = ([0, a]; \oplus_a, {}^{-_a}, {}^{\sim_a}, 0, a)$ is a pseudo MV-algebra; to verify that, we use the group representation of pseudo MV-algebras. We define two unary operators λ_a and ρ_a on $M_a = [0, a]$ by

$$\lambda_a(x) := \min\{z \in [0, a] \colon z \oplus x = a\}, \quad x \in [0, a], \tag{2.3}$$

and

$$\rho_a(x) := \min\{z \in [0, a] \colon x \oplus z = a\}, \quad x \in [0, a]. \tag{2.4}$$

Using (2.1)–(2.2), we have for each $x \in [0, a]$

$$x^{-a} = \lambda_a(x), \quad x^{\sim_a} = \rho_a(x).$$

In other words, if $a \in B(M)$, then $([0, a]; \oplus_a, \lambda_a, \rho_a, 0, a) = ([0, a]; \oplus, \lambda_a, \rho_a, 0, a)$ is a pseudo MV-algebra, and $\lambda_a(x) + x$ and $x + \rho_a(x)$ are defined in M_a and $\lambda_a(x) + x = a = x + \rho_a(x)$. In addition, if $x + z_1 = a = z_2 + x$ in M_a, then $z_1 = \rho_a(x)$ and $z_2 = \lambda_a(x)$. These algebras will play an important role in the definition of pseudo EMV-algebras in the next section.

3 Pseudo EMV-algebras

In the section, we introduce pseudo EMV-algebras which generalize EMV-algebras defined in [18], pseudo MV-algebras from [26, 37] and generalized Boolean algebras. It is not assumed that a pseudo EMV-algebra contains a top element; if it is a case, then it is termwise equivalent to a pseudo MV-algebra. We establish the basic properties and present some useful examples. We note that the first definition of pseudo EMV-algebras was given in [23] in the frame of generalized pseudo EMV-effect algebras.

Let $(M;\oplus,0)$ be a monoid with a neutral element 0. The monoid M is not assumed to be commutative. An element $a\in M$ is said to be an *idempotent* if $a\oplus a=a$. We denote by $\mathcal{I}(M)$ the set of idempotents of M; then (i) $0\in\mathcal{I}(M)$, (ii) if $a,b\in\mathcal{I}(M)$ and $a\oplus b=b\oplus a$, then $a\oplus b\in\mathcal{I}(M)$. We say that a monoid $(M;\oplus,0)$ endowed with a partial order $\le$ is (i) *ordered* if $x\le y$ implies $z_1\oplus x\oplus z_2\le z_1\oplus y\oplus z_2$ for all $z_1,z_2\in M$, (ii) *naturally ordered* if $x\le y$ iff there are $z_1,z_2\in M$ such that $x\oplus z_1=y=z_2\oplus x$.

Definition 3.1. An algebra $(M;\vee,\wedge,\oplus,0)$ of type $(2,2,2,0)$ is called a *pseudo EMV-algebra* if it satisfies the following conditions:

(E1) $(M;\vee,\wedge,0)$ is a distributive lattice with the least element 0;

(E2) $(M;\oplus,0)$ is an ordered monoid with a neutral element 0;

(E3) for each $a\in\mathcal{I}(M)$, the elements

$$\lambda_a(x):=\min\{z\in[0,a]\colon z\oplus x=a\},\quad \rho_a(x):=\min\{z\in[0,a]\colon x\oplus z=a\}$$

exist in M for all $x\in[0,a]$, and the algebra $([0,a];\oplus,\lambda_a,\rho_a,0,a)$ is a pseudo MV-algebra;

(E4) for each $x\in M$, there is $a\in\mathcal{I}(M)$ such that $x\le a$.

We note that order in (E1) and (E2) are the same. Property (E3) implies directly $\lambda_a(x)\oplus x=a=x\oplus\rho_a(x)$, $x\in[0,a]$. We will write also $M=(M;\vee,\wedge,\oplus,0)$.

Important examples of pseudo EMV-algebras are the following algebras.

3.1 EMV-algebras

We note that if $\oplus$ is commutative, then the pseudo EMV-algebra is an *EMV-algebra* in the sense of [18], and conversely, every EMV-algebra is a pseudo EMV-algebra. We remind that the lattice operation $\wedge$ and $\vee$ from (E1) determine a partial order $\le$ on M via $x\le y$ iff $x=x\wedge y$. Then 0 is a bottom element and a top element is not assumed, in general.

3.2 Pseudo MV-algebras

If $(M; \oplus, ^-, ^\sim, 0, 1)$ is a pseudo MV-algebra, then $(M; \vee, \wedge, \oplus, 0)$ is a pseudo EMV-algebra with a top element 1. Conversely, if a pseudo EMV-algebra $(M; \vee, \wedge, \oplus, 0)$ has a top element 1, then M can be converted into a pseudo MV-algebra whose algebraic pseudo MV-structure is compatible with the pseudo EMV-algebraic structure of M. Indeed, the top element 1 is an idempotent, so that $M = [0, 1]$ and $(M; \oplus, \lambda_1, \rho_1, 0, 1)$ is a pseudo MV-algebra due to definition of pseudo EMV-algebras. In addition, it is easy to show that pseudo MV-algebras are termwise equivalent to pseudo EMV-algebras with top element; the equivalence is given by $(M; \vee, \wedge, \oplus, 0)$ with a top element 1 is equivalent to $(M; \oplus, \lambda_1, \rho_1, 0, 1)$ and vice-versa.

3.3 Finite Pseudo EMV-algebras

A pseudo EMV-algebra is *proper* if it does not contain a top element. If M is finite, then it has a top element, and therefore, $\oplus$ is commutative since every finite pseudo MV-algebra is commutative, see [12, Thm 4.2].

3.4 Generalized Boolean Algebras

Any generalized Boolean algebra $(M; \vee, \wedge, 0)$ (studied also as a Boolean ring, see [33]) forms an EMV-algebra $(M; \vee, \wedge, \oplus, 0)$, where $\oplus = \vee$ and if $a \leq b$, then $\lambda_a(x)$ is the unique relative complement of x in the interval $[0, a]$. Generalized Boolean algebras without top element can be used to construct many examples of pseudo EMV-algebras without top element. Indeed, take a generalized Boolean algebra B (e.g. a ring of subsets of a set $\Omega \neq \emptyset$) without top element and a pseudo MV-algebra M, then the direct product $B \times M$ is a pseudo EMV-algebra without top element, and $\mathcal{I}(B \times M) = B \times \mathcal{I}(M)$.

3.5 Idempotents $\mathcal{I}(M)$

The set of idempotents $\mathcal{I}(M)$ of a pseudo EMV-algebra M is a generalized Boolean algebra, so that it is an EMV-algebra.

3.6 Sum of Pseudo EMV-algebras

Let $\{M_i : i \in I\}$ be a family of pseudo EMV-algebras. We can easily show that $\sum_{i \in I} M_i := \{(x_i)_{i \in I} \in \prod_{i \in I} M_i : x_i = 0$ for all but a finite number of $i \in I\}$ is a pseudo EMV-algebra with componentwise operations. For example, let $A_1 = M$ be a pseudo EMV-algebra with top element and $A_i = \{0, 1\}$ for all $i \in \mathbb{N}$, then $\sum_{i \in I} A_i$ is a pseudo EMV-algebra.

Now we present basic properties of pseudo EMV-algebras. First we start with comparing two partial orders on $[0,a]$ for each $a \in \mathcal{I}(M)$.

Let $(M;\vee,\wedge,\oplus,0)$ be a pseudo EMV-algebra. Its reduct $(M;\vee,\wedge,0)$ is a distributive lattice with a bottom element 0. Let $\leq$ be the partial order determined from the lattice structure of M, that is $x \leq y$ iff $x \vee y = y$ iff $x \wedge y = x$. If a is a fixed idempotent element of M, $([0,a];\oplus,\lambda_a,\rho_a,0,a)$ is a pseudo MV-algebra. Recall that, in each pseudo MV-algebra $(A;\oplus,^-,^\sim,0,1)$ there is a partial order relation $\preccurlyeq$ (induced by $\oplus,^-,^\sim,1$) defined by $x \preccurlyeq y$ iff $x^- \oplus y = 1$ iff $y \oplus x^\sim = 1$, see [26, Prop 1.9(g)]. So, the partial order on the pseudo MV-algebra $([0,a];\oplus,\lambda_a,\rho_a,0,a)$ is defined by $x \preccurlyeq y$ iff $\lambda_a(x) \oplus y = a$ iff $y \oplus \rho_a(x) = a$. Therefore from (2.4)–(2.3) and (E3), we have for each $x \in [0,a]$

$$\begin{aligned}\min{}_{\leq}\{z \in [0,a]\colon z \oplus x = a\} = \lambda_a(x) = \min{}_{\preccurlyeq}\{z \in [0,a]\colon z \oplus x = a\},\\ \min{}_{\leq}\{z \in [0,a]\colon x \oplus z = a\} = \rho_a(x) = \min{}_{\preccurlyeq}\{z \in [0,a]\colon x \oplus z = a\},\end{aligned}$$

where $\min_{\leq}$ and $\min_{\preccurlyeq}$ are minima taken with respect to $\leq$ in M and $\preccurlyeq$ taken in $[0,a]$, respectively.

In the next proposition, we show that $\leq$ and $\preccurlyeq$ coincide on $[0,a]$.

Proposition 3.2. *Let $M = (M;\vee,\wedge,\oplus,0)$ be a pseudo EMV-algebra and let $x,y \in [0,a]$. Then $x \leq y$ if and only if $x \preccurlyeq y$. In addition, $(M;\oplus,0)$ is naturally ordered.*

Proof. First, let $x \leq y$. We show that $x \preccurlyeq y$ or equivalently, $\lambda_a(x) \oplus y = a$. By definition of $\lambda_a(x)$, we have $a = \lambda_a(x) \oplus x \leq \lambda_a(x) \oplus y \leq a \oplus a \leq a$, when we have used the fact that M is an ordered monoid, see (E2). Whence, $x \preccurlyeq y$.

Conversely, let us assume that $x \preccurlyeq y$. Then $\lambda_a(x) \oplus y = a$. Set $z := \lambda_a(x)$. Then $z \oplus y = a$ entails that $\rho_a(z) \leq y$ and so $x = \rho_a(\lambda_a(x)) \leq y$.

Now, let $x,y \in M$ and $x \leq y$. There is an idempotent $a \in \mathcal{I}(M)$ which dominates x,y. By the first part of the present proof, we have $x \preccurlyeq y$, and since every pseudo MV-algebra $[0,a]$ is naturally ordered, we have that so is M. ☐

Due to the latter proposition, join and meet of two elements $x,y \in [0,a]$ are the same in M as well as in the pseudo MV-algebra $[0,a]$.

Now, we study functions λ_a and ρ_a for each $a \in \mathcal{I}(M)$.

Proposition 3.3. *Let $(M;\vee,\wedge,\oplus,0)$ be a pseudo EMV-algebra, $a,b \in \mathcal{I}(M)$ such that $a \leq b$. Then for each $x \in [0,a]$, we have*

(i) *$\lambda_b(a) = \rho_b(a)$ is an idempotent, and $\lambda_a(a) = 0 = \rho_a(a)$;*

(ii) *$\lambda_a(x) = \lambda_b(x) \wedge a$ and $\rho_a(x) = \rho_b(x) \wedge a$;*

(iii) $\lambda_b(x) = \lambda_a(x) \oplus \lambda_b(a) = \lambda_b(a) \oplus \lambda_a(x)$ *and* $\rho_b(x) = \rho_a(x) \oplus \rho_b(a) = \rho_b(a) \oplus \rho_a(x)$;

(iv) $\rho_a(\lambda_a(x)) = x = \lambda_a(\rho_a(x))$;

(v) $\lambda_a(x) \leq \lambda_b(x)$ *and* $\rho_a(x) \leq \rho_b(x)$.

Proof. For the proof see [23, Prop 3.2]. □

Now, we define another binary operation, $\odot$, using the identity $x \odot y = (y^- \oplus x^-)^\sim$ from pseudo MV-algebras.

Proposition 3.4. *Let $(M; \vee, \wedge, \oplus, 0)$ be a pseudo EMV-algebra. For all $x, y \in M$, we define*

$$x \odot y = \rho_a(\lambda_a(y) \oplus \lambda_a(x)),$$

where $a \in \mathcal{I}(M)$ and $x, y \in [0, a]$. Then $\odot : M \times M \to M$ is a well-defined binary operation on M which does not depend on $a \in \mathcal{I}(M)$ if $x, y \leq a$. Moreover, $(M; \odot)$ is an ordered semigroup. Then

$$x \odot y = \lambda_a(\rho_a(y) \oplus \rho_a(x)).$$

In addition, if $x, y \in M$, $x \leq y$, then

$$y \odot \lambda_a(x) = y \odot \lambda_b(x), \quad \rho_a(x) \odot y = \rho_b(x) \odot y \tag{3.1}$$

for all idempotents a, b of M with $x, y \leq a, b$, and

$$y = (y \odot \lambda_a(x)) \oplus x = x \oplus (\rho_a(x) \odot y). \tag{3.2}$$

If $x, y \in [0, a]$ for some idempotent $a \in M$, then

$$x \odot \lambda_a(y) = x \odot \lambda_a(x \wedge y), \quad \rho_a(y) \odot x = \rho_a(x \wedge y) \odot x, \tag{3.3}$$

and

$$(x \odot \lambda_a(y)) \oplus (x \wedge y) = x = (x \wedge y) \oplus (\rho_a(y) \odot x). \tag{3.4}$$

Finally, if $x, y \leq a \in \mathcal{I}(M)$, then

$$((x \oplus y) \odot \lambda_a(x)) \oplus x = x \oplus y = y \oplus (\rho_a(y) \odot (x \oplus y)) \tag{3.5}$$

and if $x \leq a \in \mathcal{I}(M)$, then

$$x \odot \lambda_a(x) = 0 = \rho_a(x) \odot x. \tag{3.6}$$

Proof. For the proof see [23, Prop 3.3]. □

We present the following simple but important property of idempotents which follows from [26, Prop 4.3]: If $a \in \mathcal{I}(M)$ and $x \in M$,

$$x \oplus a = x \vee a = a \oplus x, \quad x \odot a = x \wedge a = a \odot x. \tag{3.7}$$

Moreover, if

$$z \leq x \oplus y \Rightarrow \ \exists\, x_1, y_1 \in M \text{ such that } x_1 \leq x,\ y_1 \leq y, \text{ and } z = x_1 \oplus y_1. \tag{3.8}$$

This property follows from an analogous property holding in each pseudo MV-algebra.

The following proposition shows that a pseudo EMV-algebra can be defined also in a simpler way.

Proposition 3.5. *An algebra $(M; \vee, \wedge, \oplus, 0)$ of type $(2,2,2,0)$ is a pseudo EMV-algebra if and only if*

(M1) *$(M; \vee, \wedge, 0)$ is a lattice with the least element 0;*

(M2) *$(M; \oplus, 0)$ is an ordered monoid with the neutral element 0;*

(M3) *for each $x \in M$, there is $b \in \mathcal{I}(M)$ with $x \leq b$ such that $([0,b]; \oplus, \lambda_b, \rho_b, 0, b)$ is a pseudo MV-algebra, where $\lambda_b(x) := \min\{z \in [0,b] \colon z \oplus x = b\}$ and $\rho_b(x) := \min\{z \in [0,b] \colon x \oplus z = b\}$ exist for each $x \in [0,b]$.*

Proof. Let (M1)–(M3) hold.

(i) According to (M2), $(M; \oplus, 0)$ is an ordered monoid. In a similar way as in Proposition 3.2, we can show that if $x, y \leq b \in \mathcal{I}(M)$ and $([0,b]; \oplus, \lambda_b, \rho_b, 0, b)$ is a pseudo MV-algebra, then $x \leq y$ iff $\lambda_b(x) \oplus y = b$. If $x, y \leq c \in \mathcal{I}(M)$, then there is an idempotent $d \in M$ such that $x, y \leq b, c \leq d$ and $([0,d]; \oplus, \lambda_d, \rho_d, 0, d)$ is a pseudo MV-algebra.

Therefore, if $x, y, z \in M$ are arbitrary elements, there is an idempotent $b \in \mathcal{I}(M)$ such that $x, y, z \in [0,b]$ and $([0,b]; \oplus, \lambda_a, \rho_b, 0, b)$ is a pseudo MV-algebra which is a distributive lattice. Therefore, M is a distributive lattice, too.

(ii) Now, we show that for all $b \in \mathcal{I}(M)$, $([0,b]; \oplus, \lambda_b, \rho_b, 0, b)$ is a pseudo EMV-algebra. Let b be an arbitrary idempotent element of M. By the assumption, there is $v \in \mathcal{I}(M)$ such that $b \leq v$ and $([0,v]; \oplus, \lambda_v, \rho_v, 0, v)$ is a pseudo MV-algebra. It can be easily seen that $\lambda_b(x) = b \wedge \lambda_v(x)$ and $\rho_b(x) = b \wedge \rho_v(x)$ for all $x \in [0,b]$, and due to notes around (2.4)–(2.3), $([0,b]; \oplus, \lambda_b, \rho_b, 0, b)$ is a pseudo MV-algebra.

Therefore, M is a pseudo EMV-algebra.

Clearly, the converse holds. ☐

For any integer $n \geq 0$ and any x of a pseudo EMV-algebra M, we can define

$$0.x = 0, \quad 1.x = x, \quad (n+1).x = (n.x) \oplus x, \; n \geq 2,$$

and

$$x^1 = x, \quad x^n = x^{n-1} \odot x, \; n \geq 2,$$

and if M has a top element 1, we define also $x^0 = 1$.

Now we introduce Archimedean pseudo EMV-algebras using infinitesimal elements. We show that every Archimedean pseudo EMV-algebra is an EMV-algebra, i.e. $\oplus$ is commutative.

Let M be a pseudo EMV-algebra. An element $x \in M$ is said to be *infinitesimal* if, there is an idempotent $a \in \mathcal{I}(M)$ with $x \leq a$ such that $n.x \leq \lambda_a(x)$ for each integer $n \geq 1$. We denote by $\mathrm{Infinit}(M)$ the set of infinitesimal elements of M. Clearly, $0 \in \mathrm{Infinit}(M)$. A pseudo EMV-algebra M is said to be *Archimedean* if $\mathrm{Infinit}(M) = \{0\}$. In addition, a pseudo EMV-algebra M is said to be (i) *Dedekind σ-complete* if, for any sequence $\{x_n\}$ of elements of M such that there is $x \in M$ with $x_n \leq x$ for each $n \geq 1$, then the element $\bigvee_n x_n$ exists in M, (ii) *Dedekind complete* if, for any system $\{x_t \colon t \in T\}$ of elements of M such that there is an element $x \in M$ with $x_t \leq x$ for each $t \in T$, then the element $\bigvee_{t\in T} x_t$ exists in M. Clearly, if M is Dedekind complete, then it is Dedekind σ-complete. In Theorem 3.7 below we show that every Dedekind σ-complete EMV-algebra is Archimedean, and every Archimedean EMV-algebra is a commutative EMV-algebra.

Lemma 3.6. *An element x of a pseudo EMV-algebra M is infinitesimal if and only if there is an idempotent $a \in \mathcal{I}(M)$ with $x \leq a$ such that $n.x \leq \rho_a(x)$ for each integer $n \geq 1$.*

In addition, if x is an infinitesimal, then for each $b \in \mathcal{I}(M)$ with $x \leq b$, we have $n.x \leq \lambda_b(x)$ and $n.x \leq \rho_b(x)$ for each $n \geq 1$.

Proof. Let $x \in \mathrm{Infinit}(M)$. There is an idempotent $a \in M$ such that $n.x \leq \lambda_a(x)$ for each $n \in \mathbb{N}$. Since $([0,a]; \oplus, \lambda_a, \rho_a, 0, a)$ is a pseudo MV-algebra, we can define a partial addition $+_a$ defined in the pseudo MV-algebra $[0,a]$ which is given as follows: For $x, y \in [0,a]$ by $x +_a y$ is defined in $[0,a]$ iff $x \leq \lambda_a(y)$, or equivalently, $y \leq \rho_a(y)$ iff $y \odot x = 0$; in such a case, $x +_a y = x \oplus y$.

By induction, we can show $nx := x +_a \cdots +_a x$ exists in $[0,a]$ for each integer $n \in \mathbb{N}$, and $n.x = nx$. Hence, $n.x \leq \lambda_a(x)$ for each $n \geq 1$ iff $n.x \leq \rho_a(x)$ for each $n \geq 1$.

Now, assume x is an infinitesimal and let b be an idempotent with $x \leq b$. By definition, there is an idempotent $a \in \mathcal{I}(M)$ with $x \leq a$ such that $n.x \leq \lambda_a(x)$ for each $n \geq 1$. Let $c \in \mathcal{I}(M)$ such that $a, b \leq c$. For each integer $n \geq 1$, by Proposition

3.3, $n.x \le \lambda_a(x) \le \lambda_c(x)$ and so $n.x = n.x \wedge b \le \lambda_c(x) \wedge b = \lambda_b(x)$. Therefore, $n.x \le \lambda_b(x)$ as we claimed.

Similarly, we have $n.x \le \rho_b(x)$ for each $n \ge 1$. ☐

Theorem 3.7. *Any Archimedean pseudo EMV-algebra $(M; \oplus, \vee, \wedge, 0)$ is a commutative EMV-algebra. In addition, every Dedekind σ-complete EMV-algebra is an Archimedean and commutative EMV-algebra.*

Proof. Since $\mathrm{Infinit}(M) = \{0\}$, then $\mathrm{Infinit}([0,a]) = \{0\}$ for each pseudo MV-algebra $[0,a]$, $a \in \mathcal{I}(M)$. Then each pseudo MV-algebra $[0,a]$ is an Archimedean pseudo MV-algebra and by [12, Thm 4.2], $[0,a]$ is an MV-algebra, so that $\oplus$ is commutative on $[0,a]$ for each $a \in \mathcal{I}(M)$. Therefore, $(M; \oplus, 0)$ is a commutative monoid and M is an EMV-algebra.

Now, let M be a Dedekind σ-complete pseudo EMV-algebra and let $x \in M$ be an infinitesimal. That is, there is an idempotent $a \in \mathcal{I}(M)$ with $x \le a$ such that $x_n := n.x \le \lambda_a(x)$ for each $n \ge 1$. Then $x_0 := \bigvee_{n=1}^{\infty} x_n$ exists in M. Using [26, Prop 1.21], we have $x_0 \oplus x = (\bigvee_{n=1}^{\infty} x_n) \oplus x = \bigvee_{n=1}^{\infty}(x_n \oplus x) = \bigvee_{n=1}^{\infty} x_{n+1} = x_0$. Since also $x_0 \le \lambda_a(x)$, we have $x_0 +_a x$ exists in M which yields $x_0 +_a x = x_0 \oplus x = x_0$ and, consequently, $x = 0$. ☐

4 Ideals, Homomorphisms, Congruences, and the q-Variety of Pseudo EMV-algebras

In the section, we continue with study of basic properties of pseudo EMV-algebras. We introduce ideals, normal ideals, homomorphisms, congruences and we show that the class of pseudo EMV-algebras is not a variety and rather a special class very close to a variety.

Let M be a pseudo EMV-algebra. A non-empty subset I of M is said to be an *ideal* of M if (i) $x, y \in I$ implies $x \oplus y \in I$, and (ii) if $x \le y \in I$, then $x \in I$. An ideal I of M is said to be (i) *maximal* if I is a proper subset of M and if J is any proper ideal of M containing I, then $I = J$, (ii) *normal* if $x \oplus I = I \oplus x$ for each $x \in M$, where $x \oplus I = \{x \oplus y \colon y \in I\}$ and $I \oplus x = \{z \oplus x \colon z \in I\}$. We denote by $Id(M)$, $NId(N)$, $MaxId(M)$ and $NMaxId(M)$ the set of all ideals of M, normal ideals of M, maximal ideals of M, and maximal ideals of M which are normal, respectively. We note that if a pseudo EMV-algebra M has a top element, then it can happen that M has no maximal ideal that is normal, see [11].

Proposition 4.1. *An ideal I of a pseudo EMV-algebra M is normal if and only if, for each $x, y \in M$,*

$$y \odot \lambda_a(x) \in I \ \Leftrightarrow \rho_a(x) \odot y \in I, \tag{4.1}$$

where $a \in \mathcal{I}(M)$ is any idempotent such that $x, y \leq a$.

Proof. Let I be a normal ideal of M and let $u_1 := y \odot \lambda_a(x) \in I$ for $a \in \mathcal{I}(M)$ with $x \leq a$. Then $x \vee y = (y \odot \lambda_a(x)) \oplus x = u_1 \oplus x = x \oplus (\rho_a(x) \odot y)$. The normality of I implies that there is $u_2 \in I$ such that $x \vee y = x \oplus u_2$. Then $\rho_a(x) \odot y \leq \rho_a(x) \odot (x \vee y) = \rho_a(x) \odot (x \oplus u_2) = \rho_a(x) \wedge u_2 \leq u_2$ which yields $\rho_a(x) \odot y \in I$. In a similar way we can show that if $\rho_a(x) \odot y \in I$, then $y \odot \lambda_a(x) \in I$.

Conversely, let (4.1) hold. Choose $x \in M$, $a \in \mathcal{I}(M)$ with $x \leq a$, and $v_1 \in I$. If we put $y = v_1 \oplus x$, then $x \leq y$. Therefore, $(y \odot \lambda_a(x)) \oplus x = y \vee x = y = x \oplus (\rho_a(x) \odot y)$. Hence, $v_1 \oplus x = y = (y \odot \lambda_a(x)) \oplus x = x \oplus (\rho_a(x) \odot y)$. If $y \odot \lambda_a(x) \in I$, then by (4.1), $v_2 = \rho_a(x) \odot y \in I$ which yields $v_1 \oplus x = x \oplus v_2$, i.e. $I \oplus x \subseteq x \oplus I$. The reverse inclusion can be proved in a similar way. $\square$

Short notes on the latter proposition. Let M be a pseudo EMV-algebra and $x, y \in M$.

(1) Let $a, b \in \mathcal{I}(M)$ such that $x, y \leq a \leq b$. Then

$$y \odot \lambda_a(x) = y \odot (\lambda_b(x) \wedge a) = (y \odot \lambda_b(x)) \wedge (y \odot a) = (y \odot \lambda_b(x)) \wedge y = y \odot \lambda_b(x).$$

(2) Let $a, b \in \mathcal{I}(M)$ such that $x, y \leq a, b$. There exists $c \in \mathcal{I}(M)$ such that $a, b \leq c$. By (1), $y \odot \lambda_a(x) = y \odot \lambda_c(x) = y \odot \lambda_b(x)$.

(3) Similarly to (2), we can show that if $a, b \in \mathcal{I}(M)$ such that $x, y \leq a, b$, then $\rho_a(x) \odot y = \rho_b(x) \odot y$.

From (2) and (3) it follows that the result of Proposition 4.1 is true if it holds for an element $a \in \mathcal{I}(M)$ greater than x and y. In other words, I is a normal ideal of M iff, for all $x, y \in M$, there exist $a, b \in \mathcal{I}(M)$ such that $x \leq a$ and $y \leq b$ and

$$y \odot \lambda_a(x) \in I \Leftrightarrow \rho_b(x) \odot y \in I.$$

Let M_1 and M_2 be pseudo EMV-algebras. A mapping $f : M_1 \to M_2$ is said to be a *homomorphism* of pseudo EMV-algebras (pEMV-homomorphism, in short) if f preserves $\vee, \wedge, \oplus, 0$ and, for each idempotent $a \in \mathcal{I}(M)$ and for each $x \in [0, a]$, $f(\lambda_a(x)) = \lambda_{f(a)}(f(x))$ and $f(\rho_a(x)) = \rho_{f(a)}(f(x))$.

Proposition 4.2. *Let $f : M_1 \to M_2$ be a pEMV-homomorphism. Then*

(i) $f(0) = 0$.

(ii) *$f(x) \leq f(y)$ whenever $x \leq y$; in particular, $f(x) \leq f(a)$ for each $x \in [0, a]$, $a \in \mathcal{I}(M)$.*

(iii) $f(x \odot y) = f(x) \odot f(y)$.

(iv) *Let* $\mathrm{Ker}(f) := \{x \in M_1 \colon f(x) = 0\}$*. Then* $\mathrm{Ker}(f)$ *is a normal ideal of* M_1*.*

Proof. (i) It is trivial.

(ii) Let $x \leq y$. Since f preserves $\wedge$, we have $f(x) = f(x \wedge y) = f(x) \wedge f(y)$, so that $f(x) \leq f(y)$.

(iii) Let $x, y \in M$ and let $a \in \mathcal{I}(M)$ be such that $x, y \leq a$. Put $z = \lambda_a(y) \oplus \lambda_a(x)$ and check

$$\begin{aligned}
f(x \odot y) &= f(\rho_a(\lambda_a(y) \oplus \lambda_a(x))) = f(\rho_a(z)) \\
&= \rho_{f(a)}(f(z)) = \rho_{f(a)}(f(\lambda_a(y)) \oplus f(\lambda_a(x))) \\
&= \rho_{f(a)}(\lambda_{f(a)}(f(y)) \oplus \lambda_{f(a)}(f(x))) \\
&= f(x) \odot f(y).
\end{aligned}$$

(iv) Due to (ii), we have that $\mathrm{Ker}(f)$ is an ideal of M_1. Now let $x \in M$ and $y \in \mathrm{Ker}(f)$. Using (3.5), we get $x \oplus y = ((x \oplus y) \odot \lambda_a(x)) \oplus x$. So that $f((x \oplus y) \odot \lambda_a(x)) = (f(x) \oplus f(y)) \odot \lambda_{f(a)}(f(x)) = f(x) \odot \lambda_{f(a)}(f(x)) = 0$, see (3.6). That is, $x \oplus \mathrm{Ker}(f) \subseteq \mathrm{Ker}(f) \oplus x$. In a similar way, we can prove the reverse inclusion. $\square$

A subset N of a pseudo EMV-algebra M is a *subalgebra* (or a pEMV-subalgebra) of M if (i) N is closed under $\vee, \wedge, \oplus$ and 0, (ii) for each $x \in N$, there is $a \in \mathcal{I}(N)$ such that $x \leq a$, and (iii) for each $a \in \mathcal{I}(N)$, the set $[0, a]_N := [0, a] \cap N$ is a pseudo MV-subalgebra of the pseudo MV-algebra $([0, a]; \oplus, \lambda_a, \rho_a, 0, a)$. The latter condition means, for $a \in \mathcal{I}(N)$ and for all $x \in [0, a]_N$,

$$\begin{aligned}
\min\{z \in [0, a]_N \colon z \oplus x = a\} &= \lambda_a(x) = \min\{z \in [0, a] \colon z \oplus x = a\}, \\
\min\{z \in [0, a]_N \colon x \oplus z = a\} &= \rho_a(x) = \min\{z \in [0, a] \colon x \oplus z = a\}.
\end{aligned}$$

Lemma 4.3. *Let* M_1 *and* M_2 *be two pseudo EMV-algebras and* $f : M_1 \to M_2$ *be a homomorphism of pseudo EMV-algebras. If* A *is a pseudo EMV-subalgebra of* M_1*, then* $f(A)$ *is a pseudo EMV-subalgebra of* M_2*.*

Proof. Clearly, $f(A)$ is closed under $\oplus, \vee, \wedge$ and 0. Now we have to show that, for each $b \in f(A) \cap \mathcal{I}(M_2)$, the set $f(A) \cap [0, b]$ is a subalgebra of the pseudo MV-algebra $[0, b]$. By definition, since A is a pseudo EMV-subalgebra of M_1, then for each $y \in f(A)$, there is an element $b \in f(A) \cap \mathcal{I}(M_2)$ such that $y \leq b$. We only need to show that $f(A) \cap [0, b]$ is closed under λ_b and ρ_b. Put $y \in f(A) \cap [0, b]$. Then there exist $a, x \in A$ such that $f(a) = b$ and $f(x) = y$. Let $u \in \mathcal{I}(M_1) \cap A$ such that $a, x \leq u$. Then $b, y \in [0, f(u)]$ and so

$$\begin{aligned}
\lambda_b(y) &= \lambda_{f(u)}(y) \wedge b = \lambda_{f(u)}(f(x)) \wedge f(a) = f(\lambda_u(x)) \wedge f(a) = f(\lambda_u(x) \wedge a), \\
\rho_b(y) &= \rho_{f(u)}(y) \wedge b = \rho_{f(u)}(f(x)) \wedge f(a) = f(\rho_u(x)) \wedge f(a) = f(\rho_u(x) \wedge a).
\end{aligned}$$

Since A is a subalgebra of M_1, then $[0,u]\cap A$ is a subalgebra of the pseudo MV-algebra $[0,u]$ and so $\lambda_u(x)\wedge a\in A$, which implies that $\lambda_b(y)\in f(A)$. Clearly, $\lambda_b(y)\in[0,b]$. Therefore, $\lambda_b(y)\in[0,b]\cap f(A)$. In the same way, we have $\rho_b(y)\in[0,b]\cap f(A)$. That is, $f(A)$ is a pseudo EMV-subalgebra of M_2. □

Unfortunately, the class of pseudo EMV-algebras is neither a variety nor a quasivariety since even the class of EMV-algebras is not such a structure, because it is not closed under forming subalgebras with respect to the original operations $\vee,\wedge,\oplus$, and 0, see [18, Thm 3.11]. Indeed, let $M=\Gamma(\mathbb{Z}\overrightarrow{\times}\mathbb{Z},(1,0))$ (it is also known as the Chang MV-algebra). It defines an EMV-algebra with top element having a unique maximal ideal I, namely $I=\{(0,n)\colon n\geq 0\}$. The set I is closed under $\vee,\wedge,\oplus,0$, so it is a subalgebra of M, but I is not an EMV-subalgebra of the EMV-algebra M because it does not have enough idempotent elements.

Therefore, similarly as in [18, Thm 3.11], instead of the classical operators H,S,P, we define on the class of pseudo EMV-algebras new operators, qH,qS,qP, mapping classes of pseudo EMV-algebras to classes of pseudo EMV-algebras, which are analogues of H,S,P, as follows: Let $\mathcal{V}$ be a class of pseudo EMV-algebras and M be a pseudo EMV-algebra:

qH: $M\in qH(\mathcal{V})$ if there are a pseudo EMV-algebra $N\in\mathcal{V}$ and a surjective pEMV-homomorphism $h:N\to M$;

qS: $M\in qS(\mathcal{V})$ if there is $N\in\mathcal{V}$ such that M is a pEMV-subalgebra of N;

qP: $M\in qP(\mathcal{V})$ if $M=\prod_t M_t$, where $\{M_t\}$ is a system of pseudo EMV-algebras of $\mathcal{V}$,

and the class $\mathcal{V}$ of pseudo EMV-algebras is said to be a *q-variety* of pseudo EMV-algebras if it is closed under qH, qS, and qP operators. In the same way we define a *q-subvariety* of pseudo EMV-algebras.

We note that if $\mathcal{K}$ is a family of pseudo EMV-algebras, then there is the least q-subvariety $\mathbb{V}_0^q(\mathcal{K})$ of pseudo EMV-algebras containing $\mathcal{K}$. Using the same ideas as those used in the proof of the Tarski theorem, [1, Thm 9.5], we can show that $\mathbb{V}_0^q(\mathcal{K})=qHqSqP(\mathcal{K})$.

Theorem 4.4. *The class $\mathbb{PEMV}$ of pseudo EMV-algebras is a q-variety of pseudo EMV-algebras.*

Proof. Clearly the direct product of pseudo EMV-algebras is a pseudo EMV-algebra. The family $\mathbb{PEMV}$ is closed under subalgebras and by Lemma 4.3, homomorphic images are pseudo EMV-subalgebras. Hence, $\mathbb{PEMV}$ is closed under qH,qS,qP, so it is a q-variety of pseudo EMV-algebras. □

An equivalence relation θ on a pseudo EMV-algebra $(M;\oplus,\vee,\wedge,0)$ is called a *congruence relation* or simply a *pEMV-congruence* if it satisfies the following conditions:

(i) θ is compatible with $\vee$, $\wedge$ and $\oplus$;

(ii) for all $b\in\mathcal{I}(M)$, $\theta\cap([0,b]\times[0,b])$ is a congruence on the pseudo MV-algebra $([0,b];\oplus,\lambda_b,\rho_b,0,b)$.

In the next result we establish that to prove that an equivalence is a congruence it is enough to verify only special cases.

Proposition 4.5. *An equivalence relation θ on a pseudo EMV-algebra M is a pEMV-congruence if and only if it is compatible with $\vee$, $\wedge$ and $\oplus$, and for all $(x,y)\in\theta$, there exists $b\in\mathcal{I}(M)$ such that $x,y\le b$ and $(\lambda_b(x),\lambda_b(y))\in\theta$.*

Proof. For the sufficiency, assume that the condition holds and let $b\in\mathcal{I}(M)$ be an arbitrary idempotent. We show that $\theta\cap([0,b]\times[0,b])$ is a congruence on the pseudo MV-algebra $([0,b];\oplus,\lambda_b,\rho_b,0,b)$. Let $x,y\in[0,b]$ such that $(x,y)\in\theta$. Then by the assumption, there exists $u\in\mathcal{I}(M)$ such that $x,y\in[0,u]$ and $(\lambda_u(x),\lambda_u(y))\in\theta$. Suppose that $v\in\mathcal{I}(M)$ such that $u,b\le v$. By Proposition 3.3(iii), $\lambda_v(x)=\lambda_u(x)\oplus\lambda_v(u)$, $\lambda_v(y)=\lambda_u(y)\oplus\lambda_v(u)$ and $\rho_v(x)=\rho_u(x)\oplus\rho_v(u)$, $\rho_v(y)=\rho_u(y)\oplus\rho_v(u)$. Since $(\lambda_u(x),\lambda_u(y))\in\theta$, $(\rho_u(x),\rho_u(y))\in\theta$ and θ is compatible with $\oplus$, we get that $(\lambda_v(x),\lambda_v(y)),(\rho_v(x),\rho_v(y))\in\theta$. It follows that $(\lambda_v(x)\wedge b,\lambda_v(y)\wedge b),(\rho_v(x)\wedge b,\rho_v(y)\wedge b)\in\theta$ and so by Proposition 3.3(ii), $(\lambda_b(x),\lambda_b(y)),(\rho_b(x),\rho_b(y))\in\theta$.

The proof of the converse is clear. $\square$

Proposition 4.6. *Let θ be a pEMV-congruence on a pseudo EMV-algebra M, and let $M/\theta=\{[x]\colon x\in M\}$ be the quotient class induced by θ, where $x/\theta=\{y\in M\colon (x,y)\in\theta\}$. Then M/θ can be organized into a pseudo EMV-algebra, and the mapping $x\mapsto x/\theta$, $x\in M$, is a pEMV-homomorphism.*

Proof. Consider the induced operations $\vee$, $\wedge$ and $\oplus$ on M/θ defined by

$$x/\theta\vee x/\theta=(x\vee y)/\theta,\quad x/\theta\wedge y/\theta=(x\wedge y)/\theta,\quad x/\theta\oplus x/\theta=(x\oplus y)/\theta,\quad x,y\in M.$$

Clearly, $(M/\theta;\vee,\wedge,[0])$ is a distributive lattice and $(M/\theta;\oplus,[0])$ is a commutative monoid with neutral element $[0]$. The partial order on M/θ is given by $x/\theta\le x/\theta$ iff $x/\theta\wedge y/\theta=x/\theta$. Let $u_1/\theta,u_2/\theta\in M/\theta$ and $x/\theta\le y/\theta$. Then for $x_0=x\wedge y$ we have $x_0/\theta=x/\theta$. Therefore, $u_1\oplus x_0\oplus u_2\le u_1\oplus y\oplus u_2$, so that $(u_1/\theta\oplus x/\theta\oplus u_2/\theta)\wedge(u_1/\theta\oplus x/\theta\oplus u_2/\theta)=(u_1/\theta\oplus x_0/\theta\oplus u_2/\theta)\wedge(u_1/\theta\oplus y/\theta\oplus u_2/\theta)=u_1/\theta\oplus x_0/\theta\oplus u_2/\theta=u_1/\theta\oplus x/\theta\oplus u_2/\theta$. This proves that M/θ is an ordered monoid.

Now, we show that for each $x/\theta \in M/\theta$, there exists an idempotent element $a/\theta \in M/\theta$ such that $x/\theta \leq a/\theta$ and $([0/\theta, a/\theta]; \oplus, \lambda_{a/\theta}, \rho_{a/\theta}, 0/\theta, a/\theta)$ is a pseudo MV-algebra. Take $x/\theta \in M/\theta$. There is $a \in \mathcal{I}(M)$ such that $x \leq a$. Clearly, $x/\theta \leq a/\theta$ and a/θ is idempotent. Also, $([0, a]; \oplus, \lambda_a, \rho_a, 0, a)$ is a pseudo MV-algebra and $\theta_a := \theta \cap ([0, a] \times [0, a])$ is a congruence relation on $[0, a]$, so $[0, a]/\theta_a$ (with the quotient operations) is a pseudo MV-algebra, where

$$\lambda_{a/\theta_a}(x/\theta_a) = \min\{z/\theta_a \colon z \in [0, a], z/\theta_a \oplus x/\theta_a = a/\theta_a\} = \lambda_a(x)/\theta_a, \tag{4.2}$$

and

$$\rho_{a/\theta_a}(x/\theta_a) = \min\{z/\theta_a \colon x \in [0, a], x/\theta_a \oplus z/\theta_a = a/\theta_a\} = \lambda_a(x)/\theta_a, \tag{4.3}$$

for each $x/\theta_a \in [0, a]/\theta_a$. First, we show that for all $x/\theta \in [0/\theta, a/\theta]$, $\lambda_a(x)/\theta$ is the least element of the set $\{z/\theta \in [0/\theta, a/\theta] \colon z/\theta \oplus x/\theta = a/\theta\}$. For each $x/\theta \in [0/\theta, a/\theta]$, we have $x/\theta = x/\theta \wedge a/\theta = (x \wedge a)/\theta$ and $x \wedge a \in [0, a]$. So, we can assume that $x \in [0, a]$. If $z/\theta \in [0/\theta, a/\theta]$ such that $z/\theta \oplus x/\theta = a/\theta$, then $(z \oplus x, a) \in \theta$ and $x, z, a \in [0, a]$, thus $(z \oplus x, a) \in \theta_a$, that is $z/\theta_a \oplus x/\theta_a = a/\theta_a$ (which implies that $z/\theta \oplus x/\theta = a/\theta$). Hence by (4.2), $y/\theta_a \geq \lambda_a(x)/\theta_a$ and so $y/\theta \geq \lambda_a(x)/\theta$. Also, $\lambda_a(x)/\theta \oplus x/\theta = a/\theta$. Thus $\lambda_{a/\theta}(x/\theta)$ exists and is equal to $\lambda_a(x)/\theta$. In a similar way, we can prove that $\rho_{a/\theta}(x/\theta)$ exists and is equal to $\rho_a(x)/\theta$.

Now, it is straightforward to check that $\lambda_{a/\theta}$ and $\rho_{a/\theta}$ satisfy conditions in order to be the quotient algebra $([0/\theta, a/\theta]; \oplus, \lambda_{a/\theta}, \rho_{a/\theta}, 0/\theta, a/\theta)$ a pseudo MV-algebra. Applying Proposition 3.5, $(M/\theta; \vee, \wedge, \oplus, 0/\theta)$ is a pseudo EMV-algebra, and the mapping $x \mapsto x/\theta$ is a pEMV-homomorphism from M onto M/θ. □

Theorem 4.7. *If θ is a congruence on a pseudo EMV-algebra $(M; \vee, \wedge, \oplus, 0)$, then $I_\theta := 0/\theta$ is a normal ideal of M.*

Conversely, let I be a normal ideal of a pseudo EMV-algebra $(M; \vee, \wedge, \oplus, 0)$. Then the relation θ_I defined by

$$(x, y) \in \theta_I \Longleftrightarrow \Big(\exists\, b \in \mathcal{I}(M) : x, y \leq b \quad \& \quad \lambda_b(x \oplus \rho_b(y)), \rho_b(\lambda_b(y) \oplus x) \in I\Big) \tag{4.4}$$

is a congruence on M. In addition, the mapping $I \mapsto \theta_I$ is a bijection from the set $NId(M)$ of normal ideals of M onto the set of congruences on M.

Proof. Let θ be a congruence on a pseudo EMV-algebra $(M; \vee, \wedge, \oplus, 0)$. Then it can be easily shown that $I_\theta := [0] = 0/\theta$ is an ideal of M. To show that I_θ is a normal ideal, we use (3.5). Thus, let $x \in M$ and $y \in [0]$. There is an idempotent $b \in \mathcal{I}(M)$ such that $x, y \leq b$. Then $x \oplus y = ((x \oplus y) \odot \lambda_b(x)) \oplus x$, so that $(x \oplus y) \odot \lambda_b(x) \in [0]$ which shows $x \oplus [0] \subseteq [0] \oplus x$. In a similar way we prove the opposite inclusion.

Now, let I be a normal ideal of M. Then for each $b \in \mathcal{I}(M)$, the set $I_b := I \cap [0,b]$ is a normal ideal of the pseudo MV-algebra $([0,b]; \oplus, \lambda_b, \rho_b, 0, b)$. Define a relation θ_I on M by (4.4).

By [26, Thm 3.8], the normal ideal I_b defines a congruence θ_{I_b} on the pseudo MV-algebra $[0,b]$ induced by I_b which is given by $(x,y) \in \theta_{I_b}$ iff $x, y \le b \in \mathcal{I}(M)$ and $\lambda_b(x \oplus \rho_b(y)), \rho_b(\lambda_b(y) \oplus x) \in I$. Then $\theta_I = \bigcup\{\theta_{I_b} \colon b \in \mathcal{I}(M)\}$. Since if $x, y \le b \le c \in \mathcal{I}(M)$, then $(x,y) \in \theta_{I_c}$, we see that θ_I is a congruence on the whole pseudo EMV-algebra M, see also Proposition 4.5.

Using [26, Cor 3.1], we see that the mapping $I \mapsto \theta_I$ is a bijection between $NId(M)$ and the set of congruences on M. $\square$

If I is a normal ideal of M, then by M/I we will denote the quotient EMV-algebra M/θ_I, where θ_I is the unique congruence on M corresponding to the normal ideal I by Theorem 4.7. For any $x \in M$ we will denote the co-set corresponding x by $[x]_I$ or x/I.

Proposition 4.8. *Let I be a normal ideal of a pseudo EMV-algebra M and let $z \in M$. Then the ideal $I_0(I,z)$ of M generated by I and z is the set*

$$\begin{aligned} I_0(I,z) &= \{x \in M \colon x \le h \oplus n.z \text{ for some } h \in I,\ n \in \mathbb{N}\} \\ &= \{x \in M \colon x \le m.z \oplus k \text{ for some } k \in I,\ m \in \mathbb{N}\}. \end{aligned}$$

Proof. Clearly, $I \subseteq I_0(I,z)$ and $z \in I_0(I,z)$. Let $x, y \in I_0(I,z)$. Then $x \le h_1 \oplus n.z$ and $y \le h_2 \oplus m.z$. We have $x \oplus y \le h_1 \oplus n.z \oplus h_2 \oplus m.z = h_1 \oplus h_2' \oplus n.z \oplus m.z$ when we have used that I is normal and h_2 is an appropriate element in I. Clearly, $x \le y \in I_0(I,z)$ implies $x \le I_0(I,z)$, so that $I_0(I,z)$ is an ideal of M in question.

In the same way we prove the second equality. $\square$

A dual notion to an ideal of a pseudo EMV-algebra is a filter. We say that a non-empty subset F of a pseudo EMV-algebra is a *filter* if (i) $x, y \in F$ implies $x \odot y \in F$, and (ii) if $x \le y \in M$, $x \in F$, then $y \in F$. A filter F is *normal* if $x \odot F = F \odot x$ for each $x \in M$; here $x \odot F := \{x \odot y \colon y \in F\}$ and $F \odot x = \{z \odot y \colon z \in F\}$. The set M is filter of M. A proper filter F is *maximal* if $F \subseteq G$, where G is a proper filter of M, then $F = G$. Let $Fil(M)$, $NFil(M)$, $MFil(M)$, $NMFil(M)$ be the set of all filters, normal filters, maximal filters, and maximal filters which are also normal, respectively.

If a is an idempotent of M, then by Proposition 3.3(i) and Proposition 3.4, we have $a \odot a = a$, so that $F(a) = \{x \in M \colon x \ge a\}$ is a filter of M. If M is non-trivial, i.e. $M \ne \{0\}$, then using Zorn's Lemma, we can see that $MFil(M) \ne \emptyset$.

The following result is dual to Proposition 4.1.

Proposition 4.9. *A filter F of a pseudo EMV-algebra M is normal if and only if, for each $x, y \in M$,*

$$\lambda_a(x) \oplus y \in F \quad \Leftrightarrow \quad y \oplus \rho_a(x) \in F, \tag{4.5}$$

where $x \leq a \in \mathcal{I}(M)$.

Proof. First assume b is an idempotent such that $x, y, a \leq b$. Then we have $x \wedge y = x \odot (\lambda_b(x) \oplus y) = x \odot ([\lambda_a(x) \vee \lambda_b(a)] \oplus y)) = x \odot ([(\lambda_a(x) \oplus y) \vee (\lambda_b(a) \oplus y]) = (x \odot (\lambda_a(x) \oplus y)) \vee (x \odot (\lambda_b(a) \oplus y))$. Since $\lambda_b(a)$ is an idempotent, we have $x \odot (\lambda_b(a) \oplus y) = x \odot (\lambda_b(a) \vee y) = (x \odot \lambda_b(a)) \vee (x \odot y) = 0 \vee (x \odot y) = x \odot y$. So that $x \odot (\lambda_b(x) \oplus y) = (x \odot (\lambda_a(x) \oplus y)) \vee (x \odot y) = x \odot (\lambda_a(x) \oplus y)$. Finally, we have

$$x \odot (\lambda_b(x) \oplus y) = x \wedge y = x \odot (\lambda_a(x) \oplus y).$$

In the same way, we have

$$(y \oplus \rho_b(x)) \odot x = x \wedge y = (y \oplus \rho_a(x)) \odot x.$$

Now, let F be a normal filter and $x, y \in F$. Put $v_1 = \lambda_a(x) \oplus y \in F$ for $x \leq a \in \mathcal{I}(M)$. By the just proved equalities, we have $x \wedge y = x \odot (\lambda_a(x) \oplus y) = x \odot v_1$. Normality of F implies there is $v_2 \in F$ such that $x \wedge y = v_2 \odot x$. Then $y \oplus \rho_a(x) \geq (x \wedge y) \oplus \rho_a(x) = (v_2 \odot x) \oplus \rho_a(x) = v_2 \vee \rho_a(x) \geq v_2$ which yields $y \oplus \rho_a(x) \in F$. In the same way, we can establish that $y \oplus \rho_a(x) \in F$ implies $\lambda_a(x) \oplus y \in F$.

Conversely, let (4.5) hold. Choose $x \in M$, $a \in \mathcal{I}(M)$ with $x \leq a$, and $u_1 \in F$. If we put $y = u_1 \odot x$, then $x \geq y$. Therefore, $x \odot (\lambda_a(x) \oplus y) = x \wedge y = y = (y \oplus \rho_a(x)) \odot x$. Hence, $u_1 \odot x = y = (y \oplus \rho_a(x)) \odot x$. If $\lambda_a(x) \oplus y \in F$, then by (4.5), $u_2 = y \oplus \rho_a(x) \in F$ which yields $u_1 \odot x = x \odot u_2$, i.e. $F \odot x \subseteq x \odot F$. The reverse inclusion can be proved in a similar way. □

Proposition 4.10. *Let F be a normal filter of a pseudo EMV-algebra M and let $z \in M$. Then the filter $F_0(F, z)$ of M generated by F and z is the set*

$$\begin{aligned} F_0(F,z) &= \{x \in M \colon x \geq h \odot z^n \text{ for some } f \in F,\ n \in \mathbb{N}\} \\ &= \{x \in M \colon x \geq z^m \odot g \text{ for some } g \in F,\ m \in \mathbb{N}\}. \end{aligned}$$

Proof. It is dual to the proof of Proposition 4.8. □

Proposition 4.11. *Let F be a proper normal filter of a pseudo EMV-algebra M. The following statements are equivalent:*

(i) *F is a maximal filter of M.*

(ii) *For each $x \notin F$, there are an idempotent $a \in \mathcal{I}(M)$ with $x \le a$ and an integer $n \in \mathbb{N}$ such that $\lambda_a(x^n) \in F$.*

(iii) *For each $x \notin F$, there are an idempotent $a \in \mathcal{I}(M)$ with $x \le a$ and an integer $n \in \mathbb{N}$ such that $\rho_a(x^n) \in F$.*

Proof. (i) $\Rightarrow$ (ii), (iii). Let $x \notin F$ and $x \le a \in \mathcal{I}(M)$. By Proposition 4.10, there are elements $f, g \in F$ and integer $n, m \ge 1$ such that $f \odot x^n = 0 = x^m \odot g$ which gives $f \le \rho_a(x^n)$ and $g \le \lambda_a(x^m)$, so that $\rho_a(x^n), \lambda_a(x^m) \in F$.

(ii), (iii) $\Rightarrow$ (i). Let $x \notin F$, $x \le a \in \mathcal{I}(M)$. By the assumption, there is an integer $n \ge 1$ such that $\lambda_a(x^n) \in F$. Since $0 \ge x^n \odot \lambda_a(x^n) \in F_0(F, x)$ saying that F is a maximal filter. In the same way, we prove the second implication. ☐

Proposition 4.12. *If I is a maximal ideal that is also normal of a pseudo EMV-algebra $(M; \vee, \wedge, \oplus, 0)$, then for each $b \in \mathcal{I}(M)$, $I_b := I \cap [0, b]$ is either equal to $[0, b]$ or I_b is a normal and maximal ideal of the pseudo MV-algebra $([0, b]; \oplus, \lambda_b, \rho_b, 0, b)$.*

In addition, if $B = \mathcal{I}(M)$, then $I \cap \mathcal{I}(M)$ is a maximal ideal of B.

Proof. Let $b \in \mathcal{I}(M)$. Then I_b is a normal ideal of the pseudo MV-algebra $[0, b]$; normality of I_b follows from the criterion (4.1). If $I_b = [0, b]$, we are ready. Thus let $I_b \ne [0, b]$ and choose $x \in [0, b] \setminus I_b$. Then $I_0(I, x) = M$. By Proposition 4.8, for each $z \in [0, b]$, there exist $n \in \mathbb{N}$ and $h \in I$ such that $z \le h \oplus n.x$ and so $z = z \wedge (h \oplus n.x)$. Since $([0, b]; \oplus, \lambda_b, \rho_b, 0, b)$ is a pseudo MV-algebra, then by [26, Prop. 1.17(1)], we have

$$z = z \wedge b \le (h \oplus n.x) \wedge b \le (h \wedge b) \oplus (n.x \wedge b) \le (h \wedge b) \oplus n.x.$$

Hence, z belongs to the ideal of $[0, b]$ generated by $I_b \cup \{x\}$. Therefore, I_b is a maximal ideal of the MV-algebra $([0, b]; \oplus, \lambda_b, \rho_b, 0, b)$ that is also normal.

Since I is a proper subset of M, there are $x \in M \setminus I$ and $a \in B$ such that $x \le a$, whence, $a \notin I \cap \mathcal{I}(M)$, which says that $I \cap \mathcal{I}(M)$ is a proper ideal of B. Now let $b \in B \setminus I \cap \mathcal{I}(M)$. Then $b \notin I$. Hence, for each idempotent $a \in B$, there are an element $c \in I$ and an integer n such that $a \le c \oplus n.b = c \vee b$. Then $a = a \wedge (c \vee b) = (a \wedge c) \vee (a \wedge b)$ so that, $\lambda_a(a \wedge b) \le a \wedge c \in I$. But according to Proposition 3.3(i), $\lambda_a(a \wedge b)$ is an idempotent of M, and thus $\lambda_a(a \wedge b) \in I \cap \mathcal{I}(M)$ which yields, $a \le \lambda_a(a \wedge b) \vee b$, and finally, $I \cap B$ is a maximal ideal of B. ☐

Proposition 4.13. *Let I be a normal ideal of a pseudo EMV-algebra M. The following statements are equivalent:*

(i) *I is a maximal normal ideal of M.*

(ii) *For each $x \notin I$ and for each idempotent $a \in \mathcal{I}(M)$ with $x \le a$, there is an integer $n \in \mathbb{N}$ such that $\lambda_a(n.x) \in I$.*

(iii) *For each $x \notin I$ and for each idempotent $a \in \mathcal{I}(M)$ with $x \le a$, there is an integer $n \in \mathbb{N}$ such that $\rho_a(n.x) \in I$.*

Proof. (i) $\Rightarrow$ (ii), (iii). Let $x \notin I$. Let $a \in \mathcal{I}(M)$ be an idempotent such that $x \le a$. Put $I_a = [0,a] \cap I$. Then $x, a \in [0,a] \setminus I_a$. By Proposition 4.12, I_a is a maximal and normal ideal of the pseudo MV-algebra $[0,a]$. Applying [26, Prop 3.5], there are integers $n, m \ge 1$ such that $\lambda_a(n.x), \rho_a(m.x) \in I_a \subseteq I$.

(ii) $\Rightarrow$ (i). Let $x \in M \setminus I$ and let $a \in \mathcal{I}(M)$ be an arbitrary idempotent such that $x \le a$. We assert that the ideal $I_0(I,x)$ generated by $I \cup \{x\}$ is equal to M. Indeed, for x and $a \ge x$, there is an integer $n \ge 1$ such that $\lambda_a(n.x) \in I$. Then $a = \lambda_a(n.x) \oplus n.x$ which by Proposition 4.8 means that $a \in I_a(I,x)$. Since this is true for each $a \in \mathcal{I}(M)$ with $a \ge x$, then for each $a \in \mathcal{I}(M)$, we have $a \in I_0(I,x)$, so that $I_0(I,x) = M$.

(iii) $\Rightarrow$ (i). It follows the same steps as the proof of the latter implication. ☐

Proposition 4.14. *Let F be a filter of a proper pseudo EMV-algebra $(M; \vee, \wedge, \oplus, 0)$. Then the sets*

$$I_F^\lambda := \{x \in M \colon \exists\, a \in \mathcal{I}(M), x \le a, \lambda_a(x) \in F\}$$

and

$$I_F^\rho := \{x \in M \colon \exists\, a \in \mathcal{I}(M), x \le a, \rho_a(x) \in F\}$$

are ideals of M, and

$$I_F^\lambda = \{\rho_a(x) \colon x \in F, \exists\, a \in \mathcal{I}(M), x \le a\}, \quad I_F^\rho = \{\lambda_a(x) \colon x \in F, \exists\, a \in \mathcal{I}(M), x \le a\}.$$

In addition:

(i) *If F is a proper filter, $a \in \mathcal{I}(M) \cap F$, then $a \notin I_F^\lambda \cup I_F^\rho$, and I_F^λ and I_F^ρ are proper ideals of M.*

(ii) *If $a \in \mathcal{I}(M) \cap F$, then for each $b \in \mathcal{I}(M)$ with $a < b$, we have $\lambda_b(a) \in I_F^\lambda$ and $\rho_b(a) \in I_F^\rho$.*

(iii) *If F is a maximal filter, then for each $a \in \mathcal{I}(M)$, $a \notin I_F^\lambda$ implies $a \in F$. Moreover, for each $a \in \mathcal{I}(M)$, $a \notin I_F^\rho$ implies $a \in F$.*

Proof. Clearly, $I_F^\lambda \neq \emptyset$. Let $x, y \in M$ such that $x \in I_F^\lambda$ and $y \leq x$. Then there exists $a \in \mathcal{I}(M)$ such that $x \leq a$ and $\lambda_a(x) \in F$. Since $x, y \in [0, a]$, then $\lambda_a(x) \leq \lambda_a(y)$ and so by the assumption, $\lambda_a(y) \in F$. It follows that $y \in I_F^\lambda$.

Now, suppose that $x, y \in I_F^\lambda$. Then there exist $a, b \in \mathcal{I}(M)$ such that $x \leq a$ and $y \leq b$ and $\lambda_a(x) \in F$ and $\lambda_b(y) \in F$. Put $c \in \mathcal{I}(M)$ such that $a, b \leq c$. Then by Proposition 3.3, $\lambda_c(x), \lambda_c(y) \in F$ and so $\lambda_c(y) \odot \lambda_c(x) \in F$. Since $\lambda_c(x), \lambda_c(y) \leq c$, $\lambda_c(y) \odot \lambda_c(x) = \lambda_c(x \oplus y)$, hence $x \oplus y \in I_F^\lambda$. Therefore, I_F^λ is an ideal of M.

In the same way, we can prove that also I_F^ρ is an ideal of M.

Now let $x = \rho_a(w)$ for $w \in F$ and $x \leq a \in \mathcal{I}(M)$. Then $w = \lambda_a(x) \in F$ and $x \in I_F^\lambda$. Conversely, if $x \in I_F^\lambda$, then there exists $a \in \mathcal{I}(M)$ with $x \leq a$ such that $\lambda_a(x) \in F$. Put $w = \lambda_a(x)$, then $w \in F$ and $x = \rho_a(w) \in \{\rho_a(x)\colon x \in F, \exists\, a \in \mathcal{I}(M), x \leq a\}$.

In the same way we prove the second equality.

(i) Now, let $a \in \mathcal{I}(M) \cap F$. Suppose the converse, i.e. $a \in I_F^\lambda$. Then there is $b \in \mathcal{I}(M)$ with $a < b$ and $\lambda_b(a) \in F$ which gives $0 = a \odot \lambda_b(a) \in F$. Analogously, we have $a \notin I_F^\rho$.

(ii) It follows from definition of I_F^λ and I_F^ρ.

(iii) Let $a \in \mathcal{I}(M)$ such that $a \notin I_F^\lambda$. Then by definition, for all $b \in \mathcal{I}(M)$ with $a \leq b$, $\lambda_b(a) \notin F$. If $a \notin F$, then $F_0(F, a) = M$ (since F is maximal). Since a is an idempotent, it is easy to verify that $F_0(F, a) = \{x \in M\colon x \geq f \wedge a, f \in F\}$. In addition, there is $b \in \mathcal{I}(M)$ with $a < b$. Hence, there is $f \in F$ such that $0 = f \wedge a = f \odot a$ which gives $f \leq \lambda_b(a) \in F$, which is a contradiction. Hence, $a \in F$.

In the same way, we prove the second half of (iii). $\square$

Proposition 4.15. *Let M be a proper pseudo EMV-algebra.*

(i) *Let I be a maximal normal ideal of M. Then*

$$\forall\, a \in \mathcal{I}(M) \setminus I \Longrightarrow (\forall\, b \in \mathcal{I}(M),\ a < b)\ \lambda_b(a) \in I \tag{4.6}$$

(ii) *Let I be an ideal of M satisfying* (4.6), *then the sets*

$$F_I^\lambda := \{x \in M\colon \exists\, a \in \mathcal{I}(M) \setminus I, x \leq a, \lambda_a(x) \in I\},$$

$$F_I^\rho := \{x \in M\colon \exists\, a \in \mathcal{I}(M) \setminus I, x \leq a, \rho_a(x) \in I\}$$

are filters of M. In addition,

$$F_I^\lambda = \{\rho_a(x)\colon x \in I, \exists\, a \in \mathcal{I}(M) \setminus I, x < a\}$$
$$F_I^\rho = \{\lambda_a(x)\colon x \in I, \exists\, a \in \mathcal{I}(M) \setminus I, x < a\}.$$

(iii) *If I is a proper ideal of M, $a \in \mathcal{I}(M) \cap I$, then $a \notin F_I^\lambda \cup F_I^\rho$, and F_I^λ and F_I^ρ are proper filters of M.*

Proof. (i) Let $a \notin I$ and $a < b \in \mathcal{I}(M)$. Then $b \notin I$, so that by Proposition 4.8, there are an element $h \in I$ and an integer $n \geq 1$ such that $\lambda_b(a) \leq h \oplus n.a = h \oplus a = h \vee a$, see (3.7). Then

$$\lambda_b(a) = \lambda_b(a) \wedge (h \vee a) = (\lambda_b(a) \wedge h) \vee (\lambda_b(a) \wedge a) = (\lambda_a(x) \wedge h) \leq h,$$

which yields $\lambda_b(a) \in I$.

In the same way we establish $\rho_b(a) \in I$.

(ii) Let $x, y \in M$. If $y \geq x \in F_I^\lambda$, then there exists $a \in \mathcal{I}(M) \setminus I$ such that $x < a$ and $\lambda_a(x) \in I$. Let $b \in \mathcal{I}(M)$ such that $a, y < b$. Then $\lambda_b(y) \leq \lambda_b(x) = \lambda_a(x) \oplus \lambda_b(a)$. By the assumption, $\lambda_b(a) \in I$, so $\lambda_a(x) \oplus \lambda_b(a) \in I$, which implies that $\lambda_b(y) \in I$ and $y \in F_I^\lambda$.

Now, if $x, y \in F_I^\lambda$, then there exist $a, b \in \mathcal{I}(M) \setminus I$ such that $x < a$ and $y < b$ and $\lambda_a(x), \lambda_b(y) \in I$. Let $c \in \mathcal{I}(M)$ such that $a, b < c$. Then by the assumption, $\lambda_c(a), \lambda_c(b) \in I$ and hence by Proposition 3.3, we have $\lambda_c(x) = \lambda_a(x) \oplus \lambda_c(a) \in I$ and $\lambda_c(y) = \lambda_b(y) \oplus \lambda_c(b) \in I$. It follows that $\lambda_c(y) \oplus \lambda_c(x) \in I$. But $\lambda_a(x \odot y) = \lambda_a(y) \oplus \lambda_a(x) \in I$ which gives $x \odot y \in F_I^\lambda$.

We proceed with F_I^ρ in an analogous way; we have to take into account that $\lambda_b(a) = \rho_b(a)$ for idempotents $a, b \in \mathcal{I}(M)$ with $a < b$.

(iii) Let $a \in \mathcal{I}(M) \cap I$. Assume the converse, i.e. $a \in F_I^\lambda$. Then there is $b \in \mathcal{I}(M) \setminus I$ such that $\lambda_b(a) \in I$. Then $b = \lambda_b(a) \oplus a \in I$ which is a contradiction.

Using the same steps, we can prove that also $a \in F_I^\rho$. □

By a way, (4.6) is a special case of Proposition 4.13.

Using the Zorn Lemma, it is easy to show that every non-trivial pseudo MV-algebra admits a maximal ideal. However if a pseudo EMV-algebra M does not have a top element, we cannot apply the Zorn Lemma to show that M has at least one maximal ideal. The following result shows that also every proper pseudo EMV-algebra has at least one maximal ideal.

Theorem 4.16. *Every proper pseudo EMV-algebra M admits at least one maximal ideal. In addition, if F is a maximal filter of M, then $I_F^\lambda = I_F^\rho$ is a maximal ideal of M.*

Proof. Since M is proper, it means that $M \neq \{0\}$. Using the Zorn Lemma, we can easily show that M admits a maximal filter. Thus, let F be any maximal filter of M. Define I_F^λ and I_F^ρ by Proposition 4.14. Then by Proposition 4.14(ii),

$$I_F^\lambda = \{\rho_a(x) \colon x \in F, \exists\, a \in \mathcal{I}(M), x \leq a\}, \quad I_F^\rho = \{\lambda_a(x) \colon x \in F, \exists\, a \in \mathcal{I}(M), x \leq a\}.$$

By Proposition 4.14, I_F^λ and I_F^ρ are ideals of M. Since F is a proper filter of M, there is an idempotent $a \in \mathcal{I}(M)$ such that $a \in F$ which by Proposition 4.14(i) means $a \notin I_F^\lambda \cup I_F^\rho$, so that I_F^λ and I_F^ρ are proper ideals of M.

First, we show that $I_F^\lambda = I_F^\rho$. Let J be a proper ideal of M containing $I_F^\lambda \cap I_F^\rho$. For each $a \in \mathcal{I}(M)$, if $a \notin J$, then $a \notin I_F^\lambda$ or $a \notin I_F^\rho$. In either case, by Proposition 4.14(iii), $a \in F$. It follows that $\rho_b(a) = \lambda_b(a) \in I_F^\rho$ and $\lambda_b(a) = \rho_b(a) \in I_F^\lambda$ for all $b \in \mathcal{I}(M)$ with $b > a$, so that $\rho_b(a) = \lambda_b(a) \in I_F^\rho \cap I_F^\lambda \subseteq J$. Define F_J^λ and F_J^ρ, so by Proposition 4.15(ii),

$$F_J^\lambda := \{\rho_a(x) \in M \colon x \in J, \exists\, a \in \mathcal{I}(M) \setminus J, x \le a\},$$
$$F_J^\rho := \{\lambda_a(x) \in M \colon x \in J, \exists\, a \in \mathcal{I}(M) \setminus J, x \le a\}.$$

They are proper filters of M.

Let x be an arbitrary element of F. Since J is a proper ideal, then there is an idempotent element $v \in M$ which is not in J (otherwise, $J = M$). Put $w \in \mathcal{I}(M)$ such that $x, v < w$. Then $w \notin J$ and since $x = \lambda_w(\rho_w(x))$, we have $\rho_w(x) \in I_F^\lambda \subseteq J$, hence $x \in F_J^\lambda$, see Proposition 4.15(ii). Similarly, due to $x = \rho_w(\lambda_w(x))$, we have $x \in F_J^\rho$.

That is, $F \subseteq F_J^\lambda \cap F_J^\rho$. Since F is a maximal filter and J is a proper ideal, then F_J^λ is a proper filter of F, so that $F_J^\lambda \cap F_J^\rho = F$. The maximality of F implies

$$F_J^\lambda = F = F_J^\rho. \tag{4.7}$$

For the ideal J, condition (4.6) holds. Let $x \in J$. Then there is $a \in \mathcal{I}(M) \setminus J$ such that $x < a$ and $\rho_a(x) \in F_J^\lambda \subseteq F$. It follows that $x = \lambda_a(\rho_a(x)) \in I_F^\rho$, which shows that $J \subseteq I_F^\rho$. Due to $x = \rho_a(\lambda_a(x))$, in a similar way, we can show that $x \in I_F^\lambda$, so that $J \subseteq I_F^\lambda$. Hence, $J \subseteq I_F^\lambda \cap I_F^\rho \subseteq J$, that is $J = I_F^\lambda \cap I_F^\rho$. The latter equality holds for $J = I_F^\lambda$ as well as for $J = I_F^\rho$, which yields $I_F^\lambda = I_F^\lambda \cap I_F^\rho = I_F^\rho$.

Now we show that $I_F^\lambda = I_F^\rho$ is a maximal ideal of M. Let J be a proper ideal of M containing I_F^λ and let $x \in J$. Then there is $a \in \mathcal{I}(M) \setminus J$ such that $x < a$ and $\rho_a(x) \in F_J^\lambda = F$. It follows that $x = \lambda_a(\rho_a(x)) \in I_F^\rho = I_F^\lambda$, which shows that $J \subseteq I_F^\rho = I_F^\lambda \subseteq J$ and therefore, $I_F^\rho = J$. Whence, $I_F^\lambda = I_F^\rho$ is a maximal ideal of M. □

According to the latter theorem, if F is a maximal filter of a pseudo EMV-algebra, then $I_F^\lambda = I_F^\rho$ is a maximal ideal of M; we denote simply $I_F = I_F^\lambda$.

Theorem 4.17. *Let M be a proper pseudo EMV-algebra. If I is a maximal ideal of M, then $F_I^\lambda = F_I^\rho$ and $F_I^\lambda = F_I^\rho$ is a maximal filter of M.*

In addition, the mapping $\phi : MFil(M) \to MaxId(M)$ defined by $\phi(F) = I_F$, $F \in MFil(M)$, is bijective.

Moreover, a maximal filter F of M is normal if and only if I_F is a normal and maximal ideal of M.

Proof. Let F and G be two maximal filters of M such that $I_F = I_G$. By (4.7), we conclude $F = G$.

Now, let I be a maximal ideal of M and by Proposition 4.15(ii), we set $F_I^\lambda = \{\rho_a(x)\colon x \in I, \exists\, a \in \mathcal{I}(M) \setminus I, x < a\}$. Then F_I^λ is a proper filter of M, so there is a maximal filter F containing F_I^λ. By Theorem 4.16, I_F is a maximal ideal of M. If $x \in I$, then $\rho_a(x) \in F_I^\lambda \subseteq F$, where $a \in \mathcal{I}(M) \setminus I$ satisfies $x < a$. Then $x = \lambda_a(\rho_a(x)) \in I_F$, i.e. $I \subseteq I_F$. Maximality of I entails $I = I_F$. We can apply (4.7), so that, we get $F_I^\lambda = F = F_I^\rho$, and $F_I^\lambda = F_I^\rho$ is a maximal filter of M. Hence, we set $F_I := F_I^\lambda = F_I^\rho$. Then $I = I_{F_I}$.

Now let J and K be maximal ideals of M such that $F_J = F = F_K$. Then $J = I_{F_J} = I_{F_K} = K$. Let G be a maximal filter of M and let $x \in G$. Then $\rho_a(x) \in I_G$, where $x < a \in \mathcal{I}(M) \setminus I_G$ and $x = \lambda_a(\rho_a(x)) \in F_{I_G}$ which yields $G \subseteq F_{I_G}$. Maximality of G entails $G = F_{I_G}$. Then this equality and $I = I_{F_I}$ imply ϕ is a bijective mapping.

Finally, let I be a maximal and normal ideal of M. Then $x \oplus I = I \oplus x$ for each $x \in M$. Let $y_1, y_2 \in I$ and $x \in M$ be such that $\rho_a(x) \oplus y_1 = y_2 \oplus \rho_a(x)$, where $x, y_1, y_2 \leq a \in \mathcal{I}(M)$. Then $\lambda_a(y_1) \odot x = x \odot \lambda_a(y_2)$, that is F_I is a normal and maximal filter of M. In the same way we prove that if F is a normal and maximal filter of M, then I_F is a maximal and normal ideal of M. □

Lemma 4.18. (i) *If F is a normal filter of a proper pseudo EMV-algebra M, then I_F^λ and I_F^ρ are normal ideals of M and $I_F^\lambda = I_F^\rho$.*

(ii) *If I is a normal ideal of a proper pseudo EMV-algebra M and if I satisfies* (4.6)*, then F_I^λ and F_I^ρ are normal filters of M and $F_I^\lambda = F_I^\rho$.*

Proof. (i) If $F = M$, then, for each $x \in M$ and $a \in \mathcal{I}(M)$ with $x \leq a$, we have $\lambda_a(x) \in F$, so that $x = \rho_a(\lambda_a(x)) \in I_F^\lambda$, i.e. $I_F^\lambda = F = M$ and I_F^λ is a normal ideal. In a similar way, we have $I_F^\rho = F = M$.

Now, let F be a proper ideal of M. Then I_F^λ and I_F^ρ are proper ideals of M. Since F is normal, then $x \odot F = F \odot x$ for each $x \in M$. Let $\lambda_a(x) \odot z_1 = z_2 \odot \lambda_a(x)$ for $z_1, z_2 \in F$ and $x \in M$ such that $x, z_1, z_2 \leq a \in \mathcal{I}(M)$. Then $\rho_a(z_1) \oplus x = x \oplus \rho_a(z_2)$ which implies I_F^λ is a normal ideal of M. In the same way, we can verify that I_F^ρ is a normal ideal of M.

Assume $x \in F$ and $x \leq a \in \mathcal{I}(M)$. From $\lambda_a(x) \oplus x = a = x \oplus \rho_a(x)$ we have that if $\rho_a(x) \in I_F^\lambda$, then there is $x_1 \in I_F^\lambda$ such that $x_1 \oplus x = a$, so that $\lambda_a(x) \leq x_1$ and $\lambda_a(x) \in I_F^\lambda$. Consequently, $I_F^\rho \subseteq I_F^\lambda$. In a dual way, we have $I_F^\lambda \subseteq I_F^\rho$ which gives $I_F^\lambda = I_F^\rho$.

(ii) In a dual way, we can show that if I is a normal ideal, then F_I^λ and F_I^ρ are normal filters of M. Now let $x \in I$ and $a \in \mathcal{I}(M) \setminus I$ be such that $x < a$. Then we have $x \odot \lambda_a(x) = 0 = \rho_a(x) \odot x$. For $\rho_a(x) \in F_I^\lambda$, there is $y_1 \in F_F^\lambda$ such that $0 = x \odot y_1$, so that $y_1 \le \lambda_a(x)$ which gives $\lambda_a(x) \in F_I^\lambda$, i.e. $F_I^\rho \subseteq F_I^\lambda$. In a similar way, we have the converse inclusion, which gives $F_I^\lambda = F_I^\rho$. $\square$

Remark 4.19. *Let M be a proper pseudo EMV-algebra and let F be a proper filter of M. Without use of* (4.6) *for ideals I_F^λ and I_F^ρ, we can show that $F^\rho_{I_F^\lambda} := \{\lambda_a(x) \colon x \in I_F^\lambda, x \le a \in \mathcal{I}(M)\} = F$ and $F^\lambda_{I_F^\lambda} := \{\rho_a(x) \colon x \in I_F^\lambda, x \le a \in \mathcal{I}(M)\} = \{\rho_a^2(x) \colon x \in F, x \le a \in \mathcal{I}(M)\}$ is a proper filter of M. The latter follows from two facts:* (i) *$\rho_a^2(x) \odot \rho_a^2(y) = \rho_a^2(x \odot y)$ and $\lambda_a^2(x) \odot \lambda_a^2(y) = \lambda_a^2(x \odot y)$ for $x, y \le a \in \mathcal{I}(M)$.* (ii) *If $\rho_a^2(x) = 0$ for some $x \in F$ and $x \le a \in \mathcal{I}(M)$, then $x = 0 \in F$ which is a contradiction.*

In the same way, we have $F^\lambda_{I_F^\rho} := \{\rho_a(x) \colon x \in I_F^\rho, x \le a \in \mathcal{I}(M)\} = F$ and $F^\rho_{I_F^\rho} := \{\lambda_a(x) \colon x \in I_F^\rho, x \le a \in \mathcal{I}(M)\} = \{\lambda_a^2(x) \colon x \in F, x \le a \in \mathcal{I}(M)\}$ is a proper filter of M.

Remark 4.20. *Let M be a proper pseudo EMV-algebra and let I be a proper ideal of M satisfying* (4.6). *Then $I^\rho_{F_I^\lambda} = F = I^\lambda_{F_I^\rho}$, and $I^\lambda_{F_I^\lambda} = \{\rho^2(x) \colon x \in F, x \le a \in \mathcal{I}(M)\}$ and $I^\rho_{F_I^\rho} = \{\lambda^2(x) \colon x \in F, x \le a \in \mathcal{I}(M)\}$ are proper ideals of M. Here we use $\rho_a^2(x) \oplus \rho_a^2(y) = \rho_a^2(x \oplus y)$ and $\lambda_a^2(x) \oplus \lambda_a^2(y) = \lambda_a^2(x \oplus y)$ for $x, y \le a \in \mathcal{I}(M)$.*

Lemma 4.21. *Let M be a pseudo EMV-algebra and $a \in \mathcal{I}(M)$. If I_a is a proper ideal of the pseudo MV-algebra $[0, a]$, then there is a maximal ideal of M containing I_a.*

Proof. If M has a top element, then trivially I_a can be embedded into a maximal ideal of M. Thus, let M be a proper pseudo EMV-algebra. Clearly, I_a is also a proper ideal of M. Applying the proof of Lemma 4.18, we can show that $F_a := \{\lambda_a(x) \colon x \in I_a\}$ is a proper filter of $[0, a]$, consequently, it is a proper filter of M and there is a maximal filter F of M containing F_a. By Theorem 4.16, $I_F^\lambda = I_F^\rho$ is a maximal ideal of M containing I_a. $\square$

5 Prime Ideals and q-Varieties of Pseudo EMV-algebras

In the section, we will study prime ideals more in detail and we describe the class of representable pseudo EMV-algebras. We note that there are pseudo MV-algebras without any maximal ideal (filter) that is also normal. On the other side, the class of pseudo MV-algebras M such that every maximal ideal of M is normal forms a variety [13, Prop 6.2] which contains the variety of representable pseudo MV-algebras as well as the variety of normal-valued ones, [13]. Inspired by these ideas from pseudo MV-algebras, we define the class of normal-valued pseudo EMV-algebras, the class of pseudo EMV-algebras where every maximal ideal is normal to show that they are q-varieties of pseudo EMV-algebras. Whereas the lattice of q-subvarieties of EMV-algebras is countably infinite, see [18, Thm 5.22], we show that the lattice of q-subvarieties of pseudo EMV-algebras is uncountable.

The following proposition shows that every linearly ordered pseudo EMV-algebra has a unique maximal ideal, it is normal, and the EMV-algebra has a top element.

Proposition 5.1. *Every linearly ordered pseudo EMV-algebra $M \neq \{0\}$ has a unique maximal ideal, it is normal, and M possesses a top element.*

Proof. The set of proper ideals of M is non-empty (it contains the ideal $\{0\}$) and is linearly ordered under the set-theoretical inclusion. Indeed, if I, J are two proper ideals, assume that neither $I \subseteq J$ nor $J \subseteq I$. Then there are $x \in I \setminus J$ and $y \in J \setminus I$. Since $x \leq y$ or $y \leq x$, we obtain a contradiction. According to Theorem 4.17, M possesses at least one maximal ideal. Consequently, M has a unique maximal ideal.

The set of idempotents $\mathcal{I}(M)$ is a linearly ordered pseudo EMV-algebra where $\oplus$ is commutative. According to [18, Thm 5.6], $\mathcal{I}(M)$ has a maximal ideal. Since $\mathcal{I}(M)$ is linearly ordered, by the previous paragraph, $\mathcal{I}(M)$ has a unique maximal ideal. Using [18, Thm 3.25], $\mathcal{I}(M)$ has a top element and this top element is also a top element of M. The pseudo EMV-algebra M is termwise equivalent to the linearly ordered pseudo MV-algebra $(M; \oplus, \lambda_1, \rho_1, 0, 1)$. Due to [11, Prop 5.4], I is a unique maximal ideal of the pseudo MV-algebra and it is normal. Hence, I is a normal and maximal ideal of the pseudo EMV-algebra M. $\square$

Now we exhibit the lattice structure of the set $Id(M)$ of ideals of M. We remind, that if K is a subset of M, then $I_0(K)$ is the ideal generated by K.

Proposition 5.2. *Let M be a pseudo EMV-algebra.*

(i) *The system $(Id(M); \cap, \vee)$ is a Brouwerian lattice with respect to the set-theoretical inclusion $\subseteq$, i.e. the meet $\wedge$ of ideals is their intersection, $\bigcap_t I_t$, the join $\vee$ is $\bigvee_t I_t = I_0(\bigcup_t I_t)$, and $I \cap \bigvee_t I_t = \bigvee_t (I \cap I_t)$.*

(ii) *Let $I_0(x)$ be the ideal of M generated by an element $x \in M$. Then $I_0(x) \cap I_0(y) = I_0(x \wedge y)$.*

Proof. (i) It is routine.

(ii) Clearly $I_0(x \wedge y) \subseteq I_0(x) \cap I_0(y)$. Now let $z \in I_0(x) \cap I_0(y)$. Then by Proposition 4.8, there are integers $n, m \geq 1$ such that $z \leq n.x$ and $z \leq m.y$. Without loss of generality, we can assume $n = m$. Using [26, Prop 1.17], $z \le n.x \wedge n.y \le n(x \wedge n.y) \le n^2.(x \wedge y)$, so that $z \in I_0(x \wedge y)$. $\square$

We say that an ideal P of a pseudo EMV-algebra M is *prime* if, for $x, y \in M$, $x \wedge y \in P$ implies $x \in P$ or $y \in P$. Let $x > 0$ be an element of M. Using the Zorn Lemma, there is an ideal V such that V is maximal under the condition $x \notin V$; the ideal V is said to be a *value* of x and the ideal $V^* := I_0(V, x)$ is a *cover* of V. We note that if $z \in V^* \setminus V$, then $V \vee I_0(z) = V^*$; if not, then $x \notin V \vee I_0(z)$ which contradicts maximality of V. A pseudo EMV-algebra M is said to be *normal-valued* if every ideal V of M is normal in its cover V^*, that is $y \oplus V = V \oplus y$ for each $y \in V^*$.

Proposition 5.3. *Let P be an ideal of a pseudo EMV-algebra M. The following statements are equivalent:*

(i) *P is a prime ideal of M.*

(ii) *$x \wedge y = 0$ implies $x \in P$ or $y \in P$.*

(iii) *For all $x, y \in M$ and every $a \in \mathcal{I}(M)$ with $x, y \le a$, $x \odot \lambda_a(y) \in P$ or $y \odot \lambda_a(x) \in P$.*

(iv) *For all $x, y \in M$ and every $a \in \mathcal{I}(M)$ with $x, y \le a$, $\rho_a(x) \odot y \in P$ or $\rho_a(y) \odot x \in P$.*

(v) *The set of ideals $\{I \in Id(M) \colon P \subseteq I\}$ is linearly ordered under the set theoretical inclusion.*

In addition, every value and every maximal ideal of M is prime. If P is prime and I is an ideal of M containing P, then I is prime.

Proof. (i) $\Rightarrow$ (ii). If $x \wedge y = 0$, then clearly $x \in P$ or $y \in P$.

(ii) $\Rightarrow$ (iii). Let $x, y \in M$ with $x, y \le a \in \mathcal{I}(M)$ be given. Then x and y belong to the pseudo MV-algebra $[0, a]$ and due to [26, Prop 1.24], we have $(x \odot \lambda_a(x)) \wedge (y \odot \lambda_a(y)) = 0$ which gives the result.

(iii) $\Rightarrow$ (i). Let $x \wedge y \in P$. Choose an idempotent $a \in \mathcal{I}(M)$ with $x, y \le a$. Assume e.g. $x \odot \lambda_a(y) \in P$. Then $x \le (x \odot \lambda_a(y) \oplus x) \wedge (x \vee y) = (x \odot \lambda_a(y) \oplus x) \wedge (x \odot \lambda_a(y) \oplus y) = x \odot \lambda_a(y) \oplus (x \wedge y) \in P$.

(ii) $\Rightarrow$ (iv). Let $x, y \in M$ with $x, y \le a \in \mathcal{I}(M)$ be given. Then x and y belong to the pseudo MV-algebra $[0, a]$ and due to [26, Prop 1.24], we have $(\rho_a(x) \odot y) \wedge (\rho_a(y) \odot x) = 0$ which gives the result.

(iv) $\Rightarrow$ (i). Let $x \wedge y \in P$. Choose an idempotent $a \in \mathcal{I}(M)$ with $x, y \leq a$. Assume e.g. $\rho_a(x) \odot y \in P$. Since $y \leq (x \vee y) \wedge (y \oplus \rho_a(x) \odot y) = (x \oplus \rho_a(x) \odot y) \wedge (y \oplus \rho_a(x) \odot y) = (x \wedge y) \oplus \rho_a(x) \odot y \in P$ which entails $y \in P$.

(iii) $\Rightarrow$ (v). Let I, J be two ideals of M containing P. If they are not linearly ordered, there are $x \in I \setminus J$ and $y \in J \setminus I$. Let a be an idempotent which dominates both x and y. Then $x \odot \lambda_a(y) \in P$ or $y \odot \lambda_a(x) \in P$. Since $(x \odot \lambda_a(y)) \oplus y = x \vee y = (y \odot \lambda_a(x)) \oplus x$, we have $x \in J$ or $y \in I$, a contradiction.

(v) $\Rightarrow$ (i). Let $x \wedge y \in P$. Then for $I_0(x) \vee P$ and $I_0(y) \vee P$ we have by Proposition 5.2, $(I_0(x) \vee P) \cap (I_0(y) \vee P) = (I_0(x) \cap I_0(y)) \vee P = I_0(x \wedge y) \vee P = P$. Whence, by (v), $I_0(x) \vee P \subseteq I_0(y) \vee P$ or $I_0(x) \vee P \subseteq I_0(x) \vee P$ and thus $x \in P$ or $y \in P$.

Now let I be a maximal ideal of M and let $x \wedge y \in I$. If $x \notin I$ and $y \notin I$, then $I_0(x) \vee I = M = I_0(y) \vee I$, and by Proposition 5.2, we have $M = (I_0(x) \vee I) \cap (I_0(y) \vee I) = I_0(x \wedge y) \vee I = I$, a contradiction.

In the same way we can prove that if $z > 0$ and V is a value of z, then for $x, y \in M$ with $x \wedge y \in V$, then $x \in V$ or $y \in V$.

Let I be an ideal of M containing a prime ideal. Using criterion (ii), we see that I is prime. $\square$

Proposition 5.4. *Let P be a proper prime ideal of a proper pseudo EMV-algebra M. Then condition* (4.6) *holds for P.*

In addition, every proper prime ideal is contained in a unique maximal ideal of M.

Proof. Let $a \in \mathcal{I}(M) \setminus P$ and $a < b \in \mathcal{I}(M)$. Since $\lambda_b(a) \wedge a = 0$ from Proposition 5.3(ii), we have $\lambda_b(a) \in P$, so (4.6) holds.

Let P be a proper prime ideal of M. By Proposition 4.15, F_P^λ is a proper filter of M, hence, there is a maximal filter F of M containing F_P^λ. Due to Theorem 4.16, I_F^ρ is a maximal ideal of M. Let $x \in P$ and let $a \in \mathcal{I}(M) \setminus P$ be such that $x < a$. Then $\rho_a(x) \in F_P^\lambda \subseteq F$ and $\rho_a(x) \in F$ which yields $x = \lambda_a(\rho_a(x)) \in I_F^\rho$, that is P is contained in some maximal ideal of M. If P is contained in two maximal ideals I_1 and I_2, by the criterion (v) of Proposition 5.3, $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$ and maximality of I_1 and I_2 establishes $I_1 = I_2$. $\square$

Proposition 5.5. *Let M be a pseudo EMV-algebra, I an ideal of M, and $z \in M \setminus I$. Then there is an ideal P of M that is maximal under the conditions $I \subseteq P$ and $z \notin P$. In addition, P is prime.*

Proof. Using Zorn's Lemma, we can show that there is an ideal P that is maximal with respect to the condition $I \subseteq P$ and $z \notin P$. We show that P is prime. Let $x, y \in M$, $a \in \mathcal{I}(M)$, $x, y \leq a$, and suppose that P is not prime, i.e. $x \odot \lambda_a(y), y \odot \lambda_a(x) \notin P$.

Then the ideals generated by $P \cup \{x \odot \lambda_a(y)\}$ and $P \cup \{y \odot \lambda_a(x)\}$, respectively, must contain the element z. It is straightforward to see that

$$z \leq \bigoplus_{i=1}^{k}(p_i \oplus n_i.x_0)$$

and similarly,

$$z \leq \bigoplus_{j=1}^{m}(q_j \oplus m_j.y_0),$$

where $p_1,\dots,p_k,q_1,\dots,q_m \in P$, $n_i, m_j \geq 1$ are integers, and $x_0 = x\odot\lambda_a(y)$ and $y_0 = y\odot\lambda_a(x)$. Without loss of generality, we can assume $k=m$. Let $p = p_1\vee\cdots\vee p_k \in P$, $q = q_1 \vee \cdots \vee q_k \in P$, and $n_0 = \max\{n_1,\dots,n_k\}$, $m_0 = \max\{m_1,\dots,m_k\}$. We can also assume that $n_0 = m_0$ and if we put $u = p\oplus q \in P$, we have $z \leq k.(u \oplus n_0.x_0) \wedge k.(u \oplus n_0.y_0)$. Using [26, Prop 1.17], we have $z \leq k^2.(u \oplus (n_0.x_0 \wedge n_0.y_0))$. Since x_0, y_0, x, y belong to the pseudo MV-algebra $[0,a]$, due to [26, Lem 1.32], we have $n_0.x_0\wedge n_0.y_0 = n_0.(x_0\wedge y_0) = n_0.0 = 0$. Therefore, $z \leq k^2.u \in P$, a contradiction. $\square$

Proposition 5.6. *Let P be a prime and normal ideal of a pseudo EMV-algebra M. Then the quotient pseudo EMV-algebra M/P is linearly ordered and it has a top element.*

Proof. Let $x,y \in M$ and let $x,y \leq a \in \mathcal{I}(M)$. By Proposition 5.3(iv), we have $\rho_a(x)\odot y \in P$ or $\rho_a(y)\odot x \in P$. Then $[\rho_a(x)]_P\odot[y]_P = [0]_P$ or $[\rho_a(y)]_P\odot[x]_P = [0]_P$, i.e. $[y]_P \leq [x]_P$ or $[x]_P \leq [y]_P$ which says that M/P is a linearly ordered pseudo EMV-algebra. Applying Proposition 5.1, we conclude M/P has a top element. $\square$

Proposition 5.7. *A normal ideal I of a pseudo EMV-algebra M is maximal if and only if the quotient EMV-algebra M/I possesses only two ideals, $\{0/I\}$ and M/I.*

Proof. Let I be a normal and maximal ideal of M, and let J be a proper ideal of M/I. Define $J_0 := \{x \in M\colon x/I \in J\}$. Then J_0 is a proper ideal of M containing I. The maximality of I entails $J = 0/I$.

Conversely, let M/I have only two ideals and let I_1 be a proper ideal of M containing I. The set $\{a/I\colon a \in aI_1\}$ is an ideal of M/I. Therefore, this set coincides with the zero-ideal $\{0/I\}$ of M/I, i.e. $I_1 = I$, which proves that I is a maximal ideal of M. $\square$

Theorem 5.8. *If I is a maximal and normal ideal of a pseudo EMV-algebra M, then M/I is an Archimedean commutative EMV-algebra with top element. In addition, there is a unique subgroup $\mathbb{R}_0$ containing the number 1 of the group of real numbers $\mathbb{R}$ such that $M/I \cong (\mathbb{R}_0; \max, \min, \oplus, 0)$, where $s \oplus t := \min\{1, s+t\}$, $s,t \in \mathbb{R}_0$.*

Proof. Since every maximal ideal is prime, Proposition 5.3, the quotient EMV-algebra is a linearly ordered pseudo EMV-algebra with top element, see Proposition 5.6. Therefore, it can be reorganized into a pseudo MV-algebra. In addition, there is an idempotent $a_1 \in M$ such that a_1/I is a top element of M/I.

We assert that M/I is Archimedean. Given $x \in M$, there is an idempotent $a \in M$ such that $x, a_1 \le a$. Then a/I is a top element of M/I. Suppose that $n.(x/I) \le \lambda_{a/I}(x/I) = \lambda_a(x)/I$ holds in M/I for each integer $n \ge 1$. The ideal J_x of M/I generated by x/I coincides by Proposition 4.8 with the set $\{y/I \colon y/I \le n.(x/I), n \ge 1\}$. By Proposition 5.7, the quotient pseudo EMV-algebra M/I possesses only two ideals, and since $a/I \notin J_x$, we have $x/I = 0/I$, i.e. M/I is Archimedean. Theorem 3.7 guarantees that M/I is a commutative pseudo EMV-algebra, i.e. an EMV-algebra. Then the set M/I can be reorganized such that M/I is termwise equivalent to a pseudo MV-algebra that is Archimedean and linearly ordered. Therefore, M/I is isomorphic to an MV-subalgebra $\Gamma(\mathbb{R}_0, 1)$ of the pseudo MV-algebra $\Gamma(\mathbb{R}, 1)$, and $\mathbb{R}_0$ is a unique subgroup of $\mathbb{R}$ containing 1 such that M/I is isomorphic to $\Gamma(\mathbb{R}_0, 1)$, see [4, Prop 7.2.5]. $\square$

We say that a pseudo EMV-algebra M is *representable* if there is a system of linearly ordered pseudo EMV-algebras $\{M_t \colon t \in T\}$ such that there is an injective pEMV-homomorphism $\phi : M \hookrightarrow \prod_{t\in T} M_t$ with $\pi_t \circ \phi(M) = M_t$ for each $t \in T$, where π_t is the t-th projection from $\prod_{t\in T} M_t$ onto M_t.

Lemma 5.9. *A pseudo EMV-algebra is representable if and only if there is a system $\{P_t \colon t \in T\}$ of normal prime ideals of M such that $\bigcap_{t\in T} P_t = \{0\}$.*

Proof. If M is representable, then there are a system $\{M_t \colon t \in T\}$ and an injective pEMV-homomorphism $\phi : M \hookrightarrow \prod_{t\in T} M_t$ such that $\pi_t \circ \phi(M) = M_t$ for each $t \in T$. Define $P_t = \mathrm{Ker}(\pi_t \circ \phi)$ for each $t \in T$. Then P_t is by Proposition 4.2 a normal ideal of M. Let $x \in \bigcap_{t\in T} P_t$. Then $\pi_t \circ \phi(x) = 0$, so that $\phi(x) = 0$ and injectivity of ϕ yields $x = 0$. Let $x, y \in M$ and $a \in \mathcal{I}(M)$ with $x, y \le a$ be given. Then $\pi_t \circ \phi(x) \le \pi_t \circ \phi(y)$ or $\pi_t \circ \phi(y) \le \pi_t \circ \phi(x)$, so that $\pi_t \circ \phi(x \odot \lambda_a(y)) \le \pi_t \circ \phi(y) \odot \pi_t \circ \phi(\lambda_a(x)) = 0$, i.e. $x \odot \lambda_a(y) \in P_t$ or $y \odot \lambda_a(x) \in P_t$.

Conversely, let $\{P_t\}$ be a system of normal prime ideals of M with $\bigcap_{t\in T} P_t = \{0\}$. Then $M_t = M/P_t$ is a linearly ordered pseudo EMV-algebra and the mapping $\phi(x) = (x/P_t)_t$, $x \in M$, is a subdirect embedding in question. $\square$

Theorem 5.10. *Every representable pseudo EMV-algebra can be embedded into a representable pseudo EMV-algebra with top element.*

Proof. Since M is a representable pseudo EMV-algebra, by Lemma 5.9, there is a system of normal prime ideals $\{P_t \colon t \in T\}$ with $\bigcap_{t\in T} P_t = \{0\}$. Then M can be subdirectly embedded into $M_0 := \prod_{t\in T} M/P_t$. Since every M/P_t is by Proposition 5.6 linearly ordered, by Proposition 5.1, M/P_t possesses a top element 1_t. Then $1 = (1_t)_t$ is a top element of M_0. $\square$

We say that (i) a pseudo EMV-algebra with top element M is *symmetric* if $\lambda_1(x) = \rho_1(x)$ for each $x \in M$, (ii) a pseudo EMV-algebra M is *strongly symmetric* if for each $a \in \mathcal{I}(M)$ and each $x \le a$, we have $\lambda_a(x) = \rho_a(x)$. We note that this does not mean that $\oplus$ is commutative. Indeed, let G be a non-commutative ℓ-group. Then $M = \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1,0)) = (\{0\} \times G^+) \cup (\{1\} \times G^-)$ is a pseudo EMV-algebra with a top element $(1,0)$. Then $\mathcal{I}(M) = \{(0,0),(1,0)\}$, M is strongly symmetric but $\oplus$ is not commutative. If G is a non-commutative linearly ordered group, then $M = \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1,0))$ is an example of a linearly ordered strongly symmetric pseudo EMV-algebra that is not an EMV-algebra.

Let M be a pseudo EMV-algebra and $a \in \mathcal{I}(M)$ such that for each $x \in [0,a]$, $\lambda_a(x) = \rho_a(x)$. Then by Proposition 3.3, for all idempotent elements $b \in \mathcal{I}(M)$ with $b \le a$, we can easily see that $\lambda_b(x) = \rho_b(x)$ for all $x \in [0,b]$. Therefore, if for each $x \in M$, there exists $a \in \mathcal{I}(M)$ such that $\lambda_a(y) = \rho_a(y)$ for all $y \in [0,b]$, then M is a strongly symmetric pseudo EMV-algebra. In particular, every symmetric pseudo EMV-algebra with top element is strongly symmetric.

The following result gives a representation of any proper strongly symmetric representable pseudo EMV-algebra as a maximal normal ideal in a symmetric representable pseudo EMV-algebra with top element.

Theorem 5.11. *Let M be a proper strongly symmetric representable pseudo EMV-algebra. Then there is a symmetric representable pseudo EMV-algebra N_0 with a top element 1 such that M can be embedded into N_0 as a normal and maximal ideal of N_0.*

Proof. From the proof of Theorem 5.10, we know that there is a system $\{P_t : t \in T\}$ of normal prime ideals of M such that $\bigcap_{t\in T} P_t = \{0\}$, and M can be subdirectly embedded into $N := \prod_{t\in T} M/P_t$. Without loss of generality, we can assume that M is a pseudo EMV-subalgebra of N.

For every idempotent $a \in \mathcal{I}(M)$ and every $x \le a$ we have $\lambda_a(x) = \rho_a(x)$, so that $\lambda_{a/P_t}(x/P_t) = \lambda_a(x)/P_t = \rho_a(x)/P_t = \rho_{a/P_t}(a/P_t)$. Hence, if 1_t is a top element of M/P_t, $x/P_t \in M/P_t$, there is an idempotent $a_t \in \mathcal{I}(M)$ such that $x \le a_t$ and $a_t/P_t = 1_t$. Then we have $\lambda_{1_t}(x/P_t) = \lambda_{a/P_t}(x/P_t) = \rho_{a/P_t}(x/P_t) = \rho_{1_t}(x/P_t)$. Therefore, for each $(x_t)_t \in N$, we have $\lambda_{(1_t)_t}((x_t)_t) = \rho_{(1_t)_t}((x_t)_t)$. We denote simply $x^* := \lambda_1(x)$ for each $x \in N$.

Define

$$N_0 := \{x \in N : \exists\, x_0 \in M \text{ such that either } x = x_0 \text{ or } x = x_0^*\}. \tag{5.1}$$

We assert N_0 is a representable pseudo EMV-subalgebra of N that is generated by M and 1. Actually, we show that N_0 is in fact termwise equivalent to a symmetric pseudo MV-algebra.

Clearly N_0 contains M and 1. Let $x, y \in N_0$. We have four cases. (i) $x = x_0, y = y_0 \in M$. Then $x \vee y, x \wedge y, x \oplus y \in N_0$. (ii) $x = x_0^*$, $y = y_0^*$ for some $x_0, y_0 \in M$. Then $x \vee y = x_0^* \vee y_0^* = (x_0 \wedge y_0)^*$, $x \wedge y = (x_0 \vee y_0)^*$ and $x \oplus y = x_0^* \oplus y_0^* = (y_0 \odot x_0)^* \in N_0$. (iii) $x = x_0$ and $y = y_0^*$ for some $x_0, y_0 \in M$. Then

$$x \oplus y = x_0 \oplus y_0^* = (y_0 \odot x_0^*)^* = (y_0 \odot (x_0 \wedge y_0)^*)^* = (y_0 \odot \lambda_b(x_0 \wedge y_0))^*,$$

where b is an idempotent of M such that $x_0, y_0 \leq b$; for the last equality we use equality (3.1) of Proposition 3.4. Using again Proposition 3.4, we have $y_0 \odot \lambda_b(x_0 \wedge y_0) \in M$ so that $x \oplus y \in N_0$. (iv) $x = x_0^*$ and $y = y_0$ for some $x_0, y_0 \in M$. Analogously as in case (iii), we have

$$x \oplus y = x_0^* \oplus y_0 = (y_0^* \odot x_0)^* = ((x_0 \wedge y_0)^* \odot x_0)^* = (\rho_b(x_0 \wedge y_0) \odot x_0)^*,$$

which yields $x \oplus y \in N_0$. Hence, N_0 is termwise equivalent to the pseudo MV-algebra generated by M and 1. Due to [11, Thm 6.11], the class of representable pseudo MV-algebras forms a variety. Therefore, N_0 is a symmetric representable pseudo EMV-algebra with top element.

Now, we prove that M is a maximal ideal of N_0. Since M is a proper pseudo EMV-algebra, M is a proper subset of N_0. To show that M is an ideal it is sufficient to assume $y \leq x \in M$. If $y = y_0^*$, this is impossible while $1 \notin M$. Therefore, M is a proper ideal of N_0. Now let $y \in N_0 \setminus M$, then $y = y_0^*$ for some $y_0 \in M$. Let $I_0(M, y)$ be the ideal of N_0 generated by M and y. Then $1 = y_0 \oplus y_0^* \in I_0(M, y)$ which gives $I_0(M, y) = N_0$, showing M is a maximal ideal of the EMV-algebra N_0. Again applying [11, Thm 6.11], we see that the variety of representable pseudo MV-algebras is a subvariety of the variety of normal-valued pseudo MV-algebras. Therefore, M is a maximal and normal ideal of N_0. □

We can summarize our representation result as follows.

Theorem 5.12. *For every strongly symmetric representable pseudo EMV-algebra M, either M a representable pseudo EMV-algebra with top element or M can be embedded into a symmetric representable pseudo EMV-algebra N_0 with top element as a maximal and normal ideal of N_0.*

Proof. If M has a top element, then we are ready. If M has no top element, the result follows from Theorem 5.11. □

This result will be generalized in [22, Thm 6.4].

Using the Zorn Lemma and Proposition 5.3(v), it is possible to show that there is a minimal prime ideal in every pseudo EMV-algebra. To present a criterion when a pseudo EMV-algebra is representable, we use the notion of a minimal prime ideal and we present a relation between them and polars.

Let X be a non-empty subset of a pseudo EMV-algebra. The set $X^\perp := \{x \in M \colon x \wedge z = 0, \forall z \in X\}$ is a *polar* of M; for any $z \in M$, we put $z^\perp := \{z\}^\perp$. It is easy to verify that every polar of M is an ideal of M.

We note that due to Proposition 5.3(ii), if P is a prime ideal, then the set $F = M \setminus P$ has the property $x \wedge y > 0$ for each $x, y \in P$. A subset F of $M \setminus \{0\}$ maximal under this condition is said to be an *ultrafilter*. In the following we use ideas of [26, Thm 2.20].

Proposition 5.13. *Let P be a proper ideal of a pseudo EMV-algebra M. The following statements are equivalent:*

(i) *P is a minimal prime ideal.*

(ii) *$M \setminus P$ is an ultrafilter.*

(iii)
$$P = \bigcup \{z^\perp \colon z \notin P\}. \tag{5.2}$$

(iv) *P is prime and for all $x \in P$, $x^\perp \not\subseteq P$.*

Proof. (i) $\Rightarrow$ (ii). If $x, y \notin P$, then $x \wedge y \notin P$. Therefore, the Zorn Lemma guarantees that there is an ultrafilter F containing $M \setminus P$. We assert $F = M \setminus P$. Let $x \in F$, and $z \in x^\perp$. Then $z \wedge x = 0$ entails $z \notin F$, otherwise $z, x \in F$ and $z \wedge x > 0$, a contradiction. Hence,
$$x^\perp \subseteq M \setminus F \subseteq P.$$

Put $Q = \bigcup \{x^\perp \colon x \in F\}$. Then $Q \subseteq P$ and Q is an ideal of M. Indeed, clearly $0 \in Q$. Let $u, v \in Q$. There are $x, y \in F$ such that $u \in x^\perp$ and $v \in y^\perp$. Let a be an idempotent such that $u, v, x, y \le a$. Due to [26, Prop 1.17] applied to the pseudo MV-algebra $[0, a]$, we have $x \wedge y \wedge (u \oplus v) \le (x \wedge y \wedge u) \oplus (x \wedge y \wedge v) = 0$, that is, $u \oplus v \in (x \wedge y)^\perp$. Since $y \in F$ and $x \wedge y = (x \wedge y) \wedge y > 0$, we have $x \wedge y \in F$, so that $u \oplus v \in Q$. Finally, let $u \le v \in Q$. Then there is $x \in F$ such that $v \in x^\perp$ which yields $u \in x^\perp$. Hence, Q is an ideal of M.

To prove Q is a prime ideal, let $u, v \in M$ be such that $u \wedge v = 0$ and let $u \notin Q$. Since F is an ultrafilter, $u \in F$ and $v \in u^\perp \subseteq Q$. In other words, we have established Q is prime such that it is contained in P and minimality of P entails $Q = P$.

The proof of the remaining implications follows the same steps as the proof of [26, Thm 2.20], therefore, we omit it. $\square$

Proposition 5.14. *A pseudo EMV-algebra M is representable if and only if every polar $z^{\perp}$ is normal, $z \in M \setminus \{0\}$.*

Proof. If M has a top element, then it is termwise equivalent to a pseudo MV-algebra and the desired result follows from [26, Prop 3.13].

Now let M have no top element. If M is representable, there is a system of linearly ordered pseudo EMV-algebras $\{M_t \colon t \in T\}$ such that M is a subdirect product of $\{M_t \colon t \in T\}$. Let us assume M is a pseudo EMV-subalgebra of $\prod_{t\in T} M_t$. Given $x = (x_t)_t \in M$, we denote by $\operatorname{supp}(x) := \{t \in T \colon x_t \neq 0\}$. Let $P = z^{\perp}$. Since every M_t is linearly ordered, then

$$x \wedge z = 0 \quad \Leftrightarrow \quad \operatorname{supp}(x) \cap \operatorname{supp}(z) = \emptyset.$$

Let $u, v \in M$ and choose an idempotent $a \in \mathcal{I}(M)$ such that $u, v \leq a$. Then $u \odot \lambda_a(v) = 0$ iff $u \leq v$ iff $\rho_a(v) \odot u = 0$, so that $\operatorname{supp}(u \odot \lambda_a(v)) = \operatorname{supp}(\rho_a(v) \odot v)$ which establishes that $z^{\perp}$ is normal.

Conversely, let every $z^{\perp}$ be a normal ideal. Due to (5.2), P is normal, and the intersection of all minimal prime filters of M is equal to $\{0\}$. Hence, M is a subdirect product of $\{M/P \colon P \text{ is minimal prime}\}$. $\square$

Inspired by [32, Thm 3.4], we present equations which characterize representable pseudo EMV-algebras.

Theorem 5.15. *A pseudo EMV-algebra M is representable if and only if, for each idempotent element $a \in \mathcal{I}(M)$ and for all $x, y, z \leq a$, we have*

$$(x \odot \lambda_a(y)) \wedge ([z \oplus (y \odot \lambda_a(x))] \odot \lambda_a(z)) = 0,$$
$$(\rho_a(y) \odot x) \wedge (\rho_a(z) \odot [(\rho_a(x) \odot y) \oplus z]) = 0.$$

Proof. If M is a linear pseudo EMV-algebra, then it has a top element, and it satisfies both equations. Consequently, every representable pseudo EMV-algebra satisfies the equations.

Conversely, assume that M satisfies the above equations. We show that every polar $x^{\perp}$, $x \in M \setminus \{0\}$, is normal. Let $y \in x^{\perp}$ and assume $x, y, z \leq a$ for some $a \in \mathcal{I}(M)$. Then $\lambda_a(x) = \lambda_a(x) \oplus 0 = \lambda_a(x) \oplus (x \wedge y) = (\lambda_a(x) \oplus x) \wedge (\lambda_a(x) \oplus y) = a \wedge (\lambda_a(x) \oplus y) = \lambda_a(x) \oplus y$. In a similar way, $\lambda_a(y) = \lambda_a(y) \oplus x$, whence $x = \rho_a(y) \odot x$ and $y = \rho_a(x) \odot y$. Similarly, $y = y \odot \lambda_a(x)$ and $x = x \odot \lambda_a(x)$. Then we have

$$x \wedge \big((z \oplus y) \odot \lambda_a(z)\big) = 0,$$
$$x \wedge \big(\rho_a(z) \odot (y \oplus z)\big) = 0,$$

which implies $(z\oplus y)\odot\lambda_a(z) \in x^\perp$ and $\rho_a(z)\odot(y\oplus z) \in x^\perp$. Since $y\oplus z = (y\oplus z)\vee z = z \oplus (\rho_a(z) \odot (y \oplus z))$ and $z \oplus y = ((z \oplus y) \odot \lambda_a(z)) \oplus z$, we see that $x^\perp$ is a normal ideal. By Proposition 5.14, M is representable. □

As a corollary we have that a pseudo EMV-algebra M is representable iff the pseudo MV-algebra $([0,a];\oplus,\lambda_a,\rho_a,0,a)$ is representable for each $a \in \mathcal{I}(M)$. This principle will be extended in Theorem 5.17.

We recall the q-variety of EMV-algebras has countably infinitely many q-subvarieties of EMV-algebras, see [18, Thm 5.22]. In the next result, we show that the q-variety of pseudo EMV-algebras has uncountably many q-subvarieties. First, we introduce the following notation: Given a variety $\mathcal{V}$ of pseudo MV-algebras, let $\mathcal{V}^*$ be the class of pseudo EMV-algebras $(M;\vee,\wedge,\oplus,0)$ with top element such that $(M;\oplus,\lambda_1,\rho_1,0,1) \in \mathcal{V}$.

Theorem 5.16. *Given a variety $\mathcal{V}$ of pseudo MV-algebras, let $\mathbb{V}_0(\mathcal{V})$ be the class of pseudo EMV-algebras $M \in \mathbb{PEMV}$ such that, for each $a \in \mathcal{I}(M)$, the pseudo MV-algebra $([0,a];\oplus,\lambda_a,\rho_a,0,a)$ belongs to $\mathcal{V}$. Then $\mathbb{V}_0(\mathcal{V})$ is a q-subvariety of pseudo EMV-algebras containing $\mathcal{V}^*$. Conversely, given a q-subvariety $\mathbb{V}$ of pseudo EMV-algebras, let $\mathcal{V}_0(\mathbb{V})$ be the class of pseudo MV-algebras $(M;\oplus,^-,^\sim,0,1)$ such that $(M;\vee,\wedge,\oplus,0)$ belongs to $\mathbb{V}$. Then $\mathcal{V}_0(\mathbb{V})$ is a variety of pseudo MV-algebras. The mappings $\mathcal{V} \mapsto \mathbb{V}_0(\mathcal{V})$ and $\mathbb{V} \mapsto \mathcal{V}_0(\mathbb{V})$ are bijective mappings which are mutually invertible and preserving the set-theoretical inclusion.*

In particular, the q-variety $\mathbb{PEMV}$ has uncountably many subvarieties.

Proof. First we establish the following claim.

Claim. Let $a \le b$ be two idempotents of a pseudo EMV-algebra M. The mapping $f : [0,b] \to [0,a]$ defined by $f(x) = x \wedge a$, $x \in [0,b]$, is a homomorphism of pseudo MV-algebras that maps the pseudo MV-algebra $([0,b];\oplus,\lambda_1,\rho_b,0,b)$ onto the pseudo MV-algebra $([0,a];\oplus,\lambda_a,\rho_a,0,a)$.

Clearly, $f(b) = a$. Let $x,y \in [0,b]$. Then $(x \wedge a) \oplus (y \wedge a) \le x \oplus y$ and $(x\wedge a)\oplus(y\wedge a) \le a\oplus a = a$. Using [26, Prop 1.17], $(x\wedge a)\oplus(y\wedge a) \le (x\oplus y)\wedge a \le (x \wedge a) \oplus (y \wedge a)$, i.e. $f(x \oplus y) = f(x) \oplus f(y)$. In addition, if $x \in [0,b]$, then by Proposition 3.3(ii), $\lambda_a(x \wedge a) = \lambda_b(x \wedge a) \wedge a = (\lambda_b(x) \vee \lambda_b(a)) \wedge a = \lambda_b(x) \wedge a$. Similarly, $\rho_a(x \wedge a) = \rho_b(x) \wedge a$. Hence, f is a homomorphism from the pseudo MV-algebra $[0,b]$ onto the pseudo MV-algebra $[0,a]$.

Given a variety $\mathcal{V}$ of pseudo MV-algebras, let $\mathbb{V}_0(\mathcal{V})$ be the class of pseudo EMV-algebras $M \in \mathbb{PEMV}$ such that, for each $a \in \mathcal{I}(M)$, the pseudo MV-algebra $([0,a]; \oplus, \lambda_a, \rho_a, 0, a)$ belongs to $\mathcal{V}$. Let M be a pseudo MV-algebra belonging to $\mathcal{V}$. If $a \in \mathcal{I}(M)$, then by Claim, we have $[0,a]$ is a homomorphic image of M, so that $[0,a] \in \mathcal{V}$ and $(M; \vee, \wedge, \oplus, 0) \in \mathcal{V}^* \subseteq \mathbb{V}_0(\mathcal{V})$.

Using qH,qS,qP technique, we show that $\mathbb{V}_0(\mathcal{V})$ is a q-subvariety of pseudo EMV-algebras containing $\mathcal{V}^*$.

qH: For a pseudo EMV-algebra N, let $N = f(M)$, where $M \in \mathbb{V}_0(\mathcal{V})$ and f is a pEMV-homomorphism from M onto N. If $a \in \mathcal{I}(M)$, then $f(a) \in \mathcal{I}(N)$ and the mapping f_a, the restriction of f onto the pseudo MV-algebra $[0,a]$, is a homomorphism from the pseudo MV-algebra $[0,a]$ onto the pseudo MV-algebra $[0, f(a)]$. Therefore, $[0, f(a)] \in \mathcal{V}$. Now let b be an arbitrary idempotent from $\mathcal{I}(N)$. There are an element $x_0 \in M$ with $f(x_0) = b$ and $a \in \mathcal{I}(M)$ with $x_0 \leq a$. Then $[0,b] \subseteq [0, f(a)] \in \mathcal{V}$ and by Claim, also $[0,b] \in \mathcal{V}$. Therefore, $N \in \mathbb{V}_0(\mathcal{V})$.

qS: Let $M \in \mathbb{V}_0(\mathcal{V})$ and N be a pEMV-subalgebra of M. If $a \in \mathcal{I}(N)$, then $[0,a] \cap N \subseteq [0,a] \in \mathcal{V}$, i.e. $[0,a] \cap N \in \mathcal{V}$ and $N \in \mathbb{V}_0(\mathcal{V})$.

qP: Finally, let $M_t \in \mathbb{V}_0(\mathcal{V})$ for $t \in T$. Then $M = \prod_{t \in T} M_t$ is a pseudo EMV-algebra. If $a \in \mathcal{I}(M)$, then $a = (a_t)_t$, where $a_t \in \mathcal{I}(M_t)$ for each $t \in T$. Then $[0,a] = \prod_{t \in T} [0, a_t] \in \mathcal{V}$, so that $M \in \mathbb{V}_0(\mathcal{V})$.

Summarizing up, $\mathbb{V}_0(\mathcal{V})$ is a q-subvariety of pseudo EMV-algebras.

Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two varieties of pseudo MV-algebras such that $\mathcal{V}_1 \neq \mathcal{V}_2$. There is a pseudo MV-algebra $M \in \mathcal{V}_1 \setminus \mathcal{V}_2$ (or $M \in \mathcal{V}_2 \setminus \mathcal{V}_1$), and clearly, $(M; \vee, \wedge, \oplus, 0) \in \mathbb{V}_0(\mathcal{V}_1) \setminus \mathbb{V}_0(\mathcal{V}_2)$. The mapping $\mathcal{V} \mapsto \mathbb{V}_0(\mathcal{V})$ preserves the set-theoretical inclusion.

Now, let $\mathbb{V}$ be a q-subvariety of pseudo EMV-algebras. It is straightforward to see that $\mathcal{V}_0(\mathbb{V})$ is a variety of pseudo MV-algebras. If $M \in \mathbb{V}$, then $([0,a]; \oplus, \lambda_a, \rho_a, 0, a)$ belongs to $\mathcal{V}_0(\mathbb{V})$ for each $a \in \mathcal{I}(M)$ because $([0,a]; \vee, \wedge, \oplus, 0)$ is a pEMV-subalgebra of M, which yields

$$\mathbb{V} \subseteq \mathbb{V}_0(\mathcal{V}_0(\mathbb{V})). \tag{5.3}$$

Now, we show that $\mathbb{V}_0(\mathcal{V}_0(\mathbb{V})) \subseteq \mathbb{V}$. Let $M \in \mathbb{V}_0(\mathcal{V}_0(\mathbb{V}))$. If $a \in \mathcal{I}(M)$, then $([0,a]; \oplus, \lambda_a, \rho_a, 0, a) \in \mathcal{V}_0(\mathbb{V})$ and so, for all $a \in \mathcal{I}(M)$, we conclude $([0,a]; \vee, \wedge, \oplus, 0)$ is from $\mathbb{V}$. Consequently, the pseudo EMV-algebra $\prod_{a \in \mathcal{I}(M)} [0,a]$ is with top element and it belongs to $\mathbb{V}$. Define $\varphi : M \to \prod_{a \in \mathcal{I}(M)} [0,a]$, by $\varphi(x) = (x \wedge a)_{a \in \mathcal{I}(M)}$. Similarly to the proof of the claim, we can establish that φ is a homomorphism of pseudo EMV-algebras, which is clearly one-to-one. Indeed, φ preserves $\vee, \wedge, \oplus$ and 0. We show that it preserves also λ_b and ρ_b for each $b \in \mathcal{I}(M)$.

Given $x \in M$, there is $b \in \mathcal{I}(M)$ such that $x \leq b$. First we establish that, for all

$a \in \mathcal{I}(M)$, we have $\lambda_b(x) \wedge a = \lambda_{(b\wedge a)}(x \wedge a)$. Put $a \in \mathcal{I}(M)$. Since $x \le b$, then

$$\begin{aligned}\lambda_{(b\wedge a)}(x\wedge a) &= \lambda_b(x\wedge a)\wedge(a\wedge b)\\ &= \lambda_b(x\wedge b\wedge a)\wedge(a\wedge b)\\ &= \big(\lambda_b(x)\vee\lambda_b(b\wedge a)\big)\wedge(a\wedge b) = \lambda_b(x)\wedge(a\wedge b) = \lambda_b(x)\wedge a.\end{aligned}$$

Then, we have

$$\varphi(\lambda_b(x)) = (\lambda_b(x)\wedge a)_{a\in\mathcal{I}(M)}$$

and

$$\lambda_{\varphi(b)}(\varphi(x)) = \lambda_{(b\wedge a)_{a\in\mathcal{I}(M)}}((x\wedge a)_{a\in\mathcal{I}(M)}) = \big(\lambda_{(b\wedge a)}(x\wedge a)\big)_{a\in\mathcal{I}(M)},$$

which implies $\varphi(\lambda_b(x)) = \lambda_{\varphi(b)}(\varphi(x))$.

In the analogous way we deal also with ρ_b.

Hence, $\varphi(M)$ is a subdirect product in $\prod_{a\in\mathcal{I}(M)}[0,a] \in \mathbb{V}$ which entails $\varphi(M) \in \mathbb{V}$, so that $M \in \mathbb{V}$. Consequently,

$$\mathbb{V} = \mathbb{V}_0(\mathcal{V}_0(\mathbb{V})). \tag{5.4}$$

If $\mathbb{V}_1$ and $\mathbb{V}_2$ are q-subvarieties of pseudo EMV-algebras with $\mathcal{V}_0(\mathbb{V}_1) = \mathcal{V}_0(\mathbb{V}_2)$, then $\mathbb{V}_1 = \mathbb{V}_0(\mathcal{V}_0(\mathbb{V}_1)) = \mathbb{V}_0(\mathcal{V}_0(\mathbb{V}_2)) = \mathbb{V}_2$. Thus the mapping $\mathbb{V} \mapsto \mathcal{V}_0(\mathbb{V})$ is injective, and clearly it preserves the set-theoretical inclusion.

Let M be any pseudo MV-algebra belonging to a given variety of pseudo MV-algebras $\mathcal{V}$. Using definitions of $\mathbb{V}_0$ and $\mathcal{V}_0$, we see that $M \in \mathcal{V}_0(\mathbb{V}_0(\mathcal{V}))$, i.e. $\mathcal{V} \subseteq \mathcal{V}_0(\mathbb{V}_0(\mathcal{V}))$. Using (5.4), we get

$$\mathbb{V}_0(\mathcal{V}) \subseteq \mathbb{V}_0(\mathcal{V}_0(\mathbb{V}_0(\mathcal{V}))) = \mathbb{V}_0(\mathcal{V}),$$

that is, $\mathcal{V} = \mathcal{V}_0(\mathbb{V}_0(\mathcal{V}))$. Therefore, both mentioned mappings are mutually invertible.

Due to [7, Thm 7.2], the variety of pseudo MV-algebras $\mathcal{PMV}$ has uncountably many subvarieties, therefore, the q-variety $\mathbb{PEMV}$ of pseudo EMV-algebras has also uncountably many q-subvarieties. $\square$

We denote by $\mathbb{RPEMV}$ the class of representable pseudo EMV-algebras. In what follows, we show as an easy corollary of the latter theorem that $\mathbb{RPEMV}$ is a q-variety of pseudo EMV-algebras.

Theorem 5.17. *The class $\mathbb{RPEMV}$ of representable pseudo EMV-algebras is a q-variety of pseudo EMV-algebras.*

Proof. It is well known that the class of representable pseudo MV-algebras is a variety, see e.g. [32]. Due to Theorem 5.15, a pseudo EMV-algebra M is representable iff it satisfies two equations from Theorem 5.15 for each $a \in \mathcal{I}(M)$. Due to [32, Thm 3.4], if $a = 1$, it is equivalent that a pseudo MV-algebra is representable.

The mentioned two equations say that, for every $a \in \mathcal{I}(M)$, the interval algebra $([0,a]; \oplus, \lambda_a, \rho_a, 0, a)$ is a representable pseudo MV-algebra which by Theorem 5.16 means that $\mathbb{RPEMV}$ is a q-variety. $\square$

Proposition 5.18. *If M is a representable pseudo EMV-algebra which is subdirectly irreducible, then M is linearly ordered. The q-variety $\mathbb{RPEMV}$ coincides with the q-variety generated by all linear pseudo EMV-algebras, and every representable pseudo EMV-algebra is a subdirect product of subdirectly irreducible linear pseudo EMV-algebras.*

Proof. If M is trivial, i.e. $M = \{0\}$, the statement is evident. Suppose M is not trivial. By Lemma 5.9, there is a system $\{P_t \colon t \in T\}$ of normal prime ideals of M such that $\bigcap_{t\in T} P_t = \{0\}$. By Theorem 5.10, M can be subdirectly embedded into $\prod_{t\in T} M/P_t$. Since M is subdirectly irreducible, there is $t_0 \in T$ such that M can be embedded onto M/P_t which means that M is linearly ordered. By Proposition 5.1, M is linearly ordered and it has a top element 1. Consequently, M is termwise equivalent to the representable pseudo MV-algebra $(M; \oplus, \lambda_1, \rho_1, 0, 1)$.

According to [25, p. 258], every representable pseudo MV-algebra is a subdirect product of subdirectly irreducible pseudo MV-algebras which are therefore linearly ordered.

Therefore, due to the Birkhoff Subdirect Representation Theorem, see [1, Thm 8.6], which holds also for pseudo EMV-algebras, every representable pseudo EMV-algebra M is a subdirect product of subdirectly irreducible linearly ordered pseudo EMV-algebras. $\square$

We note that in [18, Thm 5.22], we have proved that the lattice of q-subvarieties of EMV-algebras is countably infinite, and passage to non-commutative pseudo EMV-algebras gives the uncountable lattice of q-subvarieties of $\mathbb{PEMV}$. Since there are uncountably many varieties of representable ℓ-groups, see e.g. [6, Thm 61.24], using [7, Thm 5.5], we can show that there are uncountably many q-subvarieties of representable pseudo EMV-algebras.

Let us denote by $\mathrm{Lat}(\mathbb{PEMV})$ the lattice of q-subvarieties of the q-variety $\mathbb{PEVM}$. The equation $x = 0$ defines the q-variety $\mathbb{O}$ which is a singleton and it is the least q-subvariety of pseudo EMV-algebras. The equation $x \oplus x = x$ defines the q-variety $\mathbb{B}$ of generalized Boolean algebras which in its own right is a variety of generalized Boolean algebras. It is an atom in the lattice $\mathrm{Lat}(\mathbb{PEMV})$ because the variety of

Boolean algebras is an atom in the lattice of subvarieties of the variety of pseudo MV-algebras, see [14, Thm 3.1] and apply Theorem 5.16. In other words, for each q-subvariety $\mathbb{V}$ of pseudo EMV-algebras, we have

$$\mathbb{O} \subsetneq \mathbb{B} \subseteq \mathbb{V}. \tag{5.5}$$

It is well-known that the variety of Abelian ℓ-groups and the variety of normal-valued ℓ-groups is the least non-trivial variety and the largest proper variety of ℓ-groups, see [28, Thm 10C]. Due to Theorem 2.2, there is a categorical equivalence between the category of pseudo MV-algebras and the category of unital ℓ-groups which gives a more finer structure of the variety lattice of pseudo MV-algebras: Due to Komori [30], the lattice of subvarieties of the variety of MV-algebras is countably infinite. According to [7, Prop 6.2], the class $\mathcal{M}$ of pseudo MV-algebras M such that either $M = \{0\}$ or every maximal ideal of M is normal is a proper subvariety of the variety of pseudo MV-algebras. The class of pseudo MV-algebras M such that either $M = \{0\}$ or M is normal-valued is also a variety, see [11, Thm 6.8]. In addition, $\mathcal{N}$ is a proper subvariety of $\mathcal{M}$. Indeed, let G be an ℓ-group that is not normal-valued. Then $M = \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$ is a pseudo MV-algebra that has only one maximal ideal, namely $I = \{0\} \times G^+$ and it is normal and not normal-valued. Pseudo EMV-algebras connected with $\mathcal{M}$ and $\mathcal{N}$ will be more detailed studied at the end of Section 6 of the part II of present paper.

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Pseudo EMV-algebras. II. Representation and States

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Abstract

The paper is a continuation of the research on pseudo EMV-algebras which started in [13]. We show that every pseudo EMV-algebra without top element can be embedded into a pseudo EMV-algebra with top element as a maximal and normal ideal of the latter one. We present a categorical equivalence of the category of pseudo EMV-algebras without top element with a special category of pseudo MV-algebras or with a special category of ℓ-groups. Finally, we study states as finitely additive mappings as well as state-morphisms on pseudo EMV-algebras. We show that each state, if it exists, is a weak limit of a net of convex combinations of state-morphisms, and every state can be represented as an integral with respect to a unique regular σ-additive Borel probability measure.

In this paper we continue the work from [13], where we introduced new algebraic structures called pseudo EMV-algebras and we have presented their basic properties. Sections, theorems, propositions, lemmas, examples, and equations are numbered in continuation of [13].

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6 Representation of Pseudo EMV-algebras

In the present section, we give a basic representation theorem of pseudo EMV-algebras by pseudo EMV-algebras with top element. We show that every pseudo EMV-algebra M is either a pseudo EMV-algebra with top element or it can be embedded into a pseudo EMV-algebra N with top element as a maximal and normal ideal of N. It generalizes the representation theorem for representable pseudo EMV-algebras from [13, Thm 5.12], the representation theorem [10, Thm 5.21] for EMV-algebras and the representation theorem [5, Thm. 2.2] for generalized Boolean algebras.

We start with the following result.

Theorem 6.1. *Let M be a proper pseudo EMV-algebra. If there is a pseudo EMV-algebra N with top element such that M can be embedded into N, then there is a pseudo EMV-subalgebra N_0 of N, N_0 is with top element, such that M can be embedded into N_0 as a maximal and normal ideal of N_0.*

Proof. Let N be a pseudo EMV-algebra with top element 1 such that M can be embedded into N. Without loss of generality, we can assume that M is a pseudo EMV-subalgebra of N. We define

$$N_0 = \{x \in N\colon \text{ either } x = x_0 \in M \text{ or } x = \rho_1(x_0) \text{ for some } x_0 \in M\}.$$

Claim. If $x_0 \in M$ and $x_0 \leq a \in \mathcal{I}(M)$, then $\lambda_1^2(x_0), \rho_1^2(x_0) \in M$. Moreover, $\lambda_1^2(x_0) = \lambda_a^2(x_0) =: \varphi_\lambda(x_0)$, $\rho_1^2(x_0) = \rho_a^2(x_0) =: \varphi_\rho(x_0)$, and $\varphi_\lambda(x_0)$ and $\varphi_\rho(x_0)$ do not depend on $a \geq x_0$.

Let $a \in \mathcal{I}(M)$ be an idempotent such that $x_0 \leq a$. By Proposition 3.3(iii), we have $\lambda_1(x_0) = \lambda_a(x_0) \oplus \lambda_1(a)$, so that by (3.3) and (3.1), we have $\lambda_1^2(x_0) = \lambda_1(\lambda_a(x_0) \oplus \lambda_1(a)) = a \odot \lambda_1(\lambda_a(x_0)) = a \odot (\lambda_a^2(x_0) \oplus \lambda_1(a)) = a \wedge (\lambda_a^2(x_0) \vee \lambda_1(a)) = (a \wedge \lambda_a^2(x_0)) \vee (a \odot \lambda_1(a)) = a \wedge \lambda_a^2(x_0) = \lambda_a^2(x_0) = \varphi_\lambda(x_0) \in M$. In the same way we establish $\rho_1^2(x_0) = \varphi_\rho(x_0) \in M$, and $\varphi_\lambda(x_0)$ and $\varphi_\rho(x_0)$ do not depend on $a \in \mathcal{I}(M)$ such that $x_0 \leq a$.

In what follows, we show that N_0 is a pseudo EMV-algebra with top element. Since $(N; \oplus, \lambda_1, \rho_1, 0, 1)$ is a pseudo MV-algebra that is termwise equivalent to the pseudo EMV-algebra N, we will use $x^- := \lambda_1(x)$ and $x^\sim := \rho_1(x)$, $x \in N$. First we note that according to Claim, we have $x_0^- = (x_0^{--})^\sim = \varphi_\lambda(x_0)^\sim \in N_0$ for each $x_0 \in M$, so that, N_0 is closed under $^-$ and $^\sim$.

Clearly, $M \subseteq N_0$ and $1 \in N_0$. Let $x, y \in N_0$. Using ideas from the proof of Theorem 5.11, we show that $x \oplus y \in N_0$. There are four cases. Case (i): $x = x_0, y = y_0 \in M$. Then $x \vee y, x \wedge y, x \oplus y \in N_0$.

Case (ii): $x=x_0^\sim$, $y=y_0^\sim$ for some $x_0,y_0\in M$. Then $x\vee y=x_0^\sim\vee y_0^\sim=(x_0\wedge y_0)^\sim$, $x\wedge y=(x_0\vee y_0)^\sim$ and $x\oplus y=x_0^\sim\oplus y_0^\sim=(y_0\odot x_0)^\sim\in N_0$.

Case (iii): $x=x_0$ and $y=y_0^\sim$ for some $x_0,y_0\in M$. Then

$$x\oplus y=x_0\oplus y_0^\sim=(y_0\odot x_0^-)^\sim=(y_0\odot(x_0\wedge y_0)^-)^\sim=(y_0\odot\lambda_b(x_0\wedge y_0))^\sim,$$

where b is an idempotent of M such that $x_0,y_0\le b$; for the last equality we use equality (3.1) of Proposition 3.4. Using again Proposition 3.4, we have $y_0\odot\lambda_b(x_0\wedge y_0)\in M$ so that $x\oplus y\in N_0$.

Case (iv): $x=x_0^\sim$ and $y=y_0$ for some $x_0,y_0\in M$. There is $b\in\mathcal{I}(M)$ such that $x_0,y_0\le b$. Check

$$\begin{aligned}x\oplus y&=x_0^\sim\oplus y_0=(y_0^\sim\odot x_0^{\sim\sim})^-=(y_0^\sim\odot\varphi_\rho(x_0))^-=\big((y_0\wedge\varphi_\rho(x_0))^\sim\odot\varphi_\rho(x_0)\big)^-\\&=\big((\rho_b(y_0\wedge\varphi_\rho(x_0))\vee b^\sim)\odot\varphi_\rho(x_0)\big)^{--\sim}=\big(\rho_b(y_0\wedge\varphi_\rho(x_0))\odot\varphi_\rho(x_0)\big)^{--\sim}\\&=\big(\varphi_\lambda(\rho_b(y_0\wedge\varphi_\rho(x_0))\odot\varphi_\rho(x_0))\big)^\sim\in N_0.\end{aligned}$$

Now let $x_0\in M$. Then $x_0^\sim\in N_0\setminus M$ and if we set $y_0=x_0^{--}=\varphi_\lambda(x_0)\in M$, then $x_0^-=y_0^\sim\in N_0\setminus M$. Claim implies $x_0^{\sim\sim}=\varphi_\rho(x_0)\in M$ and $x_0^{--}=\varphi_\lambda(x_0)\in M$. Hence, $(N_0;\oplus,^-,^\sim,0,1)$ is a pseudo MV-algebra, so that $(N_0;\vee,\wedge,\oplus,0)$ is its termwise equivalent pseudo EMV-algebra with top element.

The set M is a proper subset of N_0 and M is closed under $\oplus$. Let $y\le x_0\in M$. Then y cannot be from $N_0\setminus M$, otherwise $1\in M$. Hence $y\in M$ and M is an ideal of N_0. If we take $y\in N_0\setminus M$, then the ideal I of N_0 generated by $M\cup\{y\}$ contains 1, so that $I=N_0$ which establishes M is a maximal ideal of N_0. To prove normality of M in N_0, let $y\oplus x_0\in y\oplus M$, $y\in N_0$. It is sufficient to assume $y=y_0^\sim$ for some $y_0\in M$. Then $y_0^\sim\oplus x_0=(y_0^\sim\oplus x_0)\vee y_0^\sim=((y_0^\sim\oplus x_0)\odot y_0)\oplus y_0^\sim$. Since $(y_0^\sim\oplus x_0)\odot y_0\le y_0\in M$, we have $(y_0^\sim\oplus x_0)\odot y_0\in M$, so that $y_0^\sim\oplus M\subseteq M\oplus y_0^\sim$. In a similar way, we prove the opposite inclusion. $\square$

It is clearly from the proof of the latter theorem that if M is a proper pseudo EMV-algebra and N_1,N_2 are two pseudo EMV-algebras with top elements such that $\phi_i:M\to N_i$, $i=1,2$, is a pEMV-embedding of M into N_i such that $\phi_i(M)$ is a maximal and normal ideal of N_i and every element of N_i either belongs to $\phi_i(M)$ or it is a complement of some element from $\phi_i(M)$, then N_1 and N_2 are isomorphic pseudo EMV-algebras, see also Proposition 7.2 below.

Now, we apply the latter result to the case of representable pseudo EMV-algebra and it will generalize Theorem 5.11.

Theorem 6.2. *Let M be a representable pseudo EMV-algebra. Then either M has a top element or there is a representable pseudo EMV-algebra N_0 with top element such that M can be embedded into N_0 as a normal and maximal ideal of N_0.*

Proof. If M has a top element, the statement is trivial. Thus assume that M is a proper pseudo EMV-algebra. By Theorem 5.10, there is a representable pseudo EMV-algebra N with top element such that M can be embedded into N. Applying Theorem 6.1, there is a pseudo EMV-subalgebra N_0 of M such that M can be embedded into N_0 as a normal and maximal ideal of N_0. Since N is representable and N_0 is a pseudo EMV-subalgebra, N_0 is also representable. ☐

Motivated by the proof of Theorem 5.16, we present the following useful proposition which will be used also for the Basic Representation Theorem.

Proposition 6.3. *Every proper pseudo EMV-algebra can be embedded into a pseudo EMV-algebra with top element.*

Proof. For each idempotent $a \in \mathcal{I}(M)$, $([0,a]; \vee, \wedge, \oplus, 0)$ is a pseudo EMV-algebra with top element. Therefore, $N = \prod_{a\in\mathcal{I}(M)}[0,a]$ is a pseudo EMV-algebra with top element. According to the proof of Theorem 5.16, the mapping $\varphi : M \to N$ defined by $\varphi(x) = (x \wedge a)_{a\in\mathcal{I}(M)}[0,a]$, $x \in M$, is an embedding of M into a pseudo EMV-algebra N with top element which finishes the proof. ☐

Finally, we present the main result of the section.

Theorem 6.4. [Basic Representation Theorem] *Every pseudo EMV-algebra M is either a pseudo EMV-algebra with top element or M can be embedded into a pseudo EMV-algebra N_0 with top element as a maximal and normal ideal of N_0. In the second case, every element $x \in N_0$ is either from the image of M or there is a unique x_0 from the image of M such that $x = \rho_1(x_0)$.*

Proof. If M has a top element, then the statement holds trivially.

Thus, assume that M is a proper pseudo EMV-algebra. According to Proposition 6.3, there is a pseudo EMV-algebra N with top element into which M can be embedded. The final conclusion follows applying Theorem 6.1 and its proof of the set N_0. ☐

The pseudo EMV-algebra N_0 with top element from Theorem 6.4 is said to be the pseudo EMV-algebra *representing* M. According to a note just after Theorem 6.1, all pseudo EMV-algebras with top element representing M are isomorphic, see also Proposition 7.2 below.

It is worthy of noting that due to Theorem 6.4 if a pseudo EMV-algebra with top element has no maximal and normal ideal, then it cannot be a representing one of any pseudo EMV-algebra without top element. Such a pseudo EMV-algebra with top element is e.g. in Theorem 5.22, see [9, Ex 5.3].

To illustrate how does the Basic Representation Theorem work, take a pseudo EMV-algebra M from Example 3.5, i.e. let $\{M_i\colon i \in I\}$ be an infinite system of pseudo MV-algebras where 0_i and 1_i are least and top elements of M_i for each $i \in I$. Let M be the system of all $(x_i)_i \in \prod_{i\in I} M_i$ such that $x_i = 0_i$ for all but finitely many indices $i \in I$ and let N be the system of all $(x_i)_i \in \prod_{i\in I} M_i$ such that either $x_i = 0_i$ for all but finitely many indices i or $x_i = 1_i$ for all but finitely many indices i. Then M is a proper pseudo EMV-algebra and N is its representing pseudo EMV-algebra with top element.

Theorem 6.5. *Let $\mathbb{V}$ be a q-subvariety of pseudo EMV-algebras. If $M \in \mathbb{V}$, then every its representing pseudo EMV-algebra N_0 with top element also belongs to $\mathbb{V}$.*

Proof. If M has a top element, then $N_0 = M$ satisfies the theorem. Thus assume that M does not have a top element. According to Proposition 6.3, the pseudo EMV-algebra $N = \prod_{a\in\mathcal{I}(M)}[0,a]$ is a pseudo EMV-algebra with top element such that M can be embedded into N under the embedding φ from the proof of Proposition 6.3. In addition, $(N; \oplus, \lambda_1, \rho_1, 0, 1)$ is the termwise equivalent pseudo MV-algebra.

According to Theorem 6.1, there is a pseudo EMV-algebra N_0 with top element such that M can be embedded into N_0 as a maximal and normal ideal of N_0. Then $(N_0, \oplus, ^-, ^\sim, 0, 1)$ is a pseudo MV-algebra, see the proof of Theorem 6.1; let $\psi : M \to N_0$ be the embedding in question. Then

$$N_0 = \psi(M) \cup \psi(M)^\sim,$$

where $\psi(M)^\sim = \{(\psi(x))^\sim\colon x \in M\}$. Put $\widetilde{N_0} := \varphi(M) \cup \rho_1(\varphi(M))$, with $\rho_1(\varphi(M)) := \{\rho_1(\varphi(x))\colon x \in M\}$. Then $(\widetilde{N}_0; \oplus, \lambda_1, \rho_1, 0, 1)$ can be understood as a pseudo MV-algebra isomorphic to $(N_0, \oplus, ^-, ^\sim, 0, 1)$, and $(\widetilde{N}_0; \oplus, \lambda_1, \rho_1, 0, 1)$ is a pseudo subalgebra of $(N; \oplus, \lambda_1, \rho_1, 0, 1)$.

Using (5.4) from Theorem 5.16, we see that since $M \in \mathbb{V}$, we have $M \in \mathbb{V}_0(\mathcal{V}_0(\mathbb{V}))$, so that for each $a \in \mathcal{I}(M)$, the pseudo MV-algebra $([0,a]; \oplus, \lambda_a, \rho_a, 0, a)$ belongs to $\mathcal{V}_0(\mathbb{V})$ and every pseudo EMV-algebra with top element $([0,a]; \vee, \wedge, \oplus, 0)$ is an element of $\mathbb{V}$. Then $N \in \mathbb{V}$ and $\widetilde{N}_0 \in \mathbb{V}$, finally $N_0 \in \mathbb{V}$. $\square$

In the following result we characterize subdirectly irreducible pseudo EMV-algebras by their representing pseudo EMV-algebras with top element. First we note that if N is a pseudo EMV-algebra with top element, it is termwise equivalent to a pseudo MV-algebra $(N; \oplus, ^-, ^\sim, 0, 1)$ and due to [8], it can be represented by a unital ℓ-group (G, u) such that $N \cong \Gamma(G, u)$, and 1 corresponds to u. So that, we can represent $x^\sim$ as $-x + 1$ and $x^\sim \oplus j$ is $(-x + 1 + j) \wedge 1$.

Proposition 6.6. *Let M be a proper pseudo EMV-algebra and N be its representing pseudo EMV-algebra with top element. Then M is subdirectly irreducible if and only if N is subdirectly irreducible.*

Proof. Let M be subdirectly irreducible. Due to Theorem 4.8, there is the least non-trivial normal ideal J_0 of M. We assert J_0 is also a normal ideal of N. Indeed, let $x \in M$ and $j \in J_0$, then $x^{\sim} \oplus j = (x^{\sim} \oplus j) \vee x^{\sim} = ((x^{\sim} \oplus j) \odot x) \oplus x^{\sim}$ and, for $j_0 := (x^{\sim} \oplus j) \odot x$, we have $j_0 = ((-x+1+j) \wedge 1) \odot x = [((-x+1+j) \wedge 1) - 1 + x] \vee 0 = ((-x+1+j-1+x) \wedge x) \vee 0 = (-x + \varphi_\lambda(j) + x) \vee 0 = -x + \varphi_\lambda(j) + x \in J_0$. In a similar way, we establish also the second equality.

Let I be any non-trivial ideal of N and let $I_0 = I \cap M$. Then I_0 is an ideal of M and if $x \in M$ and $i \in I_0$, we have $x \oplus i = ((x \oplus i) \odot x^-) \oplus x$ and $(x \oplus i) \odot x^- = (x + i - x) \wedge (1 - x) \in I \cap M$. Dually we have the second property, which proves I_0 is normal. Therefore, $J_0 \subseteq I_0$ and J_0 is the least non-trivial normal ideal of N.

Conversely, let J_0 be the least non-trivial normal ideal of N. Then $I_0 := J_0 \cap M$ is a normal and non-trivial ideal of M. Now let I be a normal non-trivial ideal of M. Then it is a normal ideal of N, so that $J_0 \subseteq I$ and $I_0 \subseteq I$ proving M is subdirectly irreducible. ☐

Now we apply the Basic Representation Theorem to describe some q-subvarieties of pseudo EMV-algebras. In the following result we show how we can find proper pseudo EMV-algebras in each q-subvariety of pseudo EMV-algebras. We will use notations from [13, Thm 5.18].

Proposition 6.7. *Given a q-subvariety $\mathbb{V}$ of pseudo EMV-algebras, there is a unique subvariety $\mathcal{V}$ of pseudo MV-algebras such that $\mathbb{V} = \mathbb{V}_0(\mathcal{V})$, where $\mathbb{V}_0(\mathcal{V})$ is the class of pseudo EMV-algebras M such that, for each $a \in \mathcal{I}(M)$, $([0,a]; \oplus, \lambda_a, \rho_a, 0, a) \in \mathcal{V}$. Let $\mathbb{V}_0^p(\mathcal{V})$ be the class of proper EMV-algebras from $\mathbb{V}_0(\mathcal{V})$. Then $\mathbb{V}^p := \mathbb{V}_0^p(\mathcal{V})$ is non-empty and there is an injective mapping $\zeta : \mathcal{V} \to \mathbb{V}^p$.*

Proof. If $\mathbb{V}$ is a q-subvariety, due to [13, Thm 5.18], there is a unique subvariety $\mathcal{V}$ of pseudo MV-algebras such that $\mathbb{V} = \mathbb{V}_0(\mathcal{V})$. Let M_1 be the EMV-algebra of all finite subsets of the set of natural numbers $\mathbb{N}$; it is a proper EMV-algebra. Its representing EMV-algebra is the set N of all subsets of $\mathbb{N}$ which are finite or co-finite. Take an arbitrary pseudo MV-algebra $M_2 \in \mathcal{V}$, and now let it be understood as a pseudo EMV-algebra with top element. Then $M = M_1 \times M_2$ is a proper pseudo EMV-algebra belonging to $\mathbb{V}^p$ and its representing pseudo EMV-algebra with top element is $N \times M_2$. As a pseudo MV-algebra, $N \times M_2$ belongs to $\mathcal{V}$. The mapping $\zeta : \mathcal{V} \to \mathbb{V}^p$ defined by $\zeta(M) = M_1 \times M$, $M \in \mathcal{V}$, and M is understood as a pseudo EMV-algebra with top element, is an injective mapping in question. ☐

At the end of the section, we present two interesting q-subvarieties of pseudo EMV-algebras, namely the q-subvariety $\mathbb{MPEMV}$ of pseudo EMV-algebras M whose each maximal ideal is normal or $M=\{0\}$, and the q-subvariety $\mathbb{NPEMV}$ of normal-valued ones and compare them with other q-subvarieties.

Proposition 6.8. *Let M be a proper pseudo EMV-algebra such that every maximal ideal of M is normal. If N is its representing pseudo EMV-algebra with top element, then every maximal ideal of N is normal.*

Proof. Without loss of generality, we can assume that $M\subseteq N$. Let I be any maximal ideal of N. Set $I_0=I\cap M$; it is an ideal of M. If $M\cap I=M$, then $M\subseteq I$ and maximality of M in N entails that $I=M$ and I is a normal ideal of I since M is a normal and maximal ideal of N, see Theorem 6.2

Assume $I_0\neq M$. Then I_0 is a proper ideal of M. We assert that I_0 is a maximal ideal of M. Take $x_0\in M\setminus I_0$. Then $x_0\notin I$, so that the ideal $I_0(I,x_0)$ of N generated by I and x_0 has to be equal to N. Given $x\in M$, there are $i_1,\dots,i_k\in I$ and integers $n_1,\dots n_k\geq 0$ such that $x\leq i_1\oplus n_1.x_0\oplus\cdots\oplus i_k\oplus n_k.x_0$. Due to [13, (3.8)], we have $x=j_1\oplus x_1\oplus\cdots\oplus j_k\oplus x_k$, where $j_s\leq i_s$ and $x_s\leq n_s.x_0$ for $s=1,\dots,k$. Then every $j_s\in I_0$ which proves that the ideal of M generated by I_0 and x_0 is M, so that I_0 is maximal and, consequently, I_0 is a normal ideal of M.

According to the proof of Theorem 6.1, if $x\in N\setminus M$, there are $x_1,x_2\in M$ such that $x_1^-=x=x_2^\sim$.

Now let $x\in N$ and $y\in I$. We assert that there are $z_1,z_2\in I$ such that $x\oplus y=z_1\oplus x$ and $y\oplus x=x\oplus z_2$. There are four cases.

(i) $x,y\in M$. Then trivially the claim is satisfied.

(ii) $x\in M$, $y\in N\setminus M$. Then $x=x_0$ and $y=y_0^\sim$ for $x_0,y_0\in M$, and there is $a\in\mathcal{I}(M)$ such that $x_0,y_0\leq a$. Using [13, Prop 3.3], we have

$$
\begin{aligned}
x\oplus y=x_0\oplus y_0^\sim&=(y_0\odot x_0^-)^\sim=(y_0\odot\lambda_a(x_0))^\sim=\rho_a(y_0\odot\lambda_a(x_0))\vee a^\sim\\
&=(x_0\oplus\rho_a(y_0))\vee a^\sim=(z_0\oplus x_0)\vee a^\sim=(z_0\oplus x_0)\oplus a^\sim=(z_0\oplus a^\sim)\oplus x_0\\
&=z_1\oplus x,
\end{aligned}
$$

where $\rho_a(y_0),z_0\in I_0$ and $z_1=a^\sim\vee z_0\in I$ because $y=y_0^\sim=(y_0\wedge a)^\sim=y_0^\sim\vee a^\sim\in I$, so that $a^\sim\in I$.

To find $z_2\in I$, assume that $y=y_0^-$ for some $y_0\in M$. Again let $x_0,y_0\leq a\in\mathcal{I}(M)$. Then

$$
\begin{aligned}
y\oplus x=y_0^-\oplus x_0&=(x_0^\sim\odot y_0)^-=(\rho_a(x_0)\odot y_0)^-=\lambda_a(\rho_a(x_0)\odot y_0)\vee a^-=\\
&=(\lambda_a(y_0)\oplus x_0)\vee a^-=(x_0\oplus z_0)\vee a^-=x_0\oplus z_2=x\oplus z_2,
\end{aligned}
$$

where $\lambda_a(y_0) \leq y_0^- \in I$, so that $\lambda_a(y_0) \in I_0$, and $z_0 \in I_0$ and $z_2 = z_0 \vee a^- \in I$ because $y = y_0^- = (y_0 \wedge a)^- = y_0^- \vee a^- \in I$, so that $a^- \in I$.

(iii) $x \in N \setminus M$, $y \in I_0$. Then $x = x_0^-$ and $y = y_0$ for some $x_0, y_0 \in M$. There is $a \in \mathcal{I}(M)$ such that $x_0, y_0 \leq a$. Then we have

$$\begin{aligned} x \oplus y = x_0^- \oplus y_0 &= (y_0^\sim \odot x_0)^- = (\rho_a(y_0) \odot x_0)^- = \lambda_a(\rho_a(y_0) \odot x_0) \vee a^- \\ &= (\lambda_a(x_0) \oplus y_0) \vee a^- = (z_1 \oplus \lambda_a(x_0)) \vee a^- = z_1 \oplus x, \end{aligned}$$

where $z_1 \in I_0$ and $x = x_0^- = \lambda_a(x_0) \vee a^-$.

For the second z_2, we assume $x = x_0^\sim$ and $y = y_0$ for some $x_0, y_0 \leq a \in \mathcal{I}(M)$. Then

$$\begin{aligned} y \oplus x = y_0 \oplus x_0^\sim &= (x_0 \odot y_0^-)^\sim = (x_0 \odot \lambda_a(y_0))^\sim = \rho_a(x_0 \odot \lambda_a(y_0)) \vee a^\sim \\ &= (y_0 \oplus \rho_a(x_0)) \vee a^\sim = (\rho_a(x_0) \oplus z_2) \vee a^\sim = x \oplus z_2, \end{aligned}$$

where $z_2 \in I_0$ and $x = x_0^\sim = \rho_a(x_0) \vee a^\sim$.

(iv) $x, y \in N \setminus M$. Then $x = x_0^-$ for some $x_0 \in M$. Using [13, (3.3)], we have

$$\begin{aligned} x \oplus y = x_0^- \oplus y &= (y^\sim \odot x_0)^- = ((y \wedge x_0)^\sim \odot x_0)^- = (\rho_a(y \wedge x_0) \odot x_0)^- \\ &= \lambda_a(\rho_a(y \wedge x_0) \odot x_0) \vee a^- = (\lambda_a(x_0) \oplus (y \wedge x_0)) \vee a^- = (z_1 \oplus \lambda_a(x_0)) \vee a^- \\ &= z_1 \oplus x, \end{aligned}$$

where $y \wedge x_0, z_1 \in I_0$ and $x = x_0^- = \lambda_a(x_0) \vee a^-$.

To find z_2, assume $x = x_0^\sim$ for some $x_0 \in M$. Using again [13, (3.3)], we obtain

$$\begin{aligned} y \oplus x = y \oplus x_0^\sim &= (x_0 \odot y^-)^\sim = (x_0 \odot (x_0 \wedge y)^-)^\sim = (x_0 \odot \lambda_a(x_0 \wedge y))^\sim \\ &= \rho_a(x_0 \odot (\lambda_a(x_0 \wedge y))) \vee a^\sim = ((x_0 \wedge y) \oplus \rho_a(x_0)) \vee a^\sim = (\rho_a(x_0) \oplus z_2) \vee a^\sim \\ &= x \oplus z_2, \end{aligned}$$

where $x_0 \wedge y, z_2 \in I_0$ and $x = x_0^\sim = \rho_a(x_0) \vee a^\sim$.

Hence, every maximal ideal of N is normal. □

As it was already mentioned in the first part, according to [6, Prop 6.2], the class $\mathcal{M}$ of pseudo MV-algebras M such that either $M = \{0\}$ or every maximal ideal of M is normal is a proper subvariety of the variety of pseudo MV-algebras.

Corollary 6.9. *Let M be a pseudo EMV-algebra such that every maximal ideal of M is normal. For each $a \in \mathcal{I}(M)$, the pseudo MV-algebra $([0,a]; \oplus, \lambda_a, \rho_a, 0, a)$ belongs to the variety $\mathcal{M}$ of pseudo MV-algebras.*

Proof. Due to Proposition 6.8, the representing pseudo EMV-algebra N with top element has the property that every maximal ideal of N is normal, so that $N \in \mathcal{M}$. Define a mapping $h_a : N \to [0,a]$ given by $h_a(x) = x \wedge a$, $x \in N$. According to Claim from the proof of [13, Thm 5.18], h_a is a surjective homomorphism of pseudo MV-algebras, then $([0,a]; \oplus, \lambda_a, \rho_a, 0, a)$ belongs to the variety $\mathcal{M}$ of pseudo MV-algebras whose every maximal ideal is normal. $\square$

Now we describe a q-subvariety $\mathbb{MPEMV}$ of pseudo EMV-algebras which is related to the variety $\mathcal{M}$. For more details on q-subvarieties, see [13, Sec 4–5]. We start with the following result.

Proposition 6.10. *If I is a maximal ideal of a pseudo EMV-algebra M, then for each $a \in \mathcal{I}(M)$, $I_a := I \cap [0,a]$ is either $[0,a]$ or a maximal ideal of the pseudo MV-algebra $[0,a]$.*

Proof. If $I_a = [0,a]$, we are ready. Thus, assume that $I_a \subsetneq [0,a]$. Then I_a is a proper ideal of the pseudo MV-algebra $[0,a]$ and there is $x \in [0,a] \setminus I_a$. Then $I_0(I,x) = M$. Consequently, for every $y \in [0,a]$, we have $y \le f_1 \oplus n_1.x \oplus \cdots \oplus f_k \oplus n_k.x$, where $f_1, \ldots, f_k \in I$ and $n_1, \ldots, n_k \ge 0$ are integers. Due to [13, (3.8)], $y = g_1 \oplus x_1 \oplus \cdots \oplus g_k \oplus x_k$, where $g_i \le f_i$ and $x_i \le n_i.x$ for each $i = 1, \ldots, k$. Then $g_i \le y \le a$ so that $g_i \in I_a$, and every $y \in [0,a]$ belongs to the ideal of $[0,a]$ generated by I_a and x. In other words, I_a is a maximal ideal of $[0,a]$. $\square$

Theorem 6.11. *The class $\mathbb{MPEMV}$ of pseudo EMV-algebras M such that either $M = \{0\}$ or every maximal ideal of M is normal is a q-variety that is a proper q-subvariety of the variety $\mathbb{PEMV}$. Moreover, $\mathbb{MPEMV} = \mathbb{V}_0(\mathcal{M})$.*

Proof. Let $\mathcal{M}$ be the variety of pseudo MV-algebras M such that either $M = \{0\}$ or every maximal ideal of M is normal. By [13, Thm 5.18], $\mathbb{V}_0(\mathcal{M})$ is a q-subvariety of pseudo EMV-algebras M such that $[0,a] \in \mathcal{M}$ for each $a \in \mathcal{I}(M)$. Now, let $M \in \mathbb{V}_0(\mathcal{M})$ and I be a maximal ideal of M. Choose an idempotent $a \in \mathcal{I}(M)$. Then $[0,a] \in \mathcal{M}$. Due to Proposition 6.10, $I_a := I \cap [0,a]$ is either equal to $[0,a]$ or it is a maximal ideal of the pseudo MV-algebra $[0,a]$. In the latter case, I_a is a maximal and normal ideal of $[0,a]$. If M has a top element, then I is a normal ideal. Suppose that M has no top element. Hence, if $x \in M$ and $y \in I$, we can find $a \in \mathcal{I}(M)$ such that $x, y \le a$. Since I is maximal, we can find an idempotent a such that $a \notin I$ and $x, y \le a$. Then $[0,a] \cap I$ is a maximal ideal of the pseudo MV-algebra $[0,a]$. Hence, there is $z \in I_a$ such that $x \oplus y = z \oplus x$, i.e. $x \oplus I \subseteq I \oplus x$. In a similar way we establish the opposite inclusion. In other words, we have shown that $\mathbb{V}_0(\mathcal{M}) \subseteq \mathbb{MPEMV}$.

Conversely, let $M \in \mathbb{MPEMV}$. Without loss of generality, we can assume that M has no top element. Take $a \in \mathcal{I}(M)$, $a > 0$, and let I_a be a maximal ideal of $[0,a]$. By Corollary 6.9, every pseudo MV-algebra $([0,a]; \oplus, \lambda_a, \rho_a, 0, a)$ belongs to the variety $\mathcal{M}$. Then $M \in \mathbb{V}_0(\mathcal{M})$, i.e. $\mathbb{MPEMV} = \mathbb{V}_0(\mathcal{M})$, see [13, Thm 5.18], which confirms that $\mathbb{MPEMV}$ is a q-subvariety that is a proper q-subvariety of $\mathbb{PEMV}$ because $\mathcal{M}$ is a proper subvariety of the variety $\mathcal{PMV}$ of pseudo MV-algebras. $\square$

Theorem 6.12. *Let* $\mathbb{NPEMV}$ *be the class of pseudo EMV-algebras* M *such that either* $M = \{0\}$ *or* M *is normal-valued. Then* $\mathbb{NPEMV}$ *is a proper q-subvariety of the q-variety* $\mathbb{MPEMV}$*. A pseudo EMV-algebra* M *is in* $\mathbb{NPEMV}$ *if and only if*

$$(x \oplus y) \wedge (y^2 \oplus x^2) = x \oplus y, \quad x, y \in M. \tag{6.1}$$

Moreover, $\mathbb{NPEMV}$ *contains the variety* $\mathbb{RPEMV}$ *of representable pseudo EMV-algebras as a proper q-subvariety.*

Proof. First we establish the following claim:

Claim. Let $x > 0$*,* V *be a value of* x*, and* V^* *its cover. Let* $a \in \mathcal{I}(M)$ *be such that* $x < a$*. Then* $V_a := V \cap [0,a]$ *is a value of* x *in the pseudo MV-algebra* $[0,a]$ *and its cover is* $V_a^* = V^* \cap [0,a]$*.*

To prove the claim, we see that V_a is an ideal of $[0,a]$ such that $x \notin V_a^*$. Then the ideal $I_0(V,x)$ is equal to V^*. Take $y \in V_a^*$. Then $y \in V^*$ and therefore, there are $f_1, \dots, f_k$ and integers $n_1, \dots, n_k \geq 0$ such that we have $y \leq f_1 \oplus n_1.x \oplus \cdots \oplus f_k \oplus n_k.x$, where $f_1, \dots, f_k \in I$ and $n_1, \dots, n_k \geq 0$ are integers. In view of [13, (3.8)], $y = g_1 \oplus x_1 \oplus \cdots \oplus g_k \oplus x_k$, where $g_i \leq f_i$ and $x_i \leq n_i.x$ for each $i = 1, \dots, k$. Then $g_i \leq y \leq a$ so that $g_i \in V_a^*$, and every $y \in V_a^*$ belongs to the ideal of $[0,a]$ generated by V_a and x. This proves V_a^* is a cover of V_a.

Let $\mathcal{N}$ be the variety of pseudo MV-algebras M such that either $M = \{0\}$ or M is normal-valued. By [13, Thm 5.18], $\mathbb{V}_0(\mathcal{N})$ is a q-subvariety of pseudo EMV-algebras M such that $[0,a] \in \mathcal{N}$ for each $a \in \mathcal{I}(M)$. Now, let $M \in \mathbb{V}_0(\mathcal{N})$, $x > 0$ be an arbitrary element of M, V be a value of x, and let V^* be its cover. Without loss of generality, we can assume that M has no top element. For any idempotent $a \in \mathcal{I}(M)$, the pseudo MV-algebra $[0,a]$ belongs to $\mathcal{N}$. Take $y_1, y_2 \in V^*$ and choose an idempotent $a \in \mathcal{I}(M)$ such that $x, y_1, y_2 < a$. By Claim, $V_a := V \cap [0,a]$ is a value of x in the pseudo MV-algebra $[0,a]$ and $V_a^* := V^* \cap [0,a]$ is its cover. In addition, V_a is normal in V_a^* and $y_1, y_2 \in V_a^*$. Normality of V_a in V_a^* implies there are $z_1, z_2 \in V_a^*$ such that $x \oplus y_1 = z_1 \oplus x$ and $y_2 \oplus x = x \oplus z_2$. Hence, V is normal in V^*, and $\mathbb{V}_0(\mathcal{N}) \subseteq \mathbb{NPEMV}$.

Conversely, let $M \in \mathbb{NPEMV}$. Without loss of generality, we can assume that M has no top element. Take $a \in \mathcal{I}(M)$, $a > 0$, and let $x > 0$ be an element such that $x < a$. Choose a value V_a of x in the pseudo MV-algebra $[0,a]$ and let V_a^* be its value in $[0,a]$. Then V_a is also an ideal of M not containing x. According to [13, Prop 5.5], there is an ideal V of M that is maximal under the conditions $x \notin V$ and V contains V_a. In other words, V is a value of x in M and let V^* be its cover in M. Since M is a normal-valued pseudo EMV-algebra, V is normal in V^*. Due to Claim, $V \cap [0,a]$ is a value of x in $[0,a]$ and it contains V_a. The maximality of V_a entails $V_a = V \cap [0,a]$ and $V_a^* = V^* \cap [0,a]$. If $u \in V_a^*$ and $v_1, v_2 \in V_a$, there are $y_1, y_2 \in V$ such that $u \oplus v_1 = y_1 \oplus u$ and $v_2 \oplus u = u \oplus y_2$. Then $y_1 \leq u \oplus v_1$, $y_2 \leq v_2 \oplus u$ which yields $y_1, y_2 \in V_a$ and $[0,a]$ is a normal-valued pseudo MV-algebra. Then $M \in \mathbb{V}_0(\mathcal{N})$.

Since $\mathbb{NPEMV} = \mathbb{V}_0(\mathcal{N})$, [13, Thm 5.18] guarantees $\mathbb{NPEMV}$ is a q-subvariety of pseudo EMV-algebras. Since $\mathcal{N} \subsetneq \mathcal{M}$, we have by [13, Thm 5.18] that $\mathbb{NPEMV} \subsetneq \mathbb{MPEMV}$.

The equational base (6.1) is by [7, Thm 6.8] an equational base for the variety $\mathcal{N}$ of pseudo MV-algebras. By the above, it is also an equational base for $\mathbb{NPEMV}$.

Finally, let M be a linearly ordered pseudo EMV-algebra. Due to [13, Prop 5.1], M has a top element, so that it is termwise equivalent to a pseudo MV-algebra. According to [7, Prop 6.6], it is a normal-valued pseudo MV-algebra. Consequently, every representable pseudo EMV-algebra is a normal-valued pseudo EMV-algebra, and the variety of representable pseudo MV-algebras is a proper subvariety of $\mathcal{N}$. Therefore, the q-variety of representable pseudo EMV-algebras is a proper q-subvariety of the q-variety $\mathbb{NPEMV}$. $\square$

A famous Chang's Completeness Theorem, [2, 3], says: If an equation holds in the MV-algebra $\Gamma(\mathbb{R}, 1)$, then the equation holds in every MV-algebra. Now we present a similar statement for pseudo EMV-algebras. First, we introduce the following notion. Let $\mathrm{Aut}(\mathbb{R})$ be the set of automorphisms $f : \mathbb{R} \to \mathbb{R}$ preserving the ordering on $\mathbb{R}$. Then $\mathrm{Aut}(\mathbb{R})$ becomes an ℓ-group where the group operation is composition of two functions, and $(f \wedge g)(t) := \min\{f(t), g(t)\}$, $t \in \mathbb{R}$. The identity function is a neutral element.

The following result was proved in [9, Cor 4.9].

Theorem 6.13. *Let $u \in \mathrm{Aut}(\mathbb{R})$ be the translation $u(t) = t + 1$, $t \in \mathbb{R}$, and*

$$\mathrm{BAut}(\mathbb{R}) := \{g \in \mathrm{Aut}(\mathbb{R}) \colon \exists\, n \geq 1, u^{-n} \leq g \leq u^n\}.$$

If an equation in the language of pseudo MV-algebras holds in the pseudo MV-algebra $\Gamma(\mathrm{BAut}(\mathbb{R}), u)$, then it holds in every pseudo MV-algebra.

Corollary 6.14. *If a q-subvariety $\mathbb{V}$ of pseudo EMV-algebras contains the pseudo EMV-algebra with top element $M := \Gamma(\mathrm{BAut}(\mathbb{R}), u)$, then $\mathbb{V} = \mathbb{PEMV}$.*

Proof. Let $\mathbb{V}(M)$ be the q-subvariety of pseudo EMV-algebras generated by M. Since M as the pseudo MV-algebra generates the variety of pseudo MV-algebras, we conclude from Tarski's theorem, [1, Thm 9.5], holding also for pseudo EMV-algebras, that $\mathbb{V}(M)$ contains all pseudo EMV-algebras with top element. Now let M' be an arbitrary pseudo EMV-algebra. According to Basic Representation Theorem 6.4, there is a pseudo EMV-algebra N' with top element such that M' can be embedded into N'. Since $N' \in \mathbb{V}(M)$, we have $M' \in \mathbb{V}(M)$. $\square$

As a result of the previous statements about some q-subvarieties of pseudo EMV-algebras, we have the following chain of inclusions of q-subvarieties

$$\mathbb{O} \subsetneq \mathbb{B} \subsetneq \mathbb{EMV} \subsetneq \mathbb{RPEMV} \subsetneq \mathbb{NPEMV} \subsetneq \mathbb{MPEMV} \subsetneq \mathbb{PEMV}, \tag{6.2}$$

where $\mathbb{O}$ is the q-subvariety containing only the one-element EMV-algebras and $\mathbb{B}$ is the q-subvariety of EMV-algebras containing generalized Boolean algebras characterized also by the equation $x \oplus x = x$.

7 Categorical Equivalences

We show that the system of proper pseudo EMV-algebras forms a category and we present one category of pseudo MV-algebras with special properties and a category of unital ℓ-groups with special properties which are mutually categorically equivalent. A basic tool will be Theorem 6.4. We will follow ideas from [10, Sec 6] where such a question was investigated for proper EMV-algebras.

Let $\mathcal{PPEMV}$ be the category of proper pseudo EMV-algebras whose objects are proper pseudo EMV-algebras and morphisms are homomorphisms of pseudo EMV-algebras. On the other hand, let $\mathcal{PPMV}$ be the category whose objects are pairs (M, I), where M is a pseudo MV-algebra with a fixed maximal and normal ideal I of M such that (i) I has enough idempotents, i.e. for each $x \in I$, there is an idempotent $a \in \mathcal{I}(I)$ such that $x \leq a$, (ii) no $x \in I$ is a top element of I, and (iii) $I \cup I^\sim = M$, where $I^\sim := \{x^\sim \colon x \in I\}$. Morphisms $\phi : (M_1, I_1) \to (M_2, I_2)$ in the category $\mathcal{PPMV}$ are homomorphisms of pseudo MV-algebras $\phi : M_1 \to M_2$ such that $\phi(I_1) \subseteq I_2$. Then $I \cap I^\sim = \emptyset$: If $x = y^\sim$ for $x, y \in I$, then $1 = x^- \oplus x = y \oplus x \in I$.

For example, if N is the Chang MV-algebra, i.e. isomorphic to $\Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (1, 0))$, then $I = \{(0, n) \colon n \geq 0\}$ is a unique maximal ideal of N but I does not have enough idempotents. So (N, I) does not belong to $\mathcal{PPMV}$.

Define a mapping $\Phi : \mathcal{PPMV} \to \mathcal{PPEMV}$ as follows: For any object $(N, I) \in \mathcal{PPMV}$, let

$$\Phi(N, I) := I$$

and if (N_1, I_1) and (N_2, I_2) are objects of $\mathcal{PPMV}$ and $\phi : (N_1, I_1) \to (N_2, I_2)$ is a morphism, then

$$\Phi(\phi)(x) := \phi(x), \quad x \in I_1.$$

Proposition 7.1. Φ *is a well-defined functor that is faithful and full from the category* $\mathcal{PPMV}$ *into the category* $\mathcal{PPEMV}$.

Proof. We start to show that Φ is a well-defined functor. That is, we show if $\phi : (N_1, I_1) \to (N_2, I_2)$ is a morphism in the category $\mathcal{PPMV}$, then $\Phi(\phi)$ is a morphism in $\mathcal{PPEMV}$. At any rate, the mapping $\Phi(\phi)$ is in fact a pEMV-homomorphism from the proper pseudo EMV-algebra I_1 into the pseudo proper EMV-algebra I_2.

Let ϕ_1 and ϕ_2 be two morphisms from (N_1, I_1) into (N_2, I_2) such that $\Phi(\phi_1) = \Phi(\phi_2)$. Then $\phi_1(x) = \phi_2(x)$ for each $x \in I_1$. If $x \in N_1 \setminus I_1$, there is a unique element $x_0 \in I_1$ such that $x = x_0^\sim$. Then $\phi_1(x) = \phi_1(x_0^\sim) = (\phi_1(x_0))^\sim = (\phi_2(x_0))^\sim = \phi_2(x)$ which entails $\phi_1 = \phi_2$, i.e. Φ is a faithful functor.

Now, we show Φ is a full functor: Let $h : I_1 \to I_2$ be a morphism from $\mathcal{PPEMV}$, i.e. h is a pEMV-homomorphism. By Theorem 6.4, there are pseudo EMV-algebras N_1 and N_2 with top elements such that I_1 and I_2 can be embedded into N_1 and N_2, respectively, as their maximal ideals. We can assume that I_i is a pseudo EMV-subalgebra of N_i for $i = 1, 2$. We claim that there is a morphism $\phi : (N_1, I_1) \to (N_2, I_2)$ such that $\Phi(\phi) = h$. To show that, we extend h to a homomorphism of pseudo MV-algebras ϕ from N_1 into N_2 for some objects (N_1, I_1) and (N_2, I_2) from $\mathcal{PPMV}$. By the proof of Theorem 6.1, $N_1 = I_1 \cup I_1^\sim$ and $N_2 = I_2 \cup I_2^\sim$. If $x \in I_1$, we put $\phi(x) = h(x)$, and if $x \in N_1 \setminus I_1$, there is a unique element $x_0 \in I_1$ such that $x = x_0^\sim$. Then we put $\phi(x) = h(x_0)^\sim$. We have $\phi(1) = 1$, $\phi(x^\sim) = (\phi(x))^\sim$, $x \in I_1$. Since for $x \in I_1$, $x^- = \varphi_\lambda(x)^\sim$, we have $\phi(x^-) = \phi(\varphi_\lambda(x)^\sim) = h(\varphi_\lambda(x))^\sim = (\varphi_\lambda(h(x)))^\sim = \phi(x)^-$.

If $x \in N_1 \setminus I_1$, then $x = x_0^\sim$ for a unique $x_0 \in I_1$. Whence,

$$\begin{aligned}
\phi(x^\sim) &= \phi(x_0^{\sim\sim}) = \phi(\varphi_\rho(x_0)) = h(\varphi_\rho(x_0)) = \varphi_\rho(h(x_0)) = h(x_0)^{\sim\sim} = \phi(x)^\sim, \\
\phi(x^-) &= \phi(x_0^{\sim-}) = \phi(x_0) = h(x_0) = h(x_0)^{\sim-} = \phi(x_0^\sim)^- = \phi(x)^-.
\end{aligned}$$

Now let $x, y \in N_1$. By Theorem 6.1, there are four cases: (1) $x, y \in I_1$, then clearly $\phi(x \oplus y) = \phi(x) \oplus \phi(y)$. (2) $x = x_0^\sim$ and $y = y_0^\sim$ for some $x_0, y_0 \in I_1$. Then $\phi(x \oplus y) = \phi(x_0^\sim \oplus y_0^\sim) = \phi((y_0 \odot x_0)^\sim) = (\phi(y_0 \odot x_0))^\sim = (\phi(y_0) \odot \phi(x_0))^\sim = \phi(x_0)^\sim \oplus \phi(y_0)^\sim = \phi(x) \oplus \phi(y)$. (3) $x = x_0$ and $y = y_0^\sim$ for some $x_0, y_0 \in I_1$. Let $b \in \mathcal{I}(I_1)$ be an idempotent such that $x_0, y_0 \leq b$. Using (3.1) of Proposition 3.4, we have

$$x \oplus y = x_0 \oplus y_0^\sim = (y_0 \odot x_0^-)^\sim = (y_0 \odot (x_0 \wedge y_0)^-)^\sim = (y_0 \odot \lambda_b(x_0 \wedge y_0))^\sim,$$

which entails

$$\begin{aligned}
\phi(x \oplus y) &= \phi((y_0 \odot \lambda_b(x_0 \wedge y_0))^{\sim}) = (\phi(y_0 \odot \lambda_b(x_0 \wedge y_0)))^{\sim} \\
&= (\phi(y_0) \odot \phi(\lambda_b(x_0 \wedge y_0)))^{\sim} = (\phi(y_0) \odot \lambda_{\phi(b)}(\phi(x_0 \wedge y_0)))^{\sim} \\
&= (\phi(y_0) \odot \lambda_{\phi(b)}(\phi(x_0) \wedge \phi(y_0)))^{\sim} \\
&= (\phi(y_0) \odot (\phi(x_0) \wedge \phi(y_0))^{-})^{\sim} = (\phi(y_0) \odot \phi(x_0)^{-})^{\sim} \\
&= \phi(x_0) \oplus \phi(y_0)^{\sim} = \phi(x) \oplus \phi(y).
\end{aligned}$$

(4) $x = x_0^{\sim}$ and $y = y_0$ for some $x_0, y_0 \in I_1$. As in Case (iv) of the proof of Theorem 6.1, we have

$$x \oplus y = (y_0 \odot \lambda_b(x_0 \wedge y_0))^{\sim}.$$

Applying ϕ to this equality, we obtain as in (3), $\phi(x \oplus y) = \phi(x) \oplus \phi(y)$.

Therefore, ϕ is a homomorphism of pseudo MV-algebras which is an extension of h. Whence, $\Phi(\phi) = h$, $\phi(I_1) \subseteq I_2$, and Φ is a full functor. □

Proposition 7.2. *Let M be a proper EMV-algebra and $h_i : M \to N_i$ be an embedding of M into a pseudo EMV-algebra N_i with top element for $i = 1, 2$. Set*

$$N_i^0 = \{x \in N_i\colon \ \textit{either } x = h_i(x_0) \textit{ or } x = (h_i(x_0))^{\sim} \textit{ for } x_0 \in M\}$$

for $i = 1, 2$. Then $(N_1^0; \oplus, ^-, ^\sim, 0, 1)$ and $(N_2^0; \oplus, ^-, ^\sim, 0, 1)$ are isomorphic pseudo MV-algebras such that $(N_i^0, h_i(M))$ are objects in $\mathcal{PPMV}$ for $i = 1, 2$.

Proof. By Theorem 6.1, N_i^0 is a pseudo MV-algebra and $h_i(M)$ is its maximal and normal ideal for $i = 1, 2$. The mapping $\gamma : M_1^0 \to M_2^0$ defined by $\gamma(h_1(x)) = h_2(x)$ and $\gamma((h_1(x))^{\sim}) = (h_2(x))^{\sim}$ for $x \in M$ is an isomorphism of pseudo MV-algebras. Clearly, $(M_1^0, h_1(M))$ and $(M_2^0, h_2(M))$ are objects of the category $\mathcal{PPMV}$. □

Proposition 7.3. *The functor $\Phi : \mathcal{PPMV} \to \mathcal{PPEMV}$ has a left-adjoint.*

Proof. We show that, for a proper pseudo EMV-algebra M, there is a universal arrow $((N, I), f)$ i.e., (N, I) is an object in $\mathcal{PPMV}$ and f is a morphism from M into $\Phi(N, I) = I$ such that if (N', I') is an object from $\mathcal{PPEMV}$ and f' is a morphism from M into $\Phi(N', I')$, then there exists a unique morphism $f^* : (N, I) \to (N', I')$ such that $\Phi(f^*) \circ f = f'$.

By Basic Representation Theorem 6.4 and Proposition 7.2, there are a unique (up to isomorphism) pseudo EMV-algebra N with top element and an injective pEMV-homomorphism $f : M \to N$ such that $f(M)$ is a maximal and normal ideal of N. We assert that $((N, I), f)$ is a universal arrow for M. Let (N', I') be an object from $\mathcal{PPMV}$ and let f' be a morphism from M into $\Phi(N', I')$. We can define a mapping

$f^* : N \to N'$ such that $f^*(f(x)) := f'(x)$ if $x \in M$ and if $y \in N \setminus f(M)$, there is $y_0 \in M$ such that $y = (f(y_0))^\sim$, and we set $f^*(y) = (f'(y_0))^\sim$. Then $f^* : N \to N'$ is a unique homomorphism of pseudo MV-algebras such that $\Phi(f^*) \circ f = f'$.

Define a mapping $\Psi : \mathcal{PPEMV} \to \mathcal{PPMV}$ by $\Psi(M) := (N, I)$ whenever $((N, I), f)$ is a universal arrow for M and if $f' : M \to M'$ is a pEMV-homomorphism, there is a unique morphism $f^* : (N, I) \to (N', I')$, where $\Phi(N', I') = M'$, then we define $\Psi(f') := f^*$. Using Theorem 6.4, we have that Ψ is a left-adjoint functor of the functor Φ. $\square$

Theorem 7.4. *The functor Φ defines a categorical equivalence between the category $\mathcal{PPMV}$ and the category of proper pseudo EMV-algebras $\mathcal{PPEMV}$.*

In addition, if $h : \Phi(N, I) \to \Phi(N', I')$ is a morphism of proper pseudo EMV-algebras, then there is a unique homomorphism $\phi : N \to N'$ of pseudo MV-algebras with $\phi(I) \subseteq I'$ such that we have $h = \Phi(\phi)$, and

(i) *if h is surjective, so is ϕ;*

(ii) *if h is injective, so is ϕ.*

Proof. By Proposition 7.1 Φ is faithful and full, therefore, according to [21, Thm IV.4.1 (i),(iii)], it is necessary to show that, for any proper pseudo EMV-algebra M there is an object (N, I) in $\mathcal{PPMV}$ such that $\Phi(N, I)$ is isomorphic to M. To prove that, it is sufficient to take any universal arrow $((N, I), f)$ of M.

Surjectivity and injectivity of h follows from the fact $N = I \cup I^\sim$ and $N' = I \cup (I')^\sim$. $\square$

Now we extend the last results for unital ℓ-groups not necessarily Abelian.

Let (G, u) be a unital ℓ-group. An *ℓ-ideal* is a normal convex ℓ-subgroup I of G. An convex ℓ-subgroup I of G is *maximal* if it is a value of the strong unit u, i.e. a maximal proper convex ℓ-subgroup of (G, u) not containing u. Using categorical equivalence between the category of pseudo MV-algebras and the category of unital ℓ-groups, Theorem 2.2, we have by [7, Thm 6.1]: (i) If I is a (maximal) convex ℓ-subgroup of (G, u), then $I_0 = I \cap [0, u]$ is a (maximal) ideal of the pseudo MV-algebra $N = \Gamma(G, u)$; (ii) If I_0 is a (maximal) ideal of $\Gamma(G, u)$, then $I = \{x \in G\colon |x| \wedge u \in I_0\}$ is a (maximal) ℓ-ideal of (G, u) such that $I \cap [0, u] = I_0$. In addition,

$$I = \{x \in G\colon \exists\, x_i, y_j \in I_0, x = x_1 + \cdots + x_n - y_1 - \cdots - y_m\}. \tag{7.1}$$

The set I_0 is a normal ideal of M iff I defined by (7.1) is an ℓ-ideal of (G, u).

Any ℓ-group G has the following form of the Riesz Decomposition Property (RDP_0 in short): If $a, b, c \in G^+$ such that $a \le b + c$, there are b_1, c_1 in G^+ such that $b_1 \le b$, $c_1 \le c$ and $a = b_1 + c_1$.

Proposition 7.5. *Let I_0 be a maximal and normal ideal of the pseudo MV-algebra $N = \Gamma(G,u)$ and let I be a unique maximal ℓ-ideal of (G,u) generated by I_0. We define $I^u = \{(-y+nu)+x \colon x \in I^+, n \geq 1,\, y \in I,\, 0 \leq y < nu\}$. The following statements are equivalent:*

(i) $I_0 \cup I_0^{\sim} = N$.

(ii) $G^+ = (I^+) \cup I^u$.

(iii) $(G/I, u/I) \cong (\mathbb{Z}, 1)$.

In any case, $I \cap I^u = \emptyset$.

Proof. Suppose $y < nu$. Using RDP_0 holding in ℓ-groups and (7.1), we have $y = y_n' + \cdots + y_1'$, where $y_i' \in I_0$. Due to normality of I_0, there are $y_1, \ldots, y_n \in I_0$ such that $(-y+nu)+x = ((-y_1+u)+\cdots+(-y_n+u))+x = y_1^{\sim}+\cdots+y_n^{\sim}+x$. That is,

$$I^u = \{x \in G^+ \colon \exists\, z \in I^+, \exists\, y_1, \ldots, y_n \in I_0,\, x = (y_1^{\sim}+\cdots+y_n^{\sim})+z\}.$$

(i) $\Rightarrow$ (ii). Let $I_0 \cup I_0' = \Gamma(G,u)$ and choose $x \in G^+$. There is an integer $n \geq 1$ such that $x \leq nu$. Due to the Riesz Decomposition Property holding in ℓ-groups, $x = x_1 + \cdots + x_n$, where each $x_i \in N$. If all x_i's belong to I_0, then $x \in I^+$. If, say $x_1, \ldots, x_k$, are all those x_i's that belong to $I_0^{\sim}$, then $x_i = -y_i + u$, where $y_i \in I_0$, $1 \leq i \leq k$. Then $x = ((-y_1+u)+\cdots+(-y_k+u)) + x_{k+1} + \cdots + x_n \in I^u$.

In addition, $I \cap I^u = \emptyset$.

(ii) $\Rightarrow$ (i). Conversely, let $G^+ = I^+ \cup I^u$. Then $N = N \cap G^+ = (N \cap I^+) \cup (N \cap I^u) = I_0 \cup (N \cap I^u)$. Let $x \in N \cap I^u$, then $x = (-y+nu)+z$ for some $0 \leq y < nu$, $y, z \in I^+$. Since $0 \leq z \leq x \leq u$. Using RDP_0, we have $y = y_1 + \cdots + y_n$, where $y_i \in I_0$. Due to normality of I_0, there are $z_1, \ldots, z_n \in I_0$ such that $x = (-y+nu)+z = (((-z_1+u)+\cdots+(-z_n+u))+z) \wedge u = (z_1^{\sim} \oplus \cdots \oplus z_n^{\sim}) \oplus z = (z_n \odot \cdots \odot z_1)^{\sim} \oplus z$, but $y_0 := z_n \odot \cdots \odot z_1 \in I_0$ and $z \in I_0$ so that $x \in I_0^{\sim}$ because $I_0^{\sim}$ is an upset which yields $N \cap I^u = I_0^{\sim}$. Then $N = I_0 \cup I_0^{\sim}$.

(i) $\Rightarrow$ (iii). Let $I_0 \cup I_0^{\sim} = N$. Since I_0 is a maximal ideal of N, then $N/I_0 \cong \{0,1\}$, the two-element Boolean algebra, and the mapping $x \to x/I_0$, $x \in N$, is a two-valued state on the pseudo MV-algebra N, i.e. a mapping $s : N \to [0,1]$ such $s(x \oplus y) = s(x) + s(y)$ whenever $y \odot x = 0$ and $s(1) = 1$. For more details on states on pseudo MV-algebras, see [7]. Hence, using the categorical equivalence of pseudo MV-algebras and unital ℓ-groups, see [8], we have $(G/I, u/I) \cong (\mathbb{Z}, 1)$, where I is the maximal ℓ-ideal of (G,u) generated by I_0.

(iii) $\Rightarrow$ (i). Conversely, if $(G/I, u/I) \cong (\mathbb{Z}, 1)$, then the mapping $x \mapsto x/I$, $x \in N$, is a two-valued state s such that $\mathrm{Ker}(s) = I_0$, so that $N = I_0 \cup I_0^{\sim}$. ☐

We denote by $\mathcal{PULG}$ the category of unital ℓ-groups not necessarily commutative with a fixed maximal ℓ-ideal with a special property: The objects of $\mathcal{PULG}$ are triples (G,u,I) such that (G,u) is a unital ℓ-group and I is a fixed maximal ℓ-ideal of (G,u) such that $G^+ = I^+ \cup I^u$, where I^u was defined in Proposition 7.5, and the ideal $I_u = I \cap [0,u]$ of $\Gamma(G,u)$ has enough idempotent elements but no top element. By Proposition 7.5, the property $G^+ = I^+ \cup I^u$ is equivalent to the condition that the maximal ℓ-ideal I of (G,u) has the property $(G/I, u/I) \cong (\mathbb{Z}, 1)$.

If (G_1,u_1,I_1) and (G_2,u_2,I_2) are two objects of $\mathcal{PULG}$, then a mapping $f: (G_1,u_1,I_1) \to (G_2,u_2,I_2)$ is a morphism if f is a homomorphism of unital ℓ-groups such that $f(I_1) \subseteq I_2$. Our goal is to prove that $\mathcal{PULG}$ is categorically equivalent to the category $\mathcal{PPMV}$. We will follow steps used in the previous categorical equivalence.

Let us define a functor $\Gamma_I : \mathcal{PULG} \to \mathcal{PPMV}$ as follows: If (G,u,I) is an object of $\mathcal{PULG}$, then

$$\Gamma_I(G,u,I) := (\Gamma(G,u), I \cap [0,u]),$$

and if f is a morphism from an object (G_1,u_1,I_1) into another one (G_2,u_2,I_2), then

$$\Gamma_I(f)(x) := f(x), \quad x \in \Gamma(G,u).$$

Proposition 7.6. *Γ_I is a well-defined functor that is faithful and full.*

Proof. We have $\Gamma_I(G,u,I) = (\Gamma(G,u), I\cap[0,u]) \in \mathcal{PPMV}$. If $f:(G_1,u_1,I_1) \to (G_2,u_2,I_2)$ is a morphism, then the restriction of f onto $\Gamma(G_1,u_1)$ gives in fact a homomorphism of pseudo MV-algebras with $f(I_1) \subseteq I_2$, so that $\Gamma_I(f)(I_1 \cap [0,u_1]) \subseteq I_2 \cap [0,u_2]$, and by Proposition 7.5, Γ_I is a well-defined functor.

Let f_1 and f_2 be two morphisms from (G_1,u_1,I_1) into (G_2,u_2,I_2) such that $\Gamma(f_1) = \Gamma(f_2)$. Then $f_1(x) = f_2(x)$ for each $x \in \Gamma(G_1,u_1)$. Since f_i for $i = 1,2$ is a homomorphism of unital ℓ-groups, it is easy to see, due to Proposition 7.5, that $f_1(x) = f_2(x)$ for each $x \in G_1$ and $f_1 = f_2$.

If $\kappa : \Gamma_I(G_1,u_1,I_1) = (\Gamma(G_1,u_1), I_1 \cap [0,u_1]) \to \Gamma_I(G_2,u_2,I_2) = (\Gamma(G_2,u_2), I_2 \cap [0,u_2])$ be a morphism, i.e. κ is a homomorphism of pseudo MV-algebras from the pseudo MV-algebra $\Gamma(G_1,u_1)$ into the pseudo MV-algebra $\Gamma(G_2,u_2)$ such that $\kappa(I_1 \cap [0,u_1]) \subseteq I_2 \cap [0,u_2]$. Using methods of the proof of [8, Prop 6.1], we can uniquely extend κ to a homomorphism of unital ℓ-groups $f: G_1 \to G_2$. The normal ideal $I_i \cap [0,u_i]$ can be uniquely extended to the ℓ-ideal I_i, $i = 1,2$. Therefore, $f(I_1) \subseteq I_2$ and due to Proposition 7.5, f is a morphism from (G_1,u_1,I_1) into (G_2,u_2,I_2), which proves $\Gamma_I(f) = \kappa$ and Γ_I is a full functor. $\square$

Let M be a pseudo EMV-algebra. We define a partial operation $+$ on M by $x+y$ is defined in M iff $y \odot x = 0$ and then we put $x+y = x \oplus y$. We note that $x+y$

exists iff $x \le \lambda_a(y)$ iff $y \le \rho_a(x)$ for some $a \in \mathcal{I}(M)$ such $x, y \le a$ (equivalently, for each idempotent $a \ge x, y$). The operation $+$ is associative because if $x, y, z \in M$, there is an idempotent $a \in \mathcal{I}(M)$ with $x, y, z \le a$ and the associativity of $+$ holds in the pseudo MV-algebra $([0, a]; \oplus, \lambda_a, \rho_a, 0, a)$, see [8, Prop 2.1].

We say that a pair (G, f) is a *universal group* for a pseudo MV-algebra N if (i) f is a mapping from N into a po-group G which preserves partial addition $+$ on N such that $G = G^+ - G^+$, $f(N)$ generates G^+ as a semigroup, (ii) for any group K and any $+$-preserving mapping $h : N \to K$, there is a group homomorphism $\phi : G \to K$ such that $h = \phi \circ f$. Due to [8, Thm 5.3], if $N \cong \Gamma(G, u)$, then (G, f) is a universal group for N, where f is an isomorphism $f : N \to \Gamma(G, u)$.

Theorem 7.7. *The functor Γ_I defines a categorical equivalence of the category $\mathcal{PULG}$ and the category $\mathcal{PPMV}$.*

Proof. Using the notion of a universal group, it is possible to show that the functor Γ_I from the category $\mathcal{PULG}$ into the category $\mathcal{PPMV}$ has a left-adjoint. Indeed, take the universal group (G, f) for the MV-algebra N. Then f is a bijection of pseudo MV-algebras from N onto $\Gamma(G, u)$. We assert that $((G, u, I), f)$ is a universal arrow for (N, I_0), where I is an ℓ-ideal of G generated by $f(I_0)$.

Define a mapping $\Xi_I : \mathcal{PPMV} \to \mathcal{PULG}$ by $\Xi_I(N, I_0) = (G, u, I)$ if $((G, u, I), f)$ is a universal arrow for (N, I_0) and I is a maximal ℓ-ideal of G generated by $f(I_0)$. Then Ξ_I is a left-adjoint of Γ_I in question.

Proposition 7.6 and [21, Thm IV(i),(ii)] imply the final statement on the categorical equivalence. $\square$

As a direct corollaries of Theorem 7.4 and Theorem 7.7 we have the following final statement:

Corollary 7.8. *The categories $\mathcal{PULG}$, $\mathcal{PPMV}$ and $\mathcal{PPEMV}$ are mutually categorically equivalent.*

8 States and State-Morphisms on Pseudo EMV-algebras

We note that states and state-morphisms on pseudo MV-algebras were introduced and studied in [7], states on EMV-algebras were investigated in [12], and state-morphisms on EMV-algebras were introduced in [11]. They generalize the notion of a state on an MV-algebra introduced in [20]. States are defined as finitely additive mappings which are analogues of finitely additive probability measures on Boolean algebras. In the present section, we generalize these notions also for pseudo EMV-algebras and we will follow main ideas from [7]. The absence of a top element in pseudo EMV-algebras causes some problems, therefore, the proofs from [7] cannot be applied directly.

A mapping $s : M \to [0,1]$ is said to be a *state* if (i) $s(x \oplus y) = s(x) + s(y)$ whenever $y \odot x = 0$, and (ii) there is an element $x \in M$ such that $s(x) = 1$. Property (i) means equivalently, $s(x+y) = s(x) + s(y)$ whenever $x + y$ exists in M. This means that a state on M is an additive mapping on the pseudo EMV-algebra M that despite of the fact that M does not have necessarily a top element, it resembles a finitely additive probability measure. If M is with top element 1, then the notion of a state on the pseudo EMV-algebra M coincides with the notion of a state on the pseudo MV-algebra $(M; \oplus, ^-, ^\sim, 0, 1)$, see [7]. We define the state space on M, denoted by $\mathcal{S}(M)$, as the set of all states on M. In contrast to the state-space of EMV-algebras, $\mathcal{S}(M)$ can be empty even for some non-trivial pseudo EMV-algebras. Such a situation was described in [7, Cor 7.4] for pseudo MV-algebras. We note that the pseudo EMV-algebra with top element from Theorem 5.22 is stateless, see [9, Ex 5.3].

The basic properties of states on pseudo EMV-algebras are as follows.

Proposition 8.1. *Let s be a state on a pseudo EMV-algebra M. For all $x, y \in M$, we have*

(i) $s(0) = 0$;

(ii) *if $x \le y \le a \in \mathcal{I}(M)$, then $s(x) \le s(y)$ and $s(y \odot \lambda_a(x)) = s(y) - s(x) = s(\rho_a(y) \odot x)$; in particular, $s(\lambda_a(x)) = s(a) - s(x) = s(\rho_a(x))$;*

(iii) $s(x \vee y) + s(x \wedge y) = s(x) + s(y)$;

(iv) $s(x \oplus y) + s(x \odot y) = s(x) + s(y)$;

(v) $\mathrm{Ker}(s) = \{x \in M : s(x) = 0\}$ *is a normal ideal of M and* $\mathrm{Ker}_1(s) = \{x \in M : s(x) = 1\}$ *is a normal filter of the pseudo EMV-algebra M;*

(vi) *if $s_1, s_2 \in \mathcal{S}(M)$ and $\lambda \in [0,1]$ is a real number, then the convex combination $s = \lambda s_1 + (1-\lambda)s_2$ of states s_1, s_2 is a state on M;*

(vii) *if we define a mapping $\hat{s}$ on the quotient pseudo EMV-algebra $M/\mathrm{Ker}(s)$ by $\hat{s}(x/\mathrm{Ker}(s)) := s(x)$, $(x \in M)$, then $\hat{s}$ is a state on $M/\mathrm{Ker}(s)$, and $M/\mathrm{Ker}(s)$ is a pseudo EMV-algebra with top element;*

(viii) $s(x \oplus y) = s(y \oplus x)$ *and $M/\mathrm{Ker}(s)$ is an Archimedean EMV-algebra.*

Proof. (i) From $0 = 0 + 0$, we conclude $s(0) = 0$.

(ii) Let $x \le y \le a \in \mathcal{I}(M)$. Then in the pseudo MV-algebra $[0, a]$, we have $y = y \vee x = (y \odot \lambda_a(x)) \oplus x = x \oplus (\rho_a(x) \odot y)$, which entails the result.

(iii) Given $x, y \in M$, there is an idempotent $a \in M$ with $x, y \le a$. Then in the pseudo MV-algebra $[0, a]$, we have $(x \vee y) \odot \lambda_a(y) = (x \odot \lambda_a(x)) \vee (y \odot \lambda_a(y)) = x \odot \lambda_a(y)$ and $x \odot \lambda_a(x \wedge y) = x \odot (\lambda_a(x) \vee \lambda_a(y)) = x \odot \lambda_a(y)$, when we have used (3.3), which by (ii) gives (iii).

(iv) Assume $x, y \le a \in \mathcal{I}(M)$, using [14, Prop 1.25], we have $x = ((x \oplus y) \odot \lambda_a(y)) \oplus (y \odot x)$. Hence, (ii) implies the result.

(v) From (ii), we have if $x \le y \in \mathrm{Ker}(s)$ then $x \in \mathrm{Ker}(s)$. From the identity $x = (x \oplus y) \odot \lambda_a(y) + y \odot x$, we get $\mathrm{Ker}(s)$ is closed also under $\oplus$. Normality of the ideal $\mathrm{Ker}(s)$ follows from the following. Let $x \in M$, $y \in \mathrm{Ker}(s)$, and let $x, y \le a \in \mathcal{I}(M)$. Using (iv), we have $s(x \oplus y) = s(x)$, so from $x \oplus y = ((x \oplus y) \odot \lambda_a(x)) \oplus x = ((x \oplus y) \odot \lambda_a(x)) + x$ we get $y' = (x \oplus y) \odot \lambda_a(x) \in \mathrm{Ker}(s)$. In the same way we obtain the second equality.

The proof that $\mathrm{Ker}_1(s)$ is a normal filter of M follows analogous steps as those for the proof of normality of $\mathrm{Ker}(s)$.

(vi) It is evident.

(vii) For given $x, y \in M$, we have $x/\mathrm{Ker}(s) = y/\mathrm{Ker}(s)$ iff $s(x) = s(x \wedge y) = s(y)$. Therefore, $\hat{s}$ is correctly defined. For each $x \in M$, let $[x] = x/\mathrm{Ker}(s)$. Given $x, y \in M$, there is an idempotent $a \in \mathcal{I}(M)$ such that $x, y \le a$ and $s(a) = 1$. Hence, $\lambda_a(x)/\mathrm{Ker}(s) = \lambda_{[a]}([x])$. Assume that $[x] \le [\lambda_a(y)]$. For $x_0 = x \wedge \lambda_a(y)$, we have $x_0 \le \lambda_a(y)$ and $[x_0] = [x \wedge \lambda_a(x)] = [x] \wedge [\lambda_a(y)] = [x] \le [\lambda_a(y)] = \lambda_{[a]}([y])$. Then

$$\begin{aligned} \hat{s}([x] + [y]) &= \hat{s}([x \oplus y]) = \hat{s}([x_0 + y]) = s(x_0 + y) = s(x_0) + s(y) \\ &= \hat{s}([x_0]) + \hat{s}([y]) = \hat{s}([x]) + \hat{s}([y]). \end{aligned}$$

In addition, $\hat{s}([a]) = s(a) = 1$, so that $\hat{s}$ is a state on $M/\mathrm{Ker}(s)$. Since there is an element $a \in M$ such that $s(a) = 1$, and $s(x) \le 1$, we have that the element $[a]$ is a top element for $M/\mathrm{Ker}(s)$.

(viii) According to (vii), $\hat{s}(x) = 0$ iff $[x] = [0]$. We assert that $M/\mathrm{Ker}(s)$ is a commutative pseudo EMV-algebra. Indeed, let $n[x]$ be defined in $M/\mathrm{Ker}(s)$ for each integer $n \ge 1$. Then $\hat{s}(n[x]) = n\hat{s}([x]) \le 1$ for each $n \ge 1$. Therefore, $s(x) = \hat{s}([x]) = 0$ which yields that $M/\mathrm{Ker}(s)$ is Archimedean, and Theorem 3.7 guarantees that $M/\mathrm{Ker}(s)$ is a commutative pseudo EMV-algebra, i.e. an EMV-algebra. Therefore, $s(x \oplus y) = \hat{s}([x] \oplus [y]) = \hat{s}([y] \oplus [x]) = s(y \oplus x)$. $\square$

Let $([0, 1]; \vee, \wedge, \oplus, 0)$ be the EMV-algebra with top element of the real interval $[0, 1]$ endowed with the natural ordering of real numbers and $s \oplus t = \min\{s + t, 1\}$,

$s, t \in [0,1]$. Let M be a pseudo EMV-algebra. A pEMV-homomorphism $s : M \to [0,1]$ is said to be a *state-morphism* if there is an element $x \in M$ such that $s(x) = 1$. Whence, not every pEMV-homomorphism $s : M \to [0,1]$ is a state-morphism, for example, the zero function on M is such an example. We denote by $\mathcal{SM}(M)$ the set of state-morphisms on M.

Proposition 8.2. *Let M be a pseudo EMV-algebra. Every state-morphism on M is a state. Let s be a state on M. The following statements are equivalent:*

(i) *s is a state-morphism.*

(ii) *$s(x \wedge y) = \min\{s(x), s(y)\}$, $x, y \in M$.*

(iii) *$s(x \vee y) = \max\{s(x), s(y)\}$, $x, y \in M$.*

(iv) *$s(x \oplus y) = \min\{s(x) + s(y), 1\}$, $x, y \in M$.*

In addition, if $a \in \mathcal{I}(M)$ and s is a state-morphism, then $s(a) \in \{0,1\}$.

Proof. Let s be a state-morphism on M. Given $x, y \in M$, we can find an idempotent $a \in M$ such that $x, y \le a$ and $s(a) = 1$. Since s is a pEMV-homomorphism, we have $s(y \odot x) = s(y) \odot s(x)$. Therefore, if $y \odot x = 0$, then $s(y) \odot s(x) = \max\{s(y) - 1 + s(x), 0\} = 0$ which yields $s(x + y) = s(x \oplus y) = s(x) \oplus s(y) = \min\{s(x) + s(y), 1\} = s(x) + s(y)$ showing s is a state on M.

Now, let s be a state on M. For $x, y \in M$, there is an idempotent $a \in M$ such that $x, y \le a$ and $s(a) = 1$.

(i) $\Rightarrow$ (ii). Since s is a pEMV-homomorphism, (ii) holds.

(ii) $\Leftrightarrow$ (iii). It follows from the equalities $\lambda_a(x \vee y) = \lambda_a(x) \wedge \lambda_a(y)$ and $\lambda_a(x \wedge y) = \lambda_a(x) \vee \lambda_a(y)$.

(ii) $\Rightarrow$ (iv). Check $x \oplus y = x + (\rho_a(x) \odot (x \oplus y)) = x + (\rho_a(x) \wedge y)$. Then $s(x \oplus y) = s(x) + s(\rho_a(x) \wedge y) = s(x) + \min\{1 - s(x), s(y)\} = \min\{s(x) + s(y), 1\} = s(x) \oplus s(y)$.

(iv) $\Rightarrow$ (i). First we show that $s(x \odot y) = s(x) \odot s(y)$. Indeed, $s(x \odot y) = s(\lambda_a(\rho_a(y) \oplus \rho_a(x))) = s(a) - s(\rho_a(y) \oplus \rho_a(x)) = 1 - (s(\rho_a(y)) \oplus s(\rho_a(x))) = s(x) \odot s(y)$. Therefore, $s(x \wedge y) = s(x \odot (\lambda_a(x) \oplus y)) = s(x) \odot ((1 - s(x)) \oplus s(y)) = \min\{s(x), s(y)\}$. Similarly, $s(x \vee y) = \max\{s(x), s(y)\}$. Hence, s preserves $\oplus, \vee, \wedge, \odot$. In addition, $s(\lambda_a(x)) = 1 - s(x) = \lambda_{s(a)}(s(x))$ and $s(\rho_a(x)) = 1 - s(x) = \rho_{s(a)}(s(x))$, i.e. s is a pEMV-homomorphism.

Finally, let s be a state-morphism and let $a \in \mathcal{I}(M)$. Then $s(a) = s(a \oplus a) = \max\{s(a) + s(a), 1\}$ so that $s(a)$ is an idempotent of the EMV-algebra $[0,1]$ and it has only two idempotents 0 and 1 which establishes the result. ☐

Theorem 8.3. *Let s be a state on a pseudo EMV-algebra M. The following statements are equivalent:*

(i) *s is a state-morphism.*

(ii) $\mathrm{Ker}(s)$ *is a maximal and normal ideal of M.*

(iii) $\mathrm{Ker}_1(s)$ *is a maximal and normal filter of M.*

Proof. (i) $\Rightarrow$ (ii). If s is a state-morphism, by Proposition 8.1(v), $\mathrm{Ker}(s)$ is a normal ideal of M. We show that $\mathrm{Ker}(s)$ is a maximal ideal. To show that, take $x \in M \setminus \mathrm{Ker}(s)$ and let $\mathrm{Ker}(s)_x$ be the ideal of M generated by $\mathrm{Ker}(s)$ and x. By Proposition 4.9, $\mathrm{Ker}(s)_x = \{y \in M : \ y \leq n.x \oplus h$ for some $n \geq 1$ and some $h \in \mathrm{Ker}(s)\}$. Let z be an arbitrary element of M. There exists an integer $n \geq 1$ such that $(n-1)s(x) \leq s(z) < ns(x)$. Then $s(\rho_a(n.x) \odot z) = 0$. Since $z \leq (n.x) \vee z = (n.x) \oplus \rho_a(n.x) \odot z$, we have $z \in \mathrm{Ker}(s)_x$, finally, $M = \mathrm{Ker}(s)_x$ which proves that $\mathrm{Ker}(s)$ is a maximal ideal of M.

(ii) $\Rightarrow$ (i). Conversely, let $\mathrm{Ker}(s)$ be a maximal ideal of M. Given $x, y \in M$, there is an idempotent $a \in M$ such that $x, y \leq a$ and $s(a) = 1$. Then in the pseudo MV-algebra $[0, a]$, we have $(x \odot \lambda_a(y)) \wedge (y \odot \lambda_a(x)) = 0$. Every maximal ideal is prime, see Proposition 5.5, so that $s(x \odot \lambda_a(y)) = 0$ or $s(y \odot \lambda_a(x)) = 0$. In the first case we have $0 = s(x \odot \lambda_a(y)) = s(x \odot \lambda_a(x \wedge y)) = s(x) - s(x \wedge y)$, where we have used a fact $x \odot \lambda_a(y) = x \odot \lambda_a(x \wedge y)$, see (3.3), and in the second case, we have $s(y) = s(x \wedge y)$, i.e. $s(x \wedge y) = \min\{s(x), s(y)\}$, which by Proposition 8.2 means s is a state-morphism.

(ii) $\Rightarrow$ (iii). Put $I = \mathrm{Ker}(s)$ and $F = \mathrm{Ker}_1(s)$ and define $F_I^\lambda = \{\rho_a(x) \colon \exists\, a \in \mathcal{I}(M) \setminus I, x \leq a, \lambda_a(x) \in I\}$ and $I_F^\lambda = \{\rho_a(x) \colon x \in G, \exists\, a \in \mathcal{I}(M), x \leq a\}$. Then $F_I^\lambda = \mathrm{Ker}_1(s)$ and $I_F^\lambda = \mathrm{Ker}(s)$. According to Theorem 4.18, we have that F is maximal iff I is maximal which yields equivalence of (ii) a (iii). $\square$

The proof of the following useful lemma can be found e.g. in [4, Prop 7.2.5].

Lemma 8.4. (i) *Let G_1 and G_2 be two Abelian lattice ordered subgroups of* $(\mathbb{R}; +)$ *each containing a common non-zero element g_0. If there is an injective group-homomorphism ϕ of G_1 into G_2 preserving the order such that $\phi(g_0) = g_0$, then $G_1 \subseteq G_2$ and ϕ is the identity on G_1. If, in addition, ϕ is surjective, then $G_1 = G_2$.*

(ii) *Let M_1 and M_2 be two MV-subalgebras of the standard MV-algebra* $[0, 1]$. *If there is an MV-isomorphism ψ from M_1 onto M_2, then $M_1 = M_2$, and ψ is the identity.*

Proposition 8.5. *Let s_1 and s_2 be two state-morphisms on a pseudo EMV-algebra M such that* $\mathrm{Ker}(s_1) = \mathrm{Ker}(s_2)$. *Then $s_1 = s_2$.*

Proof. According to Theorem 8.3, every $\mathrm{Ker}(s_i)$, $i = 1, 2$, is a normal and maximal ideal of M, so that by Proposition 8.1 and Theorem 5.8, $M/\mathrm{Ker}(s_i)$ is a linear, Archimedean, and commutative pseudo EMV-algebra with top element. That is, $M/\mathrm{Ker}(s_i)$ is termwise equivalent to an MV-algebra which is an MV-subalgebra of the standard MV-algebra $[0, 1]$.

Define $\hat{s}_1$ and $\hat{s}_2$ according to (vii) of Proposition 8.1. Then they are state-morphisms on the EMV-algebra $M/\mathrm{Ker}(s_1)$. Let $M_i := \hat{s}_i(M/\mathrm{Ker}(s_i))$ for $i = 1, 2$. Then M_1 and M_2 are MV-subalgebras of the standard MV-algebra $[0, 1]$. Define a mapping $\psi : M_1 \to M_2$ by $\psi(\hat{s}_1([x])) = \hat{s}_2([x])$, $x \in M$. Then ψ is an MV-homomorphism which is injective and surjective, so by (ii) of Lemma 8.4, $M_1 = M_2$ which proves $s_1 = s_2$. ☐

Proposition 8.6. *Let I be a normal and maximal ideal of a pseudo EMV-algebra M. Then there is a unique state-morphism s on M such that* $\mathrm{Ker}(s) = I$.

Proof. Due to Theorem 5.8, M/I is a linear, Archimedean commutative pseudo EMV-algebra with top element, therefore, it is termwise equivalent to a pseudo MV-algebra M/I. Due to Theorem 2.2, there is an ℓ-group G with a strong unit u such that $M/I \cong \Gamma(G, u)$. In addition, G is linear and Archimedean, so that G is Abelian, and by Hölder's theorem, [Bir, Thm XIII.12], G is an ℓ-subgroup of the real group $(\mathbb{R}; +)$. Consequently, we can assume that $M/I \subseteq [0, 1]$, and the mapping $s : a \mapsto a/I$, $a \in M$, defines a state-morphism on M such that $\mathrm{Ker}(s) = I$. The uniqueness of the state-morphism s follows from Proposition 8.5. ☐

In view of Proposition 8.1(vi), the state space $\mathcal{S}(M)$ is a convex set. Now, we introduce the notion of an extremal state. We say that a state s on a pseudo EMV-algebra M is *extremal* if from $s = \lambda s_1 + (1-\lambda)s_2$, where $s_1, s_2 \in \mathcal{S}(M)$ and $\lambda \in (0, 1)$, we conclude that $s = s_1 = s_2$. We denote by $\partial\mathcal{S}(M)$ the set of extremal states on M. In what follows, we show that $\partial\mathcal{S}(M) = \mathcal{SM}(M)$.

Theorem 8.7. *Let s be a state on a pseudo EMV-algebra M. Then the following statements are equivalent:*

(i) *s is an extremal state on M.*

(ii) *s is a state-morphism on M.*

As a corollary, we have $\partial\mathcal{S}(M) = \mathcal{SM}(M)$.

Proof. (i) $\Rightarrow$ (ii). Let s be an extremal state on M. Define $\hat{s}$ by (vii) of Proposition 8.1 on the quotient pseudo EMV-algebra $M/\mathrm{Ker}(s)$ which by Theorem 5.8 is an EMV-subalgebra of $[0, 1]$ with top element. Since it is easy to see that on such an

EMV-algebra there is a unique state that is the identity, we see that $\hat{s}$ is an extremal state and this state is a state-morphism. Consequently, by Proposition 8.1(vii), s is a state-morphism on M.

(ii) $\Rightarrow$ (i). Let s be a state-morphism and let $s = \lambda s_1 + (1-\lambda)s_2$, where $0 < \lambda < 1$ and $s_1, s_2 \in \mathcal{S}(M)$.

Then $\mathrm{Ker}(s) = \mathrm{Ker}(s_1) \cap \mathrm{Ker}(s_2)$. Since $\mathrm{Ker}(s_1)$ and $\mathrm{Ker}(s_2)$ are normal proper ideals of M, and $\mathrm{Ker}(s)$ is a normal and maximal ideal of M, we have $\mathrm{Ker}(s) = \mathrm{Ker}(s_1) = \mathrm{Ker}(s_2)$. Applying Theorem 8.3, we conclude that $s = s_1 = s_2$, so that s is an extremal state on M.

Finally, if $M = \{0\}$, then $\mathcal{S}(M) = \emptyset = \mathcal{SM}(M) = \partial\mathcal{S}(M)$.

Moreover, if $M \neq \{0\}$ is stateless, then again $\mathcal{S}(M) = \emptyset = \mathcal{SM}(M) = \partial\mathcal{S}(M)$. □

As a corollary of the latter theorem, we obtain that if s is an extremal state on M and a is an idempotent, then $s(a) \in \{0, 1\}$.

Corollary 8.8. *Let a be an idempotent of a pseudo EMV-algebra and let s be an extremal state on M. Then $s(a) \in \{0, 1\}$.*

Proof. Assume the converse, i.e. $0 < s(a) < 1$. There is an idempotent $b \in \mathcal{I}(M)$ with $a < b$ such that $s(b) = 1$. Define two mappings $s_1(x) := s(x \wedge a)/s(a)$, $x \in M$, and $s_2(x) := s(x \wedge \lambda_b(a))/(1 - s(a))$, $x \in M$. According to the claim in the proof of Theorem 5.14, we have $(x \oplus y) \wedge c = (x \wedge c) \oplus (y \wedge c)$ for all $x, y \in M$ and $c \in \mathcal{I}(M)$. In addition, $s(x) = s(x \wedge b) = s(x \wedge a) + s(x \wedge \lambda_b(a))$ for each $x \in M$. Whence, s_1 and s_2 are states on M such that if $\lambda = s(a)$, then $s(x) = \lambda s_1(x) + (1 - \lambda)s_2(x)$, $x \in M$, which contradicts that s is an extremal state, see Theorem 8.7. □

We did not yet exhibit the question when the state space $\mathcal{S}(M)$ of a non-trivial pseudo EMV-algebra M is non-empty. In the following part we exhibit this question in more details. Due to Proposition 8.6, the existence of a maximal and normal ideal is a guarantee that M has at least one state (a state-morphism). We note that if M is a non-trivial EMV-algebra, M possesses at least one maximal ideal (which is automatically normal), so that M has at least one state. We shall prove below that the existence of at least one state on M implies the existence of a maximal and normal ideal of M.

We say that a net $\{s_\alpha\}_\alpha$ of states on a pseudo EMV-algebra M *converges weakly* to a state s on M, and we write $\{s_\alpha\}_\alpha \xrightarrow{w} s$, if $\lim_\alpha s_\alpha(x) = s(x)$ for every $x \in M$. Hence, $\mathcal{S}(M)$ is a subset of $[0,1]^M$. If we endow $[0,1]^M$ with the product topology which is a compact Hausdorff space, we see that the weak topology, which is in fact the relative topology of the product topology of $[0,1]^M$, yields a Hausdorff topological space. The case $\mathcal{S}(M)=\emptyset$ is not excluded. In addition, the system of subsets of $\mathcal{S}(M)$ of the form $S(x)_{\alpha,\beta}=\{s\in\mathcal{S}(M)\colon \alpha<s(x)<\beta\}$, where $x\in M$ and $\alpha<\beta$ are real numbers, forms a subbase of the weak topology of states.

As we have already mentioned, if $(M;\vee,\wedge,\oplus,0)$ has a top element 1, then M is termwise equivalent to the pseudo EMV-algebra $(M;\oplus,^-,^\sim,0,1)$ and the notions of states and state-morphisms, respectively, on both structures coincide. Therefore, by [7, Thm 4.8], $\mathcal{S}(M)$ and $\mathcal{SM}(M)$ are either both empty or both are non-empty compact Hausdorff topological spaces.

In the same way as we have defined the weak topology of states, we define the weak topology also for the set of state-morphisms. By Proposition 8.2, $\mathcal{SM}(M)$ is a closed subset of $\mathcal{S}(M)$, and $\mathcal{SM}(M)$ is also a Hausdorff space. The spaces $\mathcal{S}(M)$ and $\mathcal{SM}(M)$ are not necessarily compact sets because if, for a net $\{s_\alpha\}$ of states (state-morphisms), there is a limit $s(x)=\lim_\alpha s_\alpha(x)$, $x\in M$, s preserves $+$ $(\oplus,\wedge,\vee)$, but there is no guarantee that there is an element $x\in M$ with $s(x)=1$ as the following example from [12, Ex 4.8] shows.

Example 8.9. *Let $\mathcal{T}$ be the set of all finite subsets of the set $\mathbb{N}$ of natural numbers. Then $\mathcal{T}=(\mathcal{T};\vee,\wedge,\oplus,0)$ is an EMV-algebra with respect to $\vee=\cup$, $\wedge=\cap$, $\oplus=\vee$, and $0=\emptyset$, and $\mathcal{SM}(\mathcal{T})=\{s_n\colon n\in\mathbb{N}\}$, where $s_n(A)=\chi_A(n)$, $A\in\mathcal{T}$. Given $A\in\mathcal{T}$, there is $s(A)=\lim_n s_n(A)=0$, but s is not a state on $\mathcal{T}$.*

If a pseudo EMV-algebra M is with top element, then M is termwise equivalent to a pseudo MV-algebra, so that $\mathcal{S}(M)$ is easily either empty or non-empty and compact, see [7, Thm 4.8]. We present a topological criterion to be a pseudo EMV-algebra with top element.

Theorem 8.10. *Let M be a pseudo EMV-algebra. Define the following statements.*

(i) *M has a top element.*

(ii) *The space $\mathcal{S}(M)$ is compact.*

(iii) *The space $\mathcal{SM}(M)$ is compact.*

Then (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii). *If M has the property that every maximal ideal of M is normal, then all statements are equivalent.*

Proof. If $M=\{0\}$, then 0 is the top element and $\mathcal{S}(M)=\emptyset=\mathcal{SM}(M)$. Thus let $M\neq\{0\}$.

(i) $\Rightarrow$ (ii), (iii). If 1 is the top element of M, then $s(1)=1$ for each state s on M. Therefore, $\mathcal{S}(M)$ and $\mathcal{SM}(M)$ are closed in the product topology on $[0,1]^M$, so that both sets are compact in the weak topology.

(ii) $\Rightarrow$ (iii). If $\mathcal{S}(M)$ is compact, then $\mathcal{SM}(M)$, which is a closed subset of $\mathcal{S}(M)$, has to be compact, too.

(iii) $\Rightarrow$ (i). Let every maximal ideal of M be normal. Given $x \in M$, let $S(x) = \{s \in \mathcal{SM}(M) \colon s(x) > 0\}$. Then each $S(x)$ is an open set of $\mathcal{SM}(M)$. Given $s \in \mathcal{SM}(M)$, there is an idempotent $a \in M$ such that $s(a) = 1$, so that $s \in S(a)$ which means that $\{S(a) \colon a \in \mathcal{I}(M)\}$ is an open cover of $\mathcal{SM}(M)$. The compactness of $\mathcal{SM}(M)$ entails there are elements $a_1, \ldots, a_n \in M$ such that $\mathcal{SM}(M) = \bigcup_{i=1}^n S(a_i) = S(a_0)$, where $a_0 = a_1 \vee \cdots \vee a_n > 0$.

Claim. If b_0 is an idempotent element of M such that $s(b_0) = 0$ for each state-morphism s on M, then $b_0 = 0$.

We have $\mathcal{SM}(M) = \{s \in \mathcal{SM}(M) \colon s(b_0) < 1\} = \{s \in \mathcal{SM}(M) \colon s(b_0) = 0\}$. We assert $b_0 = 0$. If not, then $b_0 > 0$. Let F_{b_0} be the filter of M generated by b_0. Then $F_{b_0} = \{x \in M \colon b_0 \le x\}$ and F_{b_0} is a normal proper filter of M. At any rate, F_{b_0} is closed under $\odot$ and it is an up-set. Let $x \in M$ and $y \in F_{b_0}$ and let b be an idempotent of M such that $x, y, b_0 \le b$. Then $x \odot y = ((x \odot y) \oplus \rho_b(x)) \odot x$ and $(x \odot y) \oplus \rho_b(x) \ge (x \wedge b_0) \oplus \rho_b(x) = (x \oplus \rho_b(x)) \wedge (b_0 \oplus \lambda_b(x)) \ge b \wedge b_0$, i.e. $x \odot F_{b_0} \subseteq F_{b_0} \odot x$. In a dual way we can show the opposite inclusion.

Since $b_0 > 0$, F_{b_0} is a proper filter of M, there is a maximal filter F of M containing F_{b_0}. According to Theorem 4.18, the set $I_F^\lambda := \{\rho_a(x) \colon x \in F, \exists\, a \in \mathcal{I}(M), x \le a\}$ is a maximal ideal of M, which is normal due to the assumption on M. Hence, again by Theorem 4.18, F is a maximal filter that is also normal. Quoting Theorem 8.3, we have that there is a state-morphism s such that $\mathrm{Ker}_1(s) = F \supseteq F_{b_0}$. Then $s(b_0) = 1$ as well as $s(b_0) = 0$ which is a contradiction, so that $b_0 = 0$ which proves the claim.

Now we show that a_0 is a top element. If not, there is an idempotent $b \in \mathcal{I}(M)$ such that $a_0 < b$. Then $s(b) = 1$ for each $s \in \mathcal{SM}(M)$, so that $s(\lambda_b(a_0)) = 0$ for each state-morphism s on M which by Claim entails $\lambda_b(a_0) = 0$ and $a_0 = b$. In other words, a_0 is a top element of M. $\square$

We note that we do not know whether all statements are equivalent in any non-trivial pseudo EMV-algebra.

Let M be a proper pseudo EMV-algebra and let N be its representing pseudo EMV-algebra with top element. Using Basic Representation Theorem, we describe the states spaces of M and N, respectively. Without loss of generality, we will assume that $M \subseteq N$.

Proposition 8.11. *Let M be a pseudo EMV-algebra without top element. For each $x \in M$, we put $x^- = \lambda_1(x)$ and $x^\sim = \rho_1(x)$, where 1 is the top element* *of the representing pseudo EMV-algebra N. Given a state s on M, the mapping $\tilde{s}: N \to [0,1]$, defined by*

$$\tilde{s}(x) = \begin{cases} s(x) & \text{if } x \in M, \\ 1 - s(x_0) & \text{if } x = x_0^{\sim},\ x_0 \in M, \end{cases} \qquad x \in N, \tag{8.1}$$

is a state on N, and the mapping $s_\infty : N \to [0,1]$ defined by $s_\infty(x) = 0$ if $x \in M$ and $s_\infty(x) = 1$ if $x \in N \setminus M$, is a state-morphism on N. If s is a state-morphism on M, then $\tilde{s}$ is a state-morphism on N. Moreover, $\mathcal{SM}(N) = \{\tilde{s} \colon s \in \mathcal{SM}(M)\} \cup \{s_\infty\}$ and $\mathrm{Ker}(\tilde{s}) = \mathrm{Ker}(s) \cup \mathrm{Ker}_1^*(s)$, $s \in \mathcal{SM}(M)$, where $\mathrm{Ker}_1^*(s) = \{\rho_1(x) \colon x \in \mathrm{Ker}_1(s)\}$.

A net $\{s_\alpha\}_\alpha$ of states on M converges weakly to a state s on M if and only if $\{\tilde{s}_\alpha\}_\alpha$ converges weakly to $\tilde{s}$ on N, and the mapping $\phi : \mathcal{S}(M) \to \mathcal{S}(N)$ defined by $\phi(s) = \tilde{s}$, $s \in \mathcal{S}(M)$, is injective, continuous and affine.

Proof. By Theorem 6.1, $N = M \cup \rho_1(M)$, where $\rho_1(M) = \{\rho_1(x) \colon x \in M\}$. We note that for each $x_0 \in M$, we have $\lambda_1(x_0) = x_0^-$ and $\rho_1(x_0) = x^\sim$.

Let s be a state on M and define $\tilde{s}$ by (8.1). Then $\tilde{s}(u) = 1$. Let $x, y \in N$ and $y \odot x = 0$. There are four cases: Case (i): $x = x_0, y = y_0 \in M$. Then $\tilde{s}(x + y) = s(x_0 + y_0) = s(x_0) + s(y_0) = \tilde{s}(x) + \tilde{s}(y)$.

Case (ii): $x = x_0 \in M$ and $y = y_0^\sim$ where $y_0 \in M$. Then $y \odot x = 0$ implies $x = x_0 \le y_0$. There is an idempotent $a \in \mathcal{I}(M)$ such that $x_0 \le y_0 \le a$ and $s(a) = 1$. Since $x + y = x \oplus y = x_0 \oplus y_0^\sim = (y_0 \odot x_0^-)^\sim = (y_0 \odot \lambda_a(x_0))^\sim$, (8.1) and Proposition 8.1(ii) imply $\tilde{s}(x \oplus y) = 1 - s(y_0 \odot \lambda_a(x_0)) = 1 - s(y_0) + s(x_0) = \tilde{s}(x) + \tilde{s}(y)$.

Case (iii): $x = x_0^\sim$ and $y = y_0$ where $x_0, y_0 \in M$. Choose $a \in \mathcal{I}(M)$ such that $x, y \le a$ and $s(a) = 1$. Then $y \odot x = 0$ gives $y_0 \le x_0^{\sim\sim} = \varphi_\rho(x_0)$, see Theorem 6.1. We have

$$\begin{aligned} x \oplus y = x_0^\sim \oplus y_0 &= (y_0^\sim \odot x_0^{\sim\sim})^- = (y_0^\sim \odot \varphi_\rho(x_0))^- = \big((y_0 \wedge \varphi_\rho(x_0))^\sim \odot \varphi_\rho(x_0)\big)^- \\ &= \big(\rho_a(y_0 \wedge \varphi_\rho(x_0)) \odot \varphi_\rho(x_0)\big)^{--\sim} = \big(\varphi_\lambda(\rho_a(y_0 \wedge \varphi_\rho(x_0)) \odot \varphi_\rho(x_0))\big)^\sim \in N. \end{aligned}$$

By (ii) of Proposition 8.1, we get $\tilde{s}(x \oplus y) = 1 - s(\varphi_\lambda(\rho_a(y_0 \wedge \varphi_\rho(x_0))) \odot \varphi_\rho(x_0)) = 1 - s(\rho_a(y_0 \wedge \varphi_\rho(x_0)) \odot \varphi_\rho(x_0)) = 1 - s(\varphi_\rho(x_0)) + s(\rho_a(y_0 \wedge \varphi_\rho(x_0)) = 1 - s(x_0) + s(y_0) = s(x) + s(y) = \tilde{s}(x) + \tilde{s}(y)$.

Case (iv): $x = x_0^\sim$, $y = y_0^\sim$ for some $x_0, y_0 \in M$. Then $y \odot x = 0$ entails $x_0^\sim \le y_0$, $u = x^\sim \oplus x_0 \le y_0 \oplus x_0 \in M$ which is an absurd.

Cases (i)–(iv) prove that s is a state on the representing pseudo EMV-algebra N.

If s is a state-morphism, we proceed in a similar way as for states. Let $x, y \in N$. We have again four cases. Case (i): $x = x_0$, $y = y_0$, $x_0, y_0 \in M$. This case is trivial. Case (ii): $x = x_0$, $y = y_0^\sim$ for $x_0, y_0 \in M$. Then there exists an idempotent $a \in \mathcal{I}(M)$ such that $x_0, y_0 \le a$ and $s(a) = 1$. We have $x \oplus y = x_0 \oplus y_0^\sim = (y_0 \odot x_0^-)^\sim = (y_0 \odot \lambda_a(x_0))^\sim$ which yields $\tilde{s}(x \oplus y) = 1 - s(y_0 \odot \lambda_a(x_0)) = 1 - (s(y_0) \odot (s(a) - s(x_0))) = (1 - s(y_0)) \oplus s(x_0) = \tilde{s}(x) \oplus \tilde{s}(y)$.

Case (iii): $x = x_0^\sim$, $y = y_0$ for $x_0, y_0 \in M$. We use calculation from Case (iii) above $\tilde{s}(x \oplus y) = 1 - s(\varphi_\lambda(\rho_a(y_0 \wedge \varphi_\rho(x_0))) \odot \varphi_\rho(x_0)) = 1 - s(\rho_a(y_0 \wedge \varphi_\rho(x_0)) \odot \varphi_\rho(x_0)) = 1 - s(\varphi_\rho(x_0)) + s(y_0 \wedge \varphi_\rho(x_0)) = 1 - s(x_0) + \min\{s(y_0), s(x_0)\} = \min\{1 - s(x_0) + s(y_0), 1\} = \tilde{s}(x) \oplus \tilde{s}(y)$.

Case (iv): $x = x_0^\sim$, $y = y_0^\sim$ for $x_0, y_0 \in M$. Then $x \oplus y = (y_0 \odot x_0)^\sim$, so that $\tilde{s}(x \oplus y) = 1 - s(y_0 \odot x_0) = 1 - s(y_0) \odot s(x_0) = \tilde{s}(x) \oplus \tilde{s}(y)$.

The mapping s_∞ is evidently a state-morphism on N. Now let s be any state-morphism on N. By Corollary 8.8, there are two cases: (i) For each idempotent $a \in M$, we have $s(a) = 0$. Then $s(x) = 0$ for each $x \in M$, i.e. $s = s_\infty$. (ii) There is an idempotent $a \in M$ such that $s(a) = 1$. Then the restriction of s onto M is a state-morphism on M, say s_0, so that that $s = \tilde{s}_0$.

Therefore, a net of states $\{s_\alpha\}_\alpha \stackrel{w}{\to} s$ on M iff $\{\tilde{s}_\alpha\}_\alpha \stackrel{w}{\to} \tilde{s}$ on N. The rest of the proof is straightforward. □

Now we present a necessary and sufficient condition in order $\mathcal{S}(M) \neq \emptyset$.

Theorem 8.12. *The state space $\mathcal{S}(M)$ of a pseudo EMV-algebra M is non-empty if and only if M possesses at least one maximal and normal ideal.*

Proof. If M has at least one maximal and normal ideal I, by Proposition 8.6, M has a unique state-morphism s such that $\mathrm{Ker}(s) = I$, and due to Proposition 8.2, s is a state on M so that $\mathcal{S}(M) \neq \emptyset$.

Conversely, let s be a state on M. If M has a top element 1, then s can be viewed as a state on the pseudo MV-algebra $(M; \oplus, ^-, ^\sim, 0, 1)$. Using [7, Thm 4.8], s is a weak limit of a net of convex combinations of state-morphisms on M. So that there is at least one extremal state (= state-morphism), say s_0, and due to Theorem 8.3(ii), $\mathrm{Ker}(s_0)$ is a maximal and normal ideal of M.

If M does not have a top element, Basic Representation Theorem 6.4 guarantees that there is a pseudo EMV-algebra N with top element such that M can be viewed as a maximal and normal ideal of N, and $N = M \cup \rho_1(M)$. We extend the state s on M to a state $\tilde{s}$ using (8.1). Then the state space $\mathcal{S}(N)$ contains the state $\tilde{s}$ which is not zero on M. Again using [7, Thm 4.8], we see that N possesses a state-morphism $s' \neq s_\infty$, consequently, N has a maximal and normal ideal $J = \mathrm{Ker}(s')$ such that $J \neq M$. Let $I = J \cap M$, then I is a normal ideal of M that is also a maximal ideal of M, which finishes the proof. □

We note that according to Proposition 5.1, every non-trivial linearly ordered pseudo EMV-algebra, consequently every representable one possesses at least one state. Using (5.7), we see that the same is true for each non-trivial normal-valued one or for every non-trivial pseudo EMV-algebra with the property that each its maximal ideal is normal.

Now we present an example of a proper pseudo EMV-algebra which has no state.

Example 8.13. *Let M_0 be a non-trivial pseudo MV-algebra that has no state, for example, let $M_0 = \Gamma(\mathrm{BAut}(\mathbb{R}), u)$ from Theorem* 6.13 *which is a stateless pseudo MV-algebra,* [7]*. For each $n \geq 1$, let $M_n = M_0$ and let M be the system of all $(x_n)_n \in \prod_n M_n$ such that $x_n = 0$ for all but finitely many n. We assert that M is a stateless proper pseudo EMV-algebra.*

Proof. Suppose the converse, i.e. let s_0 be a state on M. Without loss of generality, we can assume that s_0 is a state-morphism. There is an element $x = (x_n)_n$ such that $s_0(x) = 1$. We assume that only the first n coordinates of x are non-zero. Then $s_0(x_0) = 1$ for $x_0 = (1, \ldots, 1, 0 \ldots)$, where 1 is on the first n coordinates of x_0. Define $x_i = (x^i_n)$, where $x^i_n = 1$ if $n = i$ otherwise $x^i_n = 0$, $i = 1, \ldots, n$ and $n \geq 1$. Since $x_1 \vee \cdots \vee x_n = x_0$ and s_0 is a state-morphism, from $1 = s_0(x_0) = \max\{s_0(x_1), \ldots, s_0(x_n)\}$ we conclude that for some x_i, we have $s_0(x_i) = 1$. Without loss of generality, we can assume that $x_i = x_1$.

Define a mapping $s : M_0 \to [0, 1]$ by $s(x) = s_0(x, 0, \ldots)$, $x \in M_0$. Then s is a state on M_0 which contradicts the fact that M_0 is a stateless.

We note that then $\mathcal{S}(M) = \emptyset$ and $\mathcal{S}(N) = \{s_\infty\}$, where N is a pseudo EMV-algebra with top element representing M. □

We remind that a topological space Ω is *locally compact* if every point of Ω has a compact neighborhood. Local compactness of $\mathcal{SM}(M)$ for a proper EMV-algebra M was established in [11, Thm 4.10], and here we extend this result also for proper pseudo EMV-algebra. Our proof will follows basic ideas from [11, Thm 4.10].

Theorem 8.14. *If a pseudo EMV-algebra M does not have a top element, then $\mathcal{SM}(M)$ is either an empty set or is a locally compact non-empty Hausdorff space in the weak topology such that if a is an idempotent, then $S(a) = \{s \in \mathcal{SM}(M) \colon s(a) > 0\}$ is a compact clopen subset.*

Proof. Due to Basic Representation Theorem 6.4, there is a pseudo EMV-algebra N with top element such that M can be embedded into N as a normal and maximal ideal of N. Without loss of generality, we can assume that M is a subset of N. If $\mathcal{SM}(M)$ is empty, we are ready. So, let there be at least one state-morphism on M.

Given $x \in M$ and $y \in N$, let $S(x) = \{s \in \mathcal{SM}(M)\colon s(x) > 0\}$ and $S_N(y) = \{s \in \mathcal{SM}(N)\colon s(y) > 0\}$, they are open sets.

Define a mapping $\phi : \mathcal{SM}(M) \to \mathcal{SM}(N)$ by $\phi(s) = \tilde{s}$, $s \in \mathcal{SM}(M)$, where $\tilde{s}$ is defined by (8.1). Then ϕ is an injective mapping such that $\phi(S(x)) = S_N(x)$ for each $x \in M$. Take an idempotent $a \in \mathcal{I}(M)$. Using Corollary 8.8, $S(a) = \{s \in \mathcal{SM}(M)\colon s(a) > 0\} = \{s \in \mathcal{SM}(M)\colon s(a) = 1\}$, so $S(a)$ is both open and closed. The same is true for $S_N(a) = \{s \in \mathcal{SM}(N)\colon s(a) > 0\}$, in addition $S_N(a)$ is compact because $\mathcal{SM}(N)$ is compact.

For each $x \in M$ and u, v real numbers with $u < v$, the sets $S(x)_{u,v} = \{s \in \mathcal{SM}(M)\colon u < s(x) < v\}$ and $S_N(x)_{u,v} = \{s \in \mathcal{SM}(N)\colon u < s(x) < v\}$, where $x \in N$, are open and they form a subbase of the weak topologies. Then $\phi(S(x)_{u,v}) = S_N(x)_{u,v}$ if $u \geq 0$, $\phi(S(x)_{u,v}) = S_N(x)_{u,v} \setminus \{s_\infty\}$ if $u < 0$, and $\phi(S(x)) = S_N(x)$ whenever $x \in M$.

Now we show that $S(a)$ is a compact set in $\mathcal{SM}(M)$. Take an open cover of $S(a)$ in the form $\{S(x_\alpha)_{u_\alpha,v_\alpha}\colon \alpha \in A\}$, where $x_\alpha \in M$ and u_α, v_α are real numbers such that $u_\alpha < v_\alpha$ for each $\alpha \in A$. Then

$$\begin{aligned} S(a) &\subseteq \bigcup_\alpha S(x_\alpha)_{u_\alpha,v_\alpha} \\ \phi(S(a)) &\subseteq \bigcup_\alpha \phi(S(x_\alpha)_{u_\alpha,v_\alpha}) \\ S_N(a) &\subseteq \bigcup_\alpha \phi(S(x_\alpha)_{u_\alpha,v_\alpha}). \end{aligned}$$

The compactness of $S_N(a)$ entails a finite subset F of A such that

$$S_N(a) \subseteq \bigcup \{\phi(S(x_\alpha)_{u_\alpha,v_\alpha})\colon \alpha \in F\},$$

whence, $S(a) \subseteq \bigcup \{S(x_\alpha)_{u_\alpha,v_\alpha}\colon \alpha \in F\}$. Since the system of all open sets $S(x)_{u,v}$ forms a subbase of the weak topology of $\mathcal{SM}(M)$, we have by [18, Thm 5.6], $S(a)$ is compact and clopen as well. In addition, given a state-morphism $s \in \mathcal{SM}(M)$, there is an element $x \in M$ with $s(x) = 1$, and there is an idempotent $a \in M$ such that $x \leq a$ which entails $s \in S(x) \subseteq S(a)$. Whence, $\mathcal{SM}(M)$ is locally compact. $\square$

In the following result, we show that the non-empty state space $\mathcal{S}(M)$ of a proper pseudo EMV-algebra M with the property that each maximal ideal is normal cannot be locally compact. Hence, we present another topological criterion to be a pseudo EMV-algebra with top element.

Theorem 8.15. *Let M be a pseudo EMV-algebra with the property that every maximal ideal of M is normal. If the state space of a pseudo EMV-algebra M is non-empty and locally compact, then M has a top element.*

Proof. We establish that if $\mathcal{S}(M)$ is non-empty and locally compact, then it is compact. Suppose the converse, that is, $X = \mathcal{S}(M)$ is locally compact but not compact in the weak topology of states, therefore, M has no top element. Without loss of generality, we can find a pseudo EMV-algebra N with top element representing M such that M is a maximal and normal ideal of N, and for every element $x \in N$, there is an element $x_0 \in M$ such that either $x = x_0$ or $x = \rho_1(x_0)$. According to the Alexandroff theorem, see [18, Thm 4.21], there is a compact space $X^* = X \cup \{x_\infty\}$, where $x_\infty \notin X$. Define a mapping $\phi : \mathcal{S}(M) \to \mathcal{S}(N)$ given by $\phi(s) = \tilde{s}$, $s \in \mathcal{S}(M)$, where $\tilde{s}$ is defined by (8.1). Let s be a state on M. Then a net of states $\{s_\alpha\}_\alpha$ on M converges weakly to the state s on M iff $\{\tilde{s}_\alpha\}_\alpha$ converges weakly to $\tilde{s}$ on N. Therefore, ϕ maps X onto the set $\phi(X) = \{\tilde{s} \colon s \in \mathcal{S}(M)\}$, so that ϕ is a homeomorphism from X onto $\phi(X)$. Then also $\phi(X)$ has the one-point compactification $(\phi(X))^* = \phi(X) \cup \{x^*_\infty\}$, where $x^*_\infty \notin \phi(X)$. But for the state-morphism s_∞ on N given by $s_\infty(x) = 0$ if $x \in M$ and $s_\infty(x) = 1$ for $x \in N \setminus M$, there is a net $\{t_\beta\}_\beta$ of state-morphisms on M, such that $\{\tilde{t}_\beta\}_\beta$ converges weakly to s_∞ on N. Therefore, $t_\beta \in X$ and $\tilde{t}_\beta \in \phi(X)$ for each index β and $\lim_\beta t_\beta(x) = 0$ for each $x \in M$ and $\tilde{s}_0 = s_\infty$. On the other hand, since $(\phi(X))^*$ is compact, there is a subnet $\{\tilde{t}_{\beta_\alpha}\}_\alpha$ of the net $\{\tilde{t}_\beta\}_\beta$ which converges to some point $x^* \in \phi(X) \cup \{x^*_\infty\}$. Then $x^* = x^*_\infty = s_\infty$.

Now let s be any state-morphism on M and for each $\lambda \in (0,1)$ we set $t^\lambda_\beta = \lambda s + (1-\lambda)t_\beta$. Then $t^\lambda_\beta \in X$ and $\phi(t^\lambda_\beta) = \lambda\tilde{s} + (1-\lambda)\tilde{t}_\beta \in \phi(X)$ for each index β. Since $\{\phi(t^\lambda_\beta)\}_\beta$ converges weakly on N to $\lambda\tilde{s} + (1-\lambda)s_\infty$ so that $\lambda\tilde{s} + (1-\lambda)s_\infty \in (\phi(X))^* = \phi(X) \cup \{s_\infty\}$. But $\lambda\tilde{s} + (1-\lambda)s_\infty$ gives for each $\lambda \in (0,1)$ uncountably many mutually different states on N not belonging to $\phi(X)$, which says that there is no one-point compactification of $\mathcal{S}(M)$. Hence, our assumption that $\mathcal{S}(M)$ is not compact was wrong, and $\mathcal{S}(M)$ has to be compact.

Applying Theorem 8.10, we obtain the desired result, that is, M has a top element. $\square$

Remark 8.16. From the proof of Theorem 8.15 we conclude that for every pseudo EMV-algebra M, $\mathcal{SM}(M)$ is locally compact if and only if it is compact in the weak topology of states on M.

We continue with description of the set of state-morphisms on a pseudo EMV-algebra M that does not have a top element and of the set of state-morphisms of its representing pseudo EMV-algebra N with top element. We generalize the analogous result [11, Thm 4.13] holding for EMV-algebras without top elements. Here we follow steps of the proof from [11, Thm 4.13] which was necessary now improve for pseudo EMV-algebras.

Theorem 8.17. *Let M be a proper pseudo EMV-algebra and N be its representing pseudo EMV-algebra with top element. If $\mathcal{SM}(M)$ is non-empty, then $\mathcal{SM}(N)$ is the one-point compactifications of the space $\mathcal{SM}(M)$.*

Proof. According to Theorem 8.14, $\mathcal{SM}(M)$ is a locally compact Hausdorff topological space. Due to the Alexandroff theorem, see [18, Thm 4.21], there is the one-point compactification of $\mathcal{SM}(M)$. We show that the one-point compactification of $\mathcal{SM}(M)$ is the space $\mathcal{SM}(N)$.

We proceed in five steps. We note that s_∞ is a unique two-valued state-morphism on N defined in Proposition 8.11.

(1) If O_N is an open set of $\mathcal{SM}(N)$ such that $s_\infty \notin O_N$, then $O_N = \phi(O)$ for some open subset O of $\mathcal{SM}(M)$.

(2) Now take an open set O_N containing s_∞ and $O_N = S_N(x)_{u,v}$, where $x \in M$ and u, v are real numbers with $u < v$. Since $s_\infty(x) = 0$, $u < 0 < v$ and we have $S_N(x)_{u,v} = \{s_\infty\} \cup \{\tilde{s}\colon s \in \mathcal{SM}(M), s(x) < v\} = \{s_\infty\} \cup \phi(\{s \in \mathcal{SM}(M)\colon s(x) < v\})$. If $X := \mathcal{SM}(N) \setminus S_N(x)_{u,v}$, then

$$
\begin{aligned}
X &= \phi(\mathcal{SM}(M)) \setminus (\{s_\infty\} \cup \phi(\{s \in \mathcal{SM}(M)\colon s(x) < v\})) \\
&= \{s \in \mathcal{SM}(M)\colon s(x) \ge v\} \subseteq \{s \in \mathcal{SM}(N)\colon s(a) \ge v\},
\end{aligned}
$$

where $a \in \mathcal{I}(M)$ such that $x \le a$. If $v > 1$, then $X = \emptyset$ which is a compact set and if $v \le 1$, then $X \subseteq \{s \in \mathcal{SM}(M)\colon s(a) = 1\}$. Since the latter set is a compact set of $\mathcal{SM}(\mathcal{N})$, see Theorem 8.14, we see that X is closed, and consequently, X is a compact set of $\mathcal{SM}(N)$, too.

(3) Now let $s_\infty \in O_N = S_N(x)_{u,v}$, where $x \in M$ and u, v are real numbers with $u < v$ and $x = \rho_1(x_0)$, where $x_0 \in M$. Since $s_\infty(x) = 1$, we have $u < 1 < v$. Then $S_N(x)_{u,v} = \{s_\infty\} \cup \{\tilde{s}\colon s \in \mathcal{SM}(M), u < \tilde{s}(x)\} = \{s_\infty\} \cup \phi(\{s \in \mathcal{SM}(M)\colon s(x_0) < 1 - u\})$. Therefore, $X := \mathcal{SM}(N) \setminus S_N(x)_{u,v} = \phi(\mathcal{SM}(M)) \setminus (\{s_\infty\} \cup \phi(\{s \in \mathcal{SM}(M)\colon s(x_0) < 1 - u\})) = \phi(\mathcal{SM}(M) \setminus \{s \in \mathcal{SM}(M)\colon s(x_0) < 1 - u\}) = \phi(\{s \in \mathcal{SM}(M)\colon s(x_0) \ge 1 - u\})$ and $X = \phi(\{s \in \mathcal{SM}(M)\colon s(x_0) \ge 1 - u\}) = \emptyset$, which is a compact set, if $u < 0$, and $X \subseteq \phi(\{s \in \mathcal{SM}(M)\colon s(a) \ge 1 - u\}) = \phi(\{s \in \mathcal{SM}(M)\colon s(a) = 1\})$ if $u \ge 0$ and a is an idempotent of M with $x_0 \le a$. Therefore, X is a closed subset which is a subset of a compact set, and we have X is a compact set of $\mathcal{SM}(N)$.

(4) Let $s_\infty \in O_N = \bigcap_{i=1}^n S_N(x_i)_{u_i,v_i}$, where $u_i \in N$, $u_i < v_i$ and $s_\infty \in S_N(x_i)_{u_i,v_i}$ for each $i = 1, \ldots, n$. Then $S_N(x_i)_{u_i,v_i} = \{s_\infty\} \cup \phi(S(x_i')_{u_i',v_i'})$ where if $x_i \in M$, then $x_i' = x_i$ and $u_i' = u_i$, $v_i' = v_i$ and if $x_i \in N \setminus M$, then $x_i' = \rho_1(x_i)$ and $u_i' = 1 - v_i$, $v_i' = 1 - u_i$.

Then, $\phi(\mathcal{SM}(M)) \setminus \bigcap_{i=1}^n S_N(x_i)_{u_i,v_i} = \phi(\mathcal{SM}(M)) \setminus (\{s_\infty\} \cup \phi(\bigcap_{i=1}^n S(x_i')_{u_i',v_i'})) = \phi(\bigcup_{i=1}^n (\mathcal{SM}(M) \setminus S(x_i')_{u_i',v_i'}))$, so that $\bigcup_{i=1}^n (\mathcal{SM}(M) \setminus S(x_i')_{u_i',v_i'})$ is a compact set in view of (3).

(5) $O_N = \bigcup_\alpha O_\alpha^N$, where each O_α^N is the set of the form (4). Then $O_\alpha^N = \{s_\infty\}\cup\phi(O_\alpha)$ if $s_\infty \in O_\alpha^N$, otherwise $O_\alpha^N = O_\alpha$, where O_α is an open set in $\mathcal{SM}(M)$.

Then $\phi(\mathcal{SM}(M) \setminus \bigcup_\alpha O_\alpha^N) = \phi(\mathcal{SM}(M) \setminus \bigcup_\alpha O_\alpha)$, where O_α is a subset of $\mathcal{SM}(M)$ such that $O_\alpha^N = \phi(O_\alpha)$. Whence, $\phi(\mathcal{SM}(M) \setminus \bigcup_\alpha O_\alpha) = \phi(\bigcap_\alpha(\mathcal{SM}(M) \setminus O_\alpha)) \subseteq \phi(\mathcal{SM}(M) \setminus O_{\alpha_0})$, where α_0 is an index α such that $s_\infty \in O_{\alpha_0}^N$, which is by (4) a compact set, consequently, $\phi(\bigcap_\alpha(\mathcal{SM}(M) \setminus O_\alpha))$ is a compact set.

Therefore, $\mathcal{SM}(N)$ is the one-point compactification of $\mathcal{SM}(M)$. $\square$

As we have already said, states on a pseudo EMV-algebra $(M; \vee, \wedge, \oplus, 0)$ with a top element 1 are the same as states on the pseudo MV-algebra $(M; \oplus, ^-, ^\sim, 0, 1)$. In addition, every state on a pseudo MV-algebra is a weak limit of convex combinations of state-morphisms, see [7, Thm 4.8]. This statement follows easily from Krein–Mil'man theorem, see e.g. [15, Thm 5.17], holding in compact convex Hausdorff spaces. But as we have seen, Theorem 8.10, the space of a pseudo EMV-algebra without top element is not compact. Despite of this, we present the following Krein–Mil'man type representation of states even for proper pseudo EMV-algebras. We will follow almost mutatis mutandis the ideas from [12, Thm 4.12].

Theorem 8.18. *Let M be a pseudo EMV-algebra. Then*

$$\mathcal{S}(M) = (\mathrm{Con}(\mathcal{SM}(M)))^{-_M}, \tag{8.2}$$

where $^{-_M}$ and Con *denote the closure in the weak topology of states on M and the convex hull, respectively.*

Proof. If M has a top element, then $\mathcal{S}(M)$ is a compact set in the weak topology. A direct application of the Krein–Mil'man theorem yields the result. If $M = \{0\}$, then $\mathcal{S}(M) = \mathcal{SM}(M) = \emptyset$, so that the result holds also in this case.

Now, let M have no top element and let $\mathcal{S}(M) \neq \emptyset$. Due to Theorem 8.12, M admits at least one state-morphism. Using the Basic Representation Theorem for pseudo EMV-algebras, see Theorem 6.4, we can assume that there is a pseudo EMV-algebra N with top element such that M is its maximal and normal ideal and every element $x \in N$ is either $x \in M$ or $x = \rho_1(x_0)$ for some $x_0 \in M$. Then for the state space of N we have $\mathcal{S}(N) = (\mathrm{Con}(\mathcal{SM}(N)))^{-_N}$, where $^{-_N}$ is the closure in the weak topology of states on N. Take an arbitrary state s on M that is not extremal, equivalently, s is not a state-morphism on M. There is a net $\{s_\alpha\}_\alpha$ of convex combinations from $\mathcal{SM}(N)$ such that $\{s_\alpha\}_\alpha$ converges weakly to $\tilde{s}$ on N. Since also $\tilde{s}$ is not an extremal state on N, without loss of generality we can assume that each s_α is not a state-morphism.

In addition, let $s_\alpha = \lambda_0^\alpha s_\infty + \sum_{i=1}^{n_\alpha} \lambda_i^\alpha \tilde{s}_i^\alpha$, where all λ^α's are from $[0,1]$, $\sum_{i=0}^{n_\alpha} \lambda_i^\alpha = 1$, and $s_i^\alpha \in \mathcal{SM}(M)$ for $i = 1,\dots,n_\alpha$ and for each α. If there is an index α_0 such that for each $\alpha > \alpha_0$, we have $\lambda_0^\alpha = 0$ which gives $s \in (\mathrm{Con}(\mathcal{SM}(M)))^{-_M}$. Therefore, we can assume also that each $\lambda_0^\alpha > 0$, or to pass to a subnet of $\{\alpha_0^\alpha\}_\alpha$ with such a property, if necessary. In addition, we can assume $\lambda_0^\alpha < 1$ for each α, otherwise $s_\alpha = s_\infty$, $s_\alpha(x) = 0$ and $s(x) = 0$ for each $x \in M$, which is impossible.

Since s is a state on M, there is an element $a \in M$ such that $s(a) = 1$. Then $s_\alpha(a) = \sum_{i=1}^{n_\alpha} \lambda_i^\alpha \tilde{s}_i^\alpha(a) \le \sum_{i=1}^{n_\alpha} \lambda_i^\alpha = 1 - \lambda_0^\alpha \le 1$. Then $1 = \liminf_\alpha s_\alpha(a) \le \liminf_\alpha (1 - \lambda_0^\alpha) \le 1$, so that $\limsup_\alpha \lambda_0^\alpha = 1$ and similarly, $1 = \limsup_\alpha s_\alpha(a) \le \limsup_\alpha (1-\lambda_0^\alpha) \le 1$, i.e. $\liminf_\alpha \lambda_0^\alpha = 1$. Whence, $\lim_\alpha \lambda_0^\alpha$ exists and $\lim_\alpha \lambda_0^\alpha = 1$. In addition, $t_\alpha := s_\alpha/(1-\lambda_0^\alpha) = \sum_{i=1}^{n_\alpha} \lambda_i^\alpha/(1-\lambda_0^\alpha) s_i^\alpha$ and $t_\alpha \in \mathrm{Con}(\mathcal{SM}(M))$. Moreover, the net $\{t_\alpha\}_\alpha$ converges weakly to s on M, so that $s \in (\mathrm{Con}(\mathcal{SM}(M)))^{-_M}$ and finally, $\mathcal{S}(M) = (\mathrm{Con}(\mathcal{SM}(M)))^{-_M}$. □

Finally, we introduce the so-called hull-kernel topology on the set $\mathcal{NM}(M)$ of all maximal and normal ideals of a pseudo EMV-algebra M. According to Theorem 8.12, we see that $\mathcal{NM}(M)$ is non-empty iff M possesses at least one state.

For every $a \in M$, we put

$$M(a) := \{I \in \mathcal{NM}(M) :\ a \notin I\}.$$

Then (i) $M(0) = \emptyset$, $M(a) \subseteq M(b)$ whenever $a \le b$, $M(a \wedge b) = M(a) \cap M(b)$, $a, b \in M$, $M(a \vee b) = M(a) \cup M(b)$, $a, b \in M$, and $\{M(a) :\ a \in M\}$ is the base of the so-called *hull-kernel topology* $T_{\mathcal{NM}}$ on $\mathcal{NM}(M)$.

Proposition 8.19. *Let M be a pseudo MV-algebra. Then the hull-kernel topology defines a Hausdorff topology such that the closed subspaces of $\mathcal{NM}(M)$ are exactly of the form*

$$C = C(J) := \{I \in \mathcal{NM}(M) :\ I \supseteq J\}, \tag{8.3}$$

where J is an ideal of M. Similarly, every open set O is of the form

$$O = O(J) := \{I \in \mathcal{NM}(M) :\ I \not\supseteq J\}. \tag{8.4}$$

Moreover, $T_{\mathcal{NM}}(M)$ is a Hausdorff topology on $\mathcal{NM}(M)$.

Proof. Let J be any ideal of M. Then $J = \bigvee_{a\in J} I(a)$, where $I(a)$ is an ideal of M generated by the element a. Then $O(J) := \{I \in \mathcal{NM}(M) \colon\ I \not\supseteq J\} = \bigcup_{a\in J}\{I \colon\ I \not\supseteq I(a)\} = \bigcup_{a\in J} M(a)$ is an open set of $\mathcal{NM}(M)$, and each open set is of the form (8.4). Consequently, every closed subset of $\mathcal{NM}(M)$ is of the form (8.3).

If I_1 and I_2 are two different maximal and normal ideals of M, they are non-comparable, so that there are $x \in I_1 \setminus I_2$ and $y \in I_2 \setminus I_1$. Then $x \wedge y \in I_1 \cap I_2$

and take an idempotent $a \in \mathcal{I}(M)$ such that $x, y \leq a$. The elements $x \odot \lambda_a(y)$ and $y \odot \lambda_a(x)$ belong to the pseudo MV-algebra $[0, a]$. Due to $x = (x \odot \lambda_a) \oplus (x \wedge y)$, we have $x \odot \lambda_a(y) \in I_1 \setminus I_2$. In the same way, we have $y \odot \lambda_a(x) \in I_2 \setminus I_1$. We get $I_2 \in M(x \odot y^-)$, $I_1 \in M(y \odot x^-)$, and $(x \odot \lambda_a(y)) \wedge (y \odot \lambda_a(x)) = 0$ which proves that $\mathcal{NM}(M)$ is a Hausdorff space that can be even empty. $\square$

To prove that the spaces $\mathcal{SM}(M)$ and $\mathcal{MN}(M)$ are homeomorphic, we establish the following proposition which for an EMV-algebra was established in [11, Prop 4.7] and our proof will follow ideas of the proof [11, Prop 4.7].

Proposition 8.20. *Let M be a pseudo EMV-algebra and X be a non-empty subspace of state-morphisms on M that is closed in the weak topology of state-morphisms. Let t be a state-morphism such that $t \notin X$. There exists an $a \in M$ such that $t(a) > 1/2$ while $s(a) < 1/2$ for all $s \in X$. Moreover, the element $a \in M$ can be chosen such that $t(a) = 1$ and $s(a) = 0$ for each $s \in X$.*

Proof. The proof will follow the next three steps.

(1) Let t be a state-morphism such that $t \notin X$. We assert that there exists an $a \in M$ such that $t(a) > 1/2$ while $s(a) < 1/2$ for all $s \in X$.

Indeed, set $A = \{a \in M : \ t(a) > 1/2\}$, and for all $a \in A$, let

$$W(a) := \{s \in \mathcal{SM}(M) \colon \ s(a) < 1/2\},$$

which is an open subset of $\mathcal{SM}(M)$. We note that $A \neq \emptyset$ and A is downward directed and closed under $\oplus$.

We assert that these open subsets cover X. Consider any $s \in X$. Since $\mathrm{Ker}(s)$ and $\mathrm{Ker}(t)$ are non-comparable subsets of M, there exists $x \in \mathrm{Ker}(t) \setminus \mathrm{Ker}(s)$. Hence $t(x) = 0$ and $s(x) > 0$. Choose an idempotent $b \in M$ such that $x \leq b$ and $t(b) = 1$. There exists an integer $n \geq 1$ such that $s(n.x) > 1/2$. Since there is also an integer k such that $s(k.x) = k.s(x) = 1$ and $k.x \leq b$, we conclude $s(b) = 1$. Because t is a state-morphism, we have $t(n.x) = 0$. Putting $a = \lambda_b(n.x)$, we have $t(a) = 1 > 1/2$ and $s(a) < 1/2$. Therefore $\{W(a) \colon a \in A\}$ is an open covering of X.

(i) If M has a top element, the state-morphism space $\mathcal{SM}(M)$ is compact and Hausdorff, so that X is compact, and $X \subseteq W(a_1) \cup \cdots \cup W(a_n)$ for some $a_1, \ldots, a_n \in A$.

(ii) If M has no top element, embed M into a representing pseudo EMV-algebra N with top element as its maximal and normal ideal. Since $s(1) = 1$ for each state-morphism s on N, we see that $\mathcal{SM}(N)$ is a compact set in the product topology, consequently, it is compact in the weak topology of state-morphisms on N. The mapping $\phi : \mathcal{SM}(M) \to \mathcal{SM}(N)$ given by $\phi(s) = \tilde{s}$, where $\tilde{s}$ is defined through (8.1), is by Proposition 8.11 injective and continuous.

We assert the set $\phi(X)\cup\{s_\infty\}$ is a compact subset of $\mathcal{SM}(N)$. Indeed, let $\{s_\alpha\}_\alpha$ be a net of state-morphisms from $\phi(X)\cup\{s_\infty\}$. Since $\mathcal{SM}(N)$ is compact, there is a subnet $\{s_{\alpha_\beta}\}_\beta$ of the net $\{s_\alpha\}_\alpha$ converging weakly to a state-morphism s on N. If $s=s_\infty$, $s\in\phi(X)\cup\{s_\infty\}$. If $s\neq s_\infty$, there is a state-morphism $s_0\in\mathcal{SM}(M)$ such $s=\tilde{s}_0$. Then there is β_0 such that for each $\beta>\beta_0$, $s_{\alpha_\beta}\in X$. Therefore, $s_0\in X$ and $s=\phi(s_0)\in\phi(X)\cup\{s_\infty\}$. We note that $\tilde{t}\notin\phi(X)\cup\{s_\infty\}$.

For each $a\in A$, let $\widetilde{W}(a):=\{s\in\mathcal{SM}(N)\colon s(a)<1/2\}$. Then $\tilde{t}(a)=t(a)>1/2$ and $0=s_\infty(a)<1/2$, so that $s_\infty\in\widetilde{W}(a)$ for each $a\in A$. Then $\{\widetilde{W}(a)\colon a\in A\}$ is an open covering of the compact set $\phi(X)\cup\{s_\infty\}$. There are $a_1,\ldots,a_n\in A$ such that $\phi(X)\cup\{s_\infty\}\subseteq\widetilde{W}(a_1)\cup\cdots\cup\widetilde{W}(a_n)$, consequently, $X\subseteq W(a_1)\cup\cdots\cup W(a_n)$. Put $a=a_1\wedge\cdots\wedge a_n$. Then $a\in A$ and for each $s\in X$, we have $s(a)\le s(a_i)<1/2$ for $i=1,\ldots,n$, which proves $X\subseteq W(a)$, i.e., $s(a)<1/2$ for all $s\in X$.

(2) By the first part of the present proof, there exists an $a\in M$ such that $t(a)>1/2$ while $s(a)<1/2$ for all $s\in X$. In addition, there is an idempotent b of M with $a\le b$ and $t(b)=1$. Then $t(a\wedge\lambda_b(a))=t(\lambda_b(a))$ and $t(a\odot\lambda_b(a\wedge\lambda_b(a)))=t(a)-t(a\wedge\lambda_b(a))=t(a)-t(\lambda_b(a))=2t(a)-1>0$.

Now let s be an arbitrary element of X. If $s(a)=0$, then $s(a\odot\lambda_b(a\wedge\lambda_b(a)))=0$. If $s(a)>0$, there is an integer m_s such that $s(m_s.a)=m_s.s(a)=1$ and since $m_s.a\le m_s.b=b$, we have $s(b)=1$. Hence, $s(a\wedge\lambda_b(a))=s(a)$, so that $s(a\odot\lambda_b(a\wedge\lambda_b(a)))=s(a)-s(a\wedge\lambda_b(a))=0$. In any case, the element $a\odot\lambda_b(a\wedge\lambda_b(a))$ is an element of $\bigcap\{\mathrm{Ker}(s)\colon s\in X\}$ for which $t(a\odot\lambda_b(a\wedge\lambda_b(a)))>0$.

(3) From (1) and (2), we have concluded that if we use (3.3), then $a\odot\lambda_b(a\wedge\lambda_b(a))=a\odot a$ and $s(a\odot a)=0$ for each $s\in X$. In addition, $t(a\odot a)>0$. There is an integer r such that $t(r.(a\odot a))=r.t(a\odot a)=1$ and $s(r.(a\odot a))=0$ for each $s\in X$. Hence, for $x=r.(a\odot a)$, we have $s(x)=0$ for each $s\in X$ and $t(x)=1$. $\square$

Theorem 8.21. *Let M be a pseudo EMV-algebra. The mapping $\theta:\mathcal{SM}(M)\to\mathcal{MN}(M)$ given by $\theta(s)=\mathrm{Ker}(s)$, $s\in\mathcal{SM}(M)$, is a homeomorphism.*

Proof. Due to Theorem 8.3, the mapping θ is a bijection. Let $C(I)$ be any closed subspace of $\mathcal{MN}(M)$. Then

$$\theta^{-1}(C(I))=\{s\in\mathcal{SM}(M)\colon s(x)=0\text{ for all }x\in I\}$$

which is a closed subset of $\mathcal{SM}(M)$. Therefore, θ is continuous.

Given a non-empty subset X of $\mathcal{SM}(M)$, we set

$$\mathrm{Ker}(X):=\bigcap\{\mathrm{Ker}(s)\colon s\in X\}.$$

Then $\mathrm{Ker}(X)$ is a normal ideal of M. If, in addition, X is a closed subset of $\mathcal{SM}(M)$, we assert

$$\theta(X)=C(\mathrm{Ker}(X)).\tag{8.5}$$

The inclusion $\theta(X) \subseteq C(\mathrm{Ker}(X))$ is evident. By Proposition 8.20, if $t \notin X$, there is an element $a \in M$ such that $s(a) = 0$ for each $s \in X$ and $t(a) = 1$. Consequently, $t \notin X$ implies $\theta(t) \notin C(\mathrm{Ker}(X))$, and $C(\mathrm{Ker}(X)) \subseteq \theta(X)$. As a result, we conclude θ is a homeomorphism. □

As a corollary, we can reformulate all results concerning previous topological properties of the weak topology of state-morphisms into the language of the hull-kernel topology of $\mathcal{MN}(M)$. For example, if M does not have a top element, then either $\mathcal{MN}(M)$ is empty or it is a non-empty locally compact Hausdorff space in the hull-kernel topology.

9 Simplices of State Spaces and Integral Representation

In the section, we show that the state space of a pseudo EMV-algebra is either empty or a non-empty simplex. We present also an integral representation of a state on a pseudo EMV-algebra by a unique regular Borel probability measure which is a σ-additive probability measure on the Borel σ-algebra concentrated on some locally compact Hausdorff space. We remind that such a representation for states on MV-algebras was presented in [19, 22] and for EMV-algebras in [12]. Here we extend this result also for states on pseudo EMV-algebras.

We start with some notions on simplices, for more information about simplices, please consult the monograph [15, Chap 10].

A *simplex* in a linear space V is any convex subset K of V that is affinely isomorphic to a base for a lattice cone in some real linear space. A simplex K in a locally convex Hausdorff space is said to be (i) *Choquet* if K is compact, and (ii) *Bauer* if K and ∂K are compact, where ∂K is the set of extreme points of K.

Theorem 9.1. *Let M be a pseudo EMV-algebra. The state space $\mathcal{S}(M)$ is either empty or it is a non-empty simplex. In addition, if M has the property that every maximal ideal of M is normal, the following statements are equivalent:*

(i) *M has a top element.*

(ii) *$\mathcal{S}(M)$ is a Choquet simplex.*

(iii) *$\mathcal{S}(M)$ is a Bauer simplex.*

Proof. Let $I := \bigcap\{\mathrm{Ker}(s) \colon s \in \mathcal{S}(M)\}$. Due to Proposition 8.1(v), I is a normal ideal of M. We assert that the quotient M/I is an EMV-algebra. Indeed, let x/I be the quotient class corresponding to an element $x \in M$. Then if $x/I = y/I$, we have $x/\mathrm{Ker}(s) = y/\mathrm{Ker}(s)$ for each $s \in \mathcal{S}(M)$. Due to the proof of Proposition 8.1(vii), we have $x/\mathrm{Ker}(s) = y/\mathrm{Ker}(s)$ iff $s(x) = s(x \wedge y) = s(y)$. Whence, $x/I = y/I$ iff $s(x) = s(x \wedge y) = s(y)$ for each state s on M. Applying Proposition 8.1(viii), we have that M/I is Archimedean, so that M/I is an EMV-algebra.

If we define, for each $s \in \mathcal{S}(M)$, the function $\hat{s}(x/I) := s(x)$ $(x \in I)$, we can show that every $\hat{s}$ is a state on the EMV-algebra M/I. Conversely, if μ is a state on M/I, then $s_\mu : x \mapsto \mu(x/I)$, $x \in M$, is a state on M. The mapping $\Phi : \mathcal{S}(M) \to \mathcal{S}(M/I)$ defined by $\Phi(s) := \hat{s}$, $s \in \mathcal{S}(M)$, is an affine isomorphism of the state spaces, so it injectively maps state-morphisms of M onto state-morphisms of M/I.

In [12, Thm 7.1], it was shown that the state space of any non-trivial EMV-algebra is a simplex, due to the affine isomorphism Φ, the state space $\mathcal{S}(M)$ is a simplex, too.

If now M has the property that every maximal ideal of M is normal, then the state space is non-empty and applying Theorem 8.10, we see that conditions (i)–(iii) are mutually equivalent. □

Let $\mathcal{B}(K)$ be the Borel σ-algebra of a Hausdorff topological space K generated by all open subsets of K. Every element of $\mathcal{B}(K)$ is said to be a *Borel set* and each σ-additive (signed) measure on it is said to be a *Borel (signed) measure*. We recall that a Borel measure μ on $\mathcal{B}(K)$ is called *regular* if

$$\inf\{\mu(O) :\ Y \subseteq O,\ O \text{ open}\} = \mu(Y) = \sup\{\mu(C) :\ C \subseteq Y,\ C \text{ compact}\} \qquad (9.1)$$

for any $Y \in \mathcal{B}(K)$. For example, let δ_x be the Dirac measure concentrated at the point $x \in K$, i.e., $\delta_x(Y) = 1$ iff $x \in Y$, otherwise $\delta_x(Y) = 0$, then every Dirac measure is a regular Borel probability measure whenever K is compact, see e.g. [15, Prop 5.24].

Let K be a locally compact Hausdorff topological space. Due to the Alexandroff theorem, see [18, Thm 4.21], there is the one-point compactification of K, which is a space $K \cup \{x_\infty\}$, where $x_\infty \notin K$. Theorem 8.17 says that if a pseudo EMV-algebra has the property that each maximal ideal of M is normal and if M has no top element, the one-point compactification of $\mathcal{SM}(M)$ is homeomorphic to the set $\mathcal{SM}(N)$.

Theorem 9.2. [Integral Representation of States] *Let M be a pseudo EMV-algebra such that either M has a top element or M does not have a top element but M has the property that each maximal ideal of M is normal. Given a state s on M, there is a unique regular Borel probability measure μ_s on the Borel σ-algebra $\mathcal{B}(\mathcal{S}(M))$ with $\mu_s(\mathcal{SM}(M)) = 1$ such that*

$$s(x) = \int_{\mathcal{SM}(M)} \hat{x}(t)\,\mathrm{d}\mu_s(t), \quad x \in M, \qquad (9.2)$$

where $\hat{x}$ $(x \in M)$ is a continuous affine mapping from $\mathcal{S}(M)$ into the interval $[0,1]$ *such that $\hat{x}(s) := s(x)$, $s \in \mathcal{S}(M)$.*

Moreover, if M has no top element, there is a one-to-one correspondence between the set of regular Borel probability measures on $\mathcal{B}(\mathcal{SM}(M))$ and the set of regular Borel probability measures on $\mathcal{B}(\mathcal{SM}(N))$ vanishing at $\{s_\infty\}$, where N is its representing pseudo EMV-algebra with top element.

Proof. So we assume that $\mathcal{S}(M) \neq \emptyset$. Similarly as in the proof of Theorem 9.1, put $I = \bigcap\{\mathrm{Ker}(s)\colon s \in \mathcal{S}(M)\}$ and define M/I which is an Archimedean EMV-algebra. For each $s \in \mathcal{S}(M)$, the mapping $\hat{s}(x/I) := s(x)$ $(x \in M)$ is a state on M/I and the mapping $\Phi : \mathcal{S}(M) \to \mathcal{S}(M/I)$ given by $\Phi(s) = \hat{s}$, $s \in \mathcal{S}(M)$, is an affine homeomorphism in the weak topology between the state spaces $\mathcal{S}(M)$ and $\mathcal{S}(M/I)$. In addition, $\Phi(\mathcal{SM}(M)) = \mathcal{SM}(M/I)$.

Given $x \in M$, we define a mapping $\hat{x} : \mathcal{S}(M) \to [0,1]$ such that $\hat{x}(s) = s(x)$, $s \in \mathcal{S}(M)$. In a similar way, we define $\widehat{x/I} : \mathcal{S}(M/I) \to [0,1]$ such that $\widehat{x/I}(\hat{s}) = \hat{s}(x/I)$ $(s \in \mathcal{S}(M))$. Then $\hat{x}$ and $\widehat{x/I}$ are continuous and affine functions, i.e. they preserve convex combinations, and $\hat{x}(s) = s(x) = \widehat{x/I}(\hat{s})$ for each $s \in \mathcal{S}(M)$.

First assume that M has a top element 1. Then M is termwise equivalent to the pseudo EMV-algebra $(M; \oplus, ^-, ^\sim, 0, 1)$ and M/I is termwise equivalent to the MV-algebra $(M/I; \oplus, ^*, 0/I, 1/I)$. According to [19, 22], there is a unique regular Borel probability measure $\mu_{\hat{s},I}$ on the Borel σ-algebra $\mathcal{B}(\mathcal{S}(M/I))$ with $\mu_{\hat{s},I}(\mathcal{SM}(M/I)) = 1$ such that

$$s(x) = \hat{s}(x/I) = \int_{\mathcal{SM}(M/I)} \widehat{x/I}(\tau)\,\mathrm{d}\mu_{\hat{s},I}(\tau), \quad x \in M. \tag{9.3}$$

Using transformation of integrals, see e.g. [16, p. 163], we have

$$s(x) = \hat{s}(x/I) = \int_{\Phi^{-1}(\mathcal{SM}(M/I))} \widehat{x/I}(\Phi(t))\,\mathrm{d}\mu_{\hat{s},I}(\Phi(t)) = \int_{\mathcal{SM}(M)} \hat{x}(t)\,\mathrm{d}\mu_s(t),$$

where $\mu_s(A) = \mu_{\hat{s},I}(\Phi(A))$, $A \in \mathcal{B}(\mathcal{S}(M))$, is a unique regular Boreal probability measure on $\mathcal{B}(\mathcal{S}(M))$ with $\mu_s(\mathcal{SM}(M)) = 1$ satisfying (9.2).

Now, let us assume that the pseudo EMV-algebra M does not have a top element but M has the property that each maximal ideal of M is normal, and let N be its representing pseudo EMV-algebra with top element which is guaranteed by Basic Representation Theorem 6.4. Without loss of generality, we can assume that M is a maximal and normal ideal of N. We note that since $\mathcal{S}(N)$ is non-empty, compact and Hausdorff, the singleton $\{s_\infty\}$ is closed so it belongs to $\mathcal{B}(\mathcal{S}(N))$.

The quotient M/I is Archimedean, so it is an EMV-algebra. Since the state spaces $\mathcal{S}(M)$ and $\mathcal{S}(M/I)$ are affine homeomorphic, due to Theorem 8.10, the EMV-algebra M/I also does not have a top element (otherwise $\mathcal{S}(M/I)$ would be compact). Let N_I be an EMV-algebra with top element such that it is representing M/I; then M/I is a maximal ideal of N_I. Define a mapping $\phi : \mathcal{S}(M) \to \mathcal{S}(N)$ by $\phi(s) = \tilde{s}$, $s \in \mathcal{S}(M)$, where $\tilde{s}$ is a state on N defined by (8.1); let s_∞ be the two-valued state on N vanishing on M. In the same way, we define $\phi_I : \mathcal{S}(M/I) \to \mathcal{S}(N_I)$ by $\phi(\hat{s}) = \tilde{\hat{s}}$ $(s \in \mathcal{S}(M))$ and let s^I_∞ be the unique two-valued state on N_I vanishing on M/I. In addition, let us define $\Phi_I : \mathcal{S}(N) \to \mathcal{S}(N_I)$ by $\Phi_I(\phi(s)) = \phi_I(\Phi(s))$ $(s \in \mathcal{S}(M))$, and $\phi_I(s_\infty) = s^I_\infty$. Clearly if $x \in M$, then $\Phi_I(\phi(s))(x/I) = s(x) = \phi_I(\Phi(s))(x/I)$. Then Φ_I is an affine homeomorphism.

Due to [12, Thm 7.2], there is a unique regular Borel probability measure $\mu_{\hat{s},I}$ on the Borel σ-algebra $\mathcal{B}(\mathcal{S}(M/I))$ with $\mu_{\hat{s},I}(\mathcal{SM}(M/I)) = 1$ such that (9.3) holds. Then we can show as in the line under (9.3) that (9.2) holds for a unique regular Borel probability measure μ_s on $\mathcal{B}(\mathcal{S}(M))$ with $\mu_s(\mathcal{SM}(M)) = 1$ satisfying (9.2), where $\mu_s(A) = \mu_{\hat{s},I}(\Phi(A))$, $A \in \mathcal{B}(\mathcal{S}(M))$.

As we have seen, the regular Borel probability measure μ_s on $\mathcal{B}(\mathcal{S}(M))$ is concentrated on $\mathcal{SM}(M)$.

Again, by [12, Thm 7.2], there is a one-to-one correspondence between the set of regular Borel probability measures on $\mathcal{B}(\mathcal{SM}(M/I))$ and the set of regular Borel probability measures on $\mathcal{SM}(N_I)$ vanishing at s^I_∞; this correspondence is given by $\mu \leftrightarrow \mu \circ \phi_I^{-1}$.

Take a state s on M and let $\tilde{s}$ be its unique extension to a state on N defined by (8.1), and fix an element $x \in M$. We denote by $\bar{\hat{x}} : \mathcal{S}(N) \to [0,1]$ given by $\bar{\hat{x}}(u) = u(x)$, $u \in \mathcal{S}(N)$. Then $\hat{x}(s) = \bar{\hat{x}}(\tilde{s})$ for each $s \in \mathcal{S}(M)$. By the first part of the present proof, there is a unique regular Borel probability measure $\mu_{\tilde{s}}$ on $\mathcal{B}(\mathcal{S}(N))$ with $\mu_{\tilde{s}}(\mathcal{SM}(N)) = 1$ such that, for all $x \in M$, we have,

$$\begin{aligned} s(x) &= \tilde{s}(x) = \int_{\mathcal{SM}(N)} \bar{\hat{x}}(u)\,\mathrm{d}\mu_{\tilde{s}}(u) = \int_{\{s_\infty\}} \bar{\hat{x}}(u)\,\mathrm{d}\mu_{\tilde{s}}(u) + \int_{\mathcal{SM}(N)\setminus\{s_\infty\}} \bar{\hat{x}}(u)\,\mathrm{d}\mu_{\tilde{s}}(u) \\ &= \int_{\phi(\mathcal{SM}(M))} \bar{\hat{x}}(u)\,\mathrm{d}\mu_{\tilde{s}}(u) = \int_{\mathcal{SM}(M)} \hat{x}(t)\,\mathrm{d}\mu_s(t), \end{aligned}$$

because $\bar{\hat{x}}(s_\infty) = s_\infty(x) = 0$, and $\mu_s(B) = \mu_{\tilde{s}}(\phi(B))$ for each $B \in \mathcal{B}(\mathcal{S}(M))$. We assert that μ_s is a unique regular Borel probability measure on $\mathcal{B}(\mathcal{S}(M))$ with $\mu_s(\mathcal{SM}(M)) = 1$ such that (9.2) holds. Indeed, let ν be an arbitrary regular Borel probability measure on $\mathcal{B}(\mathcal{S}(M))$ with $\nu(\mathcal{SM}(M)) = 1$ and satisfying (9.2). If $x_0 \in M$, then

$$s(x_0) = \int_{\mathcal{SM}(M)} \hat{x}_0(t)\,\mathrm{d}\nu(t).$$

Now let $x = \rho_1(x_0)$ and $s \in \mathcal{S}(M)$, then

$$\hat{s}(x_0) = 1 - s(x_0) = 1 - \int_{\mathcal{SM}(M)} \hat{x}_0(t)\,\mathrm{d}\nu(t) = \int_{\mathcal{SM}(M)} (1 - \hat{x}_0(t))\,\mathrm{d}\nu(t)$$

Choose an idempotent $a \in \mathcal{I}(M)$ such that $s(a) = 1$ and put $S(a) = \{s \in \mathcal{SM}(M)\colon s(a) > 0\} = \{s \in \mathcal{SM}(M)\colon s(a) = 1\}$. Then $S(a)$ is compact and clopen by Theorem 8.14. Similarly, if $S_N(a) = \{s \in \mathcal{SM}(N)\colon s(a) > 0\}$, then $S_N(a)$ is also compact and open, $\phi(S(a)) = S_N(a)$, and $s_\infty \notin S_N(a)$. Therefore, $\bar{\hat{a}} = \chi_{S_N(a)}$, so that

$$1 = s(a) = \int_{\mathcal{SM}(N)} \bar{\hat{a}}(t)\,\mathrm{d}\mu_{\tilde{s}}(t) = \int_{\mathcal{SM}(N)} \chi_{S_N(a)}(t)\,\mathrm{d}\mu_{\tilde{s}}(t) = \mu_{\tilde{s}}(S_N(a)),$$

which implies $\mu_{\tilde{s}}(\{s_\infty\}) = 0$ for each $s \in \mathcal{SM}(M)$.

The mapping Φ defines a one-to-one correspondence $\mu \leftrightarrow \mu(\Phi^{-1})$ between regular Borel probability measures on $\mathcal{B}(\mathcal{SM}(M))$ and $\mathcal{B}(\mathcal{SM}(N))$. Due to construction of the mapping Φ_I, Φ_I is an affine homeomorphism, so that Φ_I yields a one-to-one correspondence $\nu \leftrightarrow \nu(\Phi_I^{-1})$ between regular Borel probability measures on $\mathcal{B}(\mathcal{SM}(N))$ and $\mathcal{B}(\mathcal{SM}(N_I))$ vanishing at s_∞ and s_∞^I, respectively.

Therefore, there is a one-to-one correspondence between the set of regular Borel probability measures on $\mathcal{B}(\mathcal{SM}(M))$ and the set of regular Borel probability measures on $\mathcal{B}(\mathcal{SM}(N))$ vanishing at $\{s_\infty\}$, if necessary, see also the proof of (3) in [12, Thm 7.2]. $\square$

Remark 9.3. *We note that uniqueness of μ_s in the latter theorem is guaranteed by the condition $\mu_s(\mathcal{SM}(M)) = 1$. For example, if s is a state on M, then the Dirac measure δ_s on $\mathcal{S}(M)$ concentrated in the point s is a regular Borel probability measure on $\mathcal{B}(\mathcal{S}(M))$ such that $s(x) = \int \hat{x}(t)\,\mathrm{d}\delta_s(t)$, $x \in M$. But if s is not an extremal state, then $\delta_s(\mathcal{SM}(M)) = 0 \neq 1$.*

We underline that the restriction onto $\mathcal{B}(\mathcal{SM}(M))$ of every regular Borel probability measure μ defined on $\mathcal{B}(\mathcal{S}(M))$ with $\mu(\mathcal{SM}(M)) = 1$ gives a regular Borel probability measure on $\mathcal{B}(\mathcal{SM}(M))$. Conversely, every regular Borel probability measure ν on $\mathcal{B}(\mathcal{SM}(M))$ defines by $\mu(A) = \nu(A \cap \mathcal{SM}(M))$, $A \in \mathcal{B}(\mathcal{S}(M))$, a regular Borel probability measure μ on $\mathcal{B}(\mathcal{SM}(M))$ such that μ is concentrated on $\mathcal{SM}(M)$.

It is worthy of note that the first part of Theorem 9.2 can be reformulated in the following equivalent way: For every state s on a pseudo EMV-algebra M which has the property that every maximal ideal of M is normal, there is a unique regular Borel probability measure μ_s on $\mathcal{B}(\mathcal{SM}(M))$ such that (9.2) holds for each $x \in M$, where $\hat{x}$ is a continuous function from $\mathcal{SM}(M)$ into the real interval $[0,1]$ such that $\hat{x}(s) = s(x)$, $s \in \mathcal{SM}(M)$.

Remark 9.4. *Using transformation of integral* (9.2) *and the homeomorphism* $s \in \mathcal{SM}(M) \mapsto \mathrm{Ker}(s) \in \mathcal{MN}(M)$*, see Theorem* 8.21*, Theorem* 9.2 *can be rewritten in the equivalent form: For any state s on the pseudo EMV-algebra M, there is a unique regular probability measure ν_s on the Borel σ-algebra $\mathcal{B}(\mathcal{MN}(M))$ such that*

$$s(x) = \int_{\mathcal{MN}} x^* \,\mathrm{d}\nu_s, \quad x \in M,$$

where $x^*(I) = x/I \in [0,1]$*,* $I \in \mathcal{MN}(M)$*.*

Finally we present some questions:

Problem 9.5. *Extend the Horn–Tarski theorem, see* [17]*, for pseudo EMV-algebras: Let M_1 be a pEMV-subalgebra of a pseudo EMV-algebra M_2. If s is a state on M_1, is it possible to extend s to a state on M_2? For EMV-algebras this was proved positively in* [12, Thm 8.1]*.*

Problem 9.6. *Develop logics connected with EMV-algebras as well as with pseudo EMV-algebras.*

10 Conclusion

In the paper we have introduced pseudo EMV-algebras which are a non-commutative generalization of MV-algebras, generalized Boolean algebras, EMV-algebras and of pseudo MV-algebras. The paper is divided into two parts.

Part I: We studied the basic properties of pseudo EMV-algebras. The existence of a top element is not assumed a priori. We showed that every non-trivial pseudo EMV-algebra possesses at least one maximal ideal, Theorem 4.17, and congruences correspond to normal ideals, Theorem 4.8. The class of pseudo EMV-algebras is not a variety because it is not closed under subalgebras, and it forms a more general class, a q-variety, similar to varieties. We studied the class of representable pseudo EMV-algebras, normal-valued ones and pseudo EMV-algebras whose each maximal ideal is normal; they form q-varieties, Theorem 5.15, Theorem 6.12 and Theorem 6.11. In addition, the lattice of q-subvarieties of pseudo EMV-algebras is uncountable, see Theorem 5.14.

Part II: We presented Basic Representation Theorem 6.4 saying that each pseudo EMV-algebra without top element can be embedded into a pseudo EMV-algebra with top element as a normal and maximal ideal of the latter one. It generalizes the analogous result for generalized Boolean algebras from [5, Thm. 2.2]. We have showed that the category of proper pseudo EMV-algebras is categorically equivalent to a special category of pseudo MV-algebras and to a special category of ℓ-groups, see Corollary 7.8. We have introduced the notion of a state as a finitely additive mapping on M with values in the real interval $[0,1]$ such that at some element of M the state attains the value 1. In contrast to EMV-algebras, the state space can be empty. The state space of a pseudo EMV-algebra M is non-empty if and only if M possesses at least one maximal and normal ideal, Theorem 8.12. We introduced also state-morphisms as special $[0,1]$-valued pEMV-homomorphisms. We showed that state-morphisms are only extremal states, Theorem 8.7. If a state exists, then it is a weak limit of a net of convex combinations of state-morphisms, see Theorem 8.18. If a pseudo EMV-algebra does not have a top element, then the space of state-morphisms is a locally compact Hausdorff space whose one-point compactification is affinely isomorphic to the space of state-morphisms of the representing pseudo EMV-algebras with top element, see Theorem 8.17. Finally, we have showed that every state, which is a finitely additive mapping, can be expressed as an integral through a unique regular Borel σ-additive probability measure defined on the Borel σ-algebra of the state space; the measure is concentrated on the set of state-morphisms, Theorem 9.2, which generalizes an analogous result for states on MV-algebras established in [19, 22].

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Recent Thought on *Is* and *Ought*: Connections, Confluences and Rediscoveries

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Abstract. Section 1 of this critical survey recalls the much discussed difficulty noted by A. N. Prior in a 1960 paper for a formulation of Hume's Law to the effect that no valid inference can take us from non-ethical premises to an ethical conclusion. Section 2 presents a response by Toomas Karmo from the 1980s, echoes of which surface in discussions of the problem over the past 5–10 years, also noted in this section along with some objections that have been raised to this line of thought. Section 3 reviews another, more recent (2010) contribution to the debate, from Greg Restall and Gillian Russell, and discusses its connections to the material in play in previous section, as well as aspects of the reception of this contribution by commentators. This way of organizing things makes possible a reasonably comprehensive guide to (at least the main highlights of) the recent literature. Several more detailed passages are postponed to Postscripts at the end of each section, or demoted to footnote discussion, to be skipped by those wanting a speedier overview, though of necessity that will mean that some voices go unheard and some mistakes uncorrected.

1 Introduction

What might reasonably count for present purposes as *recent* in the literature on the principle variously called Hume's Law, the Is–Ought Gap, or the thesis of the autonomy of ethics, is perhaps given by the publication date – 2010 – of the stimulating, varied, and much discussed anthology Pigden [82], with perhaps special mention due to the paper Restall and Russell [92] therein, in view of its ambitious elegance and the interest it has sparked in subsequent forays into the field. On a slightly larger time scale, recency might be dated back to 1988 and the appearance of the strikingly original Karmo [61], to which little attention gets paid in Pigden [82].[1] We should be

[1]It is, however, mentioned in Maitzen's contribution to the collection, [68]. Maitzen [66] paid it much closer – albeit unsympathetic – attention, dialectically downstream from which we have Nelson [79], Hill [38], Maitzen [67], Hill [39].

alive to the possibility these and other alternative responses to, in particular, difficulties for Hume's Law raised in Prior [88], are not, once terminological adjustments are made, mutually incompatible, and they accordingly compete for our attention rather than for our assent. The proponent of one such response wants to focus on one aspect of the situation while those favouring an alternative reaction are essentially saying, "No, let's look at things *this* way." The present discussion is not entirely neutral, expressing a particular interest in the Karmo-style approach, but with an even greater interest in looking at some of the links that emerge between various responses to Prior, touching also, if sometimes all too briefly, on several post-[82] discussions (in chronological order of publication: Brown [7] and [8], Singer [108], Wolf [117], Maguire [65], Woods and Maguire [118] and Fine [19]). Slightly less recent contributions, some before and some since Prior [88], will also be touched on. After the present introduction, Section 2 looks at the content, subsequent discussion, and sometimes unknowing re-discovery, of aspects of Karmo [61], though this theme also finds its way into a final Section 3, similarly focused on Restall and Russell [92] and its reception.

We need, therefore, to begin by recalling the nub of Prior [88]. Suppose E and F are respectively uncontroversially an ethical and a non-ethical statement, in the latter case supposing – even if one thinks that this does not hold automatically in virtue of the classification of F non-ethical – that $\neg F$ is also non-ethical. One might have wanted a version of Hume's Law saying that no ethical statement is a logical consequence of a consistent set of non-ethical statements, where "non-ethical" just means "not ethical".[2] We ask about the status of the disjunction $E \vee F$, and observe that since this follows from F it must be classified as non-ethical to avoid a violation of the envisaged law, but then from the nonethical $E \vee F$ and $\neg F$, there follows our ethical conclusion E, giving a different violation of Hume's Law. So there is no way to classify $E \vee F$ and which permits us to retain Hume's Law.[3]

[2]One would not normally have to say this, but I see that, after using 'non-ethical' for many pages, in note 12 on p. 59 of [7], Brown casually remarks: "As I use the term 'non-ethical', it is not equivalent to 'not ethical'. To say that a sentence is neither ethical nor non-ethical is, therefore, no violation of the Law of Excluded Middle."

[3]Rescher [91], note 2, mistakenly says that this argument was first given in Mavrodes [74]. Mavrodes, apparently not familiar with Prior [88], gives the argument and does indeed provide an excellent discussion of the issues it raises, with many deft moves, several appearing here in notes 6, 14 and 33. The snapshot of Prior's argument given above conceals some details brought out in the proof of Proposition 1.1 in the Postscript to this section. Forty years after Prior's paper, Sinnott-Armstrong gives the same argument in [110] (and again in a mild re-working of this material as Chapter 7 of [111]) with no mention of either Prior or Mavrodes. (Among other things, the re-working corrects a typo from line 5 of p. 171: "$2+2=5$" to "$2+2\neq 5$".) No doubt Prior's argument has occurred independently to many people; the present author thought he had discovered it in the late 1970s and was lucky enough to have a better-informed colleague (Edward

Prior's discussion features several further examples with a less artificial flavour to them than the disjunctive example just abstractly rehearsed, including several about what all undertakers ought to do or what should be done to all New Zealanders, which raise some distractions it would be helpful to be able to avoid. Before doing so, let us note that even these examples, which will be familiar to anyone who has dipped into the Prior-initiated dialectic on all this, are rather artificial. For this reason, Jackson [58][4] offers something closer to a real life example:

> Suppose Jane has serious reservations about abortion but nevertheless agrees to pay for a close friend's abortion. For her the *a priori* valid inference
>
> > I have paid for an abortion.
> > Therefore, if anyone who has paid for an abortion has done something morally wrong, I have done something morally wrong.
>
> corresponds to a line of reasoning that worries her a great deal.

Now, even if this example seems straightforward, we should recall that the nature of conditionality in deontic contexts has been notoriously problematic. Conditionals with "ought" apparently in their consequents force us to decide between constructions – using "O" as the *Ought* or *Obligation* operator of deontic logic – of the form $p \to Oq$, $O(p \to q)$ and $O(q/p)$. In the first two cases here, $\to$ is our default representation for material implication, though it could be swapped out for another (e.g., subjunctive) conditional construction, and the third features the primitive binary conditional obligation operator $O(\cdot/\cdot)$ of dyadic deontic logic, itself open to competing semantic interpretations,[5] and with status of Modus Ponens for whatever format is adopted being the subject of perennial debate.[6] When we pass to universal generalizations of the "Whatever is F ought to be G", complications ramify

Khamara) who directed him to Prior's discussion.

[4]This is one of three entries in the encyclopedia in which it appears, all of them directly addressing, in their own ways, the issue under discussion here; the other two are Elgin [17] and Pigden [84].

[5]References to many alternative semantic proposals for this connective are listed under Example 4.4.4, p. 241, in Humberstone [51].

[6]A tiny sample, in chronological order: Greenspan [33], Humberstone [43], Section 7.4 of Makinson [69], Kolodny and MacFarlane [62], Saint Croix and Thomason [97]. The 'fundamental problem' of the title of Makinson's paper concerns the problematic status of truth-based semantics for those who think of normative language as not truth-valued, rather than the specifics of conditional constructions. A good first move in the solution of that problem is made (on p. 58*f.*) of Mavrodes [74]: if that's how you feel about truth, just run the discussion in terms of an artificial predicate stipulated to behave disquotationally. In terms of this, let's say, schmuth-predicate, we have: "People ought never to lie" is schmue if and only if people ought never to lie. (Mavrodes actually uses 'right' rather than 'schmue', but this introduces distractions. A point similar to Mavrodes' is made in note 4 of Singer [108]: "one may substitute whatever analogue of truth one wishes here.") This

further: $O\forall x(Fx \to Gx)$, $\forall xO(Fx \to Gx)$, $\forall x(Fx \to O(Gx))$, $\forall xO(Gx/Fx)$,...[7] An instructive example of the issues arising from trying to assign the appropriate scope to "O" when formally representing some of these constructions with *if*, *all* and *ought* is given in the report at p. 10 of Mares [70] on a spat with a referee for that paper. After some involvement with these concepts in the following paragraph, we will accordingly do our best to steer clear of them.

Rynin [96] had considered arguments apparently of the form 'Ga, Therefore $O(Ha)$' which might be felt to be enthymematic and, with the missing premise $\forall x(Gx \to O(Hx))$ restored, would no longer be (at least blatant) counterexamples to Hume's Law. Rynin then cleverly executes a conditional proof step[8] (not that he describes it in exactly these terms) to pass from the now explicit form:

$$Ga, \forall x(Gx \to O(Hx)) \vdash O(Ha) \tag{1}$$

to

$$Ga \vdash \forall x(Gx \to O(Hx)) \to O(Ha) \tag{2}$$

In concrete terms, Rynin writes ([96], 314*f.*): "Thus if we have the argument: 'I have given my promise, and all promises ought to be kept, therefore I ought to keep my promise', we can transform it into 'I have given my promise, therefore if all promises ought to be kept, then I ought to keep my promise.'" The simplified version (2) represents "a is a promise; therefore, if all promises ought to be kept then a ought to be kept."[9] We see already with this example that the point about scope arises: should the 'all promises ought to be kept' have been $O\forall x(Gx \to Hx)$ instead, in principle undermining the validity of the pre-conditional-proof version of the argument.[10]

is a first step because in the semantics for deontic logic we need not just the absolute notion of (something like) truth, but a world-relativized notion – preferably still 'thin' enough as not to beg any questions against non-cognitivism.

[7] One may be tempted to include on this list "$O\forall x(Gx/Fx)$", thinking that it may be true that in respect of each of the Fs it would be better that it be G than not ($\forall xO(Gx/Fx)$, on one common understanding), and at the same that it would be disastrous if all Fs were G. But the envisaged addition to the list is not well formed, since dyadic O takes two formulas to make a formula: the slash separates these two, rather than being part of a restricted quantifier notation.

[8] This move is also made in Pigden [85]: see (B♯) on p. 407.

[9] In Section 2 we will encounter Searle's idea in [104] that it may be possible to drop this premise altogether from the original argument, because of an analytic connection between having made and not yet kept a promise, on the one hand, and being such that one ought to keep it, on the other. Indeed, Rynin is already sympathetic to such a view, speaking (p. 316) of a "normative principle that serves as a rule of inference to validate the derivation" – though this is not a part of Rynin's discussion that Prior takes up.

[10] Wolf [117] raises justified doubts about the example 'Lois should donate to charity if she is

In fact, rather than discussing such $\vdash$-claims as (1) and (2) here, Rynin ([96], p. 314) discusses what he calls the conditionals corresponding to the arguments thereby represented, where the main conditional is reproduced here (using "$\multimap$") as strict implication,[11] and using (possibly decorated) "N" and "F" to represent normative and factual statement represents the transition from (1) to (2) as a transition from the one conditional to the other, i.e., as from

$$(N \wedge F) \multimap N' \qquad \text{to} \qquad F \multimap (N \to N').$$

This representation is potentially problematic if the N and F here are taken as playing the E and F roles above, since, the N is already what is in the literature (and below) called a *mixed* case, the main operator not being O, which instead governs only the consequent of a (universally quantified) material conditional here.

Prior follows Rynin with variations on (2). Since all that the work the universally premise is doing here is done by the single instance with a as x, however, we might as well just simplify, both (1) and (2) by rewriting the universal premise to $Ga \to Ha$, which turns (2) into

$$Ga \vdash (Ga \to O(Ha)) \to O(Ha). \tag{3}$$

But if $\vdash$ here is taken as the consequence relation of classical logic, the right-hand side is equivalent to $O(Ha) \vee Ga$, so we are again considering essentially the $E \vee F$ case of the second paragraph above. It is not being suggested that this was Prior's own route from the rather cluttered natural Rynin-style examples to the streamlined – though less natural-seeming – disjunction case.[12] However, since, as already remarked, it is

able' and the unobviousness of whether this has the form *Lois is able to donate to charity* $\to$ O(*Lois donates to charity*), on the one hand, or O(*Lois is able to donate to charity* $\to$ *Lois donates to charity*), on the other. What may be less justified is the association of this example with p. 264*f.* of Vranas [115], where the closest case resembling this one concerns instead the sentence 'If Jane is a citizen, she ought to vote,' especially as Vranas is maintaining that something about these examples – their genuine normativity, if not their logical form – varies from case to case.

[11]In fact Rynin writes "e$\rightarrow$" here for an entailment connective clunkily defined as the conjunction of a strict implication with a conjunct saying that its antecedent and consequent are not analytic. This second conjunct (for which one might have expected – the equally clunky – "and neither the consequent nor the negation of the antecedent is analytic") can be ignored for present purposes, though. Rynin's dot notation for conjunction has also been replaced here with "$\wedge$". In Section 2 we will be discussing an approach to these matters according to which strict implication, understood as truth-preservation in all worlds, is a good conceptualization of entailment, which instead has to be taken as truth-preservation relative to all worlds and all ethical (or normative) standards.

[12]Prior explicitly thanks ([88] p. 202) one T. H. Mott for suggesting it to him, and was in any case at around the time at which [88] was written, unaware of the classical equivalence of $p \vee q$ and $(p \to q) \to q$ (or $(q \to p) \to p$), as we see from the 'Note 1960' appended (p. 229) to the discussion of deontic logic in [89], retracting his recent favourable remarks about $Op \to \big((p \to Oq) \to Oq\big)$ as a plausible deontic principle: no-one realising that this was another way of writing $Op \to (p \vee Oq)$

far from clear how to handle natural language deontic conditionals, it is safer to avoid the issue as much as possible, and stick to uncontroversially Boolean embeddings and interactions with our monadic deontic operators. The case of $\vee$-introduction is especially simple in illustrating the presence of material in the conclusion of a valid argument not present in the premises, undermining, as Rynin [96] pointed out, an attempt by P. H. Nowell-Smith in an attempt to establish Hume's Law as a special case of the supposed impossibility what is thereby illustrated. Rynin and Prior diagnose this as a case of overfamiliarity with syllogistic reasoning at the expense of the fuller picture provided by (then) contemporary logic. Pigden [85], p. 403, turns up an appeal to this same incorrect principle from a 1725 publication – though that is more understandable, since the syllogism was then the only game in town.[13]

Reactions to Prior's disjunctive syllogism argument naturally include whose querying the underlying logic assumed in delivering the claimed consequences – the $\vee$ introduction step in passing from F to $E \vee F$ (queried in Beall [4]) or the disjunctive syllogism step taking us from $E \vee F$ and $\neg F$ to E (queried in Mares [71]); the Postscript to this section begins by taking up the second of these reactions, which urges as a remedy for this unfortunate malady: a shift from classical to relevant logic.[14] The first response will be touched on at the end of Postscript

would find it at all plausible, especially with the further reformulation – again recalling that the logical background here is classical – to $(Op \to p) \vee Oq$: either all obligations are fulfilled or everything is obligatory. (The present observation is adapted from p. 476, last ten lines, in [44].)

[13]The relevant considerations do appear to take some time to absorb. Garcia [27], p. 549, reconstructing Hume's reasoning in the famous is–ought passage offers as a version of one of its premises: "No proposition with an 'ought'-operator governing some element within it can be deduced from a group of propositions none of which contains this feature." Garcia's comment on this is that it comes close to assuming the desired conclusion to begin with, rather than that, taken at face value (and with 'proposition' corrected – so that it makes better sense – to something more linguistic, such as 'sentence' or 'statement') it is simply false.

[14]Mares observes that the ingredients in Prior's argument – $\vee$-introduction (or "addition" as some of those in our bibliography say) along with disjunctive syllogism – are those involved in the C. I. Lewis/Albert of Saxony demonstration ([1], p. 164) that any contradiction has every statement as a (classical) consequence. But care is required with this observation – the care displayed by Mavrodes [74] as he considers what he lists as Objection 5 to his/Prior's argument. (In Mavrodes' presentation of the argument our F, E, become F, M, respectively, and specific but representative choices are made as to which statements these are. F is 'The Fisher Building is the tallest building in Detroit', and M is 'Men ought never to lie'. Their disjunction is called D.) The objection says that the argument trades on the controversial feature of classical logic that a mutally contradictory statements together entail everything, calling only for a revision of Hume's Law to exclude inconsistent premises. Mavrodes (p. 362) then writes concerning this objection, that "... in the form given here it is simply mistake about the structure argument which I have discussed. I have nowhere used or discussed any argument which includes both F and not-F (or any other contradiction) among its premises. I have instead pointed out that if D is normative then it is entailed by F, and hence there is a nonnormative statement which entails a normative one. On the other hand,

(i) to Section 3. Turning to responses not contesting the underlying logic, which is standardly taken to be classical logic (though for the inferences mentioned so far – not including the point about the implicational definability of disjunction, of course – could equally well be intuitionistic logic), we have what we might call *trichotomy* responses. These retain the emphasis on a failure of anything in some class – call it the 'conclusion class' – to follow from a set of statements in another class – call it the 'premise class'. Here, uncontroversially (or 'basic') ethical statements are in the conclusion class, while the similarly straightforward nonethical statements are in the premise class; but these two classes are not jointly exhaustive of all the statements.[15] For instance the premise class might be described as *factual*, the conclusion class

if D is nonnormative, then D and not-F together entail M, which again subverts the gap thesis. Now *neither* of these entailments involves any self-contradictory premises. One of them has only the single premise F, and the other has the pair of premises D and not-F. But neither of them involves the contradictory premises F and not-F."

[15]This strategy is dubbed the "No Mixed Sentences Defence" by Campbell Brown and discussed by him in Section 3, bearing that title, of [7]. At least, so it seems at the start of that section. As we proceed, however, it transpires that Brown doesn't mean by "mixed" what is usually meant by this: that we have some basic ethical statements and some basic non-ethical statements, and the mixed cases arise as combinations of the one with the other using Boolean connectives and quantifiers. (This is what "mixed" has meant in these discussions for over fifty years, occurring with this signification in Atkinson [2] and Schurz [101] from 1958 and 2010 respectively, and of course in many other contributions in between.) By the time we get to p. 58, however, we are worrying about sentences which contain, on the one hand no ethical, and on the other, no non-ethical predicates – as though being 'mixed' amounted to having both ethical and non-ethical *vocabulary*. (Here, for Proposition 1, there is an appeal to an implicational formulation of the Halldén completeness of first-order predicate logic without identity, to show that there are no implications from formulas without ethical predicates to formulas in which only ethical predicates appear, a corollary of Prop. 1 called 'NOFI 3' – No Ought from Is, Mark 3 – by Brown. This corresponds to the Barrier Lemma on p. 472 of Humberstone [42], for a propositional logic with two sets of sentence letters, one set for the basic ethical and the other for the basic non-ethical case – but the latter was not envisaged to represent statements constructed with *only* ethical and logical vocabulary.) Sentences entirely devoid of non-ethical vocabulary are surely of negligible interest from the perspective of Hume's Law, and are certainly not basic ethical sentences. (But see also note 65 below.) By contrast with basic ethical statements, for Brown, "[p]urely ethical sentences are rarely encountered in the wild, outside the philosopher's laboratory. Notice, for example, that even Prior's sentence 'All New Zealanders ought to be shot' fails to be wholly ethical (assuming 'New Zealander' is non-ethical). Two oddities with this comment: first, there is no 'even' about it – on the second page of Prior [88], we have: "I would not count as 'ethical' a statement in which only ethical and logical expressions occurred essentially." Secondly, why is only 'New Zealander' mentioned and not also '(are) shot' as non-ethical vocabulary – albeit non-ethical vocabulary embedded in the scope of a deontic operator making the whole of "ought to be shot" an ethical – though not what Brown calls a 'purely ethical' – expression?

as *ethical*,[16] and the rest *mixed*. The contrasting *dichotomy* response[17] aims at a formulation of Hume's Law in which every statement gets to be in either the premise class or the conclusion class. That seems closer to the letter of Hume's own formulation about conclusions containing 'ought' being claimed to be derivable on the basis of premises not containing 'ought', even if such an overly syntactic characterization would be a hopeless first stab at a precise articulation of the spirit of Hume's discussion. Still, one would like some exhaustive non-ethical/ethical division with a significant inferential relation that can be seen never to take us from the former to the latter. Differently put, a dichotomous approach aims at a claim of closure: the class of non-ethical statements is closed under something like entailment. Prior's $\vee$-introduction + disjunctive syllogism argument shows that any once-and-for-all way of redistributing the slack the mixed cases comprise into the one of the two classes to obtain such a dichotomy approach cannot succeed, when that inferential relation is taken to be entailment; for a more precise statement, see the Postscript. We should be alive the possibility that the way that redistribution is effected may need to influence the replacement of entailment proper – even when the latter is subject to the further requirement that the premises are consistent ('closure under consistent consequence' as it is put in the Postscript). A more promising candidate

[16]Or perhaps *evaluative* or *normative*, though these terms are generally understood to encompass much more than the specifically ethical or moral. These broader notions create a potential problem of their own for the present discussion of the validity of arguments with such-and-such premises and so-and-so conclusions, if they treat *valid* itself as an evaluative terms, as does Urmson [113], p. 223: "to call an argument valid is not merely to classify it logically, as when we say it is a syllogism or *modus ponens*; it is at least in part to evaluate or appraise it; it is to signify approval of it." See also, in this connection, Shaw [106], where considerable play is made of seemingly valid arguments *about* arguments which conclude with verdicts on the latter arguments' validity or invalidity, despite not having any of their premises evaluative. One might try to abstract from any such evaluativity, saying that for logical purposes validity is to be understood as no more than the necessary, *a priori*, or formally secured (depending on the purposes at hand) preservation of truth, but even truth itself has been held to be an evaluative or normative notion: see Horwich [40] for a discussion of several thinkers (which do not include Horwich himself) inclined to say such things. Certainly at some point along the line from 'Snow is white' through 'The proposition that snow is white is true' to 'The belief that snow is white is correct', we seem to have gone from the non-normative to the normative. This calls for comment even if the normativity is not ethical: the puzzle is formulated and addressed in Gibbard [30].

[17]The dichotomy/trichotomy terminology for marking this contrast appears in Schurz [100] and [101]. A dichotomous version of Hume's Law is called the Special Hume thesis (or 'SH') in these publications (and in [99]), in which Schurz looks at conditions on bimodal alethic–deontic logics necessary and sufficient for them to satisfy SH. The simpler monomodal version of such results appears as Lemma 5.6 in Zolin [120]; further characterizations of the class of logics concerned, called (fully) modalized logics, can be found in Humberstone [51], §4.6. Potentially confusingly, Morscher [76] uses the term *dichotomy* for the (as [76] puts it) descriptive/normative contrast even when summarizing Schurz's trichotomous SH findings.

will emerge in Section 2.

This issue of how to distribute the slack is in large part a technical problem rather than one of special meta-ethical significance, the latter more aptly applying to the unmixed cases in the trichotomous approach: the basic ethical and basic non-ethical cases. Even the significance of the latter (non-exhaustive) division was subjected to serious questioning by Peter Singer in [109], where the serious gap is taken to be that between recognising that things are thus-and-so in the world on the one hand and taking this as a reason for acting a certain way, on the other. It is not so important whether what is considered one's moral beliefs are taken as tied to the former recognition or to the latter acknowledgment – assimilations associated in the 1960s with Philippa Foot and R. M. Hare respectively, and called non-neutralism and neutralism by Singer.[18] The focus in what follows is mostly on moral language of the deontic rather than the axiological kind, and even here one faces a choice as to whether to concentrate, for example, on unnegated – or more generally unembedded – *ought*-judgments, or to include also negated such judgments, often spoken of (in the distinctive English associated with deontic logic) in terms of permissibility.[19] The latter will be the policy here: a claim that such and such is, for example, *not* morally required is just as much a moral claim as the claim that whatever is under consideration *is* morally required.[20] Those favouring a normal deontic logic will

[18]Singer cites Hare explicitly though not Foot, but p. 52, left column, lines 5–6, makes it clear he has Foot in mind by illustrating it with a principle about *clasping one's hands* three times an hour as, according to the neutralist, a candidate moral principle, held as such a principle by those ordering their lives by resolutely acting in accordance with it. The same point was later made in Jaggar [59], esp. Section V. Neutralism about the content of morality is evidently more congenial to of an internalist inclination, wanting to minimize the step from moral judgment to disposition to act.

[19] The standardly quoted passage from Hume's Treatise – to be found in many of the entries in our bibliography – is open to a respectable interpretation as specifically focusing on unnegated (etc.) *ought*-conclusions, and, for instance, the opening page of Mares [71] takes Hume's Law specifically to pertain to the underivability of formulas of the form OA from sets of formulas free of deontic vocabulary. (However, in mid-p. 123, Wolf [117] cites a case from the Treatise in which Hume queries an inference from premises about human nature to a permissibility conclusion – i.e., to a negated *ought* judgment.) This includes the cases in which A itself contains further deontic vocabulary, excluded under the rubric 'single-main-occurring O-conclusion' below. See also Mares [70], where it is shown for a relevant deontic logic favoured there that for deontic-free A, B, the implication $A \to OB$ is never provable: see what Mares calls Lemma 2.5 (though it is not actually used to prove anything else) on p. 14; on the other hand, the logic in question does contain theorems of the form $A \to \neg OB$ for deontic-free A, B, such as $\Box\neg p \to \neg Op$ – contraposing an observation from the base of p. 15. Thus here it is only the 'main O in the conclusion' form of Hume's Law that holds (and indeed the single-main-O form that is being shown to hold).

[20]Space considerations preclude a discussion of the question of moral nihilism here, which this cursory remark raises – a topic arising in several of the publications referred to; in particular: Maitzen [66], [68], Hill [38], Nelson [77], Pigden [81], and the final section (headed §8.6) of Maguire

not need consider conjunctions of *ought*-judgments as a further case, in view of the equivalence in such a logic of $OA \wedge OB$ with $O(A \wedge B)$.[21]

Another case worthy of consideration is the disjunction of two *ought*-statements, as Daniel Singer ([108], p. 196) reminds us, saying of a proposal from Gibbard [31] to consider only straight unembedded occurrences of "O":[22] "It is too strong because it excludes some arguments from the purview of the is-ought gap that it should not. For instance, it excludes an argument with the conclusion 'Either Jane ought to eat tomato soup, or Ange ought to buy garlic bread.'" For an argument with this conclusion, normal (indeed, more generally, monotone) deontic logics would have $O(A \vee B)$ as a consequence of $OA \vee OB$ despite the failure of the converse implication so typically one could still conclude to a single-main-occurring O-conclusion, but (i) this might end us up with a conclusion that few would consider ethical despite the main O (e.g., if B is $\neg A$) and (ii) this would not work with agent-relative or agent-implicating deontic operators (as arguably in the example of Jane and Ange) – though for simplicity we ignore such operators in what follows.[23] Finally, let us consider the case of conditionals – which to avoid the complications alluded to above – we may take to involve material implication of the form $OA \to OB$. We saw these emerge, above, from the 'conditional proof' move made in Rynin's dialectic from [96], and we can also see them in play in Sen [105], which is of some interest in having prompted Hare to write (replying to Geach [22]) the following ([35] p. 469):

> I have indeed been persuaded, not by Geach but by Professor Amartya Sen, that my own thesis of universalizability commits me to allowing valid inferences from non-evaluative premises to logically complex evaluative conclusions.

[65]; also relevant is the discussion of 'positively ethical' sentences in the §5 of Brown [7]. (There is a typo in the first new paragraph of p. 68 here, with "any sentence implied by an inconsistent sentence is inconsistent", presumably intended to read "any sentence implying an inconsistent sentence is inconsistent.") Maitzen [68], p. 307*f.* regards the Disjunctive Syllogism part of Prior's argument, the transition from $E \vee F$ and $\neg F$ to E, as straightforwardly refuting Hume's Law since each of the premises, but not the conclusion, is ethical by the following criterion: each is capable of being accepted by a moral nihilist (which doesn't mean, we may take it, that both could be simultaneously accepted by such a nihilist.) Sinnott-Armstrong [110], p. 161 second paragraph, endorses a similar principle.

[21] Early opposition to this equivalence can be found in [98]; for other references, see the index entries under 'aggregation' in Humberstone [51], which refers specifically to the implication from $OA \wedge OB$ with $O(A \wedge B)$, though even the converse implication has been contested – e.g. in Jackson [56].

[22] Maitzen [68] recalls with approval a broadly similar suggestion from Gewirth [26].

[23] For references to the extensive literature on them, see [51], p. 251. As to the "single" in "single-main-occurring", the intention is to set aside encoding, for example $\neg OA$ or $OA \vee OB$ as of the desired simple form by rewriting them as $O\neg OA$ or $O(OA \vee OB)$ (or even $O(OA \vee B)$) to which they would be equivalent in the logic KD45, for instance. For more details and qualifications concerning Gibbard on the present issue, see Singer's discussion, including note 9 on p. 196 of [108].

Sen ([105], esp. p. 76) is mostly concerned with good rather than ought, and with inferences from "A and B are descriptively alike" to "A being good implies B being good", though the latter can be reformulated with a slight change of meaning so the that conclusion is instead "A is as good as B," making though less readily dismissible as a Boolean compound of evaluative sentences by anyone not considering such cases to fall within the basic ethical category (not that specifically moral goodness is at issue in the cases discussed by Sen, who also presents similar examples involving *ought*). Here we are in the vicinity of the issue of the supervenience of the ethical on the non-ethical, whose connection to Hume's Law is a much discussed matter, the discussion using requiring a consideration of contrasts between metaphysical and logical (or more broadly, conceptual) necessity that it is accordingly preferable to avoid here.[24]

Here we take the 'generous' line that all of these Boolean compounds are candidates for being 'basic ethical', even if some (as in the $OA \vee \neg OA$ case no less than the non-embedded $O(A \vee \neg A)$ case mentioned above) warrant exclusion, a topic to which we return in the Postscript to this section, after Proposition 1.1 there. (Recall that the Boolean compounds at issue here do not include the problematic 'mixed' compounds.) In particular, in the case of negated *ought*-judgments this has the effect that any one-premise inference from an *ought*-premise to a non-ethical conclusion will contrapose to an inference, valid if and only the original is, from a non-ethical premise to a basic ethical conclusion, as is often remarked in the case of the 'ought'-implies-'can' principle, contraposing to such things as 'Sylvie is unable to attend her mother's funeral' to 'it is permissible for Sylvie not to attend her mother's funeral'. The point is hinted at on p. 313 of [96], where Rynin suggests that "[i]n fact, most people hold many views similar in nature so far as entailment of factual by normative or normative by factual statements is concerned. In saying this I do not mean to assert that most people use the word 'entails' or have ever heard it used, but that they would agree, say, that no one is under any obligation to do what he cannot do." Though Rynin had earlier (p. 309) noted the general point about contraposition, it is more explicitly brought to bear with ought-implies-can in the contraposition point is more was made more explicitly in Mavrodes [73].[25]

[24] An airing of some of the relevant considerations and a look at the main literature can be found in Section 8 of Humberstone [52].

[25] A (comparatively) recent discussion of the Ought-implies-Can thesis with Hume's Law very much in mind can be found in Vranas [114], which also provides an extensive survey of the literature, including its pre-history (see note 3 there). Heading (2) under note 1 of [114] lists in chronological order many who have suggested that the 'implies' in Ought-implies-Can should really be '(semantically) presupposes', in which case the contraposition step fails. The list begins with Atkinson [2], to which we can add (from the same year) Remnant [90]. The still more recent Vranas [116] on Ought-implies-Can bears less closely on our current concerns.

A minimal pertinent observation would be that while contraposing the conclusion of an argument with one of its premises preserves validity, it does not preserve the property of being a potentially explanatory argument, or the property of recording a justification for accepting the conclusion on the basis of the premises (cf. note 33 below); a good discussion of these issues is provided by Basl and Coons [3].

Section Postscript: Logical Considerations Arising. Apropos of the emphasis on disjunctive syllogism in Mares – and indeed the title of – [71], we should recall, in addition to the remarks from Mavrodes quoted in note 14, the following.[26] If the class of ethical statements, or indeed any class of statements whatever, is deemed to be closed under taking negations and under converse entailment, then it can be shown to contain all statements if it contains any, by means of a chain of reasoning appearing in diagrammatic form as Figure 3 on p. 135 of [45], headed "A Lewis-like argument immune to relevance objections," rather than "A Prior-like argument immune to relevance objections". That is because of the connections (much emphasized in [45]) between the material under considerations here and the treatment of subject matters presented in Lewis [64], touched on below in the Postscript to Section 3. This argument can also be found in note 5 (p. 192) of Maguire [65], and on p. 153 of Russell [94].[27] The caption reference is to Lewis rather than Prior for the following reason. If one thinks of the task at hand as that of moving from a trichotomous basic-ethical/basic-nonethical/remainder to a dichotomous ethical/nonethical division for a formulation of Hume's Law, then the assumption that the ethical category in the two-block division – subsuming now many mixed cases formerly housed in the 'remainder' category – will be closed both under negation and under converse

[26] Maitzen [66], note 11, also recognises the potential for an objection to such disjunctive syllogism moves on relevant-logical grounds but takes it that the particular use he wants to make of such a move will not be one that will raises relevantist objections.

[27] Russell remarks (p. 160, note 3) that she "came across this argument in Gideon Rosen's Spring 2001 graduate seminar at Princeton." Instead of trotting it out again here, I will give a variant. Suppose we have a non-empty class of statements closed under taking negations and under converse entailment. Let A be an element of this class, and B be an arbitrary statement, with a view to showing that B is also an element of the class and hence that from its non-emptiness it follows that the class contains all statements. By the negation condition $\neg A$ is in (the class) since A is. So by the converse entailment condition $\neg A \wedge \neg B$ is in; so by the negation condition $\neg(\neg A \wedge \neg B)$ is in; (a redundant step this next one, to make the reasoning easier to follow) so by the converse entailment condition $A \vee B$ is in; and so, finally, by the converse entailment condition again, B is in. All of this reasoning is fine in the system FDE of first-degree entailment, with $\neg$ taken as the favoured De Morgan negation, a common core of relevantly accepted principles before one even considers the addition of a relevant implicational connective to the language and what its logical properties might be: see §15 of [1]. (As Guevara [34] mentions, an argument along these lines, with the specific is–ought case in mind, appears already on p. 468 of Humberstone [42].)

entailment lacks the appeal that such a closure assumption might have for the initial 'basic' ethical class and factual classes. Indeed, we do not need even that assumption for the disjunctive syllogism case. We just need, to recall our $E \vee F$ case, *just a single* non-ethical statement F with a non-ethical negation, in order to pass from the ethical $F \vee E$ (so classified because if it were ethical, by the converse entailment condition – alias the one-premise version of Hume's Law – F, some chosen non-ethical statement, would be ethical after all) together with the *ex hypothesi* $\neg F$, to E, given the counterexample to Hume's Law in its two-premise form. No general 'closure under negation' principle is appealed to here, just the assumption that *some* basic non-ethical statement has a non-ethical negation.[28] By contrast, it turns out, as we shall see in Proposition 3.10, that Prior style arguments make essential appeal to a two-premise rule (disjunctive syllogism or some substitute), whereas the 'linear' Lewis-like arguments from [45] and note 27 assume negating mixed conjunctions and disjunctions keeps us on the same side of the extended ethical/nonethical divide, but appeals only to one-premise inference rules.

This last point was insufficiently emphasized in Humberstone [45], especially as the re-worked version of the account in [42] is there explicitly noted not to satisfy the general condition that the negation of anything ethical (in a world – since this is a world-relative taxonomy) is again ethical in that world.[29] In view of that and also in view of Theorem 2 – labelled 'Prior's Dilemma' – in the Formal Appendix to Fine [19], which gives something like Prior's argument in the setting of an abstract theory of propositions rather than of sentences of a formal or natural language, and explicitly assumes that the classes of descriptive and normative propositions are each closed under negations, it would seem worthwhile here to show that such global

[28]Guevara [34], p. 55, writes: "It is widely held that sentences containing 'ought,' or other normative terms, are closed under negation. But I show that this is questionable." But of course the class of sentences containing expressions on any list you care to care to come up with *is* closed under negation, because the negating the sentence leaves whatever vocabulary the original sentence contained still intact – at least for a large class of natural languages, of which English is one (with the exception of few expressions – the 'positive polarity' items). It turns out that what Guevara has in mind is that the class of normative sentences is not closed under negation, where containing 'ought' and the like is not sufficient for normativity. In pointing forward (p. 46) to the passage just quoted, Guevara remarks similarly that "the concept of guidance I press throughout also calls into doubt an assumption – widely held – that sentences containing 'ought' or other normative terms are closed under negation," meaning that an *ought*-judgment's ability to guide the agent to a specific action type is not inherited by the permissibility judgment which results from negating it.

[29]Nor is mentioned in the earlier Geach [24] where (p. 229) Prior's $\vee$-Introduction + disjunctive syllogism argument is given but with the gratuitously strong assumption that the premise class is closed under negation. (In fact Geach assumes this about the conclusion class as well, treating the two classes symmetrically from the start and thereby disposing of what he calls the theory – or range of theories – of *logical islands*.)

assumptions are not needed for at least the version of the argument as it appears in Prior [88]. Certain aspects of the argument that were left tacit in the summary given above – and indeed are left tacit in [88] – are made explicit, very much along the lines of Fine's discussion (except for the closure-under-negation assumption). It should be added also that Mares is quite right to observe that *this* reasoning would not go through in relevant logic (say, putting $\vdash_{\mathsf{FDE}}$ – see note 27 – in place of $\vdash_{\mathsf{CL}}$ below).

Consider the following suppositions we might make concerning a class of statements $\mathbb{F}$:

(1a) $F \in \mathbb{F}$ and (1b) $\neg F \in \mathbb{F}$

and we have another statement E about whose membership in $\mathbb{F}$ we make no assumption, but we do suppose,

(2) E and F are logically independent according to the consequence relation $\vdash_{\mathsf{CL}}$ of classical propositional logic, in the sense that for no binary truth-function $\#$ – notation we use now for the associated (not necessarily primitive) connective – do we have $\vdash_{\mathsf{CL}} E \# F$.

(3) $\mathbb{F}$ is closed under 'consistent consequence' in the sense for any CL-consistent $\{A_1, \ldots, A_n\} \subseteq \mathbb{F}$, if $A_1, \ldots, A_n \vdash_{\mathsf{CL}} B$ then $B \in \mathbb{F}$. (The consistency condition can be taken to mean that $A_1, \ldots, A_n \nvdash_{\mathsf{CL}} C$ for some C, or equivalently, given that $A_1, \ldots, A_n \vdash_{\mathsf{CL}} B$, that we do not also have $A_1, \ldots, A_n \vdash_{\mathsf{CL}} \neg B$.)

The letters E and F are intended to recall ethical and factual (or non-ethical), as in the presentation of Prior's argument in the main body of this section, and condition (3) with its consistency rider is taken from Prior [88] too – *pace* 'Objection 5' considered in Mavrodes [74], mentioned in note 14. (2) packs a lot into it, since (considering $\#$ and the binary first projection and negated first projection functions and likewise for the second coordinate) it implies that

$$\nvdash_{\mathsf{CL}} F \text{ and } \nvdash_{\mathsf{CL}} \neg F\text{; and } \nvdash_{\mathsf{CL}} E \text{ and } \nvdash_{\mathsf{CL}} \neg E,$$

as well as ruling out essentially binary relations: $E \nvdash_{\mathsf{CL}} F$ etc. (taking $\#$ as $\to$).[30] Finally, although we presume available the logical apparatus of classical propositional logic, any extension of that logic (by quantifiers, modal – e.g., deontic – operators, or whatever), is fine, and for sentences $C_1, \ldots, C_n, C_{n+1}$ of some such richer language, "$C_1, \ldots, C_n \vdash_{\mathsf{CL}} C_{n+1}$" means that there is a substitution s and there are formulas of the language of classical propositional logic proper, $A_1, \ldots, A_n, A_{n+1}$ with $s(A_i) = C_i$ $(1 \leq i \leq n+1)$ and $A_1, \ldots, A_n \vdash_{\mathsf{CL}} A_{n+1}$.

[30]This criterion of logical independence is that employed in Lemmon [63]; a discussion of how to adapt it to independence relative to non-classical logics can be found in Humberstone [53].

Proposition 1.1. *From assumptions (1)–(3) above, it follows that* $E \in \mathbb{F}$.

Proof. Since $F \in \mathbb{F}$ by (1a) and $\{F\}$ is consistent (by (2)), by (3) we have $E \vee F \in \mathbb{F}$. Now $\{E \vee F, \neg F\}$ is also consistent, since otherwise $E \vee F \vdash_{\mathsf{CL}} F$ and so $E \vdash_{\mathsf{CL}} F$ violating assumption (2). Therefore, since $E \vee F, \neg F \vdash_{\mathsf{CL}} E$, and we have not only $E \vee F \in \mathbb{F}$ but also (by (1b)) $\neg F \in \mathbb{F}$, by (3) we have $E \in \mathbb{F}$. $\square$

The consistent closure condition (3) above is formulated by Prior ([88], p. 201) in terms of excluding 'self-contradictory' premises in a putative counterexample to Hume's Law, because from such premises "one could deduce not only ethical conclusions but any conclusions whatever, trivially," which will again invoke suspicions that $\vdash_{\mathsf{CL}}$ is showing its weakness here, but here we raise the issue to observe that, unlike some (e.g., Fine [19]) there is no corresponding exclusion on the 'conclusion' side of conclusions B for which $\vdash B$ (or $\vdash = \vdash_{\mathsf{CL}}$ or again, any desired extension thereof): (3) does not have a further condition that B is not such a formula, even though that too would have trivialized the claim that it is a consequence of any $\{A_1, \ldots, A_n\}$. The reason is that Prior is taking it that any such B is automatically on the 'non-ethical' side of the fence (in our $\mathbb{F}$, that is), as he indicates on the previous page of [88], with the classification as non-ethical of such things as "It either is or is not the case that I should fight for my country" in which the ethical vocabulary occurs inessentially ($Op \vee \neg Op$ being such a special case of $A \vee \neg A$ with O replaceable by any sentence operator). As was mentioned in note 15, Prior goes on to add that not only should a statement to be classed as ethical contain ethical expressions (such as 'O' or *ought*, in the intended sense) essentially, but it should not contain, logical vocabulary aside, *only* such expressions, as in 'It is obligatory that what is obligatory be done' – one of Prior's favourite deontic principles, schematically: $O(OA \to A)$, the subject of Example 1.2 below. – though this is cited along with other popular candidate deontic axioms, so it is not completely clear whether here we are trading on their status as logical truths (those B for which $\vdash B$, with $\vdash$ a favoured consequence relation) or on the constituent vocabulary point officially being illustrated. For that we would have needed some such example as "It is obligatory that what is obligatory *not* be done," "If anything is permissible it is obligatory," the converse of another example Prior gives here (a vernacularized form of the famous D-for-'Deontic' axiom). Whereas the vernacular versions of candidate principles of deontic logic are explicitly on Prior's list of statements in which the presence of moral language does not occasion classification as ethical, we need to cases in which such language is 'de-activated' by appearing within belief and indirect speech contexts.[31] In one sense, the statement

[31]The active/inactive terminology here is taken from p. 201 of [101] in connection with what Schurz calls the Max Weber Thesis (the fortunes of which he charts through a range of deontic–

that Jane feared that she had done the wrong thing is ethical in content, namely in the sense that grasping its content requires the possession of ethical concepts. But this is not the notion of ethicality that is at issue with Hume's Law. Finally, let us put on the record an important point, due to Campbell Brown [8]: the use of something like disjunctive syllogism is essential to refuting Hume's Law with the likes of Proposition 1.1, where "something like" disjunctive syllogism means: like it in respect of being an at least *two*-premise rule of inference. We return to this in Proposition 3.10 in Postscript (i) to Section 3. We conclude with some words on what was described above as a favourite deontic principle of Prior's.

Example 1.2. Concerning the schema $O(OA \to A)$, or more accurately the claim that all instances of this schema are true, Prior tells use at p. 229 of [89] it "was originally suggested to me by Mrs. J. F. Bennett (in 1953 or 1954) as an example of a synthetic *a priori* proposition." For reasons of space, it has not been possible to discuss the Gideon Rosen's now well-known *flurg* argument (which can be found in Russell [94], Guevara [34], Singer [108]), but the following simplified variant of the definition of *flurg* is presented by Guevara ([34] p. 48):

> We might just as well have coined the term 'blurg' to mean 'to do something one ought not to do in any actual circumstances." This yields another valid inference from 'is' to 'ought' (...): 'Jones is in some actual circumstances. Therefore, Jones ought not to blurg.' Here we derive, apparently, a kind of categorical imperative against blurging. This confirms our sense that there is something shady about the style of counterexample, and that the problem with it must lie at least in part in the arbitrariness of the stipulated terms.

Since the reference to actual circumstances is vacuous here, let us write 'a performs action x' as 'Dax', so that 'a blurgs' is in effect defined to mean $\exists x(Dax \wedge O\neg Dax)$. Thus to say that a ought not to blurg is to say: $O\neg\exists x(Dax \wedge O\neg Dax)$, or, with some processing, $O\forall x(Dax \to \neg O\neg Dax)$, or again, $O\forall x(O\neg Dax \to \neg Dax)$, and instantiating the $\forall x$ to b, say, we have $O(O\neg Dab \to \neg Dab)$: so Guevara's categorical imperative emerges as a particular case of Mrs Bennett's synthetic *a priori* principle – all very Kantian rather than evidently calamitous, so perhaps not the reductio Guevera was hoping for. (For more on this principle, as a candidate modal axiom, see the index entry "**U**" in Humberstone [51] – that being the label associated with this axiom by Lemmon and Scott.) Similar considerations are raised by what turned out to be a contentious example in Geach [23], p. 474*f.*, that of Evan and Dewi Williams in which a crucial (though Geach says 'vacuous') premise is "Nobody

doxastic logics in §7.1 of [100]); of course, the classification of contexts which are de-activating – or as it is put in Humberstone [46], 'protective' – needs careful attention: doxastic contexts, yes, epistemic contexts, no (and so on).

ought to adopt the practice of *doing something he ought not to at least twice every day*." The example and the description of this premise as vacuous certainly seemed to puzzle Hurka and Borowski in [54] and [6], respectively; communication is then further hampered by an impatient reply, [25], on Geach's part, affecting bafflement at Borowski's (fairly standard) deontic notation and citing in its note 4 the title of Borowski [6] alongside with the publication details of Borowski [5]. ◁

2 Karmo Recalled

The proposal of Karmo [61] is in what we might call the Shorter-inspired family of responses to the problem of extending the ethical/non-ethical taxonomy from the basic cases so as to subsume the mixed cases in such a way that we end up with everything falling in line with the basic ethical statements or with the basic non-ethical statements, though which side they fall into line with depends on which contingent facts obtain. So we end up with a world-relative taxonomy which, relative to any given world, is a two-block partition and is to that extent a dichotomy style approach, though one which is, as we shall see, world-variably dichotomous. Coupled with this, one backs off from attention to arguments with the world-invariant property of validity to those with the world-relative property of soundness.[32] Shorter [107] does not actually put matters in these terms and writes of futility or uselessness rather than unsoundness, and others (perhaps Karmo, even) may find this attribution of the approach to him contentious, so we devote a footnote to its defence.[33] The

[32]Here a sound argument is a valid argument with true premises (and, therefore, a true conclusion). When the premises are contingent premises, this makes the soundness of a valid argument from them to a conclusion a contingent matter. This is not the only use of the term *sound* (as applied to arguments or inferences) one will find in the literature. For example, in Chapter 1 of Lemmon [63] "sound" is used to mean "valid"; a related example from the same period would be Shaw [106]. (In fact, Shaw uses "sound" as replacement for "valid" in case the application of latter term should be held to be an entirely non-evaluative matter: see note 16.)

[33]Shorter writes ([107], p. 286*f.*) "In A [[an F to $F \vee E$ inference, where A1 is the F premise and A2 the disjunctive conclusion]] it is clear that a specific ethical duty can be derived from A2 [[the conclusion of the inference]] only if we know that the first half of the disjunction is in fact false. If it is false then we can derive the duty (...) If it is true, then A2 is of no help to us in deciding whether [[the duty in question exists]]. But if the first half of A2 is false, then A1 is false; and if A1 is false then the inference A lends no support to the conclusion A2." So there is no world in which both the $\vee$-introduction inference and the disjunctive syllogism inference are sound. (We know this *a priori*, but of course it will typically be an *a posteriori* matter where the unsoundness lies; for example in the concrete version discussed by Mavrodes – see note 14 – the $\vee$-Introduction inference from F to $F \vee M$ is certainly unsound, whether or not the disjunctive syllogism inference is also unsound: the tallest building in Detroit at the time Mavrodes was writing [74] was the Penobscot Building, not the Fisher Building.) But knowledge and its absence are mentioned in as well as

attribution in question was originally made in [42] where an earlier world-relative taxonomy, perhaps less satisfactory than Karmo's (for reasons given in note 58 below), and about which Karmo makes some comparative remarks in note 7 of [61]. (The passage in question is quoted at the start of the Postscript to this section.) The suggestion from [42] is briefly recalled at the end of Postscript (ii) to Section 3 below.

There may even be a semi-conscious anticipation of the world-relative approach – or the rejection of taxonomic essentialism, as Maitzen [68] calls it – in Prior [88] p. 204

> If a conclusion containing an expression E is validly inferred from a certain premise or set of premises, and the inference would remain valid if E were replaced by any expression whatever of the same grammatical type, then I say that in that inference the expression E is contingently vacuous. The expression "ought to" is in this sense contingently vacuous in the inferences "Tea-drinking is common in England, therefore either tea-drinking is common in England or all New Zealanders ought to be shot"(...)

Attention to the replaceability *salva validitate* of ethical vocabulary has been the focus of much subsequent work on Hume's Law – Jackson [55], Pigden [80] and [83],

the mere truth of the premises, though in his discussion of a second example on p. 287, Shorter stresses the role of knowledge. In this connection, it is worth recalling something said by Mavrodes (p. 363*f.*) after he wields Prior's argument to show, as Prior had done, that Hume's Law (put in terms of entailment or logical consequence) is mistaken "I have not even attempted to establish the corresponding epistemological thesis, i.e., that we could come to know some normative statement on the basis of some nonnormative statement. Nor will I attempt to do so here." One explanation of this is that to acquire knowledge of one thing on the basis of knowledge of another the inference in question would need to be *sound* and not just valid – which is not to say that soundness would *suffice* in this connection: see the last sentence of (the main body of) Section 1 above. Sinnott-Armstrong [110] also repeatedly raises the issue of the soundness as opposed to the mere validity of arguments violating Hume's Law (apparently unaware – see note 3 – of Prior, Shorter, Karmo or anyone other than Nelson [77], who is similarly unaware of Karmo's earlier discussion of essentially his main argument), though again his chief concern is which the justificatory efficacy of such arguments, remarking at p. 167: "Thus, even if Hume's doctrine fails logically, if it works epistemologically, then that might be enough to serve the primary purposes of many defenders of the doctrine." Similarly Heathcote is apparently similarly unaware of the attempts to use this consideration to adjust the version of 'Hume's Law' facing Prior-style difficulties; not this undermines the content of what he says, writing ([36], p. 94): "[N]ote that Hume is concerned with what can be discovered through reasoning: thus his division is a division of *sound deductive inference*, not of merely valid deductive inference. Nowhere does Hume imply that his division corresponds to what we think of as valid deductive inference." In the present discussion, to avoid over-use of the term 'argument' since what Prior gives us is an argument (in part) about arguments, the term *inference* is used as a substitute for the 'inner' arguments or the associated argument forms ($\vee$-introduction and disjunctive syllogism), rather than to suggest that their conclusions might characteristically be arrived at by inference from their premises.

Schurz [99], Chapter 4 of [100], and [101], for example – but the point of current interest is not how precisely to formulate the relevant considerations or how they bear on apparent counterexamples (whether defusing them as objections or acknowledging them as counterexamples). The issue is, rather, Prior's choice of terminology: what is *contingent* about Prior's contingent vacuity? Of course it is contingent which expressions are of what Prior calls the same grammatical type, but this seems no more to warrant calling the occurrence of a token of such a type, relative to a given inference, 'contingently vacuous' than the fact that if the expression featuring in an inference has meant something different – as they might well have done – would warrant calling the inference 'contingently valid'. What is contingent here is the *truth* of the premise about tea-drinking, sufficing for the truth of any disjunction in which it is a disjunct, thereby nullifying the bearing of any ethical vocabulary in the remaining disjunct on the truth-value of the disjunction: the truth of the disjunction under these circumstances in no way hangs on how the application of that vocabulary. But had the first disjunct been false, everything would depend on how that vocabulary applied...

Similarly, Prior is hovering in the vicinity of a Shorter-style reaction when he writes (p. 201):

> Finally, in case my conditions are not stringent enough, I shall with all my examples proceed as follows: Wherever I claim that a certain statement is an ethical conclusion, and give a deduction of it from purely non-ethical premises, I shall also give a deduction of the same conclusion from premises which are not all non-ethical, and the deduction will be of a sort generally recognised as leading to an ethical conclusion. That is, to anyone tempted to query the "ethical" status of my conclusion, I shall say "Look, you can also get it *this* way"; and if that was where you had first met with it, you wouldn't have dreamed of denying its 'ethical' character".

But what is 'getting' the conclusion in this or that way? For his official position, this needs to be 'validly infer' – yet the persuasive effect of the examples could be due entirely to our understanding this as 'soundly infer': faced with the argument, we imagine that the premises are true and take it from there. The validity of the argument takes us overtly to the truth of the conclusion in the circumstances imagined, but perhaps more covertly to a particular verdict as to the ethicality of the conclusion in those circumstances.[34]

[34]Similarly, Pigden, whose is–ought work has concentrated, like the others mentioned alongside him in the precedent paragraph, on replaceability *salva validitate* of ethical vocabulary in the conclusions of putative counterexamples exceptions to Hume's Law, quietly shifts the focus from validity to soundness at p. 221*f.* of [83] in remarking that when we look at the conclusions on their

We need to hear from Karmo himself on all this. The references, in the following quotation, to what all parties to the debate would agree on calling ethical or agree on calling non-ethical may be taken as references to what we have been calling the basic ethical or non-ethical cases, respectively, and the examples alluded to were presented before this passage in [61], two of them originating in Prior [88]:

> To deal with such examples, we define a sentence S to be *ethical* in a possible world w just in case S is true in w with respect to one ethical standard, and false in w with respect to another ethical standard.
>
> We explain the term 'ethical standard' as follows. Call a sentence 'uncontroversially ethical' just in case all parties to the logical-autonomy-of-ethics debate would unite in calling it ethical. (There surely are sentences of this kind, for example, 'It ought to be the case that all New Zealanders are shot.' 'Everything that Alfie says is true' and 'Either tea-drinking is common in England or it ought to be the case that all New Zealanders are shot', on the other hand, are presumably not sentences of this kind: for agreement is presumably lacking on their status.) Then the ethical standard subscribed to by a person is completely determined once it is determined what truth values he assigns to all uncontroversially ethical sentences.
>
> We take it that any possible world can be uniquely picked out with some assignment of truth values to those sentences which the parties to the logical-autonomy-of-ethics debate would unite in calling non-ethical. We take it that just as some one possible world is the actual world, so some one ethical standard is the correct ethical standard. When people simply say, 'Sentence S is true', we take them to mean 'S is true in the actual world with respect to the correct ethical standard'. When people simply say, 'S is true in world w', we take them to mean 'S is true in w with respect to the correct ethical standard'.[35]

In a footnote (note 6) appended to this passage, Karmo suggests that for heuristic purposes we might think of the ethical standard as given by a set of ideal or perfect worlds in a simplified Kripke model for deontic logic, or more generally, one might

own we agree that they may contain moral vocabulary essentially ("in a certain sense" – which I take to be the sense that they are not logically or *a priori* equivalent to sentences lacking the vocabulary in question) but under certain conditions such an equivalence does hold with arbitrary same-category replacements: "namely," Pigden writes, "when the premises of the arguments are true." (In fact, with the concrete examples, Pigden substitutes the predicate "hedgehog" for the moral vocabulary, as in his [80], in order to underline the fact that a purely general logical point is involved here.) Pigden is picking up on the discussion at pp. 202–203 of Schurz [101] in which (in)essentiality figures only in an argument-relative way and there is no move from validity to soundness.

[35] Karmo [61], p. 254; I have added italics to 'ethical' in the first paragraph since this is where the term is being defined, and also italicized the world variable "w".

add, the accessibility relation of a such a model.[36] What ought to be the case is what is the case in all the ideal (more generally, in all the accessible) worlds. Such models can be thought of simply as triples $\langle W, X, V\rangle$ with $X \subseteq W$ in the simplified case (or with X replaced by $R \subseteq W \times W$ in the general case), and V assigning appropriate semantic values to the non-logical vocabulary,[37] The reader is assumed to be comfortable with the inductively defined notion of the truth of a formula A at a point $w \in W$ in such a model, notated (for approximate conformity with Restall and Russell in [92]) by writing $\mathfrak{M} \models_w A$, where $\mathfrak{M}$ is, say, $\langle W, X, V\rangle$ and $w \in W$.[38] X would of course be replaced by $R \subseteq W \times W$ for the general case in with the simplification is not wanted. For Karmo's purposes the simplified version is very much what is wanted, though, because it secures the desired independence of the ethical standard and the non-moral facts taken to distinguish one world from another.[39]

[36]We can make the simplification to a subset containing the ideal worlds when any two worlds have he same worlds accessible to them, in which case that common set will serve as the set of ideal worlds in one of the simplified – or as it is put in [51], *semi*-simplified – Kripke models. If this set is required to be non-empty, then the deontic logic determined by the collection of such models is that known as KD45.

[37] For example, in the case of propositional logic, V would map each sentence letter (or propositional variable) to a subset of W, an outright stipulation as to which worlds it is true at.

[38]In fact Restall and Russell omit the valuation component V of the models, with the result that what are supposed to be models look more like *frames*, though since their discussion is in terms of truth rather than just validity they must be somehow thinking of the elements of a model as carrying with them the kind of semantic properties normally regarded as conferred on them by V. (Many others avoid a model component like V, which is specifically there to make semantic assignments to atomic expression, and instead incorporate in its place the satisfaction relation $\models$ itself, or some equivalent, such as $\|\cdot\|$, assigning semantic values to all expressions, including formulas/sentences. But Restall and Russell include no such device, though on pp. 21 and 253, at one point they use the notation "$w \Vdash p$" without making it clear how this is supposed to be construed, given their official notion of a model. Another option, often followed in computer science and AI-related applications of Kripke semantics, is to think of the points in a model as sets of sentence letters, or the associated characteristic functions, to start with. But whatever one thinks of the merits of this for alethic and deontic interpretations of modal logic, for the common tense-logical interpretation in which the points are moments of time, it leaves no room for the idea that two moments, one strictly later than the other, might verify precisely the same atomic sentences; Section 5.3 of [92] appeals to essentially this interpretation. It is for this extra flexibility that, when Scott [103] explains the transition from matrix methodology to model-theoretic semantics using indexed bivalent valuations, it is the indices, not the valuations, that play the role of points a model.) Also, [92] uses, not the present models, which underlie the model-theoretic version of Karmo's discussion, but *pointed* models (and the above reference to frames should really be to pointed frames): we return to this in Section 3.

[39]'Desired' here means: required for Karmo's project. As we shall see below, in discussing Daniel Singer's independent rediscovery of this way of handling matters, Woods and Maguire[118] are highly critical of building in such an independence at this fundamental level, wanting an account that would leave open potentially contested meta-ethical perspectives.

Definition 2.1. *For any formula A, any model $\mathfrak{M} = \langle W, X, V\rangle$ and any $w \in W$, A is ethical at w in $\mathfrak{M}$ if and only if for some $X' \subseteq W$ with $\mathfrak{M}' = \langle W, X', V\rangle$, exactly one of the following is the case: $\mathfrak{M} \models_w A$, $\mathfrak{M}' \models_w A$.*

This definition of ethicality adapts the informal characterization given in the opening sentence of the passage quoted from Karmo above. (Following Karmo, when we are not explicitly relativizing to a model, we say "ethical in world such-and-such, but to avoid doubling the "in", when that relativization is in force, we say "ethical at such-and-such world in so-and-so model".) A more direct adaptation would put after the 'if and only if' the following:

> for some $X', X'' \subseteq W$ with $\mathfrak{M}' = \langle W, X', V\rangle$, $\mathfrak{M}'' = \langle W, X'', V\rangle$ and exactly one of the following is the case: $\mathfrak{M}' \models_w A$, $\mathfrak{M}'' \models_w A$.

But this is equivalent to ethicality as per Definition 2.1 since given the latter we get this variant by taking $\mathfrak{M}''$ as $\mathfrak{M}$ and given the variant we get the original back by noting that if $\mathfrak{M}'$ and $\mathfrak{M}''$ differ in respect of verifying A at w, one of them must agree in that respect with $\mathfrak{M}$'s treatment of A at w. Non-ethicality at w in $\mathfrak{M}$ is of course just the negation of this, and so amounts to a formula's having the same truth value at w however X – our current simple-minded incarnation of the model's ethical standard – is varied.[40] The informal use made in Section 1 of talk of basic ethical and basic non-ethical statements can be understood as represented here, for a given model, as meaning ethicality at all worlds in the model and ethicality at none of them, respectively.[41]

Karmo's own characterization of world-relative ethicality should be taken as the analogue of Definition 2.1 for natural language declarative sentences in place of for-

[40] Karmo [61], at the end of note 6 there, mentions the richer option of using instead a *betterness* relation on the worlds as playing the ethical standard role, in order to handle conditional obligation statements, and yet further variations would need to be incorporated to handle not only deontic but axiological vocabulary ('morally good' etc.), where the standard would specify the application-conditions for the predicates concerned in terms of non-moral features of the individuals or actions they apply to. But here we are concerned with the fundamental ideas of Karmo's picture and how they bear on the debate over Hume's Law (which was itself similarly formulated by Hume in deontic terms – *ought* and *ought not*).

[41] Admittedly this may not sit well with Karmo's gloss 'uncontroversially ethical', since such things as "James should visit his mother in hospital" can be understood as uncontroversially ethical – deemed ethical by all parties to the is–ought debate, that is – though obviously not true in all ideals worlds, in some if not all of which James' mother is not in hospital to begin with. It would perhaps be better to speak of fundamental moral principles rather than uncontroversially ethical statements, in this case; the 'M-class' as opposed to 'm-class' statements of Basl and Coons [3] would be another contender (to the extent that it differs from the basic principles/derived judgments distinction). A fully developed version of Karmo's position would need to address this matter more thoroughly than the rather sketchy treatment in [61] does.

mulas of a formal language, with respect to something playing the role of an intended model. The more formalized version is presented here to aid comparison in the following section with the similarly model-theoretic discussion in Restall and Russell [92]. And, as just mentioned, the only reference to the sets of ideal or permissible worlds in Karmo's suggestion in his note 6, as a simple concrete realization of the concept of an ethical standard, the main discussion being cast in the latter terminology, somewhat abstractly conceived. We stick with the concrete suggestion here, in part so that the concepts in play can be clearly illustrated in Examples 2.2. For these illustrations we concentrate on a simple deontic incarnation of the schematically presented $E \vee F$ case from the second paragraph of Section 1. F was to be 'basic' non-ethical, so we take it as a sentence letter p, and E, basic ethical, so let it be Oq (q another sentence letter, O our deontic box-style operator, as in Section 1) – these choices will work for the model in play in the examples.[42] We will actually work with the disjuncts reversed (i.e., using $F \vee E$), to avoid any risk that a reader might think of O as the main connective in $Oq \vee p$:

Examples 2.2. (i) Suppose $\mathfrak{M}$ is $\langle W, X, V\rangle$ where $W = \{w_0, w_1, w_2, w_3\}$ with $X = \{w_2, w_3\}$, and $V(p) = \{w_0\}$ while $V(q) = \{w_0, w_2, w_3\}$. Then (relative to $\mathfrak{M}$) $p \vee Oq$ is non-ethical at w_1 because however we vary X to X', calling the model resulting from such a change $\mathcal{M}'$, we have $\mathfrak{M} \models_{w_0} p \vee Oq$ iff $\mathfrak{M}' \models_{w_0} p \vee Oq$, because, since $w_0 \in V(p)$, we shall always have *both* $\mathfrak{M} \models_{w_0} p \vee Oq$ *and* $\mathfrak{M} \models_{w_0} p \vee Oq$, in virtue of the first disjunct's truth at w_0. The same verdicts would be returned for the same reason had the second disjunct been any one of $\neg Oq$, $O\neg q$ or $\neg O\neg q$.

(ii) Changing the example to $\neg p \wedge Oq$, we get another formula non-ethical at w_0 because $\mathfrak{M} \models_{w_0} \neg p \wedge Oq$ iff $\mathfrak{M}' \models_{w_0} \neg p \wedge Oq$, however we adjust the set of ideal worlds to obtain $\mathfrak{M}'$, though now this in turn holds because we have *neither* $\mathfrak{M} \models_{w_0} \neg p \wedge Oq$ *nor* $\mathfrak{M}' \models_{w_0} \neg p \wedge Oq$.

(iii) Returning to the disjunctive formula in (i), but now shifting our attention to w_1, we find that, since $w_1 \notin V(p)$, whether or not $\mathfrak{M} \models_{w_1} p \vee Oq$ depends on whether or not $V(q)$ is a subset of the set of ideal worlds, so since $V(q) \subseteq X$, we do have $\mathfrak{M} \models_{w_1} p \vee Oq$, thanks to the second disjunct, whereas shifting X to $X' = \{w_1, w_2\}$ gives $V(q) \nsubseteq X'$ and so $\mathfrak{M} \not\models_{w_1} p \vee Oq$. Thus the truth of our disjunction is sensitive

[42] One may initially think that something like Oq – admittedly not for OA in general (consider $A = q \to q$), but for $A = q$, surely? – should count as ethical at all worlds *in all models*. But no: in models $\langle W, X, V\rangle$ with $V(q) = W$, Oq is true at each $w \in W$ regardless of which subset of X is, so this formula counts as non-ethical. (The corresponding point is made in lines 8–4 from the base of p. 254 in Restall and Russell [92], whose approach will be related to Karmo's in the following section.) See also example 3.9. The same goes for the case of $V(q) = \varnothing$, at least if we are restricting attention, as [92] suggests, models (on frames) for KD45.

to what the set of ideal worlds and the disjunction is accordingly ethical at w_1 in $\mathfrak{M}$.

Ethicality on Karmo's account, as well as being literally contingent or world-relative,[43] is also a property analogous to contingency itself: for contingency proper we have variation depending on which world is under consideration, while for ethicality we must have variation depending on the ethical standard in play. 'Variation' here means in each case that there is *some* way of varying the parameter concerned – world of evaluation or ethical standard – which results in a change in truth-value, not, of course (since there are only two truth-values to go round) that *every* way of varying the given parameter results in such a change (exactly as with contingency itself, indeed).

Karmo then proves (a slightly less formal version of) Proposition 2.4 below, for which we need to introduce the notation $\models_{\mathcal{M}}$ for what is sometimes called the local consequence relation determined by the class $\mathcal{M}$ of models.[44]

Definition 2.3. *$A_1, \ldots, A_n \models_{\mathcal{M}} B$ if and only if for all $\mathfrak{M} \in \mathcal{M}$, where $\mathfrak{M} = \langle W, X, V\rangle$, for all $w \in W$, if $\mathfrak{M} \models_w A_1$, and $\ldots$, $\mathfrak{M} \models_w A_n$, then $\mathfrak{M} \models_w B$.*

Karmo's soundness-based version of Hume's Law is then as follows:

Proposition 2.4. *For any formulas $A_1, \ldots, A_n \models_{\mathcal{M}} B$ then for any model $\mathfrak{M} \in \mathcal{M}$ with $\mathfrak{M} = \langle W, X, V\rangle$ and $w \in W$, if $\mathfrak{M} \models_w A_i$ $(i = 1, \ldots, n)$ and B is ethical at w in $\mathcal{M}$, then some A_i is ethical at w in $\mathfrak{M}$.*

As with Definition 2.1, of course, Karmo's own formulation makes no reference to models.[45] However, the simple proof Karmo gives of the result carries over to the present formulation without difficulty. Of course, the approach has not found universal favour and Maitzen [66] in particular develops several criticisms, to which (as well as the other sources listed in note 1) the interested reader is referred. though

[43]No distinction is here intended between these two descriptions, though for other purposes one might want to contrast world-relativity (in the sense of not being world-invariant) with contingency, distinguishing, *à la* McTaggart, a 'B-theory' of modality from an 'A-theory'.

[44]With the notation "$\mathcal{M}$" for a class of models $\mathfrak{M}$, we continue to follow Restall and Russell [92].

[45]What Karmo has ([61], p. 256) reads as follows: "In general, if sentences S1,..., Sn (where $n > 0$) entail sentence S$(n+1)$, then for any possible world w in which S$(n+1)$ is ethical, if all of S1,..., Sn are true in w, then at least one of S1,..., Sn is ethical in w. (I have added some italics here but resisted the temptation to put the indices into subscript position.) Proposition 2.4 does not include the $n > 0$ condition because it is not needed: we can't have $\models_{\mathcal{M}} B$ (i.e., $\varnothing \models_{\mathcal{M}} B$) for B ethical at a world in and model, since B can't be false at any world in any model, so its truth-value is never sensitive to a particular ethical standard (or choice of which worlds are ideal, in the current incarnation of that notion).

here we are more concerned to call attention to connections between the ingredients of Karmo's account and ideas in play elsewhere. We turn in a moment to something of a rediscovery, in (Daniel) Singer [108], of some of those ingredients – though the recipe in which he combines these ingredients for rescuing a version of Hume's Law turns out not to be quite Karmo's, after illustrating how Karmo's approach handles an objection by Geach, whose own discussion comes closed to anticipating and rejecting that approach – or Shorter-style approaches in general.

Example 2.5. The present example comes from the hard to get hold of Geach [21]. The journal 'Open Mind' was associated with the UK's Open University philosophy course and is not to be confused with the 2017-founded MIT-based cognitive science journal of the same name. Details of the example were included on the second page of Borowski [5]. Geach is concerned with a version of Hume's Law according to which what he calls morally significant conclusions never follow logically from premises none of which is morally significant, and remarks of his refutation of this principle that "the style of argument is not at all new; I am only refurbishing a weapon already used by Prior, Mavrodes, and others." For Geach's version, we let Y be the last year in which sodomy was illegal in England and are then to consider:

1 Sodomy is either wrong or at least is illegal in England in the year Y.

2 Sodomy is either wrong or at least is illegal in England in the year $Y + 1$.

In reproducing these 'mixed disjunctions', Borowski puts '1967' and '1968' in place of 'Y' and '$Y + 1$', which makes the example easier to think about in the absence of what at least *look like* variables.[46] (This is of some incidental interest because Geach, before introducing 'Y' has said "[t]he English law against sodomy might well change," as though any such change was in the future, as of 1976 – by which time

[46]They also make the example sound more like something someone might actually say, and it was perhaps with a view to increasing naturalness on this front that Geach included the words "at least" – though this addition adds a complication. Disjunctions in which the second disjunct is prefaced by "at least" or "anyway" often present it as a fallback position introduced in the face of diminishing confidence in asserting the first disjunct outright. Jackson [57], p. 27, gives the example: "George lives in Boston or anyway somewhere in New England," – which would no doubt benefit from some additional punctuation (a comma before "or" at the very least) – and points out that learning that the first disjunct was false would not (by contrast with the case of the second disjunct, equally well introduced by *at least* in place of *anyway*) lead the speaker to retract the assertion. The at least pragmatic failure of commutativity here shows that these are no ordinary disjunctions, and so, not the clear counterexamples they might have seemed to be to 'Hurford's Constraint' (note 56 below). In Geach's case, though, neither disjunct entails (or even 'contextually implies', in the style of Ciardelli and Roelofsen [10]) the other, so the order the 'at least' invokes is not one of logical strength; perhaps we are invited to think of the relative seriousness of moral and legal obligations.

the Sexual Offences Act had already passed into law nine years previously.) Noting that if both 1 and 2 are morally significant then since 1 follows from its second disjunct we have a counterexample to Hume's Law (not that Geach uses any such crass phrase), and that if neither is morally significant, then from 2 together with the negation of its second disjunct we get a counterexample. Accordingly, Geach continues ([21], p. 12):

> The only hope of saving the 'No *ought* from an *is*' principle is to say that of the pair 1, 2, one is morally significant and the other is not; in fact, that 1 is not morally significant, since it would be inferable from a true premiss that is not morally significant, whereas 2 is morally significant, since from 2 together with a true but not morally significant premiss a morally significant conclusion would follow. This would already be very odd; 1 and 2 differ as regards the date mentioned, and how can that make one morally significant and the other not? But the case again the rule is indeed now much weightier than this. To defend the rule it was necessary to supposed that whether moral significance does or does not attach to a thesis depends not just on the logical structure and sense and force the thesis, but on such grossly empirical matters as the laws recently passed by Parliament. Clearly such considerations cannot affect the application of a proper logical rule.

The defence Geach here envisions (and rejects) on behalf of the differential classification of 1 and 2 is more is in tune with the 'enthymematic' account summarised at the end of Postscript (ii) to Section 3 than with that of Karmo [61], but let us look at how 1 and 2 fare on the latter's taxonomy. Whether the correct ethical standard endorses the first disjunct of 1 does not affect its truth-value since it is true (in the actual world) in virtue of the truth of its second disjunct however we imagine varying that ethical standard. On the other hand, since the second disjunct of 2 is false (in the actual world), the truth value of 2 depends on the ethical standard. So 1 is non-ethical and 2 is ethical, on Karmo's account. The rhetorical devices Geach employs to make this look like an untenable position are as follows. He introduces the phrase 'morally significant' in such a way that we are not quite clear as whether it is to apply to the basic ethical statements which are indeed settled in world-invariant way by the ethical standard, or to various mixed cases to the ethical/nonethical distinction has to extend to make the treatment dichotomous. In the latter case there seems nothing untoward about a statement's being *de facto* morally significant. Then there is the talk of grossly empirical matters not being the kind of thing that can affect a "proper logical rule," a phrase designed to call to mind rules of inference, perhaps, though Hume's Law is no such thing. Still, Hume's Law does concern itself with the validity of inferences, so perhaps this is not too unfair. We need to recall that Karmo is not emending rather than defending Hume's

Law so understood, in replacing the reference to validity with one to soundness – a move Geach's imaginary interlocutor does not quite get round to making – something whose evident dependence on the grossly empirical vicissitudes of life exactly matches that of the contingent taxonomy. ◁

We turn now to Singer [108] as well as some criticism that has been made of that paper. As already mentioned, Singer (unknowingly) follows Karmo not only in using some contingent ethical/nonethical taxonomy – the basic Shorter strategy – but in drawing this binary distinction in essentially the same way. What he does not do, as we shall see in detail presently, is make the shift from validity to soundness – though unlike Geach's imaginary interlocutor in Example 2.5, he does make a compensatory adjustment to the conclusions of the arguments on which Hume's Law gears. Nor is the vocabulary in which Singer's discussion is couched quite the same as Karmo's, as we have *normative* and *non-normative* rather than *ethical* and *non-ethical*, which is, as mentioned in note 16 above somewhat different, though not in ways that will prevent us from seeing the connection with Karmo's treatment. It is in these rather different terms that Singer ([108] p. 200) presents his formulation of Hume's Law:

> IS-OUGHT GAP: There are no valid arguments from non-normative premises to a relevantly normative conclusion,

and concerning which, where, Singer explains, "a conclusion of an argument is *relevantly normative* when it has substantive normative implications for the possibilities described by the premises (assuming there are some such possibilities)."[47] We need this to follow the positive proposal, articulated on p. 201:

> Hume gave us an intuitive motivation for IS-OUGHT GAP. Here I take the case one step further by showing that IS-OUGHT GAP, when properly formalized, should be seen as a theorem of normative semantics. If that is correct, the is-ought gap is not subject to Prior's or any other counterexamples. To show this, I assume that normative sentences/utterances are interpreted with respect to points of evaluation that consist of (perhaps among other things) an ordinary possible world and a normative standard.

[47] In further elaboration of this talk of substantive normative implications from later in [108] (p. 205 to be precise), we have the following, to which I have added italics at one point as a reminder of the – admittedly in need of further precisification – 'guidance' criterion we saw in note 28 had been suggested in Guevara [34]: "The key claim of IS-OUGHT GAP is this: for arguments from nonnormative premises to a normative conclusion, none of the genuinely normative aspects of the conclusion can be relevant to the possibilities described by the premises. But, since a deductive argument could only help us learn something about how things ought to be insomuch as we accept the premises, any *potential normative guidance* that could be derived from non-normative premises must only apply in possibilities where the premises fail.

This will be have a familiar sound to it. It is of course exactly the apparatus we have seen Karmo [61] introduce to formulate and justify a satisfactory version of Hume's Law (Proposition 2.4 in our somewhat formalized version). Singer remarks that the role of the normative standard – or ethical standard, as Karmo says – can be played by *plans* in the normative semantics (as Singer calls it) in Gibbard [29]. The happy consilience between Karmo's approach and Gibbard's had been pointed out by James Dreier twenty years before (see [45], p. 153), at which time Gibbard [29] had not appeared but [28] had, in which already we see this normative parameter in play, though without the somewhat de-ethicizing expository shift to talk of plans (and without Hume's Law specifically in mind).

How does Singer apply this concept in a repaired version of Hume's Law? First, on p. 202 he introduces the term *norm-invariant* in what a footnote says is to be as understood as in an unpublished paper by Mark Schroeder; this turns out to be simply Karmo's non-ethicality in all worlds. Immediately passing to the 'all worlds' case does not seem promising, but instead of what in the following section we shall call de-universalizing the notion w.r.t. worlds (though still quantifying over norms or standards, we have this on p. 203:

> The solution then is to restrict the domain of the is-ought gap to arguments in which the normative aspect of the conclusion is relevant to the possibilities being reasoned about. We can formalize this intuition in our semantic framework easily. To decide whether the conclusion of an argument makes a claim about how things ought to be in the worlds described by the premises, we decide whether the conclusion is norm-invariant when restricted only to the worlds compatible with the premises.

What is it for a world to be 'compatible' with the premises of an argument? This can only mean that we are restricting attention to worlds in which the premises are *true*. So it looks as though we are in for a Shorter-style shift of attention from valid arguments to arguments which are sound in a given world, and are headed towards exactly Karmo's position. But that is not quite how Singer proceeds (still p. 203):

> In our semantics, when the premises are norm-invariant, deciding this is equivalent to deciding whether the conjunction of the conclusion and the premises is norm-invariant. This then is a reformulated version of IS-OUGHT GAP in Gibbard's semantics:
>
> > WORLD-NORM GAP: If $\{P_i\} \vdash C$, each of $\{P_i\}$ is norm-invariant, and $P_1 \wedge P_2 \wedge \ldots$ is satisfiable, then $P_1 \wedge P_2 \wedge \ldots \wedge C$ is norm-invariant.
>
> Intuitively, WORLD-NORM GAP tells us that if the premises of an argument are norm-invariant, then the set of all world-norm pairs compatible with the

conclusion and the premises is also norm-invariant. By checking the conclusion conjoined with the premises for norm-invariance, we restrict our attention to only those worlds where the premises are true.

The condition that the conjunction of the premises should be satisfiable has been included, Singer tells us, "to avoid the special case where non-norm-invariant claims follow trivially from contradictory premises." But there is no point in doing this because WORLD-NORM GAP, as written, is equivalent to the version without the satisfiability condition, as the consequent ("$P_1 \wedge P_2 \wedge \ldots \wedge C$ is norm-invariant") would automatically be correct whatever the norm-invariance status of what we thought the conclusion of the original argument (namely C) might have been. Eliminating this extra condition brings us closer to a Karmo style formulation (Proposition 2.4), but there is still this awkward feature that we were interested in the status of the argument with premises P_i and conclusion C, and are being told to attend instead to this new argument whose conclusion conjoins the premises with C, a conjunction which is not in general equivalent to C itself.[48] The interested reader is invited to ponder the persuasiveness of Singer's own explanation as to why this aspect of his approach is, as he goes on (p. 204) to argue, "a feature, not a bug," and to decide whether or not the suggestion is in the end best seen as a rather complicated variation on Karmo's approach.

It is evident that Singer is not himself familiar with Karmo [61], or he would not have written on the second page of [108] that this was a "rough first pass as the is–ought gap":

"No normative truth is determined by any non-normative truths,"

and added in a footnote "I formulate the simple version of the claim here in terms of normative and nonnormative truths. It is thus formulated to mirror Hume's talk of propositions, which I take to be bearers of truth-values, though the use of 'truth' in the claim is unnecessary." The 'rough first pass' is indeed rough, but this is a matter of trading in the vague talk of one thing determining another for talk of one statement having another as a consequence, and the unwanted focus, when thus revised, on single-premise arguments; what is not *rough* but – as anyone impressed by

[48] A similarly disconcerting shift from one argument to another arises in Borowski [5], [6] in which the $\vee$-introduction inference from A to $A \vee B$ figuring in Prior's discussion is replaced by what Borowski calls its (without clearly defining it) that inference's 'canonical form', which is said to be the inference from $A, \neg A$ to B. (The would-be definition of the canonical form of an inference at p. 463 involves talk of replacing its conclusion by "the simplest equivalent proposition whose major connective is implication," as though this conveyed a definite instruction, even once the reference to propositions is replaced by a reference to the kind of thing that might have a major connective, and a specification of what the available logical primitives were taken to be.)

Karmo's version of Hume's Law will think – *highly sophisticated* is that the premises in question should be restricted to *truths* for purposes of invoking that law.

In view of the similarities, the contrasting treatment provided by Woods and Maguire [118] of Karmo and Singer is little surprising. Appended to a sentence (p. 420) which reads "A generation of theorists attempted to characterize the intuitive thesis with increasingly sophisticated logical versions of Hume's dictum," is a footnote describing Pigden [82] as a "*locus classicus* for these discussions" emphasizing in particular Pigden and Schurz's contributions and adding references to Karmo [61], Brown [7], Maguire [65] as offering "further critique". That is the only reference to Karmo in [118], though Singer [108] comes in for extensive criticism. Before indicating the general drift of that criticism, it should be mentioned that Woods and Maguire frequently quote Cuneo and Shafer-Landau [12] with approval on the idea that there are – or at least we should take seriously the possibility that there are – 'moral fixed points': some substantive moral principles have the status of conceptual truths. Now as Fine on the third page of his [19] notes, "it is important, if the gap principle is to have any chance of being true, that there be no normatively substantive necessities," since this would make any argument with one of them as a conclusion – certainly if the necessity is conceptual – at least informally valid, even with as uncontroversially non-ethical premises as you like.[49] Of

[49]This consideration is complicated in Singer's case by the fact that he transforms the original potentially normative conclusion into the conjunction of it with the premises. Fine's reference to 'the gap' here is to the Is–Ought gap rather than to the specific WORLD-NORM GAP of Singer's discussion. Further complicating the discussion are some remarks made by Cuneo and Shafer-Landau, not echoed by Woods and Maguire, that suggest that something's being conceptually necessary does not in fact guarantee that it is true. These conceptual truths, we are told ([12], p. 410*f.*), "hold in virtue of the essences of their constituent concepts" but on p. 413 we read "If the moral fixed points are true, then they are true of conceptual necessity," where the "if" is hard to fathom. There is no Modus Ponens in the offing, so the question of their truth seems to be left open. They follow this conditional formulation with the words: "That is, if we hold certain descriptive information fixed-such as our present human constitution and environment-the concept 'being wrong' is such that it belongs to its essence that, necessarily, if anything falls under the concept 'recreational slaughter' (of a fellow person), then it also falls under it." It is not clear how that something can belong to the essence of a concept conditionally on the state of people and their environment in the way envisaged. The unexpectedly conditional formulation recalls Maitzen's premise in a putative counterexample to Hume's Law in [66], p. 354: "If any ethical sentence is true, torturing babies just for fun is morally wrong." In fact Maitzen gives a disjunctive formulation: Either no ethical sentence, standardly construed, is true, or torturing babies just for fun is morally wrong. (The naturalness of the conditional reformulation here contrasts markedly with that of the Maitzen disjunction featuring in note 57 below, and the text to which that note is appended.) and The other premise is "Some ethical sentences, standardly construed, are true," and the conclusion is the second disjunct of the disjunctive premise. A variation is used in Maitzen [68] in which the 'other' premise is instead "At least one (non-negative, atomic) moral proposition is true." Of course, if propositions are to be the common content of logically equivalent sentences,

course some of those who would posit such necessities have no intention of saving Hume's Law (the 'gap principle'). Judith Jarvis Thomson suggested that 'Other things being equal, one ought not [to] cause others pain' is a necessary truth and if the necessity is supposed to be conceptual (or analytic), will yield Hume-violating arguments from non-moral premises about actions causing pain to moral conclusions as to their wrongness (other things being equal). References, details and discussion can be found at p. 93 of Sinnott-Armstrong [110] (or p. 138*ff.* of the book version: [111]). Indeed, we already had a foretaste of this from Rynin [96] in Section 1, note 9. See §2.2 of Sobel [112] for a discussion of G. E. Moore's view of the (non-analytic) necessity of certain fundamental moral principles.

Woods and Maguire are also sympathetic to Searle [104]'s famous (putative) derivation – or at least think that it should not be dismissed out of hand – of what someone ought to pay someone five dollars on the basis of premises about what the person said to have the obligation has said, and how linguistic conventions come to constitute this as a promise – premises one would normally take to be on the non-ethical side of the divide, with a conclusion on the ethical side. They accept the formal proof that norm-invariance is passed from the premises to the conclusion of a valid argument (at least in the way Singer makes his case, beefing up the conclusion by conjoining it with the premises, though they could equally well have discussed Karmo's version in which the argument is supposed to be sound and can leave its original conclusion unmolested), and take this to show that norm-invariance is a bad guide to non-ethicality:[50] the premises should be acknowledged to be non-ethical, say Woods and Maguire, even if they are not norm-invariant.[51] It is not entirely clear how neutral on such meta-ethical questions Woods and Maguire are entitled to insist someone drawing an ethical/non-ethical distinction has to be, though. The taxonomy is drawn up with a view to having something like Hume's Law be demonstrably correct with respect to it, so it is not surprising that it won't

they do not come dressed as negative or non-negative – or even as atomic vs. non-atomic. The difficulties posed by familiar deontic interdefinabilities for any such quasi-syntactic characterization of what it is that moral nihilist is *not* to be believed are recognised in Pigden [81], p. 452 (though Pigden think they are surmountable and himself seems to want to speak similarly of "non-negative atomic moral propositions").

[50]There are some very relaxed formulations in the discussion here: on p. 426 of [118] we read "The key fact here is that promissory behavior, given the analytic connection between it and our obligations, is not norm invariant for Searle." (Behaviour itself – as opposed to descriptions or reports of behaviour – is not the kind of thing in the running for being norm invariant.)

[51]Reading [118] is further complicated by the fact that Woods and Maguire refer (mid p. 427) to Singer's definition "of ethical facts as *just those which are norm-invariant*" (their italics), which needs "ethical" to be changed to "non-ethical", or to have a "not" inserted before "norm-invariant". This slip occurs several times, including in the subsection titles of 2.3 and 2.4, both of which begin with "Specific Worries about Ethicality as Norm Invariance".

suit the purposes of those with no interest in salvaging a repaired version of Hume's Law. We should not think of ventures such as Singer's and Karmo's (or Russell and Restall's, reviewed in the following section, a Postscript to which looks at some aspects of Woods and Maguire's discussion of it) as suasive but rather as explanatory, to use Dummett's terminology (from [15]) for marking the distinction which in the present instance is that between showing a doubter *that* a version of Hume's Law is correct on the one hand, and showing a potential sympathiser *why* it is correct, on the other.

Woods and Maguire make numerous criticisms of Singer's approach which readily transmute into criticisms of Karmo's, including worrying (p. 430) about allowing w and n to vary independently to arrive at the constellation of all $\langle w, n\rangle$ pairs, which would correspond to having what Karmo calls the ethical standard under consideration be determined in part by the world (and such norms as may be endorsed in it) with which it is paired, in accordance with what our authors call (p. 427) "conventionalist metasemantic views" and feel should be taken seriously. Such a view seems so alien to what Karmo, Gibbard and Singer are doing with the world-norm pairs that it is hard to take it seriously in the present setting, though. An ethical standard may dictate that only what is taken to be permissible according to the locally prevailing norms is genuinely permissible, and of course the former will vary – not exactly from world to world since there is not in general one single normative culture (for them to prevail in) per world – but the 'norm' (which is really a normative system or normative standard in Singer's less abbreviated formulations) in a world-norm pair refers to this transcendent set of principles rather than to any of the prevalent norms by reference to which it fixes the set of actions that are morally permissible.

On their p. 424 Woods and Maguire state the Singer-modified Hume's Law as World-Norm Gap and then parody it with a corresponding World-Octopus Gap.[52] One's immediate reaction to this is perhaps that it is just silly, since the octopus in question is (presumably) already part of the world and so plays no role in the envisaged world-octopus pairs. But it seems that Woods and Maguire's position is that exactly this reaction is not legitimately available, if the account on offer is supposed to be neutral between alternative meta-ethical positions, since it presup-

[52] Note the effect achieved by using an animal we find faintly amusing – a bit goofy but potentially endearing – just as with Pigden's references to the hedgehog (note 34). A special gold star should be awarded to Maguire [65] for managing in a single paper to cite not only Pigden's hedgehog examples but also Dworkin's book *Justice for Hedgehogs* ([16]). (The book turns out not to be a protest at their ignominious treatment in being so used in the is–ought literature. Instead, it discusses justice in general, the title picking up on the hedgehog/fox contrast in intellectual temperament made famous by Isaiah Berlin.)

poses that the ethical dimension of reality can be segregated out from everything, but not every meta-ethical position does allow for such segregation, since there may be conceptual (or even metaphysical) connections between the descriptive and the ethical.[53] If Woods and Maguire had been discussing Karmo himself, they would presumably trace this 'non-neutrality' flaw encapsulated in a single comment, already quoted above, [61]: "We take it that any possible world can be uniquely picked out with some assignment of truth values to those sentences which the parties to the logical-autonomy-of-ethics debate would unite in calling non-ethical."

Another objection, returning to Singer's discussion, is raised by Woods and Maguire in the following passage, from p. 429 of [118]:

> Now, assuming supervenience, let W be the conjunction of all the worldly facts about a possible world w. Let N be some norm that holds in some pair $\langle w, n\rangle$ in our set of factual-ethical pairs. Since we've assumed supervenience, there will be no pair $\langle w, n\rangle$ in our set such that N is not in n. Since this means that N is entailed by W, just as above, either N is descriptive or W is ethical. Neither conclusion is palatable.

First, clarifying the point being made, since we have identified N via the n of an initially given world-norm pair $\langle w, n\rangle$, it would be better to proceed by saying that there will be no pair $\langle w', n\rangle$ in "our set of factual-ethical pairs" (as Woods and Maguire put it), such that N is not in n. So holding n fixed, it would be correct to say that W strictly implies N. But, whatever the Gibbard line on such matters might be, treating this as a potential problem for Karmo's conceptual apparatus (in which n is an ethical standard, oversimplifyingly identified with a set of ideal worlds) N is a necessary truth and so strictly implied by anything you like – but not entailed, since for entailment, the relation we require to hold between the conjunction of the premises of an argument on the one hand, and its conclusion on the other, for that argument to be valid, one needs truth relative to every $\langle w'', n''\rangle$ pair to be preserved by entailment, not just for a particular choice of n'' (such as that called by Karmo the *correct* ethical standard).

Fine [19] takes up with approval several of the points made by Woods and Maguire, his remarks, quoted above, against the 'moral fixed points' endorsed in

[53] Compare the following passage, with which Section 5 of Fine [19] ends: "A related idea is implicit in the treatment in Gibbard [29] of worlds as divisible into a descriptive and a purely normative component. But, as we have seen, there is no need for us to go along with this common line of thought. For the truth of the gap principle, as we have formulated it, does not require a clean separation between the normative and descriptive facts; and we may even allow every normative truthmaker will contain a nontrivial descriptive state as a proper part." Other criticisms of Gibbard's proposed semantics for normative language (designed more for grappling with Frege–Geach than for addressing Hume) can be found in pp. 24–26 of Sinnott-Armstrong [111].

[118], notwithstanding; an example was given in note 53. [19] makes an elaborate application to the Is–Ought issue, of a truthmaker-based theory of hyperintensionally individuated propositions, about which very little can be said here, where it will serve to round off this section as well as to provide background for a further brief mention in the Postscript. Fine is on the same page as Woods and Maguire in respect of the need not to rule out Searle-style 'promising' arguments, and the account he presents allows one to hold that the conjunction of the premises of such an argument might express a descriptive (or as we would put it, non-ethical) proposition – *Promise*, let's call it, which has as a logical consequence a normative (or 'ethical') conclusion – *ShouldPay*, let's say. The premise is accordingly logically equivalent to the, again normative, *Promise* ∧ *ShouldPay*, but though equivalent, this is a distinct proposition from *Promise*, enabling us consistently to classify the latter as descriptive even though the former is normative (and these classes are mutually exclusive). Hence the need for a hyperintensional account.[54] The details of the account are somewhat provisional in [19], with Fine frankly noting occasional anomalies and possible repairs.[55]

[54]One might think, for all that has been said about Rynin [96] here, that when Remnant remarks of Rynin in the opening paragraph of [90] that "[h]e maintains furthermore that some factual and some moral statements entail each other" what Remnant means is just that according to Rynin, some factual statements entail moral statements and some moral statements entail factual statements, and not literally that according to Rynin there is some pair of statements, one ethical and the other factual, which entail each other. But no, Rynin on p. 317 seems to go somewhat off the rails in the case he makes for exactly this stronger claim. I am not saying that Fine himself is similarly confused with the analogous claim – substituting 'proposition' for 'statement' – but that he is not the first person to have held the moral/normative vs. factual/descriptive distinction to be what we now call hyperintensional.

[55]"There is an awkwardness in the present case which did not arise in the previous case. For in referring to the negative propositions $\neg Q$ and $\neg P$ we have appealed, in effect, to the falsity-makers of P and Q and it would be desirable if we could somehow say what they are in terms of the truthmakers." But at the present stage of the development of Fine's theory, this is not possible. In the middle of p. 564 of [18] Fine writes: "It is important to note that within the present semantics (and this is also true of a number of variants), two formulas A and B may have the same verifiers while $\neg A$ and $\neg B$ do not have the same verifiers. For let A be the formula $p \wedge (q \vee r)$ and B the formula $(p \wedge q) \vee (p \wedge r)\ldots$" (Here I have italicized the sentence letters which are deliberately left roman in the source, as the italic versions are used as variables over states, sets of which constitute propositions.) In fact, it is not only negation that is a bit peculiar in this semantic account, but even one aspect of conjunction, as is mentioned on the previous page of [18], but which I will put here in terms of the non-linguistic theory of propositions that takes centre stage in [19]: a proposition P and the proposition $P \wedge P$ may be different (albeit equivalent) propositions. The hyperintensionality seems to have got a bit out of control here, though Fine has suggestions as to how to fix this if it should turn out intolerable for some applications of the machinery. The issue with the $P \wedge P$ example arises, as explained in Gautam [20], from the fact that idempotence for ∧ is not, by contrast with commutativity and associativity, expressed by a linear identity (sometimes called a *regular* linear

Maguire [65], p. 201, mentions another hyperintensionality example suggested to him by Fine concerning the absorption laws. I will re-notate the statements concerned so as to match the E and F (basic ethical and basic nonethical) notation from Section 1; with that change, what Maguire writes is: "Compare F with $F \vee (F \wedge E)$. They are logically equivalent. F is non-ethical. But in the world in which E obtains, E is one of the grounds of $F \wedge E$ and of $F \vee (F \wedge E)$, which by CONVERSE METAPHYSICAL AUTONOMY is ethical."[56] Note that by contrast with Fine's position in [19], touched on above and in the Postscript below, we seem to have some world-relativity coming in here – not quite of Karmo's kind, though, if E is in the basic ethical category since that would be fixed by the ethical standard and not subject to world-to-world variation. The world-relativity gets into Maguire's version of the truthmaker account because he takes it (p. 196) that

> ... grounding is factive. Non-obtaining facts cannot ground anything. False propositions cannot ground anything.

Although Maguire is officially interested in the metaphysical autonomy rather than the logical autonomy of ethics, in that it is grounding that matters rather than entailment, because of this factivity requirement, the account looks close to a Shorter-style account in which entailment is replaced by sound entailment. But I leave the interested reader to survey Maguire's diagnostic discussion (p. 200*f.*) in terms of grounding of Prior's $\vee$-introduction + disjunctive syllogism argument and decide how significantly it differs from the Shorteresque response. The suggestion would be, as with stressing what can be known on the basis of what – see note 33 – what is doing the work is not any deep epistemological (with *knows*) or metaphysical (with *grounds*) issue, but simply the factivity of these notions. We return for a moment to the hyperintensionality issue.

Coincidentally, Maguire quotes a passage from Maitzen [68], p. 303 for a different purpose from that for which I would like to draw attention to it – or in fact a slightly

identity): an equation in which each variable occurs exactly once on each side of the "="; for more information, see note 10 of [50].

[56]The content of this principle, given on the preceding page of [65], is: Any fact partly grounded by an ethical fact is an ethical fact. For the way this is reflected in Fine's account, see the text to which note 61 is appended in the Postscript to this section. One thing making examples involving disjunctions like $F \vee (F \wedge E)$ here hard to think about in general – never mind what F and E are – is that they involve violations of what has come to be called Hurford's Constraint, whether one thinks of this as making for utterance unacceptability (one could not say: sentence ungrammaticality), or just for cognitive processing difficulty; see Ciardelli and Roelofsen [10] and references there for discussion. (Strictly, in respect of Fine, this issue about the absorption law in question bears not so much on the material in [19], in which $\vee$ and $\wedge$ are operations on propositions, as that in [18], in which they appear as sentence connectives.)

longer passage – here; Maitzen is arguing against the world-relativity/contingency aspect of Shorter-inspired positions like Karmo's:

> The contingency thesis makes us implausibly ignorant of the correct classification of disjunctions such as
>
> (GR) Goldbach's Conjecture is true, or Rothenberg's setting his son on fire was morally wrong,
>
> since we don't, and perhaps can't, know the truth-value of one of the disjuncts. The contingency thesis, therefore, implies that we don't, and perhaps can't, know whether GR is moral. Possibly (if implausibly) only its first disjunct is true, in which case GR turns out mathematical and non-moral. Perhaps only its second disjunct is true, in which case GR turns out moral and nonmathematical. Perhaps, instead, the truth of GR is overdetermined by the truth of each disjunct; what is its classification then? It seems odd to say that we can't classify a proposition all of whose components we understand without first knowing which, if any, of those components makes it true.

Whatever the complaint about (GR) is here, it can't be one about the contingency thesis, since the non-moral first disjunct is not contingent. If that disjunct is true, GR is necessarily true and if that disjunct is false (GR) necessarily equivalent to its second disjunct.[57] Since Maitzen seems to want to classify the disjunction differently from its second disjunct whether or not the first disjunct turns out to be false, he is adopting a hyperintensional position of sorts, since even if GR turns out equivalent to its second disjunct, he does not want to be forced to concede that they agree in respect of ethicality. (I say "of sorts" since the equivalence involved here is necessary equivalence but some may object that this doesn't make is logical equivalence, thereby distinguishing this case from the Fine–Maguire cases of hyperintensionality.)

Section Postscript: Committing Oneself The idea of making a statement which is ethical (in the world in which it is made) seems intimately connected the idea that the class of ethical-in-w statements should be closed under the relation *is entailed by*: this is the converse entailment closure condition in play in the Postscript to Section 1. This is because if assenting to a claim involved committing oneself morally, then assenting to any claim entailing that claim would also involve committing oneself to at least the same extent on at least the same issue as the original

[57] The second disjunct alludes to a real life incident from 1983 in California, in which a Charles Rothenberg deliberately set fire to his six-year-old son David, with near fatal consequences. Maitzen is assuming this will be familiar to readers who accordingly won't be slowed down by the presupposed – or at least backgrounded – non-moral content in this disjunct, and will realise that it is the other disjunct Maitzen has in mind when he says "can't know the truth-value of one of the disjuncts."

claim did. We will look as such special cases of this idea as the case of a conjunct B, entailed by the conjunction $A \wedge B$, so that if B *de facto* (i.e. in the world in question) committed one morally and so counted as ethical in w, then so should the stronger claim with content $A \wedge B$. This was a feature of the world-relative ethical/non-ethical taxonomy in Humberstone [42], the details of which (see Postscript (ii) to Section 3) are not important here,[58] but as Karmo explains in these comparative remarks in note 7 of [61]:

> The present account, unlike Humberstone's, has the appealing feature that if it makes a sentence S ethical at a world w, then it makes the negation of S ethical at w also. On the other hand, Humberstone's account possesses, while the present one lacks, a different appealing feature: if a sentence S entails a sentence S′, and S′ is ethical in w, then so is S. (Consider the conjunction 'Some pigs have wings, and it ought to be the case that all New Zealanders are shot'. On the present account, this sentence will be non-ethical in any world in which no pigs have wings, and this even though it entails the ethical 'It ought to be the case that all New Zealanders are shot'.)—Exercise: show that a theory having both features will make ethics non-autonomous, in the sense of admitting sound arguments from non-ethical premises to ethical conclusions.

An incidental observation: the 'Exercise' makes it sound as though an account combining the features will have both ethical and non-ethical statements (in world-relative way) but allow conclusions of the former class to be soundly implied by (sets of) premises from the latter class, whereas in fact, as was mentioned in the Postscript to Section 1, and shown in note 27, if either of these classes is non-empty, the other is empty, so no such arguments as seem to be under discussion in the final sentence of the passage just quoted can exist.[59] But let us return to the issue of commitment. The feature of his own account to which Karmo draws adverse attention here, namely that we can have a non-ethical (at w) conjunction even when one of its conjuncts is ethical (at w) is something that can get people – even Karmo himself – a bit confused. Consider, for example, this passage from p. 255 of Karmo's [61]:

> But suppose Alfie to have issued just one sentence in w, and let this sentence be uncontroversially ethical – let the sentence be 'It ought to be the case that

[58]As noted there on p. 475 (and in [45]), the account there implausibly classifies every false statement as ethical, though this is not as bad as it might seem given that, in Shorter's wake, only sound arguments are of concern. Dreier notes ([14], p. 247) what may seem a similar if somewhat less serious anomaly for Karmo: "All false statements will have Karmo-moral consequences," though again with this *soundness* perspective in mind one might take a 'who cares?' attitude to the consequences of false statements, and what is ethical in a world, on Karmo's account, is not closed under converse entailment, as the quotation about to be given emphasizes.

[59]Essentially this point was made in the second half of p. 150 of [45].

> all philosophers are vegetarians'. There will then be two ethical standards E and E′, such that with respect to E, 'Everything that Alfie says is true' is itself true in w, and with respect to E′, 'Everything that Alfie says is true' is itself false in w. (Let E prohibit meat-eating among philosophers, and let E′ refrain from prohibiting meat-eating among philosophers.) No matter what is, in fact, the correct ethical standard—whether E, or E′, or something else altogether—'Everything that Alfie says is true' will be ethical in w. This is an intuitively agreeable result. If Alfie has indeed issued just one sentence, namely 'It ought to be the case that all philosophers are vegetarians', then someone who says that everything that Alfie says is true is himself taking on an ethical commitment (whether he is aware of this or not): the truth value of his comment on Alfie turns on a substantive ethical matter, namely on the permissibility or otherwise of meat-eating among philosophers.

The worry about this passage – from Karmo's own perspective – comes from the talk of commitment. The person who is imagined to claim that everything that Alfie says is true is has supposedly taken on, perhaps unknowingly, an ethical commitment, because the truth-value of the claim about Alfie "turns on a substantive ethical matter, namely on the permissibility or otherwise of meat-eating among philosophers."[60] But wouldn't we regard taking on an ethical commitment as something, whose *truth requires* ethical matters to be a certain way, rather than, less selectively, something whose *truth-value turns on* their being a certain way? With Karmo's footnote 7 example "Some pigs have wings, and it ought to be the case that all New Zealanders are shot", hasn't the envisaged speaker – whether or not pigs fly – taken on an ethical commitment in respect of the treatment of New Zealanders, even if the truth-value of the whole conjunction does not turn on which ethical standard is in play (it being doomed by its first conjunct to falsity in worlds in which pigs do not fly)?

While Karmo's 'Alfie' example is still fresh in our minds, it is only fitting to observe that an essentially similar case has attracted some attention among those unfamiliar with Karmo's discussion:

Example 2.6. Nelson [77], p. 555, raises, as a serious potential problem for Hume's Law as standardly formulated, the argument from (slightly paraphrasing here) premises (1) Aunt Dahlia believes that Bertie ought to marry Madeline, and (2) All of Aunt Dahlia's beliefs are true, to the conclusion: Bertie ought to marry Madeline. (We meet Aunt Dahlia again in Nelson [78]. Sinnott-Armstrong [110] discusses thus

[60]It was because of examples of this kind in Karmo's discussion, that note 16 urged that to avoid entanglement in the issue of whether all ascriptions of truth are somehow normative, that we concentrate on the ethical or moral rather than the normative in any sense broad enough to subsume such ascriptions.

current example at length, and seems to think it has something to do with Aunt Dahlia being a reliable authority. But the relevant point is made by changing the premises to "Aunt Dahlia expressed one of her beliefs at time t and what she said then was true" and "Aunt Dahlia expressed the belief at t that Bertie ought to marry Madeline." Or again, change "true" in the new first premise to "false" and change the second premise to "Aunt Dahlia expressed the belief at t that it was not the case that Bertie ought to marry Madeline." Issues of reliability and arguments to authority are beside the point.) Nelson suggests that the conclusion but neither of the premises is ethical, since on the traditional dichotomous and once-and-for-all ethical/non-ethical approach, classifying (2) as ethical would seem bizarre. (For instance, one might add, suppose the premises had been ($1'$) All of Aunt Dahlia's beliefs are consequences of the proposition that there are rabbits in Australia, ($2'$) There are rabbits in Australia, and the conclusion had been (2): if (2) were classified as ethical, we would now have a different counterexample to Hume.) What Nelson does not think to do is what Karmo does, and make the classification of (2) as ethical or otherwise depend on what Aunt Dahlia believes, about which (1), taken as true, gives us crucial information. (Oddly, Wolf [117]. p. 117, cites Pigden, in the introduction to [82] for such examples, where Pigden explicitly credits them to Nelson [77], and they were discussed already – the Alfie example – in Karmo [61], which Wolf discusses elsewhere in [117].) $\triangleleft$

Though without the explicit connection to commitment, the idea that an ethical conjunct should make a conjunction ethical surfaces in as different an account as that in Fine [19], mentioned at the end of the main body of this section, in which the taxonomy applies not to linguistic expressions directly but to propositions conceived as sets of things called *states*, the states which are members of a given proposition being thought of as candidate truth-makers (more specifically, what Fine calls *exact* truth-makers) for that proposition. The states themselves come in two flavours, descriptive and normative, as well standing in a quasi-mereological containment relation to other states. This is from Section 3 of the paper:

> No descriptive state can contain a normative state. It must, in other words, be *purely descriptive*. However, there is no corresponding requirement be purely normative, i.e., contain no non-null descriptive state.[61]

[61] This is Fine's version of the *asymmetry* – as it is called in mid-p. 142 of Humberstone [45] – needed for commitment-oriented approaches to Hume's Law. But a few lines below the passage from Fine quoted above, the theme of normativity as dominant and descriptivity as recessive arises again at the level of propositions themselves and Fine writes: "We will take a proposition, considered as a set of states, to be *descriptive* if all its member states are descriptive and to be *normative* if at least one of its member states is normative." This is a very different matter, since while at the level

Returning to the linguistic setting and to the commitment issue, Maitzen also has the reaction voiced above, and writing on p. 352 of [66], after noting precisely this earlier 'commitment,' mentioning the above passage of Karmo's and its lack of fit with the point about conjunction:

> Consider, for instance, a sentence that conjoins an uncontroversially ethical clause and an uncontroversially non-ethical falsehood: 'Capital punishment is morally wrong, and Montreal is south of New York.' I would classify that sentence as ethical: anyone who assents to the sentence takes on two commitments, one of them ethical. In spite of that commitment, though, Karmo's taxonomy has the sentence come out non-ethical, since the sentence is (actually) false regardless of what the correct ethical standard is.

What is strange, in view of this, is that on p. 350 of the same paper, Maitzen remarks apropos of whether "Everything that Alfie says is true" (= A1) is an ethical premise (in the argument from it and "Alfie says that it ought to be the case that everyone is sincere" to the conclusion "It ought to be the case that everyone is sincere") that the answer

> ... depends on the contingent matter of whether Alfie has, in fact, asserted any ethical sentences: if Alfie *has*, then Al is an ethical premise, since anyone who accepts Al is committed, knowingly or not, to the truth of at least one particular ethical sentence. As Karmo himself puts it, if Alfie has in fact asserted some ethical sentence or other, then Al is an ethical sentence because its truth or falsity 'turns on a substantive ethical matter'.

Despite Karmo's 'off message' remark about commitment, Maitzen's report on Karmo here overlooks the point that that comment was ill-advised precisely because of the actual details of Karmo's treatment: from Alfie's having asserted as many ethical sentences as you like, it does not on that account follow that A1 is an ethical premise, since Alfie may also have made a false non-ethical assertion, in which case the truth-value of A1 is settled – it is false – regardless of the ethical standard in

of states, contained states, speaking very loosely, behave rather in the manner of conjuncts, all of them required for the state to obtain, whereas at the level of propositions member states behave in the manner of disjuncts, any one of them sufficing for the truth of the proposition concerned. This has nothing to do with whether or not assent registers a normative commitment, and renders the propositional analogues of 'mixed disjunctive' sentences all normative (or 'ethical', in our more customary terminology). Nor would it to say that because the latter are the essentially the negations of the conjunctive cases, they must be treated similarly, since on commitment-oriented accounts (such as that of Humberstone [42]: see the end of Postscript (ii) to Section 3, where, as well as that, we also have the world-invariant 'partly about M_{eth})' proposal) one does not, *pace* Fine, have closure under negation for the non-basic ethical or non-ethical classes.

play. This point is illustrated by a minimal variation on Karmo's example "Some pigs have wings, and it ought to be the case that all New Zealanders are shot": just imagine that Alfie makes not this assertion but the two assertions "Some pigs have wings," and "It ought to be the case that all New Zealanders are shot". Thus, this aspect of Karmo's position requires considerable care.[62]

These occasional slips by himself and commentators on it notwithstanding, Karmo seems right to say that his treatment can classify a statement as non-ethical (in a world) despite its entailing – as with a conjunction and either of its conjuncts – something ethical at that world, and that this is a prima facie disadvantage of the treatment. And he is right to say that it offers a compensatory advantage: that the negation of each statement ethical in a given world is again ethical in that world. The disadvantage, as a failure of ethicality to connect with what is seen as morally committal, we saw, led Dreier to discard the Karmo taxonomy, and similar considerations are perhaps at work in the objections raised by Wolf [117] against Karmo's taxonomy. Wolf invites us (p. 118) to consider the examples:

(BILL) Bill was right to tell the truth about Monica.

(BILL*) Bill was right *not* to tell the truth about Monica.

In particular, we are to consider first the normativity/ethicality status of BILL relative to a world in which Bill lies about Monica. Brushing aside in a footnote the suggestion that what we have here is a case in which "Bill told the truth about Monica" is presupposed by BILL – taking presupposition as the semantic relation last encountered in note 25 – Wolf quickly replaces BILL with the explicitly conjunctive *Bill told the truth about Monica and Bill ought to tell the truth about Monica*, though perhaps that should be "ought to have told" rather than "ought to tell" (the implicature from "ought to have φed" to "did not φ" being readily cancellable). He reminds

[62] Another issue is also raised by the parenthetical comment in "someone who says that everything that Alfie says is true is himself taking on an ethical commitment (whether he is aware of this or not)". If we were interested in tracking ethical the commitments of a subject, wouldn't it be the subjects beliefs about was the case, rather than what was in fact the case, that were relevant? So argues Dreier [14]. First (p. 246), he illustrates his dissatisfaction with the failure of the converse entailment closure condition on statements ethical-in-w with a disjunct to disjunction entailment rather than a conjunction to conjunct entailment: "Benito is evil or New Zealand is a Communist Republic" emerging as ethical in the actual world even though it is entailed by its non-ethical second disjunct. (This of course is a version of the motivating consideration – the status of $E \vee F$ in our opening discussion – behind Shorter-style revisions of Hume's Law: the disjunction is entailed, yes, but soundly entailed, no.) Adapting a later example (p. 252) of Dreier's, if we were wanting our taxonomy to mirror ethical commitment, and we knew that the speaker believed, firmly though falsely, that New Zealand was a communist republic and asserted the Benito disjunction on that basis, we would no doubt be dissatisfied with its classification as ethical.

us that this comes out on Karmo's account as descriptive, rather than normative, since no change in the ethical standard can change its truth-value (given the false first conjunct). Wolf regards this as obviously misclassifying BILL. Here we have the observation, conceded in the second paragraph of the first passage quote from Karmo at the start of this Postscript: it would be nice to have the normatives-in-w closed under converse entailment. The 'commitment' aspect of this is especially in evidence when what's being entailed is one conjunct of a conjunction, since that conjunct is explicitly there in the premise. (A similar sentiment can be found in Brown [8], discussed at the end of Postscript (i) to Section 3.) In fact the normative conjunct in Wolf's conjunctive reformulation of BILL is even more heavily present in BILL itself, since even if one does not have to buy into the presupposition as a semantic (= truth-condition affecting) phenomenon to concede that this normative component is foregrounded in BILL and the descriptive component backgrounded.[63]

Wolf continues (p. 118*f.*):

> Parallel reasons show that BILL* is normative anywhere that Bill doesn't tell the truth. But there is no normatively significant difference between the two—each makes a clear moral evaluation. The only difference is that at some worlds the sentences correctly describe Bill's action and in others they don't. Yet it's difficult to see how this would be relevant to assessing normativity. If it isn't relevant, Karmo's approach doesn't accurately model natural language.
>
> Some might argue that correctly describing Bill's action is normatively relevant, by comparing these cases with Prior's disjunction. Because the disjunction would be descriptive when it describes the facts about tea-drinking correctly, and normative when it doesn't, it gets a mixed treatment, like BILL and BILL*. If it's acceptable for Prior's disjunction to vary with correctness, then perhaps it really is relevant to whether a sentence is normative.
>
> Yet even if we accept the mixed treatment of Prior's disjunction—and we needn't—that would show that correct description is normatively relevant only if correctness does some work toward explaining why we accept different verdicts. Otherwise, correctness might have nothing to do with normativity. Other explanations are plausible: the mixed treatment of Prior's disjunction[64] is tolerable because of what asserting it would commit us to at different worlds. At worlds where we know that tea-drinking is common in England, we can assert the disjunction while denying that New Zealanders ought to be shot. But when we consider worlds where we know that tea-drinking is not common, asserting

[63]Discussion and references concerning the various contrasts alluded to here can be found in Büring [9]; alternatively, instead of saying 'foregrounded' – a term the present author regards as preferable to 'focused' since the distinctive aspects of focus particles need not be involved – one could follow Potts [87] and say that the normative component is *at-issue entailed* by BILL.

[64]That is, the disjunction from [88]: "Either tea-drinking is common in Britain or all New Zealanders ought to be shot."

> the disjunction commits us to saying that all New Zealanders should in fact be shot. Karmo's relativity approach reflects the fact that at some worlds we would be committed to obviously normative claims, but not so at other worlds.
>
> Notice there is no similar change in our commitments when we assert BILL or BILL*. Whatever the world, saying that Bill was right to tell the truth about Monica means that Bill ought to tell the truth about Monica. That's a reason for thinking at least some normative sentences stay that way across worlds.

Yes, one could, under pressure from these considerations about commitment, treat the mixed conjunctive cases and the mixed disjunctive cases differently, despite the fact that negation toggles us between the two, as was done in Humberstone [42]. That was what Karmo was offering an alternative to, which would preserve closure under negation for the statements deemed ethical in a given world, at the cost of sacrificing closure under converse entailment. These are really not two alternative opinions, but two taxonomies concerning which one might sensibly react as Lewis does in [64] when considering precisifications of the observational/non-observational contrast: you can have a notion of observationality which is closed under converse entailment (so that one observational conjunct observationalizes a conjunction) and you can have a notion of observationality which is closed under negation (so that negating an observational statement gives another observational statement): but if you try for a notion of observationality with both features, things will not go well.[65] These are not world-relative notions in Lewis's case, though they have world-relative analogues, as described in [45] and briefly touched on in Postscript (ii) to Section 3 below.

Wolf's own conclusion after presenting difficulties for Karmo's and other responses to Prior's argument is summarised thus:

[65] Another, earlier, venture into philosophical taxonomy prompted, like Lewis's, by logical positivism and the verification principle, not mentioned (though it should have been for the sake of comprehensiveness) in Humberstone [45] is Morgan [75], especially as its final paragraph alludes to the normative/non-normative dichotomy. Morgan says on p. 217 "For the sake of this discussion I will assume that we are concerned with a language with the syntactical structure of first order predicate calculus, which may include functions, and which includes the usual connectives", and the mention of function symbols suggests without actually entailing that we are considering first-order logic *with identity*, whose presence would vitiate some of the claim made – such as Lemma 1 on p. 220 which says that the disjunction and conjunction of two formulas sharing no predicate letters are both LC if each of the two formulas is LC, where LC ('logically contingent') formulas are those which are satisfiable and have satisfiable negations. But the disjunction of the predicate-disjoint $Fa \to Fb$ and $\exists y \exists y(x \neq y)$ is not LC even though its disjuncts are. (The criticism of §3 of Brown [7] in note 15 above notwithstanding, Brown is there is alert to the sensitivity of Halldén completeness to the presence or absence of identity. Special attention is paid to the Is–Ought implications of Halldén completeness in §5.1 and Appendix A12 of Schurz [100].) A more recent discussion prompted by the positivist motivated discussions of demarcating the empirical, which similarly notes the connection with Hume's Law considerations can be found in Diller [13].

> The general problem comes from attempting to frame a philosophically significant inference barrier around the distinction between normative and descriptive sentences, which is difficult to pin down. Moore's Law steers clear of these problems because it's a semantic barrier: no atomic normative terms are synonymous with any atomic descriptive terms, either directly or by substitution. I think Moore's Law can both stand in for the Guillotine and improve on it in an important way.

A similar principle – to the effect that moral concepts cannot be analysed or expressed in entirely nonmoral terms – is called the Moore–Price Law in Sobel [112], where its logical relations to Hume's Law are examined in some detail. Whether or not Sobel's principle coincides with that favoured by Wolf, his name for it is certainly better, as it does not evoke thoughts of the 'Moore's Law' of computing hardware fame – an unnecessary (and perhaps demeaning) distraction – especially since Wolf doesn't even use the contrasting phrase 'Hume's Law' for what he wants this principle to displace (preferring instead Max Black's terminology: Hume's Guillotine). While Sobel's discussion will not be covered in the present survey, it must be mentioned that it opens with a splendid quotation from Richard Price in which what is mostly known today as Moore's Open Question argument is shown to be have been already alive and well in the eighteenth century.

3 The Restall–Russell Approach

In [92], Restall and Russell are concerned with classes of models of various types, including in particular models (or interpretations, structures,...) for first order languages, Kripke models for intensional languages, and, potentially, models of other kinds also. What is important about such models is that they make true, satisfy, or verify certain formulas (or sentences, as we will often say here to follow the usage in [92]) and not others. If $\mathfrak{M}$ is such a model and A is a formula, we write $\mathfrak{M} \models A$ to indicate that A is true in the model $\mathfrak{M}$. One can make sense of this using the kind of Kripke models we have been mentioning in which a non-empty set W (say) is tupled up with a bit of apparatus for interpreting the intensional vocabulary – a binary accessibility relation in the case of standard Kripke models, or a distinguished subset of W in the case of the simplified Kripke models of the preceding section, or (increasing rather than reducing generality) a function assigning sets of neighbourhoods to the points, etc., – and a valuation function V to assign semantic values to the atomic non-logical expressions (in the propositional case, assigning subsets of W to the sentence letters, though, as explained in note 38, [92] does not follow this practice). While one can speak of truth in a model so conceived, and this would be taken to amount to truth throughout the model, for many purposes, including Restall and Russell's, it is better to take a Kripke-style model to be a pointed model, in which also a particular element of W is singled out, and truth in the model is taken to amount to truth at that distinguished point (relative to the model concerned).[66] Thus the simplified Kripke models of the previous section above, $\langle W, X, V\rangle$ would become instead $\langle W, X, w, V\rangle$ where $w \in W$ (or, if preferred, $\langle W, X, V, w\rangle$), so that what was formerly written as "$\langle W, X, V\rangle \models_w A$" now becomes "$\langle W, X, w, V\rangle \models A$". In the more general case suited to a normal monomodal logic – as in the case of traditional deontic logic – in place of X here we would have a binary relation on W. Notice that although in the preceding section we found the models without distinguished elements to be easier to use for such purposes as Examples 2.2, in fact Karmo's own informal discussion would favour a formal rendering using the pointed models since it places the correct moral standard, which we can think of as the X of the intended model, and the actual world, which we can (now) think of as the distinguished point of the intended model, completely on a par.

Continuing our exposition of Restall and Russell, suppose, next, that we have a collection $\mathcal{M}$ of such pointed models and a relation $\mathcal{R} \subseteq \mathcal{M} \times \mathcal{M}$. This is not quite the notation used in [92] but we choose a different font for the relation symbol to minimize the danger of confusing the inter-model relations $\mathcal{R}$ with the intra-model accessibility relations. In this notation, Definitions 3 and 4 from [92] become 3.1(i) and (ii) here, in which $\mathcal{M}$ is a collection of models:

Definitions 3.1. *A formula A of the language interpreted by $\mathcal{M}$*
(i) *$\mathcal{R}$-preserved over $\mathcal{M}$ iff:*

$$\forall \mathfrak{M} \in \mathcal{M}(\mathfrak{M} \models A \Rightarrow \forall \mathfrak{M}' \in \mathcal{M}(\mathfrak{M}\mathcal{R}\mathfrak{M}' \Rightarrow \mathfrak{M}' \models A)).$$

(ii) *$\mathcal{R}$-fragile over $\mathcal{M}$ iff:*

$$\forall \mathfrak{M} \in \mathcal{M}(\mathfrak{M} \models A \Rightarrow \exists \mathfrak{M}' \in \mathcal{M}(\mathfrak{M}\mathcal{R}\mathfrak{M}' \ \&\ \mathfrak{M}' \not\models A)).$$

As the very general terminology suggests, Restall and Russell are not concerned specifically with deontic logic and Hume's Law, but with analogous 'barriers to implication' generally ('inferential barriers' in the terminology of [42] and [19]). These they take pairs of sets of sentences from some language satisfying a condition formulated by reference to the consequence relation $\models_{\mathcal{M}}$ of Definition 2.1 though dropping the quantifier over $w \in W$ and its later subscripted appearances (since we are now working with pointed models or indeed of models as the familiar structures or interpretation in first-order model theory in which there is nothing corresponding

[66]Pointed models in which the model is generated by the distinguished point are often called rooted models, but this further condition is not imposed here.

to such internal evaluation points for formulas anyway). The condition in question for $\langle \Gamma, \Sigma \rangle$ to be a barrier is that no satisfiable subset of Γ has an element of Σ as a $\models_{\mathcal{M}}$-consequence, where 'satisfiable' means simultaneously true in some $\mathfrak{M} \in \mathcal{M}$: we will call this $\mathcal{M}$-satisfiability for greater explicitness.[67] The main observation is proved without using this terminology however, as Theorem 5.[68] What follows is a mildly reformulated version of this result (also dubbed the 'Barrier Construction Theorem' in [92], p. 248):

Proposition 3.2. *For any class of models $\mathcal{M}$, if $A_1, \ldots, A_n \models_{\mathcal{M}} B$, and the set $\{A_1, \ldots, A_n\}$ is $\mathcal{M}$-satisfiable, then there is no $\mathcal{R} \subseteq \mathcal{M} \times \mathcal{M}$ for which all the A_i are $\mathcal{R}$-preserved while B is $\mathcal{R}$-fragile.*

Restall and Russell apply this general result to standard first-order structures with $\mathcal{R}$ as the substructure relation, to conclude that no satisfiable set of substructure preserved sentences has as a first-order consequence a substructure-fragile sentence, which they regard as vindicating a claim of (Bertrand) Russell's to the effect that from no (satisfiable) set of particular premises can one validly infer a universal conclusion,[69] as well as an alethic modal analogue of this which they associated with Kant, in which $\mathcal{R}$ is taken as the submodel relation[70] and for which relation the corresponding notions of preservation and fragility are called modal particularity and modal generality (rather than modal universality, for some reason).[71] There is also a tense-logical application, touched on in note 73 below, and

[67]This is Definition 6 on p. 249 of [92]; the "$B \in \Gamma$" appearing there is a typo for $B \in \Sigma$.

[68]The point of introducing the notion of barrier is to facilitate is to show – the authors' Theorem 7 – that any barrier thesis can be seen as arising from the preservation and fragility conditions in Theorem 5: a suitable $\mathcal{R}$ can always be found.

[69]Restall and Russell in fact say, in the first order case, "semantically particular" and "semantically universal," the adverb being omitted here as *all* of the notions in play in the discussion are characterized semantically. (Russell [95], p. 150, replaces this adverb with "genuinely".) Restall and Russell, pp. 248 and 250, give the following simple example of a sentence that is neither universal nor particular: $Fa \vee \forall x(Gx)$.

[70]For Restall and Russell, one Kripke model $\mathfrak{M}$ is a *submodel* of another, $\mathfrak{M}^+$ – equivalently $\mathfrak{M}^+$ is an *extension* of the $\mathfrak{M}$ – if they have the same distinguished point, and, using the obvious notation, $W \subseteq W^+$, $R \subseteq R^+$ and V is the restrictions to W ($V(p_i) = V^+(p_i) \cap W$ for each sentence letter p_i). The authors do not require that, similarly, $R = R^+ \cap W \times W$ – i.e. do not require that $\mathfrak{M}$ is the submodel of $\mathfrak{M}^+$ generated by W. That would give a different inter-model relation but would not, as far as I can see, make a difference to which sentences were preserved or fragile w.r.t. the relation in question.

[71]The conspicuously missing reference here would be: Routley and Routley [93]; cf. also the subsequent discussion in Anderson and Belnap [1], §§ 5.2.1 and 22.1.2 (the latter by J. A. Coffa). Humberstone [41] experiments tentatively with the idea of adapting to modal ends, not the "fragile upwards" conception of universality favoured by Restall and Russell, but the "preserved downwards" characterization familiar from the Łos–Tarski Preservation Theorem to the effect that the sentences

there are two applications to deontic logic, one of them along the same lines as the alethic modal case and another which is of special current relevance to us.[72] In all cases, since, as Restall and Russell point out, there are formulas that are neither $\mathcal{R}$-preserved or $\mathcal{R}$-fragile, what Proposition 3.2 delivers are Hume-like barrier theses for (setting aside the unsatisfiable cases) threefold rather than twofold classifications: we are in the heart – and perhaps close to the technical summit – of trichotomy territory. So it will take us some further work to see how this connects up with Karmo's dichotomous approach, at least world-by-world, in which the target thesis is a closure-under-consequence condition on the set of nonethical-at-w truths.

Restall and Russell denote by $\between$ the relation, called by them *normative translation*, defined thus: $\mathfrak{M} \between \mathfrak{M}'$ iff $\mathfrak{M}$ and $\mathfrak{M}'$ differ at most in respect of their accessibility relations. In the simplified presentation of the models with distinguished subset X this amounts to differing at most over what the distinguished subset is (since the implicit accessibility relation is $W \times X$, where W is the universe of the model).

Proposition 3.2 now tells us that no satisfiable set of $\between$-preserved formulas can have as a consequence a $\between$-fragile formula. Of course for a precise statement of the applications of Proposition 3.2, this one and those alluded to in the previous paragraph, we need to know about the underlying $\mathcal{M}$ and for the present application

whose truth is preserved on passage from a first-order structure to an arbitrary substructure thereof are precisely those equivalent to formulas which when written in prenex normal form have all their quantifiers universal. An alethic modal analogue of universality of this kind is called globality in [41]. Of course, we again have a Restall–Russell barrier result for the Łoś–Tarski notion of universality: no satisfiable set of such sentences can have a substructure-fragile consequence (though [92] does not isolate these notions). It does not seem unreasonable as a notion of universality for sentences, which applies to cases such as $\forall x(x = x)$ which do not count as universal in the nomenclature of Restall and Russell. Russell [95] (p. 147) herself mentions the 'upward' version of Łos–Tarski, for formulas with only "$\exists$" in prenex normal form since it is formulas with such equivalents that are preserved under extensions that count as 'particular' in the Restall–Russell classification. ([95] even at one point (p. 146) uses the term *global* – but to characterize Restall–Russell universality rather than Łoś-Tarski universality. The main applications of the Barrier Construction Theorem from [92] are conveniently summarized in §3 of [95], before the main business is under way: finding an appropriate barrier separating indexical conclusions from the non-indexical premises. The eventual solution is a variation on what Pigden [80], p. 136*f.* calls the conservativeness of logic and regards as trivializing such barrier theses: this is essentially what the "unless" clause does in Russell's Theorem 5: "No consistent set of constant sentences X entails an indexical sentence A unless X also entails all of A's complete indexical generalisations.")

[72] Both deontic applications appear in §5.4, headed 'Normativity I'. §5.5 ('Normativity II'), not discussed here, does not pretend to be anything more than suggestive and envisages an extension relation on 'situations' conceived as a partial version of possible worlds, and of the fragility of normative judgments about them as one passes from a situation to one extending it. The issue seems reminiscent of W. D. Ross's parti-resultant/toti-resultant distinction: additional considerations of any kind, and not just the consideration of additional objects, have the potential to change one's moral assessment of a situation.

Restall and Russell suggest ([92], p. 253) that we should consider (pointed) models whose accessibility relations are transitive, Euclidean, and serial,[73] which makes $\models_{\mathcal{M}}$ the local consequence relation of the logic KD45. As is well known, this is also the logic determined by a proper subset of that class of models, namely those $\langle W, S, w, V\rangle$ for which there is X with $\varnothing \neq X \subseteq W$ and $S = W \times X$. As is also well known, we get the same logic by reducing the class of models even further – though this is not something to be exploited here – taking the w-generated submodels of such models, in which case we get the further condition satisfied that $W = X \cup \{w\}$, so that we never have more than one non-ideal world in a model.

(Readers not familiar with tense logic might skip this paragraph.) The simplifications just alluded to assumes that the only modal operators – understood in the broadest sense – are the deontic O, P; we may have additional alethic – or suchlike – operators $\Box, \Diamond$ which, when embedded may direct us from a world in X back out to any point in $W \smallsetminus X$, so we can't afford to throw away all but the initial point of evaluation from among $W \smallsetminus X$. Such additions arise in Example 3.9 below. And

[73]They add to this list the condition they call secondary reflexivity, which means that any point accessible to anything is accessible to itself, but this is redundant, following immediately from the Euclidean condition (which says, using S for the accessibility relation as they do, for all model elements x, y, z if Sxy and Sxz, then Syz – so taking z as z we get the redundant condition). The associated deontic schema (the last of those listed on p. 253 and encountered above in Example 1.2) would also be correspondingly also redundant, given the earlier listed $\neg OA \to O\neg OA$, not that the authors claim otherwise. Singer, discussing Restall and Russell ([108], p. 207), writes "They also assume that S is transitive, Euclidean, serial, and secondarily reflexive, though not all of these assumptions are necessary for their proof." Well, in view of the redundancy, not all of these assumptions are necessary for any proof of anything, but when it comes specifically to Restall and Russell's proof(s), *none* of these assumptions is necessary since Proposition 3.2 is simply being applied to the case of a particular choice of $\mathcal{M}$ and $\mathcal{R}$, and that general result is indifferent to how $\mathcal{M}$ and $\mathcal{R}$ are chosen. Another strange redundancy occurs in the middle of p. 252 of [92], where the authors are discussing the accessibility relations of their tense-logical models, and ask us to suppose that this ('earlier than') relation is transitive, irreflexive and antisymmetric. It is already odd to see antisymmetry mentioned in connection with an irreflexive relation, since it is usually cited when one wants to get as close to asymmetry as is possible for a reflexive relation. But since any irreflexive transitive relation is asymmetric, and any asymmetric relation is ('vacuously') antisymmetric, the third condition in their list is redundant either way. (This is not to say that the conditions given suffice for the correctness of the claims they make about them. In mid p. 252 p, Pp, Hp and GPp – in Prior's tense-logical notation – are said to be semantically historic, which is not true in the case of GPp. If the valuation functions, V, V' of two models $\mathfrak{M}, \mathfrak{M}'$ on a frame consisting of the real numbers with 0 as distinguished point, the usual $<$ as accessibility relation, but with $V(p)$ as the set of positive reals and $V'(p)$ as $\varnothing$, then we shall have $\mathfrak{M} \models GPp$ because every point t later than 0 has an earlier point – between t and 0 – verifying p, whereas $\mathfrak{M}' \not\models GPp$ since 0 does have points later than it but p is true at no predecessor of any of them. Yet $\mathfrak{M}$ and $\mathfrak{M}'$ stand in the inter-model relation – V, V' agreeing on the distinguished point and all earlier points – preservation of which makes a sentence semantically historic.)

in any Kripke model for deontic with an accessibility relation, that relation has a converse and the option arises of introducing operators O^{-1} and P^{-1} which quantify universally and existentially quantify over points to which the current point bears the latter relation as O and P do in the case of the former, validating Hume-inimical 'bridging principle' as it is put in Schurz [100], [99], and [101]: $p \to OP^{-1}p$. (Note that this is just the familiar tense-logical principle $p \to GPp$, with P now a past tense $\Diamond$-operator whose consequent put in an appearance in note 73, though we could equally have cited the other 'Lemmon bridging axiom', $p \to HFp$). This issue is raised in Example 4.4.29 in Humberstone [51]. Schurz's own study, as reported in the references just cited, was mainly of mixed deontic–alethic modal logic and so again, does not in general permit of the simplified models even when the deontic fragment is given by $\mathsf{KD45}$.

Let us return to our current concern, which consists in displaying the connections between Russell and Restall's approach and Karmo's.[74] So far, we have seen that both are concentrating on the same class of models. To proceed further it will help to have some terminology more vivid than that used in the opening sentence of this paragraph.

Definition 21 of [92], p. 254, introduces the term *descriptive* to apply to those sentences which are $\between$-preserved over the class $\mathcal{M}$, which is a promising start. We then expect a similarly evocative label for the $\between$-fragile cases. But Restall and Russell's Definition 22, which announces itself as 'Normativity (Sufficient Condition)' tells us just that being $\between$-fragile is a sufficient condition for counting as a normative sentence. Thus, we don't really have a definition at all.[75] One can see the reason for this: the real definition of normativity comes on the following page, in Definition 23 (see also note 78 below), which is styled simply 'Normativity' and gives as necessary and sufficient for a sentence be normative that it be either $\between$-fragile or $\Subset$-fragile, where $\Subset$ is the submodel relation (as defined in note 70).[76] Restall and

[74]Imposing this as a condition would also block one of Russell and Restall's proofs, namely that of Lemma 26 (whose content is described in note 79 below).

[75]Though we are at least half way to having one, which is more than can be said for the earlier Definition 2 on p. 247, which purports to define satisfaction (or verification) and reads: Definition 1 (Satisfaction): "Given a formal language L, for each formula A in L, the model $\mathfrak{M}$ will either satisfy that formula (written '$\mathfrak{M} \models A$') or it will not satisfy that formula ('$\mathfrak{M} \not\models A$')." This is just an instance of the law of excluded middle in the metalanguage, and not in the running to be a definition of anything. It's as though the authors had been contemplating the usual kind of inductive definition of $\models$ but decided not to get bogged down in the details, without realising that what they left behind then had no content.

[76]In their summary of this discussion, Woods and Maguire ([118], p. 431) say that Restall and Russell "define *descriptive* sentences as those not ethically fragile in either sense," though, as reported above, [92]'s Definition 21 defines descriptiveness simply as $\between$-preservation. And, leaving $\Subset$ out of it, this is not equivalent to the absence of $\between$-preservation (even if, for satisfiable sentences,

Russell never actually introduce a more user-friendly term for $\between$-fragility, using the expression "$\between$-fragile" itself in the course of proving (on p. 256) of what they call the 'normativity formulation' of Hume's Law, the latter being Corollary 25 (from the previous page), which reads: If Σ is a satisfiable set of sentences, each of which is descriptive, and A is normative, then $\Sigma \not\models A$.[77] Since the concept of normativity has been given a disjunctive definition using both $\between$-fragility and $\Subset$-fragility,[78] it seems for present purposes cleaner and more instructive to isolate the $\between$-based concepts both without bringing $\Subset$-fragility into the picture,[79] and define them separately, for which purposes we put an asterisk by the word 'normative' to distinguish it from the $\Subset$-entangled Restall–Russell concept of that name.

Definitions 3.3. (*i*) *A is descriptive iff A is $\between$-preserved over (the current) $\mathcal{M}$.* (*ii*) *A is normative* iff A is $\between$-fragile over $\mathcal{M}$.*

Then we can extract from the materials of [92] a direct analogue of the other applications of Proposition 3.2:

Corollary 3.4. *If $A_1, \ldots, A_n \models_{\mathcal{M}} B$, and the set $\{A_1, \ldots, A_n\}$ is a satisfiable set of descriptive sentences, then B is not normative*.*

As with the various $\mathcal{R}$-preservation-vs.-fragile contrasts explicitly in play in [92], there are sentences which fall into neither category and so we cannot treat Coro. 3.4 as telling us that any satisfiable set of descriptive premises has only descriptive

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it implies it). We can bring in $\Subset$-preservation if we want, by appealing to Lemma 26 of [92] – see note 79 below – which allows us to rewrite "$\between$-preserved" to the equivalent "$\between$-preserved and $\Subset$-preserved," but takes us no closer to something equivalent to "not ethically fragile in either sense", i.e., "not $\between$-fragile and not $\Subset$-fragile".

[77] Corollary 24 gave what they call the 'Ought'-formulation of Hume's Law, which applies the $\Subset$-based concepts of normative particularity and normative generality (= preservation and $\Subset$-fragility), which does not bear so directly on our theme, since we are taking a negated *Ought*-judgment to be just as much a potential ethical conclusion as an unnegated *Ought*-judgment. See note 19 and the text to which it is appended, above.

[78] Restall and Russell write "$\Supset$-fragile" in Definitions 19 and 20 on p. 254, and in Definition 23 on p. 255 (where also the "$\mathfrak{M}' \models A$" in the third line is a typo for "$\mathfrak{M}' \not\models A$"), which is understandable since it is fragility travelling upward to extensions, but by the letter of the generic definitions of $\mathcal{R}$-preservation and $\mathcal{R}$-fragility in Defs. 3 and 4 on p. 248, reproduced in our Definitions 3.1(*i*) and (*ii*), the correct formulation demands $\Subset$-fragility, and, where they write "$\Supset$-preservation", $\Subset$-preservation. The pre-hyphenated inter-model relation symbols in Restall and Russell's Definitions 8 and 9 (p. 250), 11 and 12 (p. 251) are all the wrong way round for the same reason. Fortunately, since we are concentrating on the symmetric $\between$, no such correction is required in the cases of present interest.

[79] The fact notwithstanding that, according to the interesting Lemma 26 of [92], all descriptive sentences are normatively particular – i.e., $\between$-preservation implies $\Subset$-preservation.

conclusions as consequences. For instance, $p \vee Op$ is a consequence of the descriptive p but is not itself descriptive since if we take $\mathfrak{M}$ with $\mathfrak{M} \not\models p$ although $\mathfrak{M} \models Op$, we have $\mathfrak{M}$ verifying the disjunction despite having a $\between$-related $\mathfrak{M}'$ which consists in adding the distinguished point to the set of ideal worlds of $\mathfrak{M}$, with the effect that $\mathfrak{M}' \not\models p \vee Op$. (Note that the 'translation' relation $\between$ does not change the distinguished point of these pointed models.)[80] Nor is $p \vee Op$ normative*: $\between$-fragility is out for the same reasons as in the alethic and quantificational cases mentioned in the preceding note: no adjustments to the set of ideal worlds (or the accessibility relation) will take $\mathfrak{M}$ to a $\mathfrak{M}'$ with $\mathfrak{M}' \not\models p \vee Op$, if the reason we have $\mathfrak{M}' \models p \vee Op$ is that $\mathfrak{M}' \models p$. Accordingly, as already stressed, what we get is not a Humean dichotomy, but a quasi-Humean trichotomy.

In view of such considerations, it is somewhat surprising to read Mares ([71], p. 283) saying in what purports to be a summary of the Restall–Russell account "A formula is *fragile* if and only if it is *not* preserved." Even if Restall and Russell had not explicitly disavowed any such claim (as they do: see note 69 above, for instance) – since, as one can see from Definitions 3.1, $\mathcal{R}$-preservation and $\mathcal{R}$-fragility are respectively $\forall\forall$ and $\forall\exists$ notions, it would only be under exceptional circumstances that they could end up being complementary. Probably what Mares was thinking of was not the properties of sentences or formulas of being $\mathcal{R}$-preserved or being $\mathcal{R}$-fragile, but the relations between formulas and models that results from removing the initial universal quantifier "$\forall\mathfrak{M}$" from the Definitions 3.1(i) and (ii) – or more precisely from the *definientia* involved (i.e., the parts after the "iff"); this would turn the definitions into (i) and (ii) here:

Definitions 3.5. *For any class of models $\mathcal{M}$ and any sentence A which can be interpreted in $\mathcal{M}$:*

(i) *A is $\mathcal{R}$-preserved from $\mathfrak{M} \in \mathcal{M}$ (over $\mathcal{M}$) iff*

$$\mathfrak{M} \models A \Rightarrow \forall \mathfrak{M}' \in \mathcal{M}(\mathfrak{M}\mathcal{R}\mathfrak{M}' \Rightarrow \mathfrak{M}' \models A).$$

(ii) *A is $\mathcal{R}$-fragile from $\mathfrak{M} \in \mathcal{M}$ (over $\mathcal{M}$) iff*

$$\mathfrak{M} \models A \Rightarrow \exists \mathfrak{M}' \in \mathcal{M}(\mathfrak{M}\mathcal{R}\mathfrak{M}' \;\&\; \mathfrak{M}' \not\models A).$$

Since the parts following the "$\mathfrak{M} \models A \Rightarrow$" in the defining conditions in (i) and (ii) here are equivalent to each other's negations, this would then give a two-block partition of the formulas true in $\mathfrak{M}$; such truths, that is, would then fall into

[80] We could have used instead the case of $p \vee Oq$ to illustrate this point, with a suitable choice of $V(q)$, but give the present example because of its novelty as compared with the universal and modally general examples from Restall and Russell: in those cases the point could not have been made with $Fa \vee \forall x(Fx)$ or $p \vee \Box p$, because these disjunctions are equivalent to their first disjuncts.

exactly one of the categories: $\mathcal{R}$-preserved from $\mathfrak{M}$, $\mathcal{R}$-fragile from $\mathfrak{M}$, and Mares's comment so reinterpreted would be correct. Further, since we are concentrating on the truths (in some pointed model, in the deontic application of this), we might well be in business for some kind of Shorter-inspired *soundness* version of Hume's Law. Before pondering the deontic/ethical case specifically, though, let us state the general (and easily proved) 'Shorterized' version of Restall and Russell; here $\mathcal{M}$ and $A_1, \ldots, A_n, B$ are related as are $\mathcal{M}$ and A in Definitions 3.5:

Proposition 3.6. *If $A_1, \ldots, A_n \models_{\mathcal{M}} B$, and for $\mathfrak{M} \in \mathcal{M}$ we have $\mathfrak{M} \models A_i$ (each $i = 1, \ldots, n$), then there is no $\mathcal{R} \subseteq \mathcal{M} \times \mathcal{M}$ for which all the A_i are $\mathcal{R}$-preserved from $\mathfrak{M}$ while B is $\mathcal{R}$-fragile from $\mathfrak{M}$.*

Now specializing the discussion back to the ethical case and taking $\mathcal{R}$ as Restall and Russell's $\between$, we note that in terms of the unpointed models $\langle W, X, V\rangle$ in play in Definition 2.1, A's being non-ethical at $w \in W$ in such a model amounts A's having the same truth-value at w in all of the models $\langle W, X', V\rangle$ varying the ethical standard X. Transferring this across to the framework of Restall and Russell, but with the de-universalized model-specific (or model relativized) notions of Definitions 3.5 in place, we get that being non-ethical in the pointed model, $\mathfrak{M} = \langle W, X, w, V\rangle$ amounts to A's having the same truth-value in all $\mathfrak{M}'$ which are $\between$-related to $\mathfrak{M}$. But this isn't quite what Definition 3.5(i) itself says being $\mathcal{R}$-preserved from $\mathfrak{M}$ consists in, when $\mathcal{R}$ is taken as $\between$. Rather, being $\between$-preserved from $\mathfrak{M}$ is a matter of being true in all $\mathfrak{M}'$ which are $\between$-related to $\mathfrak{M}$ if A is true in $\mathfrak{M}$, and this does not address the question of what happens if $\mathfrak{M} \not\models A$.

To arrive, as we shall after Definition 3.8(ii) below, at a de-universalized Restall–Russell formulation matching Karmo's, we need to back up for a moment with a few general remarks about the general process involved. Consider two first-order sentences:

$$\forall x \forall y(Sxy \to Syx) \qquad\qquad \forall x \forall y(Syx \to Sxy)$$

They are just two ways of saying that (the binary relation interpreting) S is symmetric. Removing from each of them the initial universal quantifier binding x gives two non-equivalent conditions for an individual (value of) x to satisfy, which we could denote by lambda expressions in an obvious way: $\lambda x \forall y(Sxy \to Syx)$ – standing for the property of being, as we might say, an S-reciprocatee, and $\lambda x \forall y(Syx \to Sxy)$ – for the property of being an S-reciprocater.[81] This is illustrates the fact that there

[81] Points in a Kripke frame with S as accessibility relation are called 1-symmetric and 2-symmetric respectively in [51], p. 188*ff.* with a similar – though three-way – distinction in the case of transitivity (p. 185*ff.*).

is no such thing as *the property predicated of everything* by a closed sentence of the form $\forall x(\varphi(x))$, if we want the property concerned not to depend on the syntactic shape of the sentence but to be shared by all logically equivalent sentences, in much the same way as there is no such thing as *the* property predicated of a by a sentence $\phi(a)$ (a being an individual constant), which was illustrated in [47] for the case of $\phi(a) = Fa$ (F a monadic predicate letter).

Intermission. Given that the example just given by 'de-universalizing' the claim that a binary relation was symmetric – though such a description must be understood to denote removing only the outermost universal quantifier, rather than all of them – one may wonder if a similar possibility arises with a universally quantified monadic predication. The answer is that it does:

Example 3.7. Take the sentence $\forall x(Fx)$, which says that everything satisfies the condition $\lambda x(Fx)$. Can we find a condition which is not equivalent to this which is such that the sentence that everything satisfies that other condition is equivalent to $\forall x(Fx)$? In classical first-order logic with identity certainly we can. On which comes to mind is the following:

$$\lambda x(\exists y(Fy) \wedge \forall z(z \neq x \rightarrow Fz)).$$

The reader is invited to check that putting $\forall$ in place of λ gives an equivalent of $\forall x(Fx)$, while predicating the two properties involved of a given individual (we again use the constant a) gives the non-equivalent Fa and $\exists y(Fy) \wedge \forall z(z \neq a \rightarrow Fz)$. $\lhd$

It would be interesting to have some idea of what the inverse image of a given universal formula is, in the sense of knowing what the set of open formulas $\phi(x)$ (as we may as well write in place of "$\lambda x(\varphi(x))$") looks like for a given closed universal formula $\forall x(\phi(x))$, all of them equivalent to that $\forall$-formula. A similar line of enquiry is opened up for the case of $\Box$-formulas in modal logic in [49], where of course the set of formulas whose necessitations are equivalent to a given $\Box$-formula will vary from one to another modal logic. ***End of Intermission.***

Here we concern ourselves with some specific cases of de-universalizing bearing on the Hume's Law theme, another such case being addressed in Postscript (ii) to this section. We continue to think of de-universalizing as a syntactic process of removing the main universal quantifier from an $\forall$-formula (binding with a 'λ', if desired, the variable thus freed[82]): applying this syntactic operation to all formulas equivalent to the that formula yields the members of its inverse $\forall$-image. So for a closer *rapprochement* with Karmo, we need go to back and replace the "preserves

[82]What if no occurrences of the quantified variable *are* thus freed? It is perhaps not immediately clear whether vacuous universal quantifiers should be excluded here.

$\mathcal{R}$" idea with something that alludes to both preserving and reflecting (as it is sometimes put) the property of being true in a model. We will use the word *copied* for this stronger property. The *definiens* in Definition 3.1(i) for $\mathcal{R}$-preservation can be re-expressed, after shifting a quantifier and 'permuting antecedents' so that it looks like this:

$$\forall \mathfrak{M}, \mathfrak{M}' \in \mathcal{M}(\mathfrak{M}\mathcal{R}\mathfrak{M}' \Rightarrow (\mathfrak{M} \models A \Rightarrow \mathfrak{M}' \models A)).$$

So all we have to do is to boost the last "$\Rightarrow$" to a "$\Leftrightarrow$" to get a Restall–Russell style condition (3.8(i) here) and then de-universalize again (3.8(ii)) to get the model-specific version:

Definitions 3.8. (i) *A is $\mathcal{R}$-copied over $\mathcal{M}$ iff:*

$$\forall \mathfrak{M}, \mathfrak{M}' \in \mathcal{M}(\mathfrak{M}\mathcal{R}\mathfrak{M}' \Rightarrow (\mathfrak{M} \models A \Leftrightarrow \mathfrak{M}' \models A));$$

(ii) *A is $\mathcal{R}$-copied from $\mathfrak{M}$ (over $\mathcal{M}$) iff:*

$$\forall \mathfrak{M}' \in \mathcal{M}(\mathfrak{M}\mathcal{R}\mathfrak{M}' \Rightarrow (\mathfrak{M} \models A \Leftrightarrow \mathfrak{M}' \models A)).$$

In general, being $\mathcal{R}$-copied is a very different property of formulas from being $\mathcal{R}$-preserved, so there may be a feeling that we are relying only on a loose analogy in connecting Restall and Russell's approach to Karmo's, but note that for symmetric $\mathcal{R}$, being $\mathcal{R}$-copied and being $\mathcal{R}$-preserved completely coincide, and $\between$ is a symmetric relation (indeed, an equivalence relation). So if Restall and Russell had chosen simply to address Hume's Law in [92] and to do so in the pure $\between$-based setting, they could equally well have done so by defining the descriptive sentences to be those $\between$-copied over the relevant $\mathcal{M}$ as they do by defining them to be those sentences which are $\between$-preserved over class $\mathcal{M}$: these are just two characterizations of the same set of sentences.[83] As with the cases touched on above, de-universalizing

[83] It is therefore surprising to read Russell [95] in Remark 1 on p. 157 contrasting her approach there with the earlier Barrier Theorem work: "Instead of looking at whether the *truth* of a sentence is always preserved over changes, the definitions of constant and indexical sentences look at whether *truth-value* is preserved over changes." But this is no contrast at all when the changes are all reversible, as changes to $\mathcal{R}$-related structures for symmetric $\mathcal{R}$ are – the structures here being models paired with contexts and $\mathcal{R}$ relates any two agreeing on the model component of the pair: a symmetric relation. Indeed on p. 159 Russell writes "On our new approach to the indexical barrier theorem, the relation remains symmetric," discussing the bearing of this on another aspect of her treatment: whether the barrier operates in the reverse direction also – not quite the same issue. The treatment in [95] is indeed a new departure, since while the constant sentences are those preserved by the $\mathcal{R}$ just mentioned, the indexicals comprise simply just the complementary class, rather than being given the fragility treatment (and there is consequently another wrinkle in the treatment – mentioned at the end of note 71).

gives non-equivalent results and in particular de-universalizing in the $\between$-copied case gives Definition 3.8(ii), yielding Karmo-style non-ethicality in $\mathfrak{M}$ (or: at w in the 'unpointed' reduct of the pointed model $\mathfrak{M}$, where w is the distinguished point of $\mathfrak{M}$). Since Karmo's discussion has a very clear conception of an intended model (with the actual world as distinguished point and the correct ethical standard as the ethical standard in place), de-universalizing the general notion to focus on relativity to this intended model, $\mathfrak{M}^*$, say is close to irresistible: ethicality at the actual world is $\between$-fragility from $\mathfrak{M}^*$. With these moves, then, we remove the appearance of a discontinuity between Karmo's treatment and the de-universalized model-relative version of the Restall–Russell account.

Corresponding to what was described after Definition 2.1 as a more direct adaptation of one of Karmo's formulations – though negating it, since it is now *non*-ethicality that is at issue, instead of defining this model-relative notion of descriptiveness or non-ethicality by saying that A has this property relative to $\mathfrak{M}$ just in case:

$$\forall \mathfrak{M}' \in \mathcal{M}(\mathfrak{M} \between \mathfrak{M}' \Rightarrow (\mathfrak{M} \models A \Leftrightarrow \mathfrak{M}' \models A)),$$

we can equivalently put this as follows, for the reasons given in the discussion after Definition 2.1:

$$\forall \mathfrak{M}', \mathfrak{M}'' \in \mathcal{M}((\mathfrak{M} \between \mathfrak{M}' \,\&\, \mathfrak{M} \between \mathfrak{M}'') \Rightarrow (\mathfrak{M}' \models A \Leftrightarrow \mathfrak{M}'' \models A)).$$

Aside from considering such de-universalized versions of the Restall and Russell concepts to make contact with Karmo's approach to Hume's Law, it is worth spending a moment on their role in [92] without reference to Karmo. In the first place, Restall and Russell in fact help themselves occasionally to these model-relative notions without explicit acknowledgement, for the sake of heuristic remarks. On p. 248, the authors are considering the inter-model relation $\mathcal{R}$ (as we shall write it, though they write simply 'R') as the substructure relation, writing:

> Take the example of $Fa \vee \forall x(Gx)$. This is sometimes $\mathcal{R}$-preserved (if you have a model in which Fa is satisfied, $Fa \vee \forall x(Gx)$ is satisfied in any extension of it). However, it is sometimes not (take a model in which Fa is false, but $\forall x(Fx)$ is true – extend it to a model in which G is false for some objects).[84]

[84] I have changed the notation to match that in use here turn the authors' "R" becoming "$\mathcal{R}$" and their "$(\forall x)Fx$" becoming "$\forall x(Fx)$". Before the passage quoted here, Restall and Russell describe the inter-model relation involved as the relation of model extension, rather than substructure. This is the mistake mentioned in note 78 surfacing again.

Of course, there isn't literally such a thing as being "sometimes $\mathcal{R}$-preserved"; the more careful way of saying this is that Fa is true in a model, then $Fa \vee \forall x(Gx)$ is substructure-preserved from that model, whereas if Fa is false, it is not. (Note the similarity to the deontic 'mixed disjunctions' of Prior's argument.) Similarly, Wolf ([117], p. 119) says at the start of his summary of what he calls the *fragility approach* of Restall and Russell that "it designates a sentence as normative if just in case there is at least one modal where replacements and additions to the set of satisfactory worlds changes its truth-value", so here was have lost the $\forall$ from the authors' official $\forall\exists$ definition and are working with *fragility from* a given model – essentially, in other words, with Karmo's ethicality at a world in a model (not fussing here too much about the "replacements and additions" formulation and taking it to amount to "changes").

Another issue with which the more refined concepts introduced in Definitions 3.5 (or the similarly model-relative variant in definition 3.8(*ii*)) promise assistance is in dealing with an objection to [92] from Vranas [115], p. 263. Vranas puts his objection in terms of Restall–Russell normativity rather than what was called normativity* in Definition 3.3(*ii*), but here, to avoid complications, we present it in the latter (purely $\between$-involving) concept:

Example 3.9. Suppose we have an alethic modal operator present $\Box$ interpreted in the deontic models under consideration by Restall and Russell, though (as Vranas acknowledges) not present in the object language they use such models to interpret, and we interpret it by universal quantification over all the model elements (not just the ideal points, as with O). Then, contrary to the application of the Barrier Construction Theorem – Proposition 3.2 in our development – the valid inference from $\Box p$ to Op takes us from the descriptive to the normative*, i.e. from the $\between$-preserved to the $\between$-fragile. (Vranas is concerned with the passage from the descriptive to the normative – no asterisk – the latter concept involving also $\Subset$-fragility, and has a diagnosis of what goes wrong involving this aspect of the case, but let's stick with the simple purely $\between$-based version.) $\Box p$ is certainly $\between$-preserved: shifting around the set of ideal worlds does nothing to change the universe (W) of the models so if a model verifies $\Box p$ before the shift (or the translation, to use Restall and Russell's favoured geometric metaphor), the same will be so after the shift. But Op is not $\between$-fragile, as we saw – and observed that Restall and Russell had already seen – in note 42 (where the example was actually Oq). That much can be said in terms of the concepts officially available in [92], where the issue is raised on p. 254 with the words "Oddly enough, important normatively general sentences such as Op are not $\between$-fragile," the explanation being as given in note 42 above, which does not entirely deal with the "oddly enough" aspect of the situation. This issue is touched on in Schurz's comments on Restall and Russell (and Vranas) [102], p. 271, with the observation that if you want the implication from $\Box p$ to Op to be respected by your logic, you need to restrict the class of models $\mathcal{M}$ for which you are taking the consequence relation $\models_{\mathcal{M}}$ as your logic, you have insist that the $\Box$-pertinent alternatives include all the O-pertinent alternatives (in the simplified case: that $W \supseteq X$) and you lose $\between$-fragility, whereas if you want $\between$-fragility you need to exclude models meeting this condition and then your consequence relation will not deliver Op as a consequence of $\Box p$. One can make a somewhat finer-grained response, though, with the model-relative notions to hand: suppose $\mathfrak{M} \models \Box p$; then we know not just that Op is not $\between$-fragile – a general claim – but that, though this is not a $\between$-preserved formula, it is $\between$-*preserved from* $\mathfrak{M}$. $\lhd$

The implication from $\Box p$ to Op, or more generally from $\Box A$ to OA under discussion in Example 3.9 has been the subject of strong hostility – with objections to the provability of such things as $O(p \vee \neg p)$ in even monomodal deontic logic (i.e. without an additional primitive $\Box$). Pertinent quotations from Jonathan Harrison and Chares Pigden, as well as pointers to suggested remedies, can be found in Remark 4.4.9 in Humberstone [51]. The implication is often called Must-implies-Ought by analogy with Ought-implies-Can, but this is potentially confusing because there is also the deontic 'must' to contend with[85] – which is what is meant in the title of Vranas [116], as well as that of Jones and Pörn [60] – one of the places just alluded to as offering a remedy, in fact, for the deontic operator (written as 'Ought') defined at the top of their p. 92.

This completes our guided tour through the recent post-Prior literature. In Section 2 we found aspects of Karmo [61], developing a Shorter-style response to the difficulties Prior raised for Hume's Law by working with soundness and a world-relative dichotomous taxonomy, re-surfacing in Singer [108], though we also sampled criticisms of Karmo and of Singer by Maitzen and by Woods and Maguire, respectively, and briefly touched on Fine [19]'s distinctive hyperintensional approach to the issues. (Some indication of how Fine approaches Hume's Law itself is given at the end of Postscript (i) to this section.) In this section we have seen that treatment of Hume's Law by Restall and Russell as a special case of their general account of barrier theses in various areas. While, again, the reception has not been uniformly favourable, we have concentrated less on the criticisms than on the connections which arise with Karmo's approach in particular, once their key concepts (of $\between$-preservation and $\between$-fragility) are simplified in a certain way – de-universalized, as we put it; further connections with work of David Lewis come up in Postscript (ii) below. Of course, the views of numerous others – and not even just those named in

[85]To say nothing of the epistemic 'must': "It must have rained in the night."

the opening paragraph of Section 1 have also been brought into the mix, but that will do by way of a concluding paragraph.

Section Postscript (i): Woods and Maguire on Restall and Russell We pick up the discussion in Section 3 [118] of Restall and Russell from note 76. The second paragraph of [118]'s §3.1 includes a proof of what looks vaguely like the main result in [92], their Barrier Construction Theorem (Theorem 5 in their paper, a formulation of which appeared as Proposition 3.2 here), though on closer inspection turns out not to be.

Recall that according to that result for any class of models $\mathcal{M}$, if B is a semantic consequence of $A_1, \ldots, A_n$ over $\mathcal{M}$, then for no $\mathcal{R} \subseteq \mathcal{M} \times \mathcal{M}$ can it be that all the A_i are $\mathcal{R}$-preserved while B is $\mathcal{R}$-fragile. The definition of $\mathcal{R}$-fragility given by Woods and Maguire in the second paragraph of 3.1 of [118] correctly captures the notion in play in Restall and Russell's discussion, but they do introduce the concept of being $\mathcal{R}$-preserved here, instead defining a sentence to be $\mathcal{R}$-*stable* iff it is not $\mathcal{R}$-fragile. This is a clue that we are not going to be shown Restall and Russell's main result, or a simplified version (with the same range of application), in case that is what Woods and Maguire hoped to do, in avoiding the concept of $\mathcal{R}$-preservation. What they claim to be proving is the following on p. 431 of Woods and Maguire [118]:

> "An $\mathcal{R}$-stable sentence does not imply any sentence that is $\mathcal{R}$-fragile."

Compare the Restall and Russell version, re-worded into talk of implication: "A satisfiable set of $\mathcal{R}$-preserved sentences does not imply any sentence that is $\mathcal{R}$-fragile."

Concerning their own claim, Woods and Maguire say "The proof is easy." There is indeed a simple proof, but Woods and Maguire's proof is not easy to follow at all; comments indicating why are included here in doubled brackets; the authors' use of φ, ψ as schematic letters is followed to facilitate checking that the source text has been accurately reproduced here (except for the R which here appears – as above – as $\mathcal{R}$):

> Let φ be $\mathcal{R}$-stable and ψ be $\mathcal{R}$-fragile. $\mathcal{R}$-stable sentences are consistent by definition. [[That step is correct, since being stable means that there is a model verifying the sentence – so the sentence is consistent – and every model it bears the relation $\mathcal{R}$ to also verifies the sentence.]] If φ and ψ are not jointly inconsistent [[that should be "not jointly *consistent*"]], then any model of φ witnesses the failure of the implication of ψ from φ. If they're jointly consistent, we have a model $\mathfrak{M}$ of both φ and ψ. Since ψ is $\mathcal{R}$-fragile, we can extend [[here meaning: pass to some $\mathcal{R}$-related model]] the model to some $\mathfrak{M}^*$ where ψ is

false. Since $\mathcal{R}$-stable sentences true in $\mathfrak{M}$ are true in $\mathfrak{M}^*$, φ is true in $\mathfrak{M}^*$ and we have our counterexample. [[If φ had been assumed to be $\mathcal{R}$-preserved, we could argue that way – "$\mathcal{R}$-stable sentences true in $\mathfrak{M}$ are true in $\mathfrak{M}^*$" would follow, but not with the mere assumption of $\mathcal{R}$-stability. All the latter means is that there is some model, $\mathfrak{M}_0$, say, verifying φ with every model $\mathcal{R}$-related to $\mathfrak{M}_0$, also verifying φ. But who says that the $\mathfrak{M}$ introduced in the course of the proof to be some model verifying both φ and ψ (assumed consistent) is such an $\mathfrak{M}_0$, all models $\mathcal{R}$-related to which continue to verify ϕ?]]

In short, this would-be proof of a result which isn't Restall and Russell's anyway, is not a great success, though the result in question is not in doubt. To see that, for the record, let us pick up the proof from the correct initial step, inferring from the $\mathcal{R}$ stability of φ – an $\exists\forall$ property, since $\mathcal{R}$-fragility is an $\forall\exists$-property – that there is a model $\mathfrak{M}$ such that

$$(1)\ \mathfrak{M} \models \varphi \qquad \text{and} \quad (2) \quad \text{for all } \mathfrak{M}' \text{ such that } \mathfrak{M}\mathcal{R}\mathfrak{M}',\ \mathfrak{M}' \models \varphi.$$

By (1) and the assumption that $\varphi \models_{\mathcal{M}} \psi$ (for some unspecified $\mathcal{M}$ containing all models under consideration in this proof), we conclude that $\mathfrak{M} \models \psi$. Now, ψ is supposed to be $\mathcal{R}$-fragile (over $\mathcal{M}$), so there is some $\mathfrak{M}^*$ $\mathcal{R}$-related to $\mathfrak{M}$ for which $\mathfrak{M}^* \not\models \psi$. In that case, since $\varphi \models_{\mathcal{M}} \psi$, we have $\mathfrak{M}^* \not\models \varphi$. But, given (1), this contradicts (2), and this contradiction shows that we could not have φ implying ψ with φ $\mathcal{R}$-stable and ψ $\mathcal{R}$-fragile after all.

To see us now see how this result differs from Restall and Russell's, recall that the latters' Barrier Construction Theorem – our formulation of which appeared as Proposition 3.2 – addresses the consequences of sets of sentences rather than of individual sentences, so Woods and Maguire were hoping for a simplified version of that result which did not use preservation, in their way of setting things out, what Woods and Maguire should have gone for a proof of was this (taking some $\mathcal{M}$ for granted in the background, with implication understood as $\models_{\mathcal{M}}$):

"A set of $\mathcal{R}$-stable sentences does not imply any sentence that is $\mathcal{R}$-fragile,"

or perhaps this with the additional qualifier 'satisfiable' (or 'consistent') on the set of $\mathcal{R}$-stable sentences. But we can easily give a 'disjunctive syllogism' counterexample to this, remembering that $\mathcal{R}$-stable simply means *not* $\mathcal{R}$-fragile; of course for a concrete counterexample, it will help to supply a definite choice of $\mathcal{R}$, so let this be the substructure relation. For this choice of $\mathcal{R}$, $\mathcal{R}$-fragility corresponds to Restall-Russell universality – any model verifying a sentence with this property can be extended to a model not verifying it. As we recall from Restall and Russell's discussion $Fa \vee \forall x(Gx)$ is not fragile with respect to this relation (and not preserved by it either, as they also remarked), since a model verifying the first disjunct cannot be extended to one which does not verify that disjunct however many new object you add (and keep outside of the extension of G). Thus $Fa \vee \forall x(Gx)$ is $\mathcal{R}$-stable, as is $\neg Fa$; this pair of sentence is consistent/satisfiable. But together they imply $\forall x(Gx)$, which is $\mathcal{R}$-fragile, contrary to the would-be theorem. (Thus by the correctness of the Woods–Maguire result, not for arbitrary n, but for the $n = 1$ case of "For any $\mathcal{M}$ and $\mathcal{R} \subseteq \mathcal{M} \times \mathcal{M}$, if $\varphi_1, \ldots, \varphi_n \models_{\mathcal{M}} \psi$, then we cannot have all the φ_i $\mathcal{R}$-stable and ψ $\mathcal{R}$-fragile," one sees that the conjunction of two $\mathcal{R}$-stable formulas is not in general $\mathcal{R}$-stable: as a counterexample take the conjunction of the two formulas just in play: $Fa \vee \forall x(Gx)$ and $\neg Fa$.)

Indeed we knew *a priori* – which, after all, originally meant "according to Prior" – that we could not have a class of sentences Σ to which some sentence and its negation both belong and such that whenever Σ_0 is a consistent subset of Σ with B as a consequence, $B \in \Sigma$, without Σ being the class of all sentences. (At least we have this subject to very weak assumptions about the existence of independent sentences, as detailed in Proposition 1.1.) This is the same reason that Russell [95] gets only the result mentioned at the end of note 71 and not an unconditional barrier theorem in the style of Restall and Russell [92]. Woods and Maguire go for a dichotomous classification by starting with a fragility notion for the 'conclusion class' and taking its complement, stability, for the premise class, respectively, as remarked, $\forall\exists$ and $\exists\forall$ notions, whereas Russell's 'premise class' comprises the sentences that are preserved by a context-shift relation ('constant sentences': an $\forall\forall$ notion) and takes its complement ('indexical sentences': an $\exists\exists$ notion) as the conclusion class – again a two-block partition and so by Prior's observation, no straight barrier thesis to be had.

We return to one aspect of Woods and Maguire's formal discussion in the following paragraph, here noting that Woods and Maguire, although [118] does not quite convey them accurately, do not contest Restall and Russell's technical results, and worrying mainly, as in the case of Singer touched on in Section 2, that the fragility notions in play – our discussion having concentrated on translation ("$\between$") fragility to the exclusion of what [92] calls normative extension – cannot be capturing any intuitive idea of ethicality or normativity. The interested reader is invited to look at the first two paragraphs of §3.3 of [118] to see the examples intended to illustrate this charge. The authors then turn to the $\Box p \rightarrow Op$ issue which exercised Vranas and Schurz, as cited in Example 3.9. Here again the interested (and preferably patient) reader is referred to their take on what the example shows, since the discussion aims to reveal inappropriate verdicts of descriptiveness delivered by the apparatus of [92], but uses the mischaracterization mentioned in note 76 of what descriptive sentence are according to Restall and Russell (which is not unconnected with the idea, above, of trying to run the basic Restall–Russell proof using stability, i.e., failure of fragility, in place of preservation).

On the subject of stability, it is instructive to pause over the fact that the conjunction of two $\mathcal{R}$-stable sentences need not be $\mathcal{R}$-stable, and that Woods and Maguire's variation on Restall and Russell does not deliver the general multi-premise version of the latters' barrier thesis (or Barrier Construction Theorem: Proposition 3.2 above). Brown [8] rightly makes the 'trichotomy' point: that this does not vindicate the similarly general version of Hume's Law – [92]'s Corollaries 24 and 25 – because "Prior is concerned with arguments from the nonmoral to the moral (. . .) where these are assumed to be exhaustive categories" (p. 3). But he also makes the useful observation that there is nothing like Prior's argument which would make corresponding difficulty for a restricted version of Hume's Law – in a genuinely dichotomous form – where the restriction is to single-premise arguments. Note that instead of saying that we have a dichotomous 'validity' (as opposed to 'soundness') version of Hume's Law applying to single-premise arguments, we can put this by saying in the terminology used in the Postscripts to Sections 1 and 2 that we can extend a basic ethical/nonethical division so that it becomes exhaustive and still satisfies the condition that the class of ethical statements are closed under converse entailment – the converse of the binary relation of entailment between statements. For our purposes, we can state Brown's observation as a comment on the earlier distillation of Prior's argument in the following way:

Proposition 3.10. *Proposition 1.1 becomes false if, keeping the conditions* (1) *and* (2) *there as they are but restricting* (3) *to the case of* $n = 1$.

Proof. We must show that we can find E, F, and $\mathbb{F}$ such that (condition (1)) $F, \neg F \in \mathbb{F}$, and (condition (2)) F is (classically) independent of another sentence E, and also (condition (3^-), say): for any CL-consistent $A \in \mathbb{F}$, if $A \vdash_{\mathsf{CL}} B$ then $B \in \mathbb{F}$, and yet, by contrast with Prop. 1.1, we have $E \notin \mathbb{F}$. No problem: just let E, F be distinct sentence letters (p, q, say) and define $\mathbb{F} = \{A \mid A \nvdash_{\mathsf{CL}} E\}$ – in other words the element of $\mathbb{F}$ are just those formulas that do not by themselves classically imply q. Conditions (1) and (2) are evidently satisfied by the choice of E, F. Checking condition (3^-), suppose for a contradiction that (i) $A \in \mathbb{F}$, (ii) if $A \vdash_{\mathsf{CL}} B$ but (iii) $B \notin \mathbb{F}$. (i) means that $A \nvdash_{\mathsf{CL}} E$, and ($iii$) means that $B \vdash_{\mathsf{CL}} E$: but these together clearly contradict (ii). Finally, since $E \vdash_{\mathsf{CL}} E$, we do have $E \notin \mathbb{F}$, as desired. □

The proof given here is what might be called a 'proof of concept' demonstration that no amount of piling up of one-premise inferences can achieve the same counter-Humean effect as Prior's argument with the one-premise rule of $\vee$-introduction and the two-premise disjunctive syllogism rule. If we wanted a more realistic way of setting up our class $\mathbb{F}$ of ('factual' or) non-ethical sentences we would collect all of its intended non-members rather than just one of them, and take $\mathbb{E}$ (let's call it) to be the set of all basic ethical sentences – the recalling the technical project as described in Section 1 of carving up the terrain of the neither-basic-ethical-nor-basic-nonethical – and setting $\mathbb{F}$ to be

$$\{A \mid \textit{ for all } E \in \mathbb{E} : A \nvdash_{\mathsf{CL}} E\}.$$

The above proof of Proposition 3.10 easily adapts to this more realistic setting. (Very little specific to classical logic was used here – basically just the notions of consistency and independence and the relation between them.)

A question is raised by the fact that a dichotomous version of Hume's Law not requiring us to trade in validity for soundness, or to make special exception concern vacuous occurrences of expressions, is available when restricted to one-premise inferences even though it is not available when to such inferences we add those licensed by disjunctive syllogism are permitted. Since classically, disjunctive syllogism is essentially, give or take a double negation equivalence, Modus Ponens for the material conditional, the question arises as to how what has just been said can survive the observation that Modus Ponens (and in fact more than one-premise rules generally) can be replaced by one-premise rules in an axiomatic presentation of a good many logics, classical logic included.[86] Readers for whom this question is of interest will be able to extract an answer from either of the papers cited in the footnote just flagged, in the case of the first reference by attending to the passage indicated by the ellipsis in the above quotation, and in the second by looking at the discussion of a number of different rules going under the name 'Modus Ponens'.

Returning more directly to Brown's discussion, recall that in Section 2, we raised an eyebrow at the World-Norm Gap thesis from Singer [108], because according to that thesis if we have $P_1, \ldots, P_n \vdash C$ for a suitable (and indeed, we may suppose, classical) consequence relation $\vdash$) where each P_i is norm-invariant and their conjunction is satisfiable/consistent, then $P_1 \wedge P_2 \wedge \ldots \wedge P_n \wedge C$ is norm-invariant. The issue was that we wanted to talk about the conclusion of the original argument C, rather than this new conjunction with all the premises as further conjuncts. However, it is less likely that one would have had this 'someone changed the subject' reaction if we had passed from the original n-premise argument to making a comment on the conjunctive 1-premise argument with the same conclusion C but the new premise

[86] Here is how Herrmann and Rautenberg [37] put this at their p. 334: "As a by-product, we obtain also the remarkable fact that the set T_2 of 2-valued tautologies $(\ldots)$ is axiomatizable by finitely many axioms and unary rules." A simpler proof of this result can be found in Humberstone [48]; as John Halleck later reminded the author, Porte [86] had long ago exhibited such an axiomatization (though admittedly one with many more axioms).

$P_1 \wedge \ldots \wedge P_n$. After all, certainly in a classical setting, whether to say the premises together entail a conclusion or instead the conjunction of the premises entails the conclusion – that's not something one would normally lose a lot of sleep over. But as Brown [8] points out, this gives rise to two readily distinguishable things to mean by Hume's Law, not because the multi-premise and single conjunctive premise arguments differ as to validity, but because whatever criterion of non-ethicality we are using has to be applied in the one case to each of the several premises and in the other to the single conjunctive premise, and thus the arguments may differ in respect of whether they conform to or violate Hume's Law.

By way of explanation as to why it might be plausible to hold, as an account along the above lines must, that two non-ethical statements can have an ethical conjunction, Brown writes ([8], p. 4):

> To illustrate, consider the property of being offensive. The sentence "You are either a genius or an idiot" is not offensive. Nor is the sentence "You are no genius"; it is compatible with your being of quite respectable intelligence. But the conjunction, "You are either a genius or an idiot, and you are no genius," is offensive. The reason is that the conjunction says something extra, over and above what is said by either conjunct, namely, that you are an idiot. The offensiveness results only from the two conjuncts combining together; it is not present in either on its own.

Brown goes on to point out that a conjunction, one conjunct of which is offensive, is itself offensive, whatever the other conjunct may be. In that respect as well as in those evident in the above passage, offensiveness is like ethicality on a commitment based view – the kind of view discussed especially in the Postscript to Section 2. We should note, though that there are several ways of giving offence, and apart from being offensive by being insulting, as in the above passage, there is the use of language found offensive by an addressee – for example by swearing, of this or that kind. A disjunction in which one disjunct is offensive in any such way is itself offensive, and one can imagine someone thinking of this as a the more appropriate parallel. The ethicality of a component would infect any compound containing it.

Indeed we do not have to imagine such a position, we can read about it in print: in their very different ways Beall [4] and Fine [19] make suggestions of this kind. In Beall's case the idea, perhaps proposed somewhat facetiously, is that one use the three-valued truth-tables associated with Dmitri Bochvar and known also under the rubric 'Weak Kleene', in which classically behaving truth and falsity is joined by a third value that infects any compound once a component has that value – and that third value will serve as a marker for ethicality. The new value is undesignated, though, which may suit the moral nihilist but is out of place in a more neutral response to Prior's criticism of Hume's Law. For this reason, Beall also considers another option, at least for ethicality with a deontic source: Kripke style models with world-relative truth in the Bochvar three-valued scheme OA being true at a world when all accessible worlds have A true, OA false[87] if all accessible worlds have A false, and OA taking the infectious third value in all other cases – these other cases now including cases in which A takes one or other of the 'classical' values at all accessible worlds, but not uniformly so.[88] It is not clear why we should be forced to give up the obligatory/permissible distinction, though.

Another option might have been to take the infectious value as designated (*à la* [11]), but this is just as inappropriately unselective as the first option, now looking favourably rather than unfavourably on all moral judgments at once. If anything finitely many-valued might be appropriate in this area, perhaps it a variation on the direct product of the of the two-valued Boolean matrix, whose elements we may call T and F with the two-valued Bochvar matrix, whose elements we may call e and $\bar{e}$ for *ethical* and *non-ethical*, the former being the infectious element. Thus the values $\langle T, e\rangle$ and $\langle T, \bar{e}\rangle$ for the ethical and non-ethical truths, resp., and $\langle F, e\rangle$ and $\langle F, \bar{e}\rangle$ for the ethical and non-ethical falsehoods, the designated values being the former pair (which is why we do not here have a traditional product matrix, which would require for designating that the first and second entries in a designated pair be designated in their respect factor matrices). All the second coordinate is doing here is keeping track of infectious ethicality; which is not to say that this, or the previous suggestion, would suit Beall's purposes, since they do not result in invalidating Prior's $\vee$-introduction inference. The 'infectious' theme we saw also with Fine's proposal in [19], described above (note 61 and the text to which it is appended) in the terminology *dominant* as opposed to *recessive*. Again there is no intention to invalidate $\vee$-introduction: Fine's way with Hume's Law (which he considers only in connection with one-premise inferences, or entailments, and, it will be recalled, at the level of – albeit structured – propositions rather than of the sentences that express them) is that although the dominance of the normative in the construction of propositions gives us cases of a descriptive proposition entailing a normative proposition, in such cases the former entails suitably de-normativized core of the latter. Successive formulations (and Fine progresses through five of these) of the resulting Humean principle further tweak the way the de-normativized core is characterized. A representative intermediate case, the third approximation to the final proposal (the latter involving too many concepts to explain here) is given as:

[87]More precisely: OA having a true negation, since any undesignated value is essentially a species of falsity – as Dummett, Suszko, and Scott have variously observed.

[88]Observe that this recipe for assigning values to OA is only consistent if every world as at least one world accessible to it – an assumption of standard deontic logic.

(***) No descriptive proposition P entails a normative proposition Q unless P entails $(Q)^{\mathrm{D}}$ or Q is necessary.

Here $(Q)^{\mathrm{D}}$ is the current incarnation of the de-normativized core of Q and is defined to be $Q \cap D$ where D is the set of descriptive states (see note 61 and adjacent text above).[89] (***) is reminiscent at the propositional level of what at the sentential level would be a kind of interpolation theorem, specifically one promising "left uniform interpolants" (because $(Q)^{\mathrm{D}}$ is chosen independently of P, the latter being in the 'left-hand' – or *premise* or *antecedent* position: for a careful definition and relevant references, see the opening paragraph of van Gool et al. [32]). The analogy is only approximate, since $(Q)^{\mathrm{D}}$ may contain descriptive material absent from P. Whether the policy of 'normative infection' is pursed sententially or propositionally in the case of the familiar basic sentence connectives (or the corresponding propositional constructions), this will surely have to stop somewhere if deactivating – or 'protective,' as it is put in note 31 – contexts are on the linguistic menu, on pain of conflating the two notions of the ethical distinguished in that note: ethicality as potentially expressive of an ethical stance vs. ethicality as involving the deployment of ethical concepts.

Section Postscript (ii): De-universalizing Aboutness Two examples of the syntactic process we called de-universalzing toward the end of this section are mentioned in Humberstone [45], the first only a suggestive analogy to introduce the second, and both of them associated with the work of David Lewis. For the first, consider an initial characterization by Lewis, with which he was not completely satisfied (because of its uninformative potential circularity rather than its incorrectness), of *intrinsic* properties as properties w.r.t. which any two (qualitative) duplicates agree – either both or neither having the property in question.[90] The "any two" here marks an $\forall\forall$ prefix, and removing the first $\forall$ gives for any property P a property of having P intrinsically: x has this new property just in case for all y, if x and y are duplicates, y has the P. Since x is in the relevant sense a duplicate of itself, having the property P intrinsically does imply having the property P, but it does not imply that P is itself an intrinsic property, since there may be other pairs of individuals which are duplicates but do not agree w.r.t. P. Thus supposing that *being circular* is an intrinsic property but *being within a metre of something circular*

[89]Fine calls $(Q)^{\mathrm{D}}$ the *disjunct descriptive content* of Q, and for a subsequent honing of the principle we are introduced to the *conjunct descriptive content* $(Q)_{\mathrm{D}}$ of Q, which throws out the normative components from all the consistent truthmakers for Q.

[90]The details of Lewis's various attempts at throwing light on this topic, together with all the relevant references, can be found in Marshall and Weatherson [72].

is not, a circular ring still has the latter property intrinsically since any duplicate of it will agree with it w.r.t. the property of being within a metre of something circular. On the other hand, an iron nail sitting next to such a ring does not have the property *being within a metre of something circular* intrinsically.

That example of de-universalizing serves as a warm-up exercise for the case of a statement's being entirely about a subject matter, as this is conceived in Lewis [64]. A subject matter here is thought of as a partition of the set of worlds: those that are alike in respect of that subject matter. If M is a subject matter then we denote the corresponding equivalence relation by $\equiv_M$.[91] A statement S is *entirely about* a subject matter M just in case for any worlds w, w', if $w \equiv_M w'$ then S is true at w iff S is true at w'. De-universalizing, we get the following property of a world w: being such that for all w', if $w \equiv_M w'$ then S is true at w iff S is true at w'. In other words, at w the truth-value of S is settled by the subject matter M. Just as a property possessed intrinsically by an object need not be an intrinsic property, so such an M-settled statement need not a statement entirely about M. One of the subject matters mentioned by Lewis in [64] is that of the seventeenth century and another, that of the eighteenth century, with associated equivalence relations of exact match of worlds over the respective time periods. The statement:

(†) There were dinosaurs in Europe in the seventeenth century but they were all extinct by the end of the eighteenth century.

is not entirely about the seventeenth century, since worlds could be relevantly equivalent in respect of their seventeenth centuries but differ in respect of whether that statements was true in them: in one, dinosaurs are extinct by 1710, and in the other, not until 2010, say. But in the actual world the truth-value of (†) is settled by the 17th century subject matter, since equivalence w.r.t. that subject matter suffices for the falsity of the conjunction in any 17th-century matching world. In this case the statement is settled as false by the subject matter, or M^--settled in the actual world, as it is put in [45], where M is the subject matter in question, as opposed to M^+-settled, or M-settled as true in w.

We can think of taking the property of being M^+-settled into the object language as a modal operator, $\Box$ with accessibility relation M, which we may write as $\Box_M$. Thus $\Box_M A$ is true at w when A is M^+-settled in w. Indeed, such an operator was suggested in Yablo [119], pp. 32–34, though the focus there is rather more strongly on the dual operator $\Diamond_M$, with Yablo's preferred reading of $\Diamond_M A$ being something

[91] This will suit our purposes here, though as one of several refinements of Lewis's account, Yablo ([119], p. 36) suggests we don't actually want partitions and equivalence relations here, since transitivity will fail for such subject-matters as *approximately how many stars there are*, even when the vagueness of "approximately" is removed (e.g., being replaced by "to within 100").

along the lines of A's being true about M at the world in question, acknowledging that this may not be of much interest if A says nothing about M – so perhaps a safer reading would be in terms of A's not being false about M in w, i.e., A's not saying anything false about M at w. (Yablo actually writes "m" rather than "M".)

If, as in Humberstone [45], one wants to use this kind of machinery to discuss statements with a Gibbard–Karmo–Singer semantics in mind, then since for the truth-evaluation of a sentence, one needs not only a world a but also an ethical standard (to use Karmo's term), the subject-matters should be partitions of the set of 'Gibbard-worlds': Singer's $\langle w, n\rangle$ pairs. And here two especially salient subject-matters called in [45] M_{nat} and M_{eth} force themselves on one's attention (the subscripts suggesting 'natural(istic)' and 'ethical' respectively – though in [45] '*eth*' appeared as '*eval*'). The associated equivalence relations $\equiv_{nat}$ and $\equiv_{eth}$ relates any $\langle w, n\rangle$ to $\langle w', n'\rangle$ if and only if, for the former $w = w'$, and, for the latter, when $n = n'$. As a fair approximation, the basic ethical and basic non-ethical statements can be taken as those entirely about M_{eth} and those entirely about M_{nat}, respectively (though since anything true at every or false at every $\langle w, n\rangle$ pair will then count as both, contrary to our expectation to have these classes of statements disjoint). And, following Lewis's lead in [64], we note that whereas the statements entirely about M_{eth} are closed under negation as are those entirely about M_{nat}, and indeed those entirely about any given subject matter, and not just closed under negation but under all Boolean operations, though not under entailment or under converse entailment. If we want to get classes of statements which are closed under converse entailment, we can do so by replacing "is entirely about M" with "entails something contingent which is entirely about M," giving essentially one of Lewis's glosses on "partly about M"[92] – though now we lose the property of being closed under negation. Recall the first passage from Karmo quoted in the Postscript to Section 2, noting the tension between these features. (Lewis [64] and Karmo [61] both appeared in the same year: 1988.) The notion of ethicality as entailing something entirely about M_{eth} is probably the simplest such notion embodying the 'commitment' idea in play in that Postscript, though the alternative to Karmo's suggestion in the quoted pas-

[92]The gloss in question is what Lewis calls the part-of-content notion of partial aboutness, though he does not seem to do the equivalent in his negative way of describing it (the content of a statement being the set of worlds at which it is false) of inserting the word *contingent*, as here: since with the classical assumptions in force here and in Lewis [64], every statement entails any logical truth and that is entirely about every subject matter, we need to exclude such cases when we say "entails a statement entirely about M" if it is not to apply across the board to all statements. Not that *contingent* is really the right word, in the first place because here we only need to exclude necessary truths rather than all non-contingent statements, a and secondly because even in making that adjust we are in the wrong modality, it being logical truths (true at all $\langle w, n\rangle$ pairs) rather than necessary truths (true at all w) that need to be excluded.

sage was, like his own, a world-relative notion. In Karmo's case, translated into the present concepts, non-ethicality at w is a matter of being M_{nat}-settled at w and thus, ethicality at w is a matter of not being thus settled. One could equivalently say (non-)ethicality at $\langle w, n\rangle$ here, since the this does not depend on any particular choice of n. On the other hand, ethicality at w according to the 'enthymematic' proposal of Humberstone [42] is a world-relativized variant on being partly about M_{eth}, but instead of being a matter of entailing something contingent entirely about M_{eth}, is a matter of being such that it together with additional premises true at w and entirely about M_{nat}, entails something contingent which is entirely about M_{eth}. This gloss on [42] is taken from Humberstone [45], in which further details on the relations between M_{eth} (or "M_{eval}") and M_{nat} are related. The imaginary interlocutor summoned up by Geach in the passage quoted in Example 2.5, with its reference to the supplementary non-ethical premise that one could reach for in the envisaged disjunctive syllogism step there, takes very much the line developed in [42] – written, as it happens, without knowledge of Geach [21].

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Finite Degradation Structures

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Abstract

Probabilistic risk and safety analyses are used in virtually all industries to assess whether the risk of operating complex technical systems is low enough to be socially acceptable. As of today, these analyses rely mainly on stochastic Boolean models such as fault trees or reliability block diagrams. These models are coarse approximations of the behavior of the systems under study.

In this article, we introduce the notion of finite degradation structure. Finite degradation structures encode the degradation order among the states of multistate systems, i.e. models in which variables can take a finite number of values rather than just two. This extension of Boolean formalisms makes it possible to increase significantly the capacity of expression without increasing significantly the complexity of the calculation of risk indicators.

Technically, finite degradation structures are finite semi-lattices associated with a random process. They form a monoidal category and provide a unified algebraic framework for Boolean reliability models and multistate systems. They shed a new light on central notions of system reliability theory such as those of coherent models and minimal cutsets.
Keywords: Multivalued logics, category theory, system reliability theory, combinatorial models, finite degradation structures

Notations and Acronyms

Throughout this article, we use the following notational conventions and acronyms.

$S \times T$: Cartesian product of the set S and T.

$X \cong Y$: X is isomorphic to Y.

$\mathcal{D} : \langle D, \leq_D, \perp_D \rangle$: Finite degradation structure $\mathcal{D}$, i.e. the semi-lattice, built over the finite set of constants D, the partial order $\leq_D$ over D and the least element $\perp_D$ of D for this partial order.

$\mathcal{A} \otimes \mathcal{B}$**:** Monoidal product of the finite degradation structures $\mathcal{A}$ and $\mathcal{B}$.

$\bigotimes_{\mathcal{X} \in \{\mathcal{X}_\infty, \ldots, \mathcal{X}_\backslash\}} \mathcal{X}$**:** $\mathcal{X}_\infty \otimes \cdots \otimes \mathcal{X}_\backslash$

FDS: Category of finite degradation structures.

$dom(V)$**:** Domain of the variable V. $dom(V)$ is a finite degradation structure.

$dom(\mathcal{V})$**:** $\bigotimes_{V \in \mathcal{V}} dom(V)$.

$var(f)$**:** Set of variables occurring in the finite degradation formula f.

$[\![f]\!]$**:** (Canonical) interpretation of the finite degradation formula f.

$[\![\mathcal{M}]\!]$**:** (Canonical) interpretation of the finite degradation model $\mathcal{M}$.

$\overline{\sigma}_{\mathcal{M}}$**:** Unique admissible extension of the assignment of the state variables of the finite degradation model $\mathcal{M}$ into an assignment of variables of $\mathcal{M}$.

$\mathrm{PI}\,(O)$**:** Set of prime implicants of an observer O.

$\lceil\!\lceil O \rceil\!\rceil$**:** Coherent hull of the observer O.

$\lfloor \pi \rfloor$ **:** Least minterm compatible with the product π.

$\mathrm{MCS}\,(O)$**:** Set of minimal cutsets of an observer O.

$\mathrm{CriticalStates}\,(U)$**:** Set of critical states of a subset U of the states of a finite degradation structure $\mathcal{D} : \langle D, <, \perp \rangle$.

1 Introduction

Probabilistic risk and safety analyses are used in virtually all industries to assess whether the risk of operating complex technical systems (aircrafts, nuclear power plants, offshore platforms...) is low enough to be socially acceptable. The WASH1400 report [1], which followed the Three Mile Island nuclear accident, is usually considered as the historical starting point of their worldwide, cross-industry adoption. As of today, these analyses rely mainly on stochastic Boolean models such as fault trees, reliability block diagrams, event trees or a combination of those. These modeling formalisms are well mastered by practitioners. Reference textbooks are available, e.g. [2, 3]. Safety standards such as IEC 61508 [4] (safety systems), ISO 26262 [5] (automotive industry), or ARP4761 [6] (avionic industry) recommend their use.

Models written in these formalisms encode however coarse approximations of the behavior of the systems under study. They do not make it possible to faithfully represent important features such as cold redundancies, resource sharing or reconfigurations. Of course, more powerful formalisms exists, e.g. Markov chains or stochastic Petri nets [7]. But the complexity of the calculation of risk indicators increases dramatically when leaving the realm of combinatorial models. This complexity frames actually the whole domain: a probabilistic risk/safety model always results of a tradeoff between the accuracy of the description and the ability one has to perform calculations on the model, within one's always limited computational resources [8]. The calculation of the main risk indicators is already #P-hard for combinatorial models [9]. For these models, it is however possible to overcome this theoretical intractability because polynomial approximation schemes exist that give very good practical results [10]. Such approximation schemes are much more delicate to design in the case of more powerful formalisms.

A good compromise would be to stay in the realm of combinatorial models, but to allow the representation of components that can be in more than two states (working or failed). In the reliability engineering literature, the term "multistate systems" designates extensions of Boolean models to the case where variables can take a finite (and in general small) number of values. This term is not very appropriate, but we shall use it here as it is widely accepted. Multistate systems have attracted over the years the attention of researchers and practitioners [11–14]. They are however seldom used in practice, probably due to the too small improvement they provide compared to Boolean formalisms. Most, if not all, published works on multistate systems assume actually that the states of a component are totally ordered, from the working state up to the failed state, going possibly through a number of degraded states.

In this article, we introduce the notion of finite degradation structure which releases this total order constraint. It does not release it fully however: the notion of degradation is kept and generalized. Namely, finite degradation structures are finite semi-lattices associated with a random process. The bottom element of the semi-lattice represents the working state. The partial order relation between elements is a degradation order. The random process describes the probability to be in a given state at a given time.

Each finite degradation structures forms a category, see e.g. [15] for a reference book. The category **FDS** of finite degradation structures is thus a category of categories. Furthermore, **FDS** is a monoidal category: it has a product that makes possible to describe systems as hierarchical assemblies of components. Epimorphisms (surjective mappings) of **FDS** encode abstractions and prove to be extremely useful in the context of reliability engineering.

Eventually, finite degradation structures provide a unified algebraic framework encompassing and extending all combinatorial models used in reliability engineering. Combined with the definition of suitable abstraction, it sheds a new light on the fundamental notions of system reliability theory such as those of coherent models, minimal cutsets and top event probability from which all practical risk indicators are calculated. Finite degradation structures characterize eventually the algebraic properties a multi-valued logic should have to be used in the reliability engineering context. They can thus be seen as a new way of defining multi-valued logics by means of algebraic properties rather than by means of axioms, as it is in usually the case.

Finite degradation structures pave the way to a significant improvement of the process of probabilistic risk and safety analyses. The idea is to proceed in two steps: first, states of components or groups of dependent components are determined and their probabilities assessed by means, for instance, of Markov chains or discrete event simulations; second, the resulting finite degradation model is assessed by means of algorithms derived from those used to assess fault trees, see e.g. [16, 17]. Under the condition that systems under study can be split into small, independent groups of dependent components, which is often verified by industrial systems, it is thus possible to marry the expressive power of discrete event systems with the computational efficiency of combinatorial formalisms. This idea generalizes assessment methods for dynamic fault trees [18] without requiring that one merges dependent components into a macro-components, which is of interest for qualitative analyses. Note that static analysis techniques exist to automatically split discrete event models into independent parts, see e.g. [19].

Starting from a seemingly minor point, the relation order between states of multistate systems, the notion of finite degradation structures led us to revisit a sizable part of system reliability theory. The contribution of this article is to present and to organize this journey through the logical foundations of reliability engineering.

The remainder of this article is organized as follows. Section 2 explains the rational for finite degradation structures by means of an example stemmed from industrial practice. Section 3 introduces them formally both from an abstract and concrete point of views. Section 4 revisits the notion of prime implicant and minimal cutset in this framework. Section 5 presents some experimental results. Finally Section 6 concludes the article.

2 Illustrative Example

But before diving into formal developments of finite degradation structures, we shall provide the reader with some intuitive ideas by means of an example.

Fig. 1 shows a high integrity pressure protection system (HIPPS) as commonly found in oil and gas industry. This HIPPS is called TA4 in the ISO TR/12489 safety standard [20].

This safety instrumented system is in charge of preventing an overpressure in the pipe that could damage equipment, e.g. separators, located downstream. It works on demand, i.e. when an overpressure occurs in the pipe (the flow of oil, gas and water extracted from wells is actually irregular). It is made of three types of elements: sensors $S1-3$ in charge of

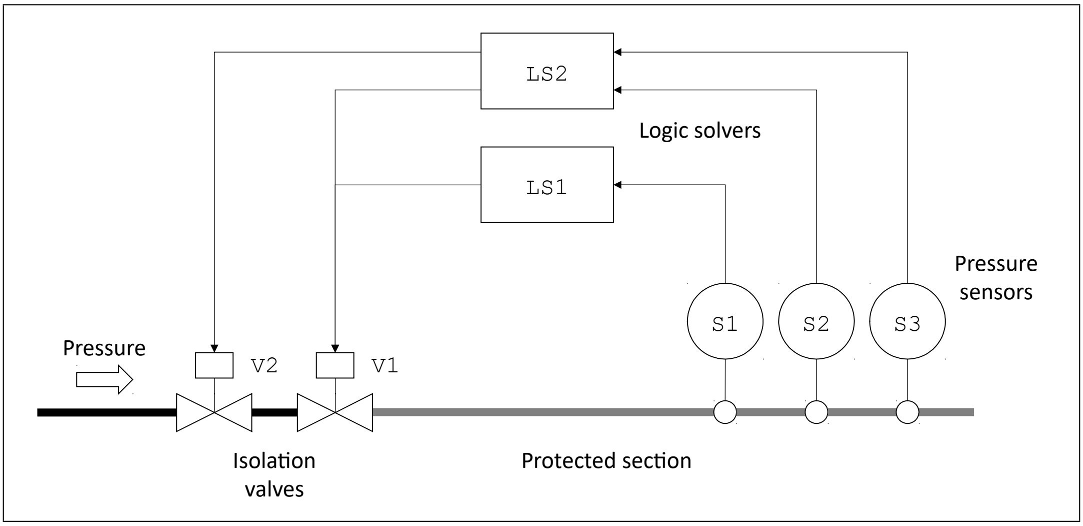

Figure 1: The high integrity pressure protection system TA4

detecting overpressure, logic solvers LS1 − 2 in charge of making the decision and the two isolation valves V1 and V2. The logic solver LS2 works according to a 1-out-of-2 logic, i.e. that it sends the order to close the valves if at least one out of two sensors S2 and S3 detects an overpressure.

According to the standard IEC61508 [4], failure modes of the components of a safety instrumented system can be classified along two directions: safe versus dangerous and detected versus undetected. In our example, safe failure modes are those which contribute to close the isolation valves, even though there is no overpressure (spurious triggers). Dangerous failure modes are those which contribute to keep the isolation valves open, even though there is an overpressure.

Logic solvers embed autotest facilities so that their failures are immediately detected. On the contrary, failure of valves remain undetected between two maintenance interventions. Failures of sensors may be detected or not.

ISO/TR 12489 makes the additional following assumptions.

– All components may fail (independently).

– Safe failures are always detected.

– Probabilities of safe and dangerous failure follow negative exponential distributions. The parameters of these distributions are given Table 1.

– Depending on the type of the component, a given ration of dangerous failures are detected.

Parameter	Sensor	Logic solver	Isolation valve
Safe failure rate	$3.00 \times 10^{-5}\ h^{-1}$	$3.00 \times 10^{-5}\ h^{-1}$	$2.90 \times 10^{-4}\ h^{-1}$
Dangerous failure rate	$5.90 \times 10^{-7}\ h^{-1}$	$5.70 \times 10^{-7}\ h^{-1}$	$2.76 \times 10^{-6}\ h^{-1}$
Detection ratio	0.9	1.0	0.0

Table 1: Reliability parameters for the HIPPS TA4

– The system is maintained once a month (once in 730 hours). The production is stopped during the maintenance. Components are as good as new after the maintenance.

Safe failures and dangerous failures are very different both in terms of frequency of occurrence and severity of consequences. Spurious triggers of the safety instrumented system have a strong economic consequences, but no impact on safety. In contrast, dangerous failures may lead to a catastrophic accident if they remain undetected. Probabilistic risk analyses aim at extracting the most probable scenarios of failure as well as at assessing the probability to be in a safe or dangerous failed state over the mission time of the system.

Our example is small enough (for pedagogical purposes) to make it possible to enumerate by hand all of the possible (global) states of the system and to calculate their probabilities. In real-life applications, such a brute-force approach is basically impossible because of the exponential blow-up of the number of states. Models have to be designed. As of today, Boolean models (fault trees and the like) are by far the most popular. They are however not well suited to represent systems like the above high integrity pressure protection system, because an accurate representation requires to consider more than two states (working or failed) for components and groups of components.

Finite degradation structures, which we shall define formally now, provide a formal algebraic setting to design and to perform risk assessment on such multivalued description.

3 Finite Degradation Stuctures

Finite degradation structures formalize an intuitive idea that is at the core of reliability engineering: components and systems can be in more or less degraded states or, to put it differently, there is a fundamental asymmetry in the possible states of a component or a system: the component or the system is "normally" working, but may degrade and eventually fail. The probability for a component or a system to be in a working state is in general much higher than the probability to be in a degraded or failed state. In other words, states of component or a system are "naturally" ordered with respect to the level of degradation.

This order is in general only a partial order, especially when considering systems made of multiple components.

3.1 Formal Definition

Recall that a *meet-semi-lattice* is a partially ordered set $\langle D, \leq \rangle$ such that any two elements $x, y \in D$ have a greatest lower bound $x \sqcap y$ in D. $x \sqcap y$ is called the meet of x and y. $x \sqcap y = x$ if and only if $x \leq y$.

If D is finite, then it has a unique least element, i.e. an element $\bot$ such that for any other element x, $\bot \leq x$. Assume for a contradiction that D has two such elements $\bot_1$ and $\bot_2$, then we have both $\bot_1 \leq \bot_2$ and $\bot_2 \leq \bot_1$, which by antisymmetry means that $\bot_1 = \bot_2$.

A *finite degradation structure* is thus a meet-semi-lattice $\langle D, \leq, \bot \rangle$ where:

- D is a finite set of constants representing the *states* of a component or a system.
- The partial order relation $\leq$ represents the *degradation order* among states.
- $\bot$, the least element of D, represents the state in which the component is as good as new.

The intuition behind this definition is that the state of a component cannot be less degraded than when it is new. Aside this state, the component may be in more or less degraded states. Some of these states may be comparable in terms of degradation level, i.e. that a state can be more degraded than another, while some other may be incomparable because they correspond to different types of degradation. States are thus organized according to a partial degradation order. As a component may have different failure modes, which are exclusive one another, there may be several distinct most degraded states. Given two states s and t, there is always at least one state that is less degraded than both s and t: the "as-good-as-new" state $\bot$.

Example 1. Fig. 2 shows the Hasse diagrams representing some finite degradation structures that play an important role in system reliability theory, either for their theoretical interest, or to describe the state space of components (they can be seen as on-the-shelf types for these components), or to characterize the state of systems. `w` stands for working, `d` for degraded and `f` for failed. The suffixes `s`, `d` and `u` stand respectively for safe, dangerous detected and dangerous undetected. On the figure, the degradation order is represented bottom-up.

The finite degradation structure **WF** is thus the "classical" Boolean domain working/-failed. In the finite degradation structure **WDF**, an intermediate degraded state is introduced. The finite degradation structure **SWF** is used to represent components in cold redundancy: the component is first in standby mode, then it is working, then it fails. We shall come back on the finite degradation structures **WFsd** and **W3F** represented on the figure.

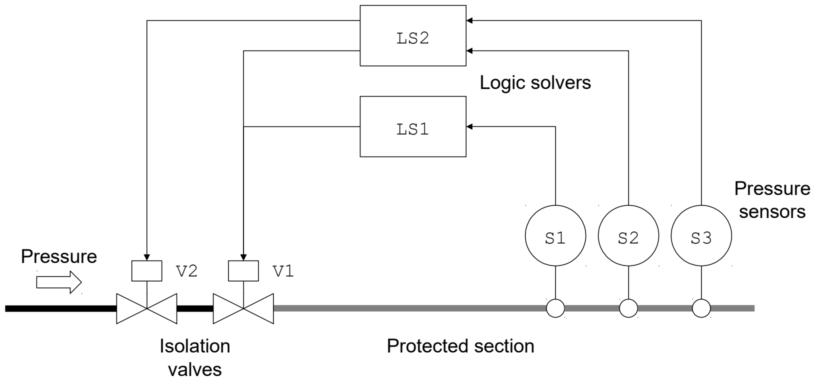

Figure 2: Some useful finite degradation structures

3.2 The Categorical Point of View

A finite degradation structure $\langle D, \leq, \perp \rangle$ is a *category*:

- The objects of this category are the states of D.
- For any two states $s, t \in D$, there is an arrow from s to t if and only if $s \leq t$. If it exists this arrow is unique (and called $\leq$).

Let $\mathcal{A} : \langle A, \leq_A, \perp_A \rangle$ and $\mathcal{B} : \langle B, \leq_B, \perp_B \rangle$ be two finite degradation structures and let ϕ be a mapping from $\mathcal{A}$ to $\mathcal{B}$. Then, we say that ϕ is structure preserving, if:

- For any two states s and t of A, $s \leq_A t$ implies that $\phi(s) \leq_B \phi(t)$.
- $\phi(\perp_A) = \perp_B$.

Structure preserving mappings are monotone functions sending the least element of their domain onto the least element of their codomain. This definition ensures that the image by a structure preserving mapping of a finite degradation structure is a finite degradation structure.

We can define the category **FDS** of finite degradation structures:

- Objects of **FDS** are finite degradation structures.

– Arrows/morphisms of **FDS** are structure preserving mappings between finite degradation structures.

It is easy to verify that **FDS** is actually a category as structure preserving mappings can be composed and it is possible to define an identity (which is indeed a structure preserving mapping) of any finite degradation structure.

Monomorphisms (injective mappings) between finite degradation structures encode extensions, i.e. operations by which states are added to a domain, in view of a finer grain analysis. For instance, we can extend **WF** into **WDF** by mapping `w` and `f` on themselves and adding the intermediate state `d`.

Epimorphisms (surjective mappings) between finite degradation structures encode abstractions: there exists an epimorphism between the finite degradation structure $\mathcal{A}$ and the finite degradation structure $\mathcal{B}$ if $\mathcal{B}$ is an abstraction of $\mathcal{A}$. We shall give in the sequel numerous examples of such abstractions.

Discussion: As we shall see, probabilistic risk assessment models involve not only morphisms between finite degradation structures but also mappings that do not preserve the structure, i.e. that are not monotone functions. Using general mappings to define **FDS** would have make this category very close to the "classical" category **FinSet** whose objects are finite sets and whose arrows are functions between finite sets. The advantage would have been to handle all operations we needed within the category. The drawback would have been to lose the centrality of the notion of degradation order, which is the important one from a reliability engineering point of view.

In any case, the most important constructions we shall use, such as the one of product defined in the next subsection and the notions related to minimal cutsets that we shall develop Section 4 work the same way if we consider structure preserving mappings or general functions.

3.3 Monoidal Product

One of the most interesting properties of **FDS** is that it has a product, i.e. the combination of two (or more) finite degradation structures is also a finite degradation structure. We shall now formalize this idea.

Let $\mathcal{A} : \langle A, \leq_A, \bot_A \rangle$ and $\mathcal{B} : \langle B, \leq_B, \bot_B \rangle$ be two finite degradation structures. We define $\mathcal{A} \otimes \mathcal{B} = \langle A \times B, \leq_{A \otimes B}, \bot_{A \otimes B} \rangle$ as follows.

– $A \times B$ is the Cartesian product of A and B.

– For all $\langle s_A, s_B \rangle, \langle t_A, t_B \rangle \in A \times B$, $\langle s_A, s_B \rangle \leq_{A \otimes B} \langle t_A, t_B \rangle$ if and only if $s_A \leq_A t_A$ and $s_B \leq_B t_B$.

– $\perp_{A\otimes B} = \langle \perp_A, \perp_B \rangle$.

It is easy to verify that $\mathcal{A} \otimes \mathcal{B}$ is a finite degradation structure.

The construction $\mathcal{A} \otimes \mathcal{B}$ comes with the two canonical projections $\pi_1 : \mathcal{A} \otimes \mathcal{B} \to \mathcal{A}$ such that $\pi_1(\langle s, t \rangle) = s$, and $\pi_2 : \mathcal{A} \otimes \mathcal{B} \to \mathcal{A}$ such that $\pi_2(\langle s, t \rangle) = t$.

The following property holds.

Proposition 1 (Product). *$\otimes$ together with the two canonical projections π_1 and π_2 is a product for the category* **FDS**, *i.e. that for any three finite degradation structures $\mathcal{A}$, $\mathcal{B}$ and C and pair of morphisms $\varphi_A : C \to \mathcal{A}$ and $\varphi_B : C \to \mathcal{B}$, there exists a unique morphism $\varphi : C \to \mathcal{A} \otimes \mathcal{B}$ such that:*

$$\begin{aligned} \varphi_A &= \pi_1 \circ \varphi \\ \varphi_B &= \pi_2 \circ \varphi \end{aligned}$$

φ is simply defined as $\varphi(s) = \langle \varphi_A(s), \varphi_B(s) \rangle$.

$\mathcal{A} \otimes \mathcal{B}$ is called the *monoidal product* of $\mathcal{A}$ and $\mathcal{B}$.

Note that $\otimes$ is still a product if we consider non-structure preserving mappings as the Cartesian product is a product in **FinSet**.

Recall that two mathematical objects X and Y are *isomorphic* if there is a morphism from X to Y and a morphism from Y to X. In this case, we note $X \cong Y$, the two objects can be considered as identical.

Proposition 2 (Properties of the monoidal product). *Let $\mathcal{A}$, $\mathcal{B}$ and C be three finite degradation structures, then the following equalities hold.*

– $\mathcal{A} \otimes \mathcal{B} \cong \mathcal{B} \otimes \mathcal{A}$ *(Commutativity).*

– $\mathcal{A} \otimes (\mathcal{B} \otimes C) \cong (\mathcal{A} \otimes \mathcal{B}) \otimes C$ *(Associativity).*

– $\mathcal{A} \otimes \mathbf{1} \cong \mathbf{1} \otimes \mathcal{A} \cong \mathcal{A}$ *(Neutral Element).*

where $\mathbf{1} = \langle \{\perp\}, \perp \leq \perp, \perp \rangle$ denotes the finite degradation structure with a unique state.

FDS is thus a symmetric monoidal category. It enjoys other nice algebraic properties, but a full exposition would go beyond the scope of this article. The important point here is that it is possible to build the finite degradation structure of a system by composing the finite degradation structures of its components. It is also possible to group finite degradation structures of a subset of components of a system, so to consider them as a single component. As explained in the introduction, this mechanism is implicitly used to compile discrete event modeling formalisms such as AltaRica into fault trees [19,21]: first, the model is split into independent groups of components by means of static analysis techniques; then, these groups are compiled separately. Finite degradation structures provide a unified algebraic framework to generalize this idea.

Example 2. Consider our illustrative example described Section 2. According to our specifications, this system is made of seven components: the three sensors, the two logic solvers and finally the two valves. We assumed that each of these components can be either working (`w`), failed safe (`fs`), failed detected (`fd`), or failed undetected (`fu`), i.e. can be described with the finite degradation structure **W3F** pictured Fig. 2. The global state of the HIPPS can thus be described by the finite degradation structure $\mathbf{W3F}^7$. Thanks to the product $\otimes$, the partial order over states of individual components is lifted-up into a partial order over the states of the system.

We can now isolate, for instance, the subsystem made of the two sensors `S2` and `S3` and consider it as a macro-component that can be studied separately. In the fault tree framework, such groups of components are called modules [22].

We can now define formulas and models built on top of finite degradation structures, i.e. eventually give a syntax to the finite degradation calculus.

3.4 Formulas

We assume given a finite set $\mathcal{S}$ of finite degradation structures and a finite set $\mathcal{O}$ of symbols called *operators*.

Each operator o of $\mathcal{O}$ is associated with a mapping $[\![o]\!]$ from $\bigotimes_{1 \leq i \leq n} s_i$, $n \geq 0$, into s, where both the s_i's and s are finite degradation structures. $\bigotimes_{1 \leq i \leq n} s_i$ is called the domain of o and is denoted $dom(o)$. s is called the codomain of o and is denoted $codom(o)$.

Together, $\mathcal{S}$ and $\mathcal{O}$ form what is called an *operad*[1] [23].

Example 3. To deal with the case study presented Section 2, it is useful to introduce parallel $\|$ and series $\varominus$ compositions. These operators are mappings from $\mathbf{W3F} \otimes \mathbf{W3F}$ into **W3F**. They are defined as shown Table 2.

It is worth noticing that $\|$ is both associative and commutative and that it is an epimorphism from $\mathbf{W3F} \otimes \mathbf{W3F}$ to **W3F**. In contrast, $\varominus$ is only associative. It is not commutative and does not preserve the partial order. If the first component is failed dangerous undetected and the second one is working then the series of these two components is failed dangerous undetected. Now, if the first component is still failed dangerous undetected, but the second one is failed safe, then the series is failed safe.

We can now define formulas of the finite degradation calculus.

Let $\mathcal{S}$ be a finite set of finite degradation structures and let $\mathcal{O}$ be a finite set of operators on $\mathcal{S}$ defined as above. Let $\mathcal{V}$ be a finite set of symbols called *variables*. Each variable V of $\mathcal{V}$ is assumed to take its value in the support set of one of the finite degradation structures of $\mathcal{S}$. This finite degradation structure is called the domain of V and is denoted $dom(V)$.

[1] We would like to thank here the reviewer who pointed out this notion.

$\|$	w	fs	fd	fu
w	w	w	w	w
fs	w	fs	fs	fs
fd	w	fs	fd	fu
fu	w	fs	fu	fu

$\ominus$	w	fs	fd	fu
w	w	fs	fd	fu
fs	fs	fs	fd	fu
fd	fd	fs	fd	fu
fu	fu	fs	fd	fu

Table 2: The operators $\|$: **W3F** $\otimes$ **W3F** $\rightarrow$ **W3F** and $\ominus$: **W3F** $\otimes$ **W3F** $\rightarrow$ **W3F**.

Then the set of *well formed (typed) formulas* over $\mathcal{S}$, $\mathcal{V}$ and $\mathcal{O}$ is the smallest set such that:

- Constants, i.e. members of finite degradation structures of $\mathcal{S}$, are well formed formulas. The type of a constant is the finite degradation structure it comes from.
- Variables of $\mathcal{V}$ are well formed formulas. The type of a variable V is simply its domain.
- If o is an operator of $\mathcal{O}$ such that $[\![o]\!] : \bigotimes_{1 \leq i \leq n} s_i \rightarrow s$, and $f_1, \ldots f_n$ are well formed formulas of types $s_1, \ldots s_n$, then $o(f_1, \ldots, f_n)$ is a well formed formula of type s.

In the sequel, we shall say simply formula instead of well formed typed formula.
The set of variables occurring in the formula f is denoted $var(f)$.

3.5 Finite Degradation Models

Finite degradation models are obtained by lifting up fault tree constructions to the finite degradation calculus. Namely, a *finite degradation model* $\mathcal{M}$ is a pair $\langle \mathcal{V} = \mathcal{S} \uplus \mathcal{F}, \mathcal{E} \rangle$ where:

- $\mathcal{S} = \{V_1, \ldots, V_m\}$, $m \geq 1$, is a finite set of *state variables*.
- $\mathcal{F} = \{W_1, \ldots, W_n\}$, $n \geq 1$, is a finite set of *flow variables*.
- $\mathcal{E} = \{e_1, \ldots, e_n\}$ is a finite set of *equations*.

Each equation e_j, $1 \leq j \leq n$ is a pair $\langle W_j, f_j \rangle$ where:

- W_j is the jth variable of $\mathcal{F}$.
- f_j is a formula built over the given sets of constants, variables and operators.

For the sake of clarity, the equation $\langle W_j, f_j \rangle$ is simply denoted as $W_j := f_j$. As there is a unique equation $W := f$ for each flow variable W, the formula f can be seen the definition of the variable W.

A finite degradation model $\mathcal{M} : \langle \mathcal{V}, \mathcal{E} \rangle$ is *well typed* if $codom(f_j) = dom(W_j)$ for each equation $W_j := f_j$ of $\mathcal{E}$.

We say that the flow variable W_j depends on the (state or flow) variable U if either $W \in var(f_j)$ or there exists a variable U' of $var(f_j)$ that depends on W.

A finite degradation model is *looped* if one of its flow variable depends on itself. It is *loop-free* or *data-flow* otherwise.

From now, we shall consider only well typed and data-flow models.

A *root variable* is a flow variable that occurs in none of the right members of equations. A finite degradation model is *uniquely rooted* if it has only one root variable. The unique root of such model represents in general the state of the system.

It is easy to see that finite degradation models generalize fault trees: state and flow variables play respectively the roles of basic and internal events, while equations play the role of gates. Moreover, the root variable plays the role of the top event. The terms “state” and “flow” comes from guarded transition systems [24].

Example 4. The high integrity pressure protection system presented Section 2 can be described by the following model.

$$
\begin{array}{rclrcl}
\mathtt{HIPPS} & := & \mathtt{SB1} \parallel \mathtt{SB2} & & & \\
\mathtt{SB1} & := & \mathtt{CL1} \otimes \mathtt{V1} & \mathtt{SB2} & := & \mathtt{CL2} \otimes \mathtt{V2} \\
\mathtt{CL1} & := & \mathtt{LSL1} \parallel \mathtt{LSL2} & \mathtt{CL2} & := & \mathtt{LSL2} \\
\mathtt{LSL1} & := & \mathtt{SL1} \otimes \mathtt{LS1} & \mathtt{LSL2} & := & \mathtt{SL2} \otimes \mathtt{LS2} \\
\mathtt{SL1} & := & \mathtt{S1} & \mathtt{SL2} & := & \mathtt{S2} \parallel \mathtt{S3}
\end{array}
$$

The state variables of this model are:

- The $\mathtt{S}i$'s that represent the states of the sensors.
- The $\mathtt{LS}i$'s that represent the states of the logic solvers.
- The $\mathtt{V}i$'s that represent the states of the valves.

The flow variables of this model are:

- $\mathtt{HIPPS}$ that describes the state of the system.
- $\mathtt{SB1}$ and $\mathtt{SB2}$ that describe respectively the states of the first and second safety barriers.

– CL1 and CL2 that describe respectively the states of the first and second command lines.

– LSL1 and LSL2 that describe respectively the states of the first and second logic solver lines.

– SL1 and SL2 that describe respectively the states of the first and second sensor lines.

It is easy to verify that the above model is data-flow and that HIPPS is its unique root variable.

Formulas and models are syntactic objects. To give them a meaning, we need to define how they are interpreted in terms of mappings from finite degradation structures.

3.6 Interpretation

Let f be a formula of the finite degradation calculus.

A *variable assignment* of f is a mapping from $var(f)$ to $\prod_{V \in var(f)} dom(V)$, i.e. a function that associates with each variable a value of its domain.

f is interpreted as a mapping $[\![f]\!] : \bigotimes_{V \in var(f)} dom(V) \rightarrow s$ where s is the codomain of the outmost operator of f, by lifting up as usual variable valuations. Let σ be a variable assignment of $var(f)$, then:

– If f is reduced to a constant c, then $[\![f]\!](\sigma) = c$.

– If f is reduced to a variable V, then $[\![f]\!](\sigma) = \sigma(V)$.

– If f is in the form $o(f_1, \ldots, f_n)$, then $[\![f]\!](\sigma) = [\![o]\!]([\![f_1]\!](\sigma), \ldots, [\![f_n]\!](\sigma))$.

A variable assignment σ is *admissible* in the model $\mathcal{M} : \langle \mathcal{V}, \mathcal{E} \rangle$ if $\sigma(v_j) = \sigma(f_j)$ for each equation $v_j := f_j$ of $\mathcal{E}$.

The following property holds, thanks to the fact that the models we consider are data-flow.

Proposition 3 (Unicity of admissible variable assignments)**.** *Let $\mathcal{M}$ be a finite degradation model and σ be a partial variable assignment that assigns values only to state variables of $\mathcal{M}$. Then there is a unique way to extend σ into an admissible total assignment σ' of variables of M.*

σ' is simply calculated bottom-up by propagating values in the set of equations.

In the sequel we shall denote by $\overline{\sigma}_{\mathcal{M}}$ the unique extension of the assignment the assignment σ of the state variables of the model $\mathcal{M}$ into an admissible assignment of the variables of $\mathcal{M}$. We shall omit the subscript when the model $\mathcal{M}$ is clear from the context and call $\overline{\sigma}$ the *admissible extension* of σ.

It follows from the above property, that we can interpret a model $\mathcal{M} : \langle \mathcal{V} = \mathcal{S} \uplus \mathcal{F}, \mathcal{E} \rangle$ as a mapping:

$$[\![\mathcal{M}]\!] \quad = \quad \bigotimes_{V \in \mathcal{S}} dom(V) \rightarrow \bigotimes_{W \in \mathcal{F}} dom(W) \tag{1}$$

Note that flow variables include in particular the root variables of the model. It is possible, by substituting bottom-up flow variables by their definitions, to transform any finite degradation model into an equivalent formula defining each root variable. This formula may however be exponentially larger than the original model, which the reason why models (in the sense we defined them) are preferred in practice to mere formulas. It remains that, if we are not interested in flow variables but the root variable R, which is often the case, then the model can be interpreted as the mapping:

$$[\![\mathcal{M}]\!] \quad = \quad \bigotimes_{V \in \mathcal{S}} dom(V) \rightarrow dom(R) \tag{2}$$

Example 5. The seven state variables of the model described Example 4 take their values in the finite degradation structure `W3F`. The root variable `HIPPS` of the model takes its value in the finite degradation structure `W3F`. The model is thus be interpreted as a mapping from $\mathbf{W3F}^7$ into **W3F**.

In the sequel, we shall denote $\bigotimes_{V \in \mathcal{V}} dom(V)$ simply by $dom(\mathcal{V})$.

3.7 Probabilities

The states of a finite degradation structure $\mathcal{D}$ can be seen as the outcomes of a random experiment. More technically, we can see (the power set of) $\mathcal{D}$ as a probability space and define a *random process*, i.e. a time indexed family X_t, $t \in \mathbb{R}^+$, of random variables over this probability space, see e.g. [25] an introduction to random processes. This random process describes the probability $p_D(s,t) = X_t(s)$ to be in state $s \in D$ at time t.

The above definition makes no assumption about how the probabilities $p_F(s,t)$ are actually obtained in practice. This can be done via analytical formulas, numerical simulations or any other convenient means. Note that random processes can be also used to associate rewards with states. By integrating such a reward over a time period, it is possible, for instance, to assess the expected production of a plant over a time period.

The following properties hold that are at the core of the calculation of probabilistic risk indicators.

Proposition 4 (Composition of probabilities)**.** *Let $\mathcal{A} : \langle A, \leq_A, \perp_A \rangle$ and $\mathcal{B} : \langle B, \leq_B, \perp_B \rangle$ be two finite degradation structures, each associated with a random process. Let p_A and p_B be the probability functions associated respectively with $\mathcal{A}$ and $\mathcal{B}$ by their associated random processes. Then,*

$$p_{\mathcal{A}\otimes\mathcal{B}}(\langle s_A, s_B \rangle, t) \quad \stackrel{def}{=} \quad p_A(s_A, t) \times p_B(s_B, t) \tag{3}$$

defines a probability measure over the monoidal product $\mathcal{A} \otimes \mathcal{B}$. The construction of this probability measure assumes that events represented by states of $\mathcal{A}$ and $\mathcal{B}$ are statistically independent.

Now let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a morphism. Then,

$$p_B(s_B, t) \quad \stackrel{def}{=} \quad \sum_{s_A \in f^{-1}(s_B)} p_A(s_A, t) \tag{4}$$

defines a probability measure over $\mathcal{B}$.

In other words, probabilized finite degradation structures compose naturally. The way we associate random processes with finite degradation structures defines actually a lax monoidal functor from **FDS** to the category of random processes, which is also a monoidal category[2].

We could have defined **FDS** by associating systematically a random process with each finite degradation structure and composing them as above. It is however convenient to be able to associate the monoidal product $\mathcal{A} \otimes \mathcal{B}$ with other probability structures than the natural one, so to take into account statistical dependencies. Note also that it is sometimes of interest to consider more complex measurable spaces, e.g. to work in the framework of Dempster-Shafer theory [26].

In the sequel, we shall omit the time when speaking about probability measures as keeping it just complexifies the notations, without bringing much to the point. Nevertheless, the above definition should be constantly borne in mind.

4 Prime Implicants and Minimal Cutsets

The notion of minimal cutset plays a central role in system reliability theory, as well as in practical probabilistic risk analyses. Intuitively, a minimal cutset is a minimal set of component failures that induces a failure of the system as a whole. In other words, minimal cutsets represent the most significant scenarios of failure. As the probability that a component is failed is in general much smaller than the probability that it is working correctly, minimal

[2]Thanks again to the reviewer would pointed out the notion of lax monoidal functor.

cutsets represent also the most probable scenarios of failure. The intuitive definition of minimal cutsets works fine for coherent (monotone) models for which the notion of minimal cutset coincide with the classical notion of prime implicant. However, it needs to be refined to handle non-coherent ones [10].

In this section, we shall generalize the notions of prime implicant and minimal cutset to multistate systems and give the latter a characterization in terms of states of finite degradation structures.

4.1 Observers

The main objective of probabilistic risk and safety analyses is to extract failure scenarios and to assess the cumulated probability of these scenarios. In fault trees, failure scenarios are represented by sets of basic events that induce the top event, i.e. combinations of values of state variables that induce a certain value of the root variable.

We can generalize this idea by considering (Boolean) observers.

Let $\mathcal{M} : \langle \mathcal{V} = \mathcal{S} \uplus \mathcal{F}, \mathcal{E} \rangle$ be a finite degradation model. An *observer* is a Boolean formula O over the values of the variables of $\mathcal{V}$.

Example 6. In the model defined Example 4, a number of observers are of interest, e.g.

- $\mathtt{HIPPS} = \mathtt{fs}$ that characterizes the states in which the system is in a safe failure mode.
- $\mathtt{HIPPS} \in \{\mathtt{fd}, \mathtt{fu}\}$ that characterizes the states in which the system is in a dangerous failure mode.
- $\mathtt{HIPPS} = \mathtt{fu}$ that characterizes the states in which the system is in a dangerous undetected failure mode.

We could also consider more complex observers, e.g.

- $\mathtt{HIPPS} \in \{\mathtt{fd}, \mathtt{fu}\} \wedge \mathtt{S1} = \mathtt{w}$ that characterizes the states in which the system is in a dangerous failure mode and the sensor S1 is working properly.

Note that observers do not need to be structure preserving mappings (assuming $0 < 1$). For instance, the observer above $\mathtt{HIPPS} = \mathtt{fs}$ is not.

Such an observer O is interpreted as a mapping from $dom(\mathcal{S})$ into $\{0, 1\}$, or equivalently, interpreting O as a characteristic function, as the subset of assignments σ of variables of $\mathcal{S}$ whose extension into admissible assignments of all variables satisfies O:

$$[\![O]\!] \quad = \quad \{\sigma \in dom(\mathcal{S}) : \overline{\sigma}(O) = 1\} \tag{5}$$

Observers are thus predicates in the logical sense.

4.2 Prime Implicants

Let us consider first the extension of the classical notion of prime implicant.

Let $\mathcal{M} : \langle \mathcal{V} = \mathcal{S} \uplus \mathcal{F}, \mathcal{E} \rangle$ be a finite degradation model.

A *product* (over $\mathcal{S}$) is a conjunct of *atoms* of the form $V = s$, where V is a variable of $\mathcal{S}$ and s is a state of $dom(V)$, such that each variable occurs at most once in the product. A *minterm* is a product in which all variables of $\mathcal{S}$ occur. Products and minterms one-to-one correspond respectively with partial and total assignments of variables of $\mathcal{S}$ (and by extension $\mathcal{V}$).

Let O be an observer of $\mathcal{M}$ and π be a product built over $\mathcal{S}$. Then, π entails O, which is denoted as usual by $\pi \models O$, if all minterms $\sigma = \pi \circ \rho$, where ρ is an assignment of the variables of $\mathcal{S} \setminus var(O)$, are such that $\overline{\sigma}(O) = 1$.

In order to lift up the definitions of prime implicant, we need first to generalize the notion of subsumption, i.e. to introduce an order relation $\sqsubseteq$ over products, so to be able to implement the idea of primality and minimality.

Let π and ρ be two products over $\mathcal{S}$. Then, $\pi \sqsubseteq \rho$ if the following conditions hold.

1. $\mathrm{var}(\pi) \subseteq \mathrm{var}(\rho)$.
2. For any atom $V = s$ of π, the atom $V = t$ of ρ verifies $s \leq t$.

Now,

- π is an *implicant* of O, if $\pi \models O$.
- π is a *prime implicant* of O, if it is an implicant of O and no product ρ such that $\rho \sqsubset \pi$ is.

The set of prime implicants of an observer O is denoted by $\mathrm{PI}(O)$. It can be interpreted as the disjunction of its elements.

Example 7. As an illustration, consider again the model of Example 4 and the observer $O_{\mathtt{fs}} : \mathtt{HIPPS} = \mathtt{fs}$.

The product $\pi = \mathtt{LS2} = \mathtt{fs} \wedge \mathtt{V1} = \mathtt{fs} \wedge \mathtt{V2} = \mathtt{w}$ is a prime implicant of $O_{\mathtt{fs}}$:

- First, it is easy to verify that, whichever way we complete π into an assignment σ of state variables, we have $\overline{\sigma}(O_{\mathtt{fs}}) = 1$.
- Second, if either we change the assignment of LS2 to w, or the assignment of V1 to w, or we remove any of the three atoms of π, then the resulting product is no longer an implicant of $O_{\mathtt{fs}}$. For instance, $\mathtt{LS2} = \mathtt{fs} \wedge \mathtt{V1} = \mathtt{fs}$ is not an implicant of $O_{\mathtt{fs}}$ because if the valve V2 is failed dangerous undetected, then the whole system is failed dangerous undetected.

Example 7 illustrates the reason why prime implicants are not used in reliability engineering: The prime implicants of observers are "polluted" by information on the states of components that do not participate to the described failure, e.g. the atom $\texttt{V2} = \texttt{w}$.

The situation can be even more awkward from a safety analysis point of view. As an illustration, consider the observer $O_{\texttt{fu}}$: $\texttt{HIPPS} = \texttt{fu}$ and the product $\pi = \texttt{S1} = \texttt{fu} \wedge \texttt{S2} = \texttt{fu} \wedge \texttt{S3} = \texttt{fu}$ that describes the catastrophic situation in which all pressure sensors are lost undetected. Then, if all other components are working properly, the system is failed undetected. But if, by chance, both logic solvers are failed safe, then the system is failed safe. Although correct from a purely logical and probabilistic point of view, it is not very reasonable to count on the safe failures of logic solvers to avoid an accident.

The problem comes from the fact that the series operator does not preserve the partial order, and consequently the definition of $\texttt{HIPPS}$ is non monotone, which is spelled non-coherent in the reliability engineering literature.

To get rid of this problem, we have to lift up to finite degradation calculus the idea introduced in reference [10] to deal with non-coherent Boolean formulas, a notion we shall now formalize.

4.3 Coherence

The notion of coherence plays an important role in system reliability theory. It captures the intuitive idea that the more components of system are degraded, the more likely the system as whole is failed. We generalize it here to the case of finite degradation structures.

Let $\mathcal{M} : \langle \mathcal{V} = \mathcal{S} \uplus \mathcal{F}, \mathcal{E} \rangle$ be a finite degradation model and let O be an observer of $\mathcal{M}$.

O is said *coherent* if, for any two assignments σ and τ of variables of $\mathcal{S}$, $\sigma < \tau$ and $\overline{\sigma}(O) = 1$ implies that $\overline{\tau}(O) = 1$. In other words, $[\![O]\!]$ is a monotone function.

Boolean models of technical systems are in general coherent. However, non-coherent models, or more exactly “almost” coherent models, are sometimes designed.

Non-coherence is used as a modeling trick to make descriptions shorter, but when the model is assessed it is interpreted as a coherent one, via the calculation of minimal cutsets (see next section and reference [27] for an in-depth discussion). With finite degradation models, the situation is slightly different, as exclusive cases can be considered (like failed safe/failed dangerous), as illustrated by Example 7. This makes the (generalization of the) notion of coherent hull, originally introduced for the Boolean case [10], even more interesting.

Let $\mathcal{M} : \langle \mathcal{V} = \mathcal{S} \uplus \mathcal{F}, \mathcal{E} \rangle$ be a finite degradation model and let O be an observer of $\mathcal{M}$. The *coherent hull* of O, denoted by $\lceil\!\lceil O \rceil\!\rceil$, is the smallest coherent set of elements of $dom(\mathcal{S})$ that contains $[\![O]\!]$. Formally,

$$\lceil\!\lceil O \rceil\!\rceil \overset{def}{=} \{\tau \in dom(\mathcal{S}); \exists \sigma \in dom(\mathcal{S}); \sigma \leq \tau \wedge \overline{\sigma}(O) = 1\}$$

The following property is a direct consequence of the definition.

Proposition 5 (Coherent hulls)**.** *Let $\mathcal{M} : \langle \mathcal{V} = \mathcal{S} \uplus \mathcal{F}, \mathcal{E}\rangle$ be a finite degradation model and let O be an observer of $\mathcal{M}$. Then, the following inclusion holds.*

$$[\![O]\!] \quad \subseteq \quad \lceil\!\lceil O \rceil\!\rceil$$

Moreover, $[\![O]\!] = \lceil\!\lceil O \rceil\!\rceil$ if and only if O is coherent.

Example 8. Consider again the observer $O_{\mathtt{fu}} : \mathtt{HIPPS} = \mathtt{fu}$ of our example. Consider the minterm π defined as follows.

$$\pi \quad = \quad \mathtt{S1} = \mathtt{fu} \wedge \mathtt{S2} = \mathtt{S3} = \mathtt{fd} \wedge \mathtt{LS1} = \mathtt{LS2} = \mathtt{fs} \wedge \mathtt{V1} = \mathtt{V2} = \mathtt{w}$$

We have $\overline{\pi}(\mathtt{HIPPS}) = \mathtt{fs}$, therefore $\pi \notin [\![O_{\mathtt{fu}}]\!]$.

Now, consider the minterm τ defined as follows.

$$\tau \quad = \quad \mathtt{S1} = \mathtt{fu} \wedge \mathtt{S2} = \mathtt{S3} = \mathtt{fd} \wedge \mathtt{LS1} = \mathtt{LS2} = \mathtt{V1} = \mathtt{V2} = \mathtt{w}$$

We have $\overline{\tau}(\mathtt{HIPPS}) = \mathtt{fu}$, therefore $\tau \in [\![O_{\mathtt{fu}}]\!]$. But as $\tau < \pi$, $\pi \in \lceil\!\lceil O_{\mathtt{fu}} \rceil\!\rceil$.

The following proposition brings us back in the realm of **FDS**.

Proposition 6 (Coherent hulls as epimorphisms)**.** *Let $\mathcal{M} : \langle \mathcal{V} = \mathcal{S} \uplus \mathcal{F}, \mathcal{E}\rangle$ be a finite degradation model and let O be an observer of $\mathcal{M}$. Then, $\lceil\!\lceil O \rceil\!\rceil$ is an epimorphism from $dom(\mathcal{S})$ into $\mathbf{2} = \langle \{0, 1\}, 0 < 1, 0\rangle$.*

The (probability of the) coherent hull provides a conservative approximation of (the probability of) the observer. In many practical cases, this turns out to be a very good approximation:

$$p(\lceil\!\lceil O \rceil\!\rceil) \quad \approx \quad \sum_{\sigma \in [\![O]\!]} p(\sigma) \tag{6}$$

4.4 Minimal Cutsets

Let π be a product built over a set of variables $\mathcal{V}$, we denote by $\lfloor \pi \rfloor$, the smallest minterm compatible with π. Formally,

$$\lfloor \pi \rfloor \quad \stackrel{def}{=} \quad \sigma \in dom(\mathcal{V});\ \pi(V) = \begin{cases} \pi(V) & \text{if } V \in var(\pi) \\ \bot_{dom(V)} & \text{otherwise} \end{cases}$$

We are now ready to lift up the notion of minimal cutset.

Let $\mathcal{M} : \langle \mathcal{V} = \mathcal{S} \uplus \mathcal{F}, \mathcal{E}\rangle$ be a finite degradation model let O be an observer of $\mathcal{M}$ and finally let π be a product built over $\mathcal{S}$. Then,

– π is a *cutset* of O if $\lfloor\pi\rfloor \in [\![O]\!]$.

– π is a *minimal cutset* of O if it is a cutset of O and no product ρ such that $\rho \sqsubset \pi$ is.

The set of minimal cutsets of an observer O is denoted by MCS (O). It can be interpreted as the disjunction of its elements.

Example 9. Consider again the observer $O_{\mathtt{fu}}$. The product $\pi = \mathtt{S1} = \mathtt{fu} \wedge \mathtt{S2} = \mathtt{fd} \wedge \mathtt{S3} = \mathtt{fd}$ is a minimal cutset of $O_{\mathtt{fu}}$.

– It is a cutset, because the minterm τ defined as in example 8 verifies $\tau \in [\![O]\!]$.

– It is minimal because no product σ smaller than π is such that $\overline{\sigma}(O) = 1$.

The following theorem establishes the relationship between prime implicants and minimal cutsets.

Theorem 7 (Prime Implicants versus Minimal Cutsets)**.** *Let $\mathcal{M} : \langle \mathcal{V} = \mathcal{S} \uplus \mathcal{F}, \mathcal{E}\rangle$ be a finite degradation model and let O be an observer of $\mathcal{M}$. Then, the following equality holds.*

$$\mathrm{MCS}\,(O) \quad = \quad \mathrm{PI}\,(\lceil\!\lceil O \rceil\!\rceil)$$

To prove the above theorem it suffices to remark that a product $\pi \in \lceil\!\lceil O \rceil\!\rceil$ if and only if $\overline{\lfloor\pi\rfloor}(O) = 1$.

We shall now give a characterization in terms of states of the notion of minimal cutsets.

4.5 Critical States

Let $\mathcal{D} : \langle D, <, \perp\rangle$ be a finite degradation structure and let $U \subseteq D$. A state $s \in D$ is *critical* for U if $s \in U$ and there is no state $t \in U$ such that $t < s$. The set of critical states of U is denoted CriticalStates (U).

Example 10. The minterm τ defined as in Example 8 is critical for the subset $[\![O]\!]$ of $dom(\mathtt{S1}) \otimes \cdots \otimes dom(\mathtt{V2})$.

The above example is by no means a coincidence, as stated by the following theorem.

Theorem 8 (Minimal Cutsets versus Critical States)**.** *Let $\mathcal{M} : \langle \mathcal{V} = \mathcal{S} \uplus \mathcal{F}, \mathcal{E}\rangle$ be a finite degradation model and let O be an observer of $\mathcal{M}$. Then, the following equality holds.*

$$\mathrm{MCS}\,(O) \quad \cong \quad \mathrm{CriticalStates}\,([\![O]\!])$$

Any minimal cutset π one-to-one corresponds with $\lfloor\pi\rfloor$. It is easy to verify that $\lfloor\pi\rfloor$ is a critical state for $[\![O]\!]$. Reciprocally, any critical state s one-to-one corresponds to a product π (once removed the variables assigned to the least state of their domain). It is easy to verify that π is a minimal cutset. The minimal cutset π that corresponds with a certain critical state σ is thus obtained by removing from σ the information about components that are working properly.

Extracting minimal cutsets of an observer O, or equivalently critical states for this observer, consists actually in defining an epimorphism $\kappa : dom(\mathcal{S}) \rightarrow \mathbf{WCF}$—where **WCF** is the finite degradation structure with three states `w` (working), `c` (failed and critical) and `f` (failed but non critical), such that `w` < `c` < `f`—as stated by the following proposition.

Proposition 9 (Minimal cutsets as epimorphisms)**.** *Let $\mathcal{M} : \langle \mathcal{V} = \mathcal{S} \uplus \mathcal{F}, \mathcal{E}\rangle$ be a finite degradation model and let O be an observer of $\mathcal{M}$. Then,* $\mathrm{MCS}\,(O)$ *is isomorphic to the epimorphism $\kappa : dom(\mathcal{S}) \rightarrow$* **WCF** *defined as follows.*

$$\kappa(\sigma) \quad = \quad \begin{cases} \mathtt{w} & \textit{if } \sigma \notin \lceil\!\lceil O \rceil\!\rceil \\ \mathtt{c} & \textit{if } \sigma \in \mathrm{CriticalStates}\,([\![O]\!]) \\ \mathtt{f} & \textit{if } \sigma \in \lceil\!\lceil O \rceil\!\rceil \setminus \mathrm{CriticalStates}\,([\![O]\!]) \end{cases} \tag{7}$$

The proof follows from the definitions and theorem 8.

Proposition 9 closes the loop: finite degradation models, when assessed via minimal cutsets, can be seen as epimorphisms of **FDS**.

4.6 Probabilities

As in the binary case, minimal cutsets can be used to approximate probabilities of formulas via the so-called *rare event approximation*, denoted by p_{REA}, and *mincut upper bound*, denoted by p_{MCUB}, which are defined as follows.

Let $\mathcal{M} : \langle \mathcal{V} = \mathcal{S} \uplus \mathcal{F}, \mathcal{E}\rangle$ be a finite degradation model let O be an observer of $\mathcal{M}$ and let finally p be a random process associated with $dom(\mathcal{S})$. Then,

$$\begin{aligned} p_{REA}(O) &\overset{def}{=} \sum_{\pi \in \mathrm{MCS}(O)} p(\pi) \\ p_{MCUB}(O) &\overset{def}{=} 1 - \prod_{\pi \in \mathrm{MCS}(O)} 1 - p(\pi) \end{aligned}$$

In practice, when the probabilities of atoms involved in minimal cutsets are sufficiently low and when the formula O is (nearly) coherent, both $p_{REA}(O)$ and $p_{MCUB}(O)$ provide good approximation of $p(O)$.

$$p_{REA}(O) \approx p_{MCUB}(O) \approx p(O) \tag{8}$$

p_{MCUB} has the advantage over p_{REA} to be always comprised between 0 and 1, but the drawback to be less easy to calculate (especially when data structures such as zero-suppressed binary decision diagrams [28] are used to encode the minimal cutsets).

5 Experimental Results

This section presents some experimental results we obtained on the safety instrumented system presented Section 2. The whole model for this system can be interpreted as a function from $\mathbf{W3F}^7$ (as there are 7 basic components) into **W3F**. However, logic solvers and valves can be only in three states: dangerous failures of logic solvers are immediately detected and dangerous failures of valves remain undetected between two tests. The system can thus be in $4^3 \times 3^4 = 5184$ states. This is indeed not very much, but adding a few components would make treatments requiring an explicit representation of the state space (like Markov chains) unfeasible (e.g. $3^{15} \approx 14 \times 10^6$ and $3^{20} \approx 3.4 \times 10^{12}$).

5.1 Assessment Techonology

For the purpose of the present article, we developed a package for Minato's zero-suppressed binary decision diagrams [28] that we adapted for the finite degradation calculus. This technique makes it possible to extract minimal cutsets as well as to calculate performance indicators.

The decision diagrams encoding the state of the HIPPS as well as the observers $O_{\mathtt{fs}}$: `HIPPS = fs`, $O_{\mathtt{fd}}$: `HIPPS = fd` and $O_{\mathtt{fu}}$: `HIPPS = fu`, which encode the states in which the HIPPS is respectively failed safe, failed dangerous detected and failed dangerous undetected. These decision diagrams are made respectively of 89, 72, 52 and 79 nodes. The decision diagrams encoding the minimal cutsets of observers $O_{\mathtt{fs}}$, $O_{\mathtt{fd}}$ and $O_{\mathtt{fu}}$ are made respectively 31, 7 and 17 nodes. All these diagrams as well as the following performance indicators presented below are calculated within in few seconds on an ordinary laptop (most of the computation time is taken by printing out results of calculations).

5.2 Minimal Cutsets

The observer $O_{\mathtt{fs}}$ has 37 minimal cutsets. Among these minimal cutsets, one finds the following ones.

$$\mathtt{S1} = \mathtt{fs} \wedge \mathtt{S2} = \mathtt{fs} \wedge \mathtt{S3} = \mathtt{fs}$$
$$\mathtt{S1} = \mathtt{fs} \wedge \mathtt{S2} = \mathtt{fs} \wedge \mathtt{S3} = \mathtt{fd}$$
$$\mathtt{S1} = \mathtt{fs} \wedge \mathtt{S2} = \mathtt{fd} \wedge \mathtt{S3} = \mathtt{fs}$$
$$\mathtt{S1} = \mathtt{fs} \wedge \mathtt{S2} = \mathtt{fd} \wedge \mathtt{S3} = \mathtt{fd}$$

The observer $O_{\mathtt{fd}}$ has the following 4 minimal cutsets.

$$\mathtt{LS1} = \mathtt{fd} \wedge \mathtt{LS2} = \mathtt{fd}$$
$$\mathtt{LS1} = \mathtt{fd} \wedge \mathtt{S2} = \mathtt{fd} \wedge \mathtt{S3} = \mathtt{fd}$$
$$\mathtt{S1} = \mathtt{fd} \wedge \mathtt{LS2} = \mathtt{fd}$$
$$\mathtt{S1} = \mathtt{fd} \wedge \mathtt{S2} = \mathtt{fd} \wedge \mathtt{S3} = \mathtt{fd}$$

Note that as the valves cannot be failed dangerous detected, no atom built over the variables V1 and V2 shows up in the minimal cutsets of observer $O_{\mathtt{fd}}$.

Finally, observer $O_{\mathtt{fu}}$ has the following 13 minimal cutsets.

$$\mathtt{V1} = \mathtt{fu} \wedge \mathtt{LS2} = \mathtt{fd}$$
$$\mathtt{V1} = \mathtt{fu} \wedge \mathtt{S2} = \mathtt{fd} \wedge \mathtt{S3} = \mathtt{fd}$$
$$\mathtt{V1} = \mathtt{fu} \wedge \mathtt{V2} = \mathtt{fu}$$
$$\mathtt{LS1} = \mathtt{fd} \wedge \mathtt{LS2} = \mathtt{fd} \wedge \mathtt{V2} = \mathtt{fu}$$
$$\mathtt{LS1} = \mathtt{fd} \wedge \mathtt{S2} = \mathtt{fu} \wedge \mathtt{S3} = \mathtt{fd}$$
$$\mathtt{LS1} = \mathtt{fd} \wedge \mathtt{S2} = \mathtt{fd} \wedge \mathtt{S3} = \mathtt{fu}$$
$$\mathtt{LS1} = \mathtt{fd} \wedge \mathtt{S2} = \mathtt{fd} \wedge \mathtt{S3} = \mathtt{fd} \wedge \mathtt{V2} = \mathtt{fu}$$
$$\mathtt{S1} = \mathtt{fu} \wedge \mathtt{LS2} = \mathtt{fd}$$
$$\mathtt{S1} = \mathtt{fu} \wedge \mathtt{S2} = \mathtt{fd} \wedge \mathtt{S3} = \mathtt{fd}$$
$$\mathtt{S1} = \mathtt{fd} \wedge \mathtt{LS2} = \mathtt{fd} \wedge \mathtt{V2} = \mathtt{fu}$$
$$\mathtt{S1} = \mathtt{fd} \wedge \mathtt{S2} = \mathtt{fu} \wedge \mathtt{S3} = \mathtt{fd}$$
$$\mathtt{S1} = \mathtt{fd} \wedge \mathtt{S2} = \mathtt{fd} \wedge \mathtt{S3} = \mathtt{fu}$$
$$\mathtt{S1} = \mathtt{fd} \wedge \mathtt{S2} = \mathtt{fd} \wedge \mathtt{S3} = \mathtt{fd} \wedge \mathtt{V2} = \mathtt{fu}$$

$Time$	$p(O_{\mathtt{fs}})$	$p(O_{\mathtt{fd}})$	$p(O_{\mathtt{fu}})$
$73h$	5.04×10^{-4}	3.34×10^{-9}	4.90×10^{-8}
$146h$	1.98×10^{-3}	1.34×10^{-8}	1.96×10^{-7}
$219h$	4.37×10^{-3}	3.01×10^{-8}	4.41×10^{-7}
$292h$	7.62×10^{-3}	5.35×10^{-8}	7.83×10^{-7}
$365h$	1.17×10^{-2}	8.36×10^{-8}	1.22×10^{-6}
$438h$	1.65×10^{-2}	1.20×10^{-7}	1.76×10^{-6}
$511h$	2.20×10^{-2}	1.64×10^{-7}	2.40×10^{-6}
$584h$	2.82×10^{-2}	2.14×10^{-7}	3.13×10^{-6}
$657h$	3.51×10^{-2}	2.71×10^{-7}	3.96×10^{-6}
$730h$	4.25×10^{-2}	3.34×10^{-7}	4.89×10^{-6}

Table 3: Probabilities of observers $O_{\mathtt{fs}}$, $O_{\mathtt{fd}}$ and $O_{\mathtt{fs}}$ at different times.

5.3 Probabilities of Failures

Table 3 shows the evolution of probabilities of observers with the mission time.

Several remarks can be made here.

First, probabilities of all observers increase with the time. This is not surprising because components are non-repairable (between two maintenance operations).

Second, the probability of safe failure is much higher than the probability of a dangerous one. Again, no surprise here, given the reliability parameters of the components. A practical consequence of that, confirmed by the industrial experience, is that most of the production down-time is due to maintenance operations and spurious triggers of safety systems.

Third, the probability of an undetected dangerous failure is one order of magnitude higher than the probability of a detected one. This is due to valves that have both a quite high (safe) failure rate and whose failures cannot be detected between inspections.

6 Conclusion

In this article, we introduced the notion of finite degradation structures. This notion provides a powerful and unified algebraic framework for Boolean and multistate models. It relies on three fundamental ideas.

First, states of Boolean and multistate models shows a finite semilattice structure. The partial order amongst the states is a degradation order. The bottom element of the semilattice is the nominal operating state.

Second, finite degradation structures can be composed, thanks to a monoidal product, which allows not only to assemble components into models, but also to reason in a uniform way on components, subsystems and systems. Technically, finite degradation structures form a symmetric monoidal category. Some very common finite degradation structures can be seen as on-the-shelf types for modeling components.

Third, epimorphisms between finite degradation structures describe abstractions between models. Many concepts of fault tree analysis can be reinterpreted and better understood by means of such epimorphisms.

We revisited here the familiar notions of coherence, minimal cutsets and importance measures from the new perspective of finite degradation structures. The nicest result we obtained is probably the isomorphism between minimal cutsets and critical states.

The objective of the present article was to present the theoretical foundations of finite degradation structures. In forthcoming articles, we shall discuss implementation issues, i.e. how to lift-up the existing algorithmic corpus on Boolean models to the finite degradation calculus, and modeling methodologies that can be deployed to take a full benefit of the new theoretical framework we introduced here.

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SETTING THE BASIS FOR HERE AND THERE MODAL LOGICS

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Abstract

We define and study a new modal extension of the logic of Here and There with operators from modal logic K. We provide a complete axiomatisation together with several results such as the non-interdefinability of modal operators, the Hennessy-Milner and the finite model properties, a bound for the complexity of the related satisfiability problem and a discussion about the canonicity of some well-known Sahlqvist formulas in our setting. We also consider the equilibrium property on this logic and we prove the theorem of strong equivalence in the resulting framework.

1 Introduction

Modal extensions of intuitionistic logic have been studied in different domains such as philosophical and formal logic [6, 8, 32, 17, 39, 28, 31, 14, 42] or computer science [15, 30, 36]. Recently, combinations of intermediate and modal logics have become very interesting due to their use in the definition of nonmonotonic formalims [33, 34]. These new contributions consist in modal extensions of *Here and There* [26] (HT), also known as Gödel G3 [24] or Smetanich [40] logics.

Two significant works on this topic are the temporal [11] and epistemic [16] extensions of HT. The former approach extends HT with operators from Linear Time Temporal Logic [37]

(LTL) while the latter consist of an orthogonal combination of HT and the modal logic $S5$. Those are specific combinations of modal frames: in the former case models are as in LTL (i.e. linear frames) but every Kripke world behaves as in HT. In case of $S5$ models are equivalence relations where every Kripke point is regarded as a HT system. Such extensions allow extending Pearce's Equilibrium Logic [33, 34] in a natural way.

However, the lack of a general theory supporting such extensions caught our attention. In this paper, we try to fill this gap by proposing a general methodology to define Here and There modal logics that allow extending Equilibrium Logic with modal operators. In our case, more precisely, we consider the combination of propositional HT and the modal logic K (denoted by KHT), where we provide detailed investigation on interdefinability of modal operators, axiomatisation and complexity of the satisfiability problem. Moreover, we also consider the canonicity problem in the new here and there setting. On KHT, we define the concept of *modal equilibrium model* (the modal extension of equilibrium models) and we study several interesting properties, such as the property of strong equivalence, which can serve as a starting point for future modal extensions. The main contribution of this paper is to pave the way for the definition of tree-like extensions of Equilibrium Logic such as *Propositional Dynamic* [25] or *Computational Tree* [13] Equilibrium Logic, which have not been considered in the literature.

Since our contribution is strongly connected with previous work in the literature on *Intuitionistic Modal Logic*, we summarise the different approaches proposed in the literature. The first one was published by Fitch [20] who axiomatise a first-order intuitionistic version of the logic T. As stated by Simpson [39], "from a modern viewpoint, the choice of axioms seems rather arbitrary". They are also remarkable Prior's intuitionistic $S5$ modal logic (named $MIPQ$) [32] and Prawitz's intuitionistic $S4$ [38].

The most popular approach was presented by Fischer Servi [17, 18, 19] (FS), who proposed a framework to determine the correct intuitionistic logic of a classical modal logic based on *birelational* models[1]. The same semantics were discovered independently by Plotkin and Stirling [36]. Our decision of building our proposal based on FS is due to the fact that other existent extensions of Here and There [2] are strongly connected with FS (note that the axiomatic system proposed in [2] extends FS with some extra axioms such as, among others, Hosoi's [27]).

However, Servi's semantic framework is not the only one that has been considered. All differ from each other although they preserve some similarities. Among others, Božić and Došen [6] and Wijesekera [41] proposed a birelational approach but they differ in the interaction between the partial order and the modal accessibility relation.

When interpreting the $\Box$ modality, Wijesekera forces the necessity operator to be inter-

[1]Birelational models are Kripke models containing two accessibility relations: a partial order to interpret the intuitionistic connectives and another accessibility relation used to interpret the modal ones.

preted with respect to both relations in an attempt to imitate the behaviour of the universal quantifier of first-order intuitionistic logic. Instead, Božić and Došen force the models to satisfy an extra confluence property involving both accessibility relations. As a consequence, regarding the $\Diamond$-free fragment, both systems induce the same intuitionistic modal logic. When adding the $\Diamond$ modality, both systems become different. Wijesekera's system, which interprets $\Diamond$ with respect to both accessibility relations, has some strange properties (see [39, page 48]). For instance, the addition of the schema $p \vee \neg p$ is not sufficient to make Wijesekera's logic to collapse to its classical counterpart. Moreover, it is easy to see that the axiom schema $\Diamond (p \vee q) \to (\Diamond p \vee \Diamond q)$, which is valid in FS, is not in Wijesekera's system. Božić and Došen interpret $\Diamond$ only with respect to the accessibility relation for the modal connectives. Under this interpretation, the resulting logic does not satisfy the disjunction property and the modal operators are not semantically independent (i. e. the schemas $\Diamond p \vee \neg \Box p$ and $\Diamond p \leftrightarrow \neg \Box \neg p$ are valid).

Layout This paper is organised as follows. In Section 2 we present syntax and two equivalent alternative semantics based on Kripke models. The former semantics (the "Here and There" semantics) is simulated by two valuation functions while the latter semantics possesses two accessibility relations to interpret implication and modal operators. In Sections 3 and 4, we present a sound and complete axiomatisation of this logic with respect to the birelational semantics. In Section 5 we discuss about the canonicity of some well-known *Sahlqvist* formulas in our setting. Section 6 defines bisimulations for our KHT-modal extensions and we use them to prove the non-interdefinability of modal operators and the Hennessy-Milner property in our framework. Section 7 defines the concept of modal equilibrium logic and shows that such definition is suitable for proving the theorem of *strong-equivalence*. We finish the paper with conclusions and future work.

2 Syntax and semantics

In this section we present the language of KHT, which coincides with the ordinary language of modal logic, and two alternative and equivalent semantics. The former is inspired by the semantics proposed in [11], while the latter is adapted from the semantics of intuitionistic modal logic described in [17, 39].

2.1 Syntax

Let VAR be a countable set of propositional variables (with typical members denoted p, q, etc). The set FOR of all formulas (with typical members denoted φ, ψ, etc) is defined as follows:

$$\varphi := p \mid \bot \mid (\varphi \wedge \psi) \mid (\varphi \vee \psi) \mid (\varphi \rightarrow \psi) \mid \Box\varphi \mid \Diamond\varphi. \tag{1}$$

As in intuitionistic logic, negation is defined in terms of implication ($\varphi \rightarrow \bot$). We will follow the standard rules for omission of the parentheses. Let $|\varphi|$ denote the number of symbols in φ. The *modal degree* of a formula φ (in symbols $deg(\varphi)$) is defined as follows:

$$deg(\varphi) \overset{def}{=} \begin{cases} 0 & if \quad \varphi = p \ (p \in VAR) \text{ or } \varphi = \bot; \\ max(deg(\psi), deg(\chi)) & if \quad \varphi = \psi \odot \chi, \text{ with } \odot \in \{\wedge, \vee, \rightarrow\}; \\ 1 + deg(\psi) & if \quad \varphi = \odot\psi, \text{ with } \odot \in \{\Box, \Diamond\}. \end{cases}$$

Given a nonempty set Γ of formulas, let $deg(\Gamma)$ be a maximal element in $\{deg(\varphi) : \varphi \in \Gamma\}$ when this set is bounded, and ω otherwise.
A *theory* is a (possibly infinite) set of formulas. A theory Σ is *closed* if it is closed under subformulas and for all formulas φ, if $\varphi \in \Sigma$ then $\neg\varphi \in \Sigma$. For all theories x, let $\Box x \overset{def}{=} \{\varphi \mid \Box\varphi \in x\}$ and $\Diamond x \overset{def}{=} \{\Diamond\varphi \mid \varphi \in x\}$.

2.2 KHT semantics

An KHT-frame is a structure $\mathbf{F} = \langle W, R \rangle$ where:

1) W is a nonempty set of worlds;

2) R is a binary relation on W.

The size of $\mathbf{F}$ is the cardinal of W. Given a nonempty set W and $H, T : W \rightarrow 2^{VAR}$, we say that H is included in T (in symbols $H \leq T$) if for all $x \in W$, $H(x) \subseteq T(x)$. We write $H < T$ when $H \leq T$ and $H(x) \neq T(x)$ for some $x \in W$. An KHT-model is a structure $\mathbf{M} = \langle W, R, H, T \rangle$ where:

1) $\langle W, R \rangle$ is an KHT-frame;

2) $H, T : W \rightarrow 2^{VAR}$ are such that $H \leq T$.

Given an KHT-model $\mathbf{M} = \langle W, R, H, T \rangle$, $x \in W$, and $\alpha \in \{h, t\}$, the satisfaction relation of a formula φ at (x, α) (in symbols $\mathbf{M}, (x, \alpha) \models \varphi$) is defined as follows:

- $\mathbf{M}, (x, h) \models p$ if $p \in H(x)$; $\mathbf{M}, (x, t) \models p$ if $p \in T(x)$;
- $\mathbf{M}, (x, \alpha) \not\models \bot$;
- $\mathbf{M}, (x, \alpha) \models \varphi \wedge \psi$ if $\mathbf{M}, (x, \alpha) \models \varphi$ and $\mathbf{M}, (x, \alpha) \models \psi$;

- $\mathbf{M}, (x, \alpha) \models \varphi \vee \psi$ if $\mathbf{M}, (x, \alpha) \models \varphi$ or $\mathbf{M}, (x, \alpha) \models \psi$;
- $\mathbf{M}, (x, \alpha) \models \varphi \to \psi$ if for all $\alpha' \in \{\alpha, t\}$ $\mathbf{M}, (x, \alpha') \not\models \varphi$ or $\mathbf{M}, (x, \alpha') \models \psi$;
- $\mathbf{M}, (x, \alpha) \models \Box\varphi$ if for all $y \in W$, if $xR\, y$ then $\mathbf{M}, (y, \alpha) \models \varphi$;
- $\mathbf{M}, (x, \alpha) \models \Diamond\varphi$ if there exists $y \in W$ such that $xR\, y$ and $\mathbf{M}, (y, \alpha) \models \varphi$.

Given a set Σ of formulas, we write $\mathbf{M}, (x, \alpha) \models \Sigma$ if for all formulas $\varphi \in \Sigma$, $\mathbf{M}, (x, \alpha) \models \varphi$.
The following lemma shows that the KHT semantics satisfy the persistence property of Intuitionistic Logic.

Lemma 1. *Let* $\mathbf{M} = \langle W, R, H, T\rangle$ *and* $\mathbf{M}' = \langle W, R, T, T\rangle$ *be two* KHT *models and let* $\mathbf{M}'' = \langle W, R, T\rangle$ *be a* K *model. For all* $x \in W$ *and for all formula* φ*,*

1) *If* $\mathbf{M}, (x, h) \models \varphi$ *then* $\mathbf{M}, (x, t) \models \varphi$*;*
2) $\mathbf{M}, (x, t) \models \varphi$ *iff* $\mathbf{M}', (x, h) \models \varphi$*;*
3) $\mathbf{M}', (x, h) \models \varphi$ *iff* $\mathbf{M}'', x \models \varphi$ *(in K).*

Proof. By induction on φ. Left to the reader. $\square$

From the 1st item of Lemma 1, we conclude that a formula φ is KHT-satisfiable iff the set of all total KHT-models of φ is nonempty. Hence, since the set of all K-models of φ corresponds to the set of all KHT-models of φ, φ is KHT-satisfiable iff φ is K-satisfiable. Moreover, K has the finite model property and K-satisfiability is PSPACE-complete [4, Chapter 6]. Consequently, KHT has the finite model property and KHT-satisfiability is PSPACE-complete.
Given two sets of formulas Σ and Γ, we say that Γ is a *local* KHT *semantic consequence* of Σ (denoted by $\Sigma \models \Gamma$) if for all KHT models $\mathbf{M} = \langle W, \leq, R, V\rangle$ and for all $(w, \alpha) \in W \times \{h, t\}$, if $\mathbf{M}, (w, \alpha) \models \Sigma$ then $\mathbf{M}, (w, \alpha) \models \Gamma$. Note that, when restricting the language to the propositional connectives, the resulting logic collapses to propositional Here and There. Therefore KHT is a conservative extension of Here and There. The following results can be proved as in Intuitionistic Modal Logic.

Lemma 2. *Let* $\mathbf{M} = \langle W, R\ , H, T\rangle$ *be an* KHT*-model. For all* $x \in W$ *and for all formulas* φ*,* $\mathbf{M}, (x, t) \models \varphi \vee \neg\varphi$*.*

Lemma 3. *Let* $\mathbf{M} = \langle W, R, H, T\rangle$ *be an* KHT*-model. For all* $x \in W$ *and for all formulas* φ*,* $\mathbf{M}, (x, h) \models \neg\neg\varphi$ *iff* $\mathbf{M}, (x, t) \models \varphi$*.*

Let $\simeq$ be the equivalence relation between formulas defined as follows: $\varphi \simeq \psi$ if for all KHT-models $\mathbf{M} = \langle W, R, H, T\rangle$ and for all $x \in W$, $\mathbf{M}, (x, h) \models \varphi$ iff $\mathbf{M}, (x, h) \models \psi$.As in Intuitionistic Logic, for all formulas φ, $\neg\neg\neg\varphi \simeq \neg\varphi$. Hence, by a straightforward induction on the structure of a formula, we obtain the following result.

Lemma 4. *Let φ be a formula. The least closed set of formulas containing φ contains at most $3|\varphi|$ equivalence classes of formulas modulo $\simeq$.*

2.3 Birelational semantics

A *birelational frame* [39] is a structure $\mathbf{F} = \langle W, \leq, R \rangle$ where

1) W is a nonempty set of worlds;
2) $\leq$ is a partial order on W;
3) R is a binary relation on W.

The size of $\mathbf{F}$ is the cardinal of W. $\mathbf{F}$ is said to be *normal* if it satisfies the following conditions for all $x, y, z \in W$:

1) if $x \leq y$ and $x \leq z$ then either $x = y$ or $x = z$ or $y = z$;
2) if $xR\,y$ and $x \leq z$ then there exists $t \in W$ such that $y \leq t$ and $zR\,t$;
3) if $xR\,y$ and $y \leq z$ then there exists $t \in W$ such that $x \leq t$ and $tR\,z$;
4) if $x \leq y$ and $yR\,z$ then there exist $t \in W$ such that $xR\,t$ and $t \leq z$.

If $\mathbf{F}$ is normal then for all $x \in W$, either x is a maximal element with respect to $\leq$, or there exists exactly one $y \in W$ such that $x \leq y$ and $x \neq y$. In the former case let $\widehat{x}$ denote x. In the latter case, let $\widehat{x}$ denote this y. From this definition, it follows that for all $x, y \in W$, $x \leq y$ iff either $y = x$, or $y = \widehat{x}$. $\mathbf{F}$ is said to be *strongly normal* if $\mathbf{F}$ is normal and for all $x, y \in W$:

5) if $\widehat{x}Ry$ then $y = \widehat{y}$;

A *birelational model* is a structure $\mathbf{M} = \langle W, \leq, R\,, V\rangle$ where

1) $\langle W, \leq, R \rangle$ is a birelational frame;
2) $V : W \to 2^{VAR}$ is such that for all $x, y \in W$ if $x \leq y$ then $V(x) \subseteq V(y)$.

Given a birelational model $\mathbf{M} = \langle W, \leq, R, V\rangle$ and $x \in W$, the satisfaction relation of a formula φ at x in $\mathbf{M}$ (in symbols $\mathbf{M}, x \models \varphi$) x is defined as follows:

1. $\mathbf{M}, x \models p$ if $p \in V(x)$;
2. $\mathbf{M}, x \not\models \bot$;
3. $\mathbf{M}, x \models \varphi \wedge \psi$ if $\mathbf{M}, x \models \varphi$ and $\mathbf{M}, x \models \psi$;
4. $\mathbf{M}, x \models \varphi \vee \psi$ if $\mathbf{M}, x \models \varphi$ or $\mathbf{M}, x \models \psi$;
5. $\mathbf{M}, x \models \varphi \to \psi$ if for all $y \in W$ if $x \leq y$ then either $\mathbf{M}, y \not\models \varphi$ or $\mathbf{M}, y \models \psi$;
6. $\mathbf{M}, x \models \Box\varphi$ if for all $y, z \in W$, if $x \leq y$ and $yR\, z$ then $\mathbf{M}, z \models \varphi$;
7. $\mathbf{M}, x \models \Diamond\varphi$ if there exists $y \in W$ such that $xR\, y$ and $\mathbf{M}, y \models \varphi$.

Note that the satisfaction of $\Box\varphi$ and $\Diamond\varphi$ are not symmetrical. Such asymmetry also occurs in Intuitionistic Modal Logic: while the satisfaction of the $\Box$ modality is accepted to be defined in terms of $\leq$ and R to preserve the monotonicity property, the satisfaction of $\Diamond$ is more controversial and several semantics have been discussed in the literature (as discussed in the introduction). One of the most accepted ones does not involve the $\leq$ relation in order to follow Kripke's spirit of keeping the satisfaction of the eventually modality locally. Given two sets of formulas x and y, we say that y is a *local KHT birelational semantic consequence* of x (denoted by $x \models y$) if for all KHT birelational models $\mathbf{M} = \langle W, \leq, R, V\rangle$ and for all $w \in W$, if $\mathbf{M}, w \models x$ then $\mathbf{M}, w \models y$.

Lemma 5. *Let $\mathbf{M} = \langle W, \leq, R, V\rangle$ be a birelational model. If $\mathbf{M}$ is normal then for all $x, y \in W$, if xRy then $\widehat{x}R\widehat{y}$.*

Proof. Suppose that $\mathbf{M}$ is normal. Let $x, y \in W$ be such that xRy. We know that $x \leq \widehat{x}$. Since xRy, there exists $z \in W$ such that $\widehat{x}Rz$ and $y \leq z$. By definition of $\widehat{y}$, either $z = \widehat{y}$ or $z \leq \widehat{y}$. In the former case, since $\widehat{x}Rz$, we conclude that $\widehat{x}R\widehat{y}$. In the latter case, since $\widehat{x}Rz$, there exists $t \in W$ such that $\widehat{x} \leq t$ and $tR\widehat{y}$. By definition of $\widehat{x}$, $\widehat{x} = t$. Thus, $\widehat{x}R\widehat{y}$. $\Box$

2.4 Equivalence between the two semantics

Let $\mathbf{M} = \langle W, R\, , H, T\rangle$ be an KHT model. We define the birelational model $\mathbf{M}' = \langle W', \leq', R', V'\rangle$ as follows:

1) $W' = W \times \{h, t\}$;
2) $(x, \alpha) \leq' (y, \beta)$ if $x = y$ and either $\alpha = h$, or $\beta = t$;
3) $(x, \alpha)R'(y, \beta)$ if $xR\, y$ and $\alpha = \beta$;
4) $V'((x, h)) = H(x)$; $V'((x, t)) = T(x)$.

Note that $\mathbf{M}'$ is strongly normal.

Lemma 6. *Let φ be a formula. For all $x \in W$ and for all $\alpha \in \{h,t\}$, $\mathbf{M},(x,\alpha) \models \varphi$ iff $\mathbf{M}',(x,\alpha) \models \varphi$;*

Proof. By induction on φ. We only consider the cases $\varphi \to \psi$, $\Diamond\varphi$ and $\Box\varphi$:

- $\varphi \to \psi$: Suppose $\mathbf{M}',(x,\alpha) \not\models \varphi \to \psi$ and let $(y,\beta) \in W'$ be such that $(x,\alpha) \leq' (y,\beta)$, $\mathbf{M}',(y,\beta) \models \varphi$ and $\mathbf{M}',(y,\beta) \not\models \psi$. By definition of $\leq'$ and the induction hypothesis, it follows that $x = y$, $\beta \in \{\alpha,t\}$, $\mathbf{M},(x,\beta) \models \varphi$ and $\mathbf{M},(x,\beta) \not\models \psi$. As a consequence, $\mathbf{M},(x,\alpha) \not\models \varphi \to \psi$. Reciprocally, let $\beta \in \{\alpha,t\}$ be such that $\mathbf{M},(x,\beta) \models \varphi$ and $\mathbf{M},(x,\beta) \not\models \psi$. By applying the induction hypothesis and the definition of $\leq'$, we conclude that $\mathbf{M}',(x,\alpha) \not\models \varphi \to \psi$.

- $\Diamond\varphi$: Suppose $\mathbf{M},(x,\alpha) \models \Diamond\varphi$. Let $y \in W$ be such that xRy and $\mathbf{M},(y,\alpha) \models \varphi$. So $\mathbf{M}',(y,\alpha) \models \varphi$. By definition of R', $(x,\alpha)R'(y,\alpha)$ and $\mathbf{M}',(x,\alpha) \models \Diamond\varphi$. Reciprocally, suppose $\mathbf{M}',(x,\alpha) \models \Diamond\varphi$. Let $(y,\beta) \in W'$ be such that $(x,\alpha)R'(y,\beta)$ and $\mathbf{M}',(y,\beta) \models \varphi$. So, by induction hypothesis, $\mathbf{M},(y,\beta) \models \varphi$. Finally, by definition of R', $\alpha = \beta$ and xRy. It follows that $\mathbf{M},(x,\alpha) \models \Diamond\varphi$.

- $\Box\varphi$: Suppose $\mathbf{M},(x,\alpha) \not\models \Box\varphi$. Let $y \in W$ be such that xRy and $\mathbf{M},(y,\alpha) \not\models \varphi$. So $\mathbf{M}',(y,\alpha) \not\models \varphi$. Finally, by the definition of R' and $\leq'$, $\mathbf{M}',(x,\alpha) \not\models \Box\varphi$. Reciprocally, suppose $\mathbf{M}',(x,\alpha) \not\models \Box\varphi$. Let (y,β) and (z,γ) in W' be such that $(x,\alpha) \leq' (y,\beta)R'(z,\gamma)$ and $\mathbf{M}',(z,\gamma) \not\models \varphi$. So, by induction hypothesis, $\mathbf{M},(z,\gamma) \not\models \varphi$. Finally, by Lemma 1 and the definitions of R' and $\leq'$, we conclude that $\mathbf{M},(x,\alpha) \not\models \Box\varphi$.

□

Let $\mathbf{M} = \langle W, \leq, R, V\rangle$ be a strongly normal birelational model. We define $\mathbf{M}' = \langle W', R', H', T'\rangle$ as follows:

1) $W' = W$;

2) $xR'y$ if xRy;

3) $H'(x) = V(x)$;

4) $T'(x) = V(\widehat{x})$.

Lemma 7. *Let φ be a formula. For all $x \in W$, $\mathbf{M}, x \models \varphi$ iff $\mathbf{M}',(x,h) \models \varphi$ and $\mathbf{M}, \widehat{x} \models \varphi$ iff $\mathbf{M}',(x,t) \models \varphi$.*

Proof. By induction on φ. We only consider the cases $\varphi \to \psi$, $\Diamond\varphi$ and $\Box\varphi$:

- $\varphi \to \psi$: Suppose $\mathbf{M}, x \models \varphi \to \psi$. Then for all $y \in \{x, \widehat{x}\}$ either $\mathbf{M}, y \not\models \varphi$, or $\mathbf{M}, y \models \psi$. By induction hypothesis, it follows that for all $\alpha \in \{h, t\}$, either $\mathbf{M}', (x, \alpha) \not\models \varphi$ or $\mathbf{M}', (x, \alpha) \models \psi$. Therefore $\mathbf{M}', (x, h) \models \varphi \to \psi$. The converse direction and the case of $\widehat{x}$ are proved in a similar way.

- $\Diamond\varphi$: Suppose $\mathbf{M}, x \models \Diamond\varphi$. Let $y \in W$ be such that xRy and $\mathbf{M}, y \models \varphi$. Hence, $xR'y$. Moreover, by induction hypothesis, $\mathbf{M}', (y, h) \models \varphi$. Thus $\mathbf{M}', (x, h) \models \Diamond\varphi$. The converse direction is proved in a similar way.

  Suppose $\mathbf{M}, \widehat{x} \models \Diamond\varphi$. Hence, for some $y \in W$, $\mathbf{M}, y \models \varphi$ and $\widehat{x}Ry$. Since $\mathbf{M}$ is strongly normal, $y = \widehat{y}$. Thus, by the induction hypothesis, we obtain that $\mathbf{M}', (x, t) \models \Diamond\varphi$. Conversely, suppose $\mathbf{M}', (x, t) \models \Diamond\varphi$. Let $y \in W$ be such that xRy and $\mathbf{M}', (y, t) \models \varphi$. By Lemma 5 and the induction hypothesis, we obtain that $\widehat{x}R\widehat{y}$ and $M, \widehat{y} \models \varphi$. Hence, $\mathbf{M}, \widehat{x} \models \Diamond\varphi$.

- $\Box\varphi$: Suppose $\mathbf{M}', (x, h) \not\models \Box\varphi$. Let $y \in W'$ be such that $xR'y$ and $\mathbf{M}', (y, h) \not\models \varphi$. By induction hypothesis, $\mathbf{M}, y \not\models \varphi$. By definition of R', xRy and $\mathbf{M}, x \not\models \Box\varphi$. Reciprocally, suppose $\mathbf{M}, x \not\models \Box\varphi$. Let $y, z \in W$ be such that $x \leq y$, yRz and $\mathbf{M}, z \not\models \varphi$. By Condition 4) on normal models, there exists $t \in W$ such that xRt and $t \leq z$. Therefore $\mathbf{M}, t \not\models \varphi$ and $\mathbf{M}', (x, h) \not\models \Box\varphi$.

  Suppose $\mathbf{M}', (x, t) \not\models \Box\varphi$. Let $y \in W'$ be such that $xR'y$ and $\mathbf{M}', (y, t) \not\models \varphi$. By induction hypothesis, $\mathbf{M}, \widehat{y} \not\models \varphi$. Since $xR'y$, by Lemma 5, $\widehat{x}R\widehat{y}$. Therefore $\mathbf{M}, \widehat{x} \not\models \Box\varphi$. Reciprocally, suppose $\mathbf{M}, \widehat{x} \not\models \Box\varphi$. Therefore there exists $x', z \in W$ such that $\widehat{x} \leq x'Rz$ and $\mathbf{M}, z \not\models \varphi$. Since $\mathbf{M}$ is strongly normal, we have that $z = \widehat{z}$. By the induction hypothesis, $\mathbf{M}', (z, t) \not\models \varphi$. It follows that $\mathbf{M}', (x, t) \not\models \Box\varphi$.

$\Box$

As a consequence of lemmas 6 and 7 we obtain the following result.

Lemma 8. *For any modal formula φ, the following statements hold:*

1) φ is satisfiable in the class of all KHT-models iff φ is satisfiable in the class of all strongly normal birelational models;

2) φ is valid in the class of all KHT-models iff φ is valid in the class of all strongly normal birelational models.

3 Axiomatisation

In this section we present an axiomatic system for KHT, which consists of the axioms of Intuitionistic Propositional Logic [12, Chapter 2] plus the following axioms and inference rules:

Hosoi axiom: $p \vee (p \to q) \vee \neg q$;

Fischer Servi Axioms:

(1) $\Box(p \to q) \to (\Box p \to \Box q)$;

(2) $\Box(p \to q) \to (\Diamond p \to \Diamond q)$;

(3) $\Diamond(p \vee q) \to \Diamond p \vee \Diamond q$;

(4) $(\Diamond p \to \Box q) \to \Box(p \to q)$;

(5) $\neg\Diamond\bot$;

(6) $\neg\neg\Diamond p \to \Diamond\neg\neg p$;

(7) $\neg\neg\Box\,(p \vee \neg p)$;

(8) $\Box\,(p \vee q) \to \Box p \vee \Diamond q$;

(9) $\Diamond p \wedge \Box q \to \Diamond\,(p \wedge q)$;

Modus ponens: $\dfrac{\varphi \to \psi,\ \varphi}{\psi}$; **Necessitation**: $\dfrac{\varphi}{\Box\varphi}$.

Proposition 1 (Soundness). *Let φ be a formula. If φ is KHT-derivable then φ is valid in the class of all KHT-models.*

Proof. Left to the reader. It is sufficient to check that all axioms are valid and that inference rules preserve validity. $\Box$

4 Completeness

In this section we prove that the axiomatic system presented in Section 3 is complete with respect to the KHT birelational semantics. We start the section by presenting several results such as the concept of prime sets (equivalent to the maximal consistent sets of classical modal logic) and the canonical model construction for our setting. These tools will be used to prove the completeness result shown at the end of this section.

From now on, KHT will also denote the set of all KHT-derivable formulas. Given two theories x and y, we say that y is a consequence of x (denoted $x \vdash y$) if there exist $\varphi_1, \ldots, \varphi_m \in x$ and $\psi_1, \ldots, \psi_n \in y$ such that $\varphi_1 \wedge \ldots \wedge \varphi_m \to \psi_1 \vee \ldots \vee \psi_n \in KHT$. We shall say that a theory x is *prime* if it satisfies the following conditions:

(1) $\bot \notin x$;

(2) for all formulas φ, ψ, if $\varphi \vee \psi \in x$ then either $\varphi \in x$, or $\psi \in x$;

(3) for all formulas φ, if $x \vdash \varphi$ then $\varphi \in x$.

The next lemma is the Lindenbaum Lemma. Its proof can be found in classical textbooks about Intuitionistic Propositional Logic such as [12, Chapter 2].

Lemma 9 (Lindenbaum Lemma)**.** *Let x and y be theories. If $x \not\vdash y$ then there exists a prime theory z such that $x \subseteq z$ and $z \not\vdash y$.*

The next Lemma shows the connection between Hosoi axiom and the relation of inclusion between prime theories.

Lemma 10. *Let x, y, z be prime theories. If $x \subseteq y$ and $x \subseteq z$ then either $x = y$, or $x = z$, or $y = z$.*

Proof. Suppose $x \subseteq y$, $x \subseteq z$, $x \neq y$, $x \neq z$ and $y \neq z$. Without loss of generality, let φ be a formula such that $\varphi \notin y$ and $\varphi \in z$. Since $x \subseteq y$ and $x \neq y$, let ψ be a formula such that $\psi \notin x$ and $\psi \in y$. By Hosoi axiom, $\psi \vee (\psi \to \varphi) \vee \neg\varphi \in x$. Consequently either $\psi \in x$ or $\psi \to \varphi \in x$ or $\neg\varphi \in x$. Since $\psi \notin x$, either $\psi \to \varphi \in x$ or $\neg\varphi \in x$. In the former case, since $x \subseteq y$, we have that $\psi \to \varphi \in y$. Since $\psi \in y$ then $\varphi \in y$: a contradiction. In the latter case, since $x \subseteq z$, $\neg\varphi \in z$. Since $\varphi \in z$, it follows that $\bot \in z$: a contradiction. $\square$

Proposition 2. *Let x be a prime theory. There exists at most one prime theory strictly containing x.*

Proof. By Lemma 10. $\square$

Hence, for all prime theories x, either x is maximal for inclusion among all prime theories, or there exists exactly one prime theory y such that $x \subseteq y$ and $x \neq y$. In the former case, let $\widehat{x} = x$. In the latter case, let $\widehat{x}$ be this y.

Lemma 11. *Let φ be a formula. For all prime theories x, $\varphi \in \widehat{x}$ iff $\neg\neg\varphi \in x$.*

Proof. Let x be a prime theory. Suppose that $\neg\neg\varphi \in x$ and $\varphi \notin \widehat{x}$. Let $y = \widehat{x} \cup \{\varphi\}$. Suppose $y \vdash \bot$. Let $\psi \in \widehat{x}$ be such that $\psi \wedge \varphi \to \bot \in KHT$. Hence $\psi \to \neg\varphi \in KHT$. Since $\psi \in \widehat{x}$ and $\neg\neg\varphi \in x$, we obtain that $\neg\varphi \in \widehat{x}$ and $\neg\neg\varphi \in \widehat{x}$. Hence, $\bot \in \widehat{x}$: a contradiction. Consequently, $y \not\vdash \bot$. By Lindenbaum Lemma, let z be a prime theory such that $\widehat{x} \subseteq z$ and $\varphi \in z$. By Lemma 10, $\widehat{x} = z$. Since $\varphi \in z$, we obtain $\varphi \in \widehat{x}$: a contradiction.

Suppose $\neg\neg\varphi \notin x$ and $\varphi \in \widehat{x}$. Let $y = x \cup \{\neg\varphi\}$. Suppose that $y \vdash \bot$. Let $\psi \in x$ be such that $\psi \wedge \neg\varphi \to \bot \in KHT$. Thus $\psi \to \neg\neg\varphi \in KHT$. Since $\psi \in x$, $\neg\neg\varphi \in x$: a contradiction. By Lindenbaum Lemma, let z be a prime theory such that $x \subseteq z$ and $\neg\varphi \in z$. Hence $\neg\varphi \in \widehat{x}$. Since $\varphi \in \widehat{x}$, we obtain that $\bot \in \widehat{x}$: a contradiction. $\square$

Lemma 12. *Let φ be a formula. For all prime theories x, either $\varphi \in \widehat{x}$ or $\neg\varphi \in \widehat{x}$.*

Proof. By Lemma 11, using the fact that $\neg\neg(\varphi \vee \neg\varphi) \in KHT$. $\square$

Lemma 13. *Let φ be a formula. For all prime theories x, if $\varphi \notin \widehat{x}$ then $\neg\varphi \in x$.*

Proof. Let x be a prime theory. Suppose that $\varphi \notin \widehat{x}$ and $\neg\varphi \notin x$. Since $\varphi \notin \widehat{x}$, by Lemma 12, $\neg\varphi \in \widehat{x}$. Let $y = x \cup \{\varphi\}$. Suppose that $y \vdash \bot$. Let $\psi \in x$ be such that $\psi \wedge \varphi \rightarrow \bot \in KHT$. Consequently, $\psi \rightarrow \neg\varphi \in KHT$. Since $\psi \in x$, $\neg\varphi \in x$: a contradiction. Thus $y \nvdash \bot$. By Lindenbaum Lemma, let z be a prime theory such that $x \subseteq z$ and $\varphi \in z$. Since $x \subseteq z$, we have that $\varphi \in \widehat{x}$: a contradiction. $\square$

Lemma 14. *Let φ be a formula. For all prime theories x, if $\Diamond\neg\varphi \in x$ then $\Box\varphi \notin x$ and if $\neg\Diamond\varphi \in x$ then $\Box\neg\varphi \in x$.*

Proof. In the first case, assume that $\Box\varphi \in x$. From $\Diamond\neg\varphi \in x$ and the derivable formula[2] $\Diamond\neg\varphi \rightarrow \neg\Box\varphi$ we conclude that $\neg\Box\varphi \in x$. Since $\Box\varphi \in x$, we obtain that $\bot \in x$: a contradiction. The second case is proved by using the derivable formula $\neg\Diamond\varphi \rightarrow \Box\neg\varphi$. $\square$

The canonical model $\mathbf{M}_c$ is defined as the structure $\mathbf{M}_c = \langle W_c, \leq_c, R_c, V_c \rangle$ where:

- W_c is the set of all prime theories;
- $\leq_c$ is the partial order on W_c defined by: $x \leq_c y$ if $x \subseteq y$;
- R_c is the binary relation on W_c defined by: $x R_c y$ if $\Box x \subseteq y$ and $\Diamond y \subseteq x$;
- $V_c : W_c \rightarrow 2^{VAR}$ is the valuation function defined by: $p \in V_c(x)$ if $p \in x$;

Lemma 15. *For all prime theories x and y, if $\widehat{x} R_c y$ then $y = \widehat{y}$.*

Proof. Suppose $\widehat{x} R_c y$ and $y \neq \widehat{y}$. Let φ be a formula such that $\varphi \notin y$ and $\varphi \in \widehat{y}$. By axiom (7), $\neg\neg\Box(\varphi \vee \neg\varphi) \in x$. Hence, by Lemma 11 $\Box(\varphi \vee \neg\varphi) \in \widehat{x}$. Since $\widehat{x} R_c y$, $\varphi \vee \neg\varphi \in y$. Hence, since $\varphi \notin y$ and $\varphi \in \widehat{y}$, we obtain that $\neg\varphi \in y$ and $\neg\varphi \notin y$: a contradiction. $\square$

Lemma 16. *Let x, y be prime theories. If $x R_c y$ then $\widehat{x} R_c \widehat{y}$.*

Proof. Suppose that $x R_c y$ but not $\widehat{x} R_c \widehat{y}$. Hence, $\Box x \subseteq y$ and $\Diamond y \subseteq x$. Moreover, either $\Box\widehat{x} \nsubseteq \widehat{y}$, or $\Diamond\widehat{y} \nsubseteq \widehat{x}$.

[2] Note that the formulas $\Diamond\neg\varphi \rightarrow \neg\Box\varphi$ and $\neg\Diamond\varphi \rightarrow \Box\neg\varphi$ are valid in Simpson's IK [39] (and, therefore, IK-derivable). Since $IK \subseteq KHT$, we conclude that both formulas are KHT-derivable.

- Case $\Box\widehat{x} \not\subseteq \widehat{y}$: Let φ be a formula such that $\Box\varphi \in \widehat{x}$ and $\varphi \notin \widehat{y}$. Hence, by Lemma 13 $\neg\varphi \in y$. Since $\Diamond y \subseteq x$, $\Diamond\neg\varphi \in x$. Thus, $\Diamond\neg\varphi \in \widehat{x}$. Finally, by Lemma 14 we conclude that $\Box\varphi \notin \widehat{x}$: a contradiction.
- Case $\Diamond\widehat{y} \not\subseteq \widehat{x}$: Let φ be a formula such that $\varphi \in \widehat{y}$ and $\Diamond\varphi \notin \widehat{x}$. Hence, by Lemma 13, $\neg\Diamond\varphi \in x$. Thanks to Lemma 14 we conclude that $\Box\neg\varphi \in x$. Since $\Box x \subseteq y$, $\neg\varphi \in y$. Hence $\neg\varphi \in \widehat{y}$: a contradiction.

$\Box$

Proposition 3. $\mathbf{M}_c$ *is a strongly normal birelational model.*

Proof. By Lemma 15, it suffices to show that $\mathbf{M}_c$ is normal. The condition 1) of normality follows from Lemma 10. Conditions 2) and 3) are consequence of Lemma 16. As for Condition 4), let $x, y, t \in W_c$ be such that $x \leq_c y$ and yR_ct and let $z_0 \stackrel{def}{=} \Box x \cup \{\neg\neg\psi \mid \psi \in t\}$. Note that the non-trivial case is when $x \subseteq y$ and $x \neq y$. Thus, $y = \widehat{x}$. Hence, by Lemma 15, $t = \widehat{t}$. In order to prove that $z_0 \not\vdash \{\chi \mid \Diamond\chi \notin x\}$, let us proceed by contradiction. Let φ, ψ and χ be formulas such that $\Box\varphi \in x$, $\psi \in t$, $\Diamond\chi \notin x$ and $\varphi \wedge \neg\neg\psi \to \chi \in KHT$. As a consequence, it follows that $\varphi \to (\neg\neg\psi \to \chi) \in x$. By necessitation and Axiom (1) we conclude that $\Box(\neg\neg\psi \to \chi) \in x$. By Axiom (2), $\Diamond\neg\neg\psi \to \Diamond\chi \in x$. Since $\psi \in t$, yR_ct and $x \leq_c y$, we have that $\Diamond\psi \in y$ and $\neg\neg\Diamond\psi \in x$. From Axiom (6) and modus ponens, it follows that $\Diamond\neg\neg\psi \in x$. Thus, since $\Diamond\neg\neg\psi \to \Diamond\chi \in x$, we obtain $\Diamond\chi \in x$: a contradiction. Thus $z_0 \not\vdash \{\chi \mid \Diamond\chi \notin x\}$. Finally, by applying the Lindenbaum Lemma, let z be a prime theory such that $z_0 \subseteq z$ and $z \not\vdash \{\chi \mid \Diamond\chi \notin x\}$. Obviously, xR_cz and $z \leq_c t$. $\Box$

Lemma 17 (Truth Lemma)**.** *For all formulas φ and for all $x \in W_c$,*

(1) If $\varphi \in x$ then $\mathbf{M}_c, x \models \varphi$;

(2) if $\varphi \notin x$ then $\mathbf{M}_c, x \not\models \varphi$.

Proof. By induction on φ. We only present the proof for $\Diamond$ and $\Box$:

- $\Diamond\psi$: Assume that $\Diamond\psi \in x$ and let $u = \Box x \cup \{\psi\}$. Suppose that $u \vdash \{\chi \mid \Diamond\chi \notin x\}$. This means that there exist two formulas φ and χ such that $\Box\varphi \in x$, $\Diamond\chi \notin x$ and $\varphi \wedge \psi \to \chi \in KHT$. By necessitation and Axiom (2) it follows that $\Diamond(\varphi \wedge \psi) \to \Diamond\chi \in x$. Since, by definition, $\Diamond\chi \notin x$ then $\Diamond(\varphi \wedge \psi) \notin x$. From Axiom (9) it follows that $\Diamond\psi \wedge \Box\varphi \notin x$: a contradiction. Therefore $u \not\vdash \{\chi \mid \Diamond\chi \notin x\}$. Thanks to Lindenbaum Lemma, let $y \in W_c$ be such that $u \subseteq y$ and $y \not\vdash \{\chi \mid \Diamond\chi \notin x\}$. Note that $\Box x \subseteq y$ and $\Diamond y \subseteq x$. Hence, xR_cy. Since $\psi \in y$, by induction hypothesis, $\mathbf{M}_c, y \models \psi$. Since xR_cy, we conclude that $\mathbf{M}_c, x \models \Diamond\varphi$. Reciprocally, assume that $\mathbf{M}_c, x \models \Diamond\psi$. Therefore there exists $y \in W_c$ such that xR_cy and $\mathbf{M}_c, y \models \psi$. By induction hypothesis $\psi \in y$. Finally, by definition of R_c, $\Diamond\psi \in x$.

- $\Box\psi$: Assume that $\Box\psi \in x$ but $\mathbf{M}_c, x \not\models \Box\psi$. From the latter assumption it follows that there exists $x', y \in W_c$ such that $x \leq_c x'$, $x' R_c y$ and $\mathbf{M}_c, y \not\models \psi$. Since $x \leq_c x'$ then $\Box\psi \in x'$. On the other hand, from $x' R_c y$, $\mathbf{M}_c, y \not\models \psi$ and the induction hypothesis we conclude that $\Box\psi \notin x'$, which is a contradiction. Reciprocally, assume that $\Box\psi \notin x$. Let $u = \Box x$. Suppose $u \vdash \{\psi\} \cup \{\chi \mid \Diamond\chi \notin x\}$. Let φ, χ be formulas such that $\Box\varphi \in x$, $\Diamond\chi \notin x$ and $\varphi \to \psi \vee \chi \in KHT$. By necessitation and Axiom (1), $\Box\varphi \to \Box(\psi \vee \chi) \in KHT$. Since $\Box\varphi \in x$, therefore, $\Box(\psi \vee \chi) \in x$. Thus, By Axiom (8), either $\Box\psi \in x$, or $\Diamond\chi \in x$: a contradiction. It follows that $u \not\vdash \{\psi\} \cup \{\chi \mid \Diamond\chi \notin x\}$. By Lindenbaum Lemma, let $y \in W_c$ be such that $u \subseteq y$ and $y \not\vdash \{\psi\} \cup \{\chi \mid \Diamond\chi \notin x\}$. Note that $\Box x \subseteq y$ and $\Diamond y \subseteq x$. Hence, $x R_c y$. Since $y \not\vdash \psi$, therefore, $\psi \notin y$ and, by induction hypothesis, $\mathbf{M}_c, y \not\models \psi$. Since $x R_c y$, we conclude that $\mathbf{M}_c, x \not\models \Box\varphi$.

$\square$

As a result, we can now state the strong completeness of KHT axiomatic system with respect to the KHT birelational semantics:

Proposition 4. *Let φ be a formula. For all sets of formulas x, if $x \models \{\varphi\}$ (i.e. φ is a local KHT birelational semantic consequence) then $x \vdash \{\varphi\}$ (i.e. φ is deducible from x).*

Proof. Suppose that $x \not\vdash \{\varphi\}$. By Lindenbaum Lemma, let x' be a prime theory such that $x \subseteq x'$ and $x' \not\vdash \{\varphi\}$. Hence $\varphi \notin x'$. From $x \subseteq x'$ and the Truth Lemma we get $\mathbf{M}_c, x' \models x$. From $\varphi \notin x'$ and the Truth Lemma it follows that $\mathbf{M}_c, x' \not\models \varphi$. As a consequence $x \not\models \varphi$: a contradiction. $\square$

Corollary 1. *The KHT axiomatic system is also complete with respect to the KHT semantics.*

5 Canonicity

As we have seen in the previous section, the canonical model construction can be transferred to KHT logic with slight modifications for obtaining the proof of completeness mentioned above. In modal logic, Sahlqvist formulas are modal formulas with remarkable properties [4, Chapter 3]: the Sahlqvist correspondence theorem says that every Sahlqvist formula corresponds to a first-order definable class of frames; the Sahlqvist completeness theorem says that when Sahlqvist formulas are used as axioms in a normal logic, the logic is complete with respect to the elementary class of frames the axioms define. Hence, a natural question is to ask whether a Sahlqvist-like theory — i.e. a theory that identifies a set of formulas that correspond to first-order definable classes of frames and that define logics complete with respect to the elementary classes of frames they correspond to — can be elaborated on the setting of KHT logic. It does not seem obvious to answer such a question and we defer attacking it till we grasp what is going on with Sahlqvist formulas in the HT setting. Simply, in this section, with respect to the birelational semantics, we address the above question for the formulas that correspond, in modal logic, to the following elementary properties: seriality and transitivity.

For a start, let us consider the class of all strongly normal birelational frames $(W, \leq, R)$ where R is serial. In modal logic, the formula $\Box p \to \Diamond p$ corresponds to the elementary property of seriality of a frame, i.e. for all frames (W, R), R is serial if (W, R) validates $\Box p \to \Diamond p$. It is also canonical, i.e. the canonical frame of $K + \Box p \to \Diamond p$ validates $\Box p \to \Diamond p$. Within the context of a birelational semantics, $\Box p \to \Diamond p$ still corresponds to seriality.

Lemma 18. *For all strongly normal birelational frames* $(W, \leq, R)$*,* R *is serial if* $(W, \leq, R)$ *validates* $\Box p \to \Diamond p$*.*

Proof. The only if direction is left to the reader. As for the if direction, suppose that R is not serial. Let $x \in W$ be such that $R(x) = \emptyset$. Let V be a valuation on W such that $V(p) = \{z \in W\colon$ there exists $y \in W$ such that $x \leq y$ and $yRz\}$. Obviously, $(W, \leq, R, V), x \models \Box p$ and $(W, \leq, R, V), x \not\models \Diamond p$. $\Box$

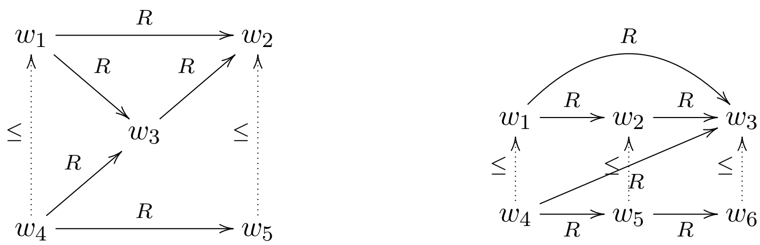
Moreover,

Lemma 19. *The canonical frame of* $KDHT = KHT + \Box p \to \Diamond p$ *is serial.*

Proof. Let x be an arbitrary prime theory. Let $y = \Box x$. Suppose $y \vdash \{\psi : \Diamond\psi \notin x\}$. Let φ, ψ be formulas such that $\Box\varphi \in x$, $\Diamond\psi \notin x$ and $\varphi \to \psi$ is in $KDHT$. By necessitation and Axiom (1), $\Box\varphi \to \Box\psi$ is in $KDHT$. Since $\Box\varphi \in x$, $\Box\psi \in x$. Consequently, $\Diamond\psi \in x$: a contradiction. Hence, $y \not\vdash \{\psi : \Diamond\psi \notin x\}$. By Lindenbaum Lemma, let z be a prime theory such that $y \subseteq z$ and $z \not\vdash \{\psi : \Diamond\psi \notin x\}$. Obviously, xR_cz. Since x was arbitrary, we conclude that R_c is serial. $\Box$

As a result, $KDHT$ is complete with respect to the class of all serial strongly normal birelational models.

Now, let us consider the class of all strongly normal birelational frames $(W, \leq, R)$ where R is transitive. In modal logic, the formula $\Box p \to \Box\Box p$ corresponds to the elementary property of transitivity of a frame. It is also canonical, seeing that $K + \Box p \to \Box\Box p$ has a transitive canonical frame. The formula $\Diamond\Diamond p \to \Diamond p$ possesses these properties as well — correspondence and canonicity. Within the context of the birelational semantics, neither $\Box p \to \Box\Box p$, nor $\Diamond\Diamond p \to \Diamond p$ corresponds to transitivity. In fact, on one hand, the strongly normal frame shown in Figure 1a is non-transitive and, moreover, it validates $\Box p \to \Box\Box p$.



(a) Frame validating $\Box p \to \Box\Box p$. (b) Frame validating $\Diamond\Diamond p \to \Diamond p$.

Figure 1: Non-transitive and strongly normal frames

On the other hand, the strongly normal frame shown in Figure 1b is non-transitive and it validates $\Diamond\Diamond p \to \Diamond p$. The thing is that we do not know if there exists a formula φ such that for all strongly normal birelational frames $(W, \leq, R)$, R is transitive if $(W, \leq, R)$ validates φ. Nevertheless,

Lemma 20. *The canonical frame of* $K4HT = KHT + (\Box p \to \Box\Box p) \wedge (\Diamond\Diamond p \to \Diamond p)$ *is transitive.*

Proof. Let x, y, z be arbitrary prime theories such that xR_cy and yR_cz. Hence, $\Box x \subseteq y$, $\Diamond y \subseteq x$, $\Box y \subseteq z$ and $\Diamond z \subseteq y$. Let φ be a formula such that $\Box\varphi \in x$. Since $\Box\varphi \to \Box\Box\varphi \in x$, $\Box x \subseteq y$, and $\Box y \subseteq z$, we have that $\Box\Box\varphi \in x$, $\Box\varphi \in y$ and $\varphi \in z$. Consequently, $\Box x \subseteq z$. Similarly, on can easily show that $\Diamond z \subseteq x$, this time using a formula of the form $\Diamond\Diamond\varphi \to \Diamond\varphi$. As a result, xR_cz. Since x, y, z were arbitrary, we conclude that R_c is transitive. $\Box$

To the best of our knowledge, the equivalent to the Sahlqvist correspondence theorem for modal extensions of here and there (and even for Simpson's intuitionistic modal logic K [39]) was not considered in the literature. Extending Sahlqvist's result to our setting would help us to prove the completeness for extensions of KHT.

6 About bisimulations

In modal logic, bisimulations are binary relations between models that relate possible worlds carrying the same modal information. However the classical definition of bisimulation must be relaxed in the case of KHT. In this section we provide a definition of bisimulation in the case of KHT, which is sufficient to prove the corresponding bisimulation lemma that states that if two models are bisimilar then they satisfy the same formulas.

6.1 Bisimulations for KHT

Let $\mathbf{M}_1 = \langle W_1, R_1, H_1, T_1 \rangle$ and $\mathbf{M}_2 = \langle W_2, R_2, H_2, T_2 \rangle$ be KHT-models. Let $D_1 = W_1 \times \{h, t\}$ and $D_2 = W_2 \times \{h, t\}$. A binary relation $\mathcal{Z}$ between D_1 and D_2 is a *bisimulation* if the following conditions are satisfied:

1) if $(x_1, \alpha_1)\mathcal{Z}(x_2, \alpha_2)$ then $\mathbf{M}_1, (x_1, \alpha_1) \models p$ iff $\mathbf{M}_2, (x_2, \alpha_2) \models p$ for all propositional variables p;

2) if $(x_1, \alpha_1)\mathcal{Z}(x_2, \alpha_2)$ then $(x_1, t)\mathcal{Z}(x_2, t)$;

3) if $(x_1, \alpha_1)\mathcal{Z}(x_2, \alpha_2)$ and $x_1 R_1 y_1$ then there exists $y_2 \in W_2$ such that $x_2 R_2 y_2$ and either $(y_1, \alpha_1)\mathcal{Z}(y_2, \alpha_2)$ or $(y_1, t)\mathcal{Z}(y_2, \alpha_2)$;

4) if $(x_1, \alpha_1)\mathcal{Z}(x_2, \alpha_2)$ and $x_2 R_2 y_2$ then there exists $y_1 \in W_1$ such that $x_1 R_1 y_1$ and either $(y_1, \alpha_1)\mathcal{Z}(y_2, \alpha_2)$ or $(y_1, \alpha_1)\mathcal{Z}(y_2, t)$;

5) if $(x_1, \alpha_1)\mathcal{Z}(x_2, \alpha_2)$ and $x_2 R_2 y_2$ then there exists $y_1 \in W_1$ such that $x_1 R_1 y_1$ and either $(y_1, \alpha_1)\mathcal{Z}(y_2, \alpha_2)$ or $(y_1, t)\mathcal{Z}(y_2, \alpha_2)$;

6) if $(x_1, \alpha_1)\mathcal{Z}(x_2, \alpha_2)$ and $x_1 R_1 y_1$ then there exists $y_2 \in W_2$ such that $x_2 R_2 y_2$ and either $(y_1, \alpha_1)\mathcal{Z}(y_2, \alpha_2)$ or $(y_1, \alpha_1)\mathcal{Z}(y_2, t)$.

Lemma 21 (Bisimulation Lemma)**.** *Let $\mathbf{M}_1 = \langle W_1, R_1, H_1, T_1 \rangle$ and $\mathbf{M}_2 = \langle W_2, R_2, H_2, T_2 \rangle$ be KHT-models. Let $D_1 = W_1 \times \{h, t\}$ and $D_2 = W_2 \times \{h, t\}$. Let $\mathcal{Z}$ be a bisimulation between D_1 and D_2 and let φ be a formula. For all $(x_1, \alpha_1) \in D_1$ and for all $(x_2, \alpha_2) \in D_2$, if $(x_1, \alpha_1)\mathcal{Z}(x_2, \alpha_2)$ then $\mathbf{M}_1, (x_1, \alpha_1) \models \varphi$ iff $\mathbf{M}_2, (x_2, \alpha_2) \models \varphi$.*

Proof. By induction on φ. We only consider the cases $\varphi \to \psi$, $\Diamond\varphi$ and $\Box\varphi$.

- $\varphi \to \psi$: Suppose $(x_1, \alpha_1)\mathcal{Z}(x_2, \alpha_2)$, $\mathbf{M}_1, (x_1, \alpha_1) \models \varphi \to \psi$. From the former assumption and Condition 2) we conclude $(x_1, t)\mathcal{Z}(x_2, t)$. From the latter assumption it follows that for all $\alpha'_1 \in \{\alpha_1, t\}$. $\mathbf{M}, (x_1, \alpha'_1) \not\models \varphi$ or $\mathbf{M}, (x_1, \alpha'_1) \models \psi$. If $\alpha'_1 = \alpha_1$ then, by $(x_1, \alpha_1)\mathcal{Z}(x_2, \alpha_2)$ and induction hypothesis we get $\mathbf{M}, (x_2, \alpha_2) \not\models \varphi$ or $\mathbf{M}, (x_2, \alpha_2) \models \psi$. If $\alpha'_1 = t$ then, by $(x_1, t)\mathcal{Z}(x_2, t)$ and the induction hypothesis, we get $\mathbf{M}, (x_2, t) \not\models \varphi$ or $\mathbf{M}, (x_2, t) \models \psi$. From this it follows that $\mathbf{M}, (x_2, \alpha_2) \models \varphi \to \psi$. The converse direction is proved in a similar way.

- $\Diamond\varphi$: Suppose $(x_1, \alpha_1)\mathcal{Z}(x_2, \alpha_2)$ and $\mathbf{M}_1, (x_1, \alpha_1) \models \Diamond\varphi$. Let $y_1 \in W_1$ be such that $x_1 R_1 y_1$ and $\mathbf{M}_1, (y_1, \alpha_1) \models \varphi$. By Lemma 1, $\mathbf{M}_1, (y_1, t) \models \varphi$ holds as well. By Condition 3) and the induction hypothesis it follows that $\mathbf{M}_2, (y_2, \alpha_2) \models \varphi$ for some $y_2 \in W_2$ satisfying $x_2 R_2 y_2$. Therefore $\mathbf{M}_2, (x_2, \alpha_2) \models \Diamond\varphi$. The converse direction is proved in a similar way, this time using Condition 4).

- $\Box\varphi$: Suppose $(x_1, \alpha_1)\mathcal{Z}(x_2, \alpha_2)$ and $\mathbf{M}_2, (x_2, \alpha_2) \not\models \Box\varphi$. Let $y_2 \in W_2$ be such that $x_2 R_2 y_2$ and $\mathbf{M}_2, (y_2, \alpha_2) \not\models \varphi$. By Lemma 1, Condition 5) and the induction hypothesis it follows that $\mathbf{M}_1, (y_1, \alpha_1) \not\models \varphi$ for some $y_1 \in W_1$ satisfying $x_1 R_1 y_1$. Therefore $\mathbf{M}_1, (x_1, \alpha_1) \not\models \Box\varphi$. The converse direction is proved in a similar way but using Condition 6).

□

Obviously, the union of two bisimulations is also a bisimulation. Hence, there exists a maximal bisimulation $\mathcal{Z}_{max}$ between D_1 and D_2.

6.2 Interdefinability of Modal Operators

As an application of bisimulation, in this section we prove that $\Box$ and $\Diamond$ are not interdefinable. To do so, we introduce the concepts of $\Box$-free and $\Diamond$-free bisimulations. Let $\mathbf{M}_1 = \langle W_1, R_1, H_1, T_1 \rangle$ and $\mathbf{M}_2 = \langle W_2, R_2, H_2, T_2 \rangle$ be KHT-models. Let $D_1 = W_1 \times \{h, t\}$ and $D_2 = W_2 \times \{h, t\}$. A binary relation $\mathcal{Z}$ between D_1 and D_2 is a *$\Box$-free-bisimulation* if it satisfies conditions 1)-4) of bisimulations. In the same way, a binary relation $\mathcal{Z}$ between D_1 and D_2 is a *$\Diamond$-free-bisimulation* if it satisfies conditions 1)-2) and 5)-6) of bisimulations.

Proposition 5. *Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be KHT-models, $D_1 = W_1 \times \{h, t\}$, $D_2 = W_2 \times \{h, t\}$ and $\mathcal{Z}$ be a binary relation between D_1 and D_2.*

- *If $\mathcal{Z}$ is a $\Diamond$-free-bisimulation then $\mathbf{M}_1$ and $\mathbf{M}_2$ satisfy the same $\Diamond$-free formulas;*
- *If $\mathcal{Z}$ is a $\Box$-free-bisimulation then $\mathbf{M}_1$ and $\mathbf{M}_2$ satisfy the same $\Box$-free formulas.*

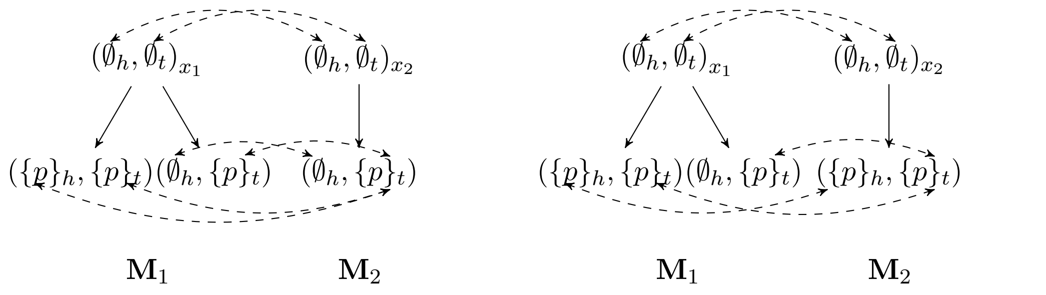
Proof. Similar to the proof of the Bisimulation Lemma. □

To show that $\Diamond$ is not definable in terms of $\Box$, let us consider the models shown in Figure 2a. It can be checked that $\mathbf{M}_1, (x_1, h) \models \Diamond p$ but $\mathbf{M}_2, (x_2, h) \not\models \Diamond p$. However, as shown in such figure, there exists a $\Diamond$-free-bisimulation between them. As a result,

Proposition 6. *There is no $\Diamond$-free formula φ such that $\models \Diamond p \leftrightarrow \varphi$.*

Proof. We proceed by contradiction. Assume that such φ exists. Let us consider the models $\mathbf{M}_1$ and $\mathbf{M}_2$ together with the points x_1 and x_2 from Figure 2a. Note that $\mathbf{M}_1, (x_1, h) \models \Diamond p$. Therefore $\mathbf{M}_1, (x_1, h) \models \varphi$. Thanks to Proposition 5 and the $\Diamond$-free bisimulation described in Figure 2a, $\mathbf{M}_2, (x_2, h) \models \varphi$. As a consequence, $\mathbf{M}_2, (x_2, h) \models \Diamond p$: a contradiction. □

To show that $\Box$ is not definable in terms of $\Diamond$, we consider the models presented in Figure 2b, in which $\mathbf{M}_2, (x_2, h) \models \Box p$ and $\mathbf{M}_1, (x_1, h) \not\models \Box p$. As shown in such figure, there exists a $\Box$-free bisimulation among these models. As a result,



(a) $\Diamond$-free bimimulation. (b) $\Box$-free bimimulation.

Figure 2: $\Diamond$-free and $\Box$-free bimimulations.

Proposition 7. *There is no $\Box$-free formula φ such that $\models \Box p \leftrightarrow \varphi$.*

Proof. We proceed by contradiction. Assume that such φ exists. Let us consider the models $\mathbf{M}_1$ and $\mathbf{M}_2$ together with the points x_1 and x_2 from Figure 2b. Note that $\mathbf{M}_1, (x_1, h) \not\models \Box p$. Consequently, $\mathbf{M}_1, (x_1, h) \not\models \varphi$. Thanks to Proposition 5 and the $\Box$-free bisimulation described in Figure 2b, $\mathbf{M}_2, (x_2, h) \not\models \varphi$. As a consequence, $\mathbf{M}_2, (x_2, h) \not\models \Box p$: a contradiction. $\Box$

6.3 Hennessy-Milner property

In order to show that our definition of bisimulation is appropriate, in this section we show that KHT possesses the Hennessy-Milner property. Our proof follows the line of reasoning suggested in [30]. Let $\mathbf{M}_1 = \langle W_1, R_1, H_1, T_1 \rangle$ and $\mathbf{M}_2 = \langle W_2, R_2, H_2, T_2 \rangle$ be finite KHT models. Let $D_1 = W_1 \times \{h, t\}$ and $D_2 = W_2 \times \{h, t\}$. We define the binary relation $\leftrightsquigarrow$ between D_1 and D_2 as follows: $(x_1, \alpha_1) \leftrightsquigarrow (x_2, \alpha_2)$ if for all formulas φ, $\mathbf{M}_1, (x_1, \alpha_1) \models \varphi$ iff $\mathbf{M}_2, (x_2, \alpha_2) \models \varphi$.

Lemma 22 (Hennesy-Milner property)**.** *The binary relation $\leftrightsquigarrow$ is a bisimulation between $\mathbf{M}_1$ and $\mathbf{M}_2$.*

Proof. Suppose that the binary relation $\leftrightsquigarrow$ is not a bisimulation. Hence, by Lemma 21, one of the conditions 1)-6) does not hold for some $(x_1, \alpha_1) \in D_1$ and some $(x_2, \alpha_2) \in D_2$ such that $(x_1, \alpha_1) \leftrightsquigarrow (x_2, \alpha_2)$.

Assume that Condition 1) is not satisfied. Hence there exists an atom p such that, without loss of generality, $\mathbf{M}_1, (x_1, \alpha_1) \models p$ and $\mathbf{M}_2, (x_2, \alpha_2) \not\models p$. Therefore not $(x_1, \alpha_1) \leftrightsquigarrow (x_2, \alpha_2)$: a contradiction.

Assume that Condition 2) is not satisfied. Thus $(x_1, \alpha_1) \leftrightsquigarrow (x_2, \alpha_2)$ and not $(x_1, t) \leftrightsquigarrow (x_2, t)$. Assume, without loss of generality, that there exists a formula φ such that $\mathbf{M}_1$,

$(x_1, t) \models \varphi$ and $\mathbf{M}_2, (x_2, t) \not\models \varphi$. Thus $\mathbf{M}_1, (x_1, \alpha_1) \not\models \neg\varphi$ and $\mathbf{M}_2, (x_2, \alpha_2) \models \neg\varphi$: a contradiction.

Assume that Condition 3) is not satisfied: Then $(x_1, \alpha_1) \leftrightsquigarrow (x_2, \alpha_2)$ and there exists $y_1 \in W_1$ such that $x_1 R_1 y_1$ and for all $y_2 \in W_2$, if $x_2 R_2 y_2$ then not $(y_1, \alpha_1) \leftrightsquigarrow (y_2, \alpha_2)$ and not $(y_1, t) \leftrightsquigarrow (y_2, \alpha_2)$. Let $R_2(x_2) \stackrel{def}{=} \{(y_2, \alpha_2) \in D_2 \mid x_2 R_2 y_2\}$. Since for all $y_2 \in W_2$, if $x_2 R_2 y_2$ then not $(y_1, \alpha_1) \leftrightsquigarrow (y_2, \alpha_2)$ and not $(y_1, t) \leftrightsquigarrow (y_2, \alpha_2)$, there exist $I, J \subseteq R_2(x_2)$ and for all $(y_2, \alpha_2) \in R_2(x_2)$ there exist formulas $\varphi(y_2, \alpha_2)$ and $\psi(y_2, \alpha_2)$ such that

1) $\mathbf{M}_1, (y_1, \alpha_1) \models \varphi(y_2, \alpha_2)$ and $\mathbf{M}_2, (y_2, \alpha_2) \not\models \varphi(y_2, \alpha_2)$ if $(y_2, \alpha_2) \in I$;

2) $\mathbf{M}_1, (y_1, \alpha_1) \not\models \varphi(y_2, \alpha_2)$ and $\mathbf{M}_2, (y_2, \alpha_2) \models \varphi(y_2, \alpha_2)$ if $(y_2, \alpha_2) \in \overline{I}$;

3) $\mathbf{M}_1, (y_1, t) \models \psi(y_2, \alpha_2)$ and $\mathbf{M}_2, (y_2, \alpha_2) \not\models \psi(y_2, \alpha_2)$ if $(y_2, \alpha_2) \in J$;

4) $\mathbf{M}_1, (y_1, t) \not\models \psi(y_2, \alpha_2)$ and $\mathbf{M}_2, (y_2, \alpha_2) \models \psi(y_2, \alpha_2)$ if $(y_2, \alpha_2) \in \overline{J}$.

Let us define $\chi(y_2, \alpha_2)$ as the following formula:

$$\chi(y_2, \alpha_2) = \begin{cases} \varphi(y_2, \alpha_2) & \text{if} \quad (y_2, \alpha_2) \in I; \\ \varphi(y_2, \alpha_2) \rightarrow \psi(y_2, \alpha_2) & \text{if} \quad (y_2, \alpha_2) \in \overline{I} \cap J; \\ \neg\psi(y_2, \alpha_2) & \text{if} \quad (y_2, \alpha_2) \in \overline{I} \cap \overline{J}. \end{cases}$$

It follows that $\mathbf{M}_1, (y_1, \alpha_1) \models \chi(y_2, \alpha_2)$ and $\mathbf{M}_2, (y_2, \alpha_2) \not\models \chi(y_2, \alpha_2)$, for all $(y_2, \alpha_2) \in R_2(x_2)$. Therefore

$$\mathbf{M}_1, (x_1, \alpha_1) \models \Diamond \bigwedge_{(y_2, \alpha_2) \in R_2(x_2)} \chi(y_2, \alpha_2)$$

while

$$\mathbf{M}_2, (x_2, \alpha_2) \not\models \Diamond \bigwedge_{(y_2, \alpha_2) \in R_2(x_2)} \chi(y_2, \alpha_2) :$$

a contradiction.

The proof for Condition 4) is similar to the proof of Condition 3)

Assume that Condition 5) is not satisfied: Then $(x_1, \alpha_1) \leftrightsquigarrow (x_2, \alpha_2)$ and there exists $y_2 \in W_2$ such that $x_2 R_1 y_2$ and for all $y_1 \in W_1$, if $x_1 R_1 y_1$ then not $(y_1, \alpha_1) \leftrightsquigarrow (y_2, \alpha_2)$ and not $(y_1, t) \leftrightsquigarrow (y_2, \alpha_2)$. Let $R_1(x_1) \stackrel{def}{=} \{(y_1, \alpha_1) \in D_1 \mid x_1 R_2 y_1\}$. Since for all $y_1 \in W_1$, if $x_1 R_1 y_1$ then not $(y_1, \alpha_1) \leftrightsquigarrow (y_2, \alpha_2)$ and not $(y_1, t) \leftrightsquigarrow (y_2, \alpha_2)$, there exist $I, J \subseteq R_1(x_1)$ and for all $(y_1, \alpha_1) \in R_1(x_1)$ there exist formulas $\varphi(y_1, \alpha_1)$ and $\psi(y_1, \alpha_1)$ such that

1) $\mathbf{M}_1, (y_1, \alpha_1) \models \varphi(y_1, \alpha_1)$ and $\mathbf{M}_2, (y_2, \alpha_2) \not\models \varphi(y_1, \alpha_1)$ if $(y_1, \alpha_1) \in I$;

2) $\mathbf{M}_1, (y_1, \alpha_1) \not\models \varphi(y_1, \alpha_1)$ and $\mathbf{M}_2, (y_2, \alpha_2) \models \varphi(y_1, \alpha_1)$ if $(y_1, \alpha_1) \in \overline{I}$;

3) $\mathbf{M}_1, (y_1, t) \models \psi(y_1, \alpha_1)$ and $\mathbf{M}_2, (y_2, \alpha_2) \not\models \psi(y_1, \alpha_1)$ if $(y_1, \alpha_1) \in J$;

4) $\mathbf{M}_1, (y_1, t) \not\models \psi(y_1, \alpha_1)$ and $\mathbf{M}_2, (y_2, \alpha_2) \models \psi(y_1, \alpha_1)$ if $(y_1, \alpha_1) \in \overline{J}$.

Let us define $\chi(y_1, \alpha_1)$ as the following formula:

$$\chi(y_1, \alpha_1) \stackrel{def}{=} \begin{cases} \varphi(y_1, \alpha_1) & \text{if} \quad (y_1, \alpha_1) \in I; \\ \varphi(y_1, \alpha_1) \to \psi(y_1, \alpha_1) & \text{if} \quad (y_1, \alpha_1) \in \overline{I} \cap J; \\ \neg\psi(y_1, \alpha_1) & \text{if} \quad (y_1, \alpha_1) \in \overline{I} \cap \overline{J}. \end{cases}$$

It follows that $\mathbf{M}_1, (y_1, \alpha_1) \models \chi(y_1, \alpha_1)$ and $\mathbf{M}_2, (y_2, \alpha_2) \not\models \chi(y_1, \alpha_1)$, for all $(y_1, \alpha_1) \in R_1(x_1)$. Therefore

$$\mathbf{M}_1, (x_1, \alpha_1) \models \Diamond \bigwedge_{(y_1, \alpha_1) \in R_1(x_1)} \chi(y_1, \alpha_1)$$

while

$$\mathbf{M}_2, (x_2, \alpha_2) \not\models \Diamond \bigwedge_{(y_1, \alpha_1) \in R_1(x_1)} \chi(y_1, \alpha_1) :$$

a contradiction.
The proof for Condition 6) is similar to the proof for Condition 5).

□

Remark how the formulas defining χ above are related to the Hosoi Axiom $p \vee (p \to q) \vee \neg q$.

7 Strong equivalence property

Pearce's Equilibrium logic [33] is the best-known logical characterization of the stable models semantics [23] and of Answer Sets [7]. It is defined in terms of the monotonic logic of Here and There [34] (HT) plus a minimisation criterion among the given models. This simple definition led to several modal extensions of *Answer Set Programming* [11, 16]. All these extensions have their roots in the corresponding modal extensions of HT-logic defined as the combination of propositional HT and any modal logic [22] that play an important role in the proof of several interesting properties of the resulting formalisms such as *strong equivalence* [10, 16, 29] and the complexity [5, 9]. Although the modal extensions of the HT-logic have been studied only in concrete cases such as the Linear Time Temporal KHT-logic [2], the lack of a general theory that allows defining such modal HT extension as well as extending the concept of equilibrium model to modal case caught our attention. In this section, we define the concept of pointed equilibrium model and prove the associated theorem of strong equivalence.

Let $\mathbf{M} = \langle W, R\ , H, T\rangle$ be a KHT model. A KHT *pointed model* is a pair $(\mathbf{M}, x)$ where $x \in W$. $(\mathbf{M}, x)$ is said to be *total* if $H = T$. Moreover, by the expression $H <^x_k T$ we mean that there exists $y \in W$ and $0 \leq k$ such that $xR^k y$ and $H(y) \neq T(y)$. A total KHT pointed model $(\mathbf{M}, x)$ is a *pointed equilibrium model* of a formula φ if

1) $\mathbf{M}, (x, h) \models \varphi$;
2) For all KHT models $\mathbf{M}' = \langle W, R, H', T\rangle$[3] and for all $0 \leq k \leq deg(\varphi)$ if $H' <^x_k T$ then $\mathbf{M}', (x, h) \not\models \varphi$.

As an example, let us consider the three models displayed in Figure 3. $(\mathbf{M}_1, x)$, $(\mathbf{M}_2, x)$ and $(\mathbf{M}_3, x)$ correspond to three different KHT pointed models of the formula $\Diamond p$. For each Kripke world, the "here" component is represented with the subscript h and the "there" part is done by the subscript t. While $(\mathbf{M}_1, x)$ is a pointed equilibrium model (i.e. it is total and minimal with respect to $<^x_0$ and $<^x_1$, $\mathbf{M}_2$ is not. Although $(\mathbf{M}_2, x)$ is a total KHT pointed model satisfying $\Diamond p$, it does not satisfy Condition 2) being $(\mathbf{M}_3, x)$ a counterexample.

$$\begin{tikzcd} x\ (\emptyset_h, \emptyset_t) \arrow[d, "R"] \\ (\{p\}_h, \{p\}_t) \end{tikzcd}$$

(a) $(\mathbf{M}_1, x)$.

$$\begin{tikzcd} & x\ (\emptyset_h, \emptyset_t) \arrow[dl, "R"'] \arrow[dr, "R"] & \\ (\{p\}_h, \{p\}_t) & & (\{p\}_h, \{p\}_t) \end{tikzcd}$$

(b) $(\mathbf{M}_2, x)$.

$$\begin{tikzcd} & x\ (\emptyset_h, \emptyset_t) \arrow[dl, "R"'] \arrow[dr, "R"] & \\ (\emptyset_h, \{p\}_t) & & (\{p\}_h, \{p\}_t) \end{tikzcd}$$

(c) $(\mathbf{M}_3, x)$.

Figure 3: Three KHT pointed models satisfying $\Diamond p$.

Lemma 23. *Let* $\mathbf{M} = \langle W, R, H, T\rangle$ *be a* KHT *model and let* $\widehat{\mathbf{M}} = \langle W, R, T, T\rangle$ *denote its corresponding total model. Let* $\Gamma_0 \stackrel{def}{=} \{\Box^k (p \vee \neg p) \mid 0 \leq k$ *and* $p \in VAR\}$ *be a theory. For all* KHT *pointed models* $(\mathbf{M}, x)$*,* $\mathbf{M}, (x, h) \models \Gamma_0$ *iff for all* $k \geq 0$*,* $H \not<^x_k T$*.*

Proof. From left to right, assume by contradiction that $H <^x_k T$ for some $k \geq 0$. Therefore, there exists $y \in W$ such that $xR^k y$ and $H(y) \neq T(y)$. Let $p \in VAR$ be such that $p \in (T(y) \setminus H(y))$. It can be checked that $\mathbf{M}, (y, h) \not\models (p \vee \neg p)$. Since $xR^k y$ we get $\mathbf{M}, (x, h) \not\models \Box^k (p \vee \neg p)$. As a consequence, $\mathbf{M}, (x, h) \not\models \Gamma_0$: a contradiction. Conversely, assume that $\mathbf{M}, (x, h) \not\models \Gamma_0$, so $\mathbf{M}, x \not\models \Box^k (p \vee \neg p)$ for some $0 \leq k$. This means that there exists $y \in W$ such that $xR^k y$ and $\mathbf{M}, (y, h) \not\models p \vee \neg p$, so $H(y) \neq T(y)$. Therefore, $H <^x_k T$: a contradiction. $\Box$

[3]Note that $\mathbf{M}'$ differs from $\mathbf{M}$ only in the valuation of the "here" component.

Two theories Γ_1 and Γ_2 are *KHT-equivalent* (in symbols $\Gamma_1 \equiv_{KHT} \Gamma_2$) if they have the same KHT pointed models. A total KHT pointed model $(\mathbf{M}, x)$ is a *pointed equilibrium model* of a theory Γ if

1) $\mathbf{M}, (x, h) \models \Gamma$;
2) For all KHT models $\mathbf{M}' = \langle W, R, H', T\rangle$ and for all $0 \leq k \leq deg(\Gamma)$, if $H' <^x_k T$ then $\mathbf{M}', (x, h) \not\models \Gamma$.

When dealing with non-monotonicity the relation of equivalence between theories depends on the context where they are considered. We say that two theories Γ_1 and Γ_2 are *strongly equivalent* [29] (in symbols $\Gamma_1 \equiv_s \Gamma_2$) if for all theories Γ, $\Gamma_1 \cup \Gamma$ and $\Gamma_2 \cup \Gamma$ have the same pointed equilibrium models.

Proposition 8. *For all theories Γ_1 and Γ_2 such that $deg(\Gamma_1) = deg(\Gamma_2)$, $\Gamma_1 \equiv_s \Gamma_2$ iff $\Gamma_1 \equiv_{KHT} \Gamma_2$.*

Proof. From right to left, suppose $\Gamma_1 \equiv_{KHT} \Gamma_2$ and let Γ be an arbitrary theory. Consequently, $(\Gamma_1 \cup \Gamma) \equiv_{KHT} (\Gamma_2 \cup \Gamma)$ so $\Gamma_1 \cup \Gamma$ and $\Gamma_2 \cup \Gamma$ have the same pointed equilibrium models.
From left to right, suppose that Γ_1 and Γ_2 are strongly equivalent but they are not KHT-equivalent. Let $\Gamma_0 \stackrel{def}{=} \{\Box^k (p \vee \neg p) \mid 0 \leq k \text{ and } p \in VAR\}$.

- **First case:** Γ_1 and Γ_2 are not K-equivalent. Without loss of generality, there exists a total KHT pointed model $(\mathbf{M}, x)$, with $\mathbf{M} = \langle W, R, T, T\rangle$, such that $\mathbf{M}, (x, h) \models \Gamma_1$ but $\mathbf{M}, (x, h) \not\models \Gamma_2$. It can be checked that $(\mathbf{M}, x)$ is a pointed equilibrium model of $\Gamma_1 \cup \Gamma_0$ but not of $\Gamma_2 \cup \Gamma_0$.

- **Second case:** Γ_1 and Γ_2 are K-equivalent. Therefore, without loss of generality, there exists a KHT model $\mathbf{M} = \langle W, R, H, T\rangle$ (whose corresponding total model is denoted by $\widehat{\mathbf{M}} = \langle W, R, T, T\rangle$) and $x \in W$ such that

  (1) $\mathbf{M}, (x, t) \models \Gamma_1$ iff $\mathbf{M}, (x, t) \models \Gamma_2$ because both Γ_1 and Γ_2 are K-equivalent;
  (2) $\mathbf{M}, (x, h) \models \Gamma_1$ and $\mathbf{M}, (x, h) \not\models \Gamma_2$ because Γ_1 and Γ_2 are not KHT-equivalent;

  Since $\mathbf{M}, (x, h) \not\models \Gamma_2$, there exists $\varphi \in \Gamma_2$ such that $\mathbf{M}, (x, h) \not\models \varphi$. Since $\mathbf{M}, (x, t) \models \Gamma_2$ then $\mathbf{M}, (x, t) \models \varphi$. Since $\mathbf{M}, (x, h) \not\models \varphi$, therefore, there exits $0 \leq k \leq deg(\varphi)$ such that $H <^x_k T$. Let $\Gamma \stackrel{def}{=} \{\varphi \rightarrow \psi \mid \psi \in \Gamma_0\}$. Note that $\mathbf{M}, (x, h) \models \Gamma_1 \cup \Gamma$ since $\mathbf{M}, (x, h) \models \Gamma_1$, $\mathbf{M}, (x, h) \not\models \varphi$ and $\mathbf{M}, (x, t) \models \Gamma_0$. By Lemma 1, $\widehat{\mathbf{M}}, (x, h) \models \Gamma_1 \cup \Gamma$. All in all, we know that: $\mathbf{M}, (x, h) \models \Gamma_1 \cup \Gamma$,

$\widehat{\mathbf{M}}, (x, h) \models \Gamma_1 \cup \Gamma$ and $H <_k^x T$, where $k \leq deg(\Gamma_1 \cup \Gamma)$. As a result, $(\widehat{\mathbf{M}}, x)$ is not a pointed equilibrium model of $\Gamma_1 \cup \Gamma$. Since Γ_1 and Γ_2 are strongly equivalent, $(\widehat{\mathbf{M}}, x)$ is not a pointed equilibrium model of $\Gamma_2 \cup \Gamma$. Since Γ_1 and Γ_2 are K-equivalent, $\Gamma_1 \cup \Gamma$ and $\Gamma_2 \cup \Gamma$ are K-equivalent. Hence, $\widehat{\mathbf{M}}, (x, h) \models \Gamma_2 \cup \Gamma$. Since $(\widehat{\mathbf{M}}, x)$ is not a pointed equilibrium model of $\Gamma_2 \cup \Gamma$, there exists $\mathbf{M}' = \langle W, R, H', T \rangle$ and $0 \leq k' \leq deg(\Gamma_2 \cup \Gamma)$ such that $H' <_{k'}^x T$ and $\mathbf{M}', (x, h) \models \Gamma_2 \cup \Gamma$. However, from $\mathbf{M}', (x, h) \models \Gamma_2 \cup \Gamma$ and the fact that $\varphi \in \Gamma_2$, it follows that $\mathbf{M}', (x, h) \models \Gamma_0$. Thus, by Lemma 23, $H' \not<_{k'}^x T$: a contradiction.

□

The theorem plays an important role in the area of Answer Set Programming [7] since it allows, under ASP semantics, to exchange two logic programs (or theories) regardless the context in which they are considered. This theorem also justifies the use of KHT as a monotonic basis supporting non-monotonicity.

8 Conclusion

In this paper, we have introduced and studied a combination of the logic of Here and There and modal logic for which we have obtained several results such as non-interdefinability of modal operators, complexity of the satisfiability problem, finite model property or axiomatisation. However, there is still a lot of open lines of research we want to study:

1) Other combinations of the logic of Here and There and modal logic: $S4 - HT$, $PDL - HT$, $CTL - HT$, etc;

2) Decision procedures, based on tableau methods, for the aforementioned logics;

3) Van Benthem characterisation theorem [3] for combinations of the logic of Here and There with modal logic;

4) An expressive completeness result similar to Kamp's result [21] in the setting of LTL-HT.

5) The logic FS was defined by G. Fisher Servi so that $\varphi \in FS$ iff the standard translation $ST_x(\varphi)$ is a tautology of first order intuitionistic logic. Checking if the same relation holds in the Here and There case will be considered in the near future.

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Review of *Immanent Reasoning or Equality in Action; A Plaidoyer for the Play Level* by Shahid Rahman, Zoe McConaughey, Ansten Klev and Nicolas Clerbout

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Immanent Reasoning or Equality in Action is the first extensive study on Dialogical Logic including, in detail, both technical and philosophical aspects of this logical framework. The principal task of the book is to explore the philosophical merits of dialogical logic (DL hereafter) by linking it with the approach of Constructive Type Theory (CTT); and in this way the main thesis is that the backbone of any reasoning is the idea of equality. In order to accomplish this task, the authors develop the following four topics:

1. A full explanation of the various technical features of DL (chapters 3, 4 and 5)

2. A detailed and clear introduction to CTT (chapter 2)

3. Combining the dialogical framework with CTT (chapters 7, 8 and 9)

4. Some discussions about the relevant philosophical issues (chapters 1, 6, 10 and 11)

Of the above four phases the first two can be studied in their own rights. In this respect the book is of a valuable contribution to the literature on DL and CTT, and so far it can be very useful for those intending to know much about these two philosophically interesting logical frameworks.

Phase 3, which is somehow the centre of the study, as the authors state in the preface, is the result, and the first presentation in book length, of a project launched some years ago among the dialogicians of *Lille School* which aims to implement CTT in dialogical perspectives, or to equip dialogical framework with the achievements of CTT. This rather technical part, not only aims to reconstruct the standard dialogical framework but also try to show new capabilities of this framework while incorporating CTT. In this way, the book provides also a dialogical demonstration of (constructive version of) the axiom of choice (ch. 8) as a proof of concept. Such a demonstration of course is interesting in its own right.

A central notion of CTT is that of judgment which is distinguished, not only conceptually but also notationally, from proposition. A judgment asserting the truth of P should be scribed in the following form:

$$a : P$$

which is to say a is a piece of evidence, or a reason, for P. Likewise the following judgment states that a and b are the same reasons for P (or generally a and b are identical within the type P):

$$a = b : P$$

The main idea of Rahman and his collaborators, developed particularly in chapter 7, is that by extending the language of the dialogical framework with the above mentioned forms some significant problems regarding the very nature of logic and reasoning can be resolved.

On one hand, if the aim of dialogical framework is to make explicit the game of giving and asking for reasons, then the above forms would be very helpful and make the interactions constituting the reasoning more manifest. On the other hand, the dialogical distinction between play-level and strategy-level makes it possible to recognize two kinds of reason: local reason (the left side of the colon in the assertion made during a particular dialogue) and strategic reason (to prove validity of a statement by providing a dialogical winning strategy for it).

A remarkable problem thus approached is what the authors call Martin-Löf's circularity problem. The problem is that if we explain demonstration as a chain of inferences and if inference is defined as making a conclusion evident on the bases of some other known judgments, then "we cannot take 'known' in the sense of demonstrated, or else we would be explaining the notion of inference in terms of demonstration when demonstration has been explained in terms of inference" (p. x). Such a problem may be considered as a challenge for the standard proof-theoretical approaches. Now, taking the dialogical interaction into account and on the basis of the distinction between local reason and strategic reason such a circularity will no more occur. In an inference we are dealing with local reasons whereas demonstration is to provide a strategic reason. Such an idea is indeed quite promising. It will help to scrutinize the nature of meaning, truth and validity; and the authors tried to draw some conclusions in those regards of their main idea.

In the following, I will focus on phase 4 mentioned above, that is, on the philosophical problems that the authors tackle in parallel to their painstaking formal studies. I distinguish and discuss here six notable theses that the authors develop in relation to the above mentioned seminal idea. Of course the detailed arguments cannot be reconstructed here but I will try to discuss how they are supposed to work.

1 Equality in action

One of the main constituents of the dialogical framework is a structural rule that prevents the proponent to assert elementary propositions unless it has been asserted by the opponent. In the other words who enters in a dialogue in order to support a thesis is allowed to state an elementary statement only if the challenger has appealed to it before. Rahman and his collaborators call this *Socratic rule*. It has been also called by some scholars *ipse dixit* (He, himself, said it). The reason for calling it Socratic is clear: it is a feature of Socratic dialectic not to claim a fact but to use what the interlocutor admits. In fact this is a feature of formal dialogue, where the framework is devised to evaluate the formal validities. In the material dialogues it should be modified in a proper way. However, in any case Socratic rule is a cornerstone for dialogic interactions. Now, the idea of the authors is that by embracing the expressive tools of CTT in dialogical framework, peculiarities of this rule and its status can be more explored. Above all, the form which expresses the identity is quite relevant here. The proponent can state explicitly that he asserts, say, P, for the same reason that the opponent has asserted it. That is if the opponent has made the judgment

$$a : P$$

the proponent, when asked, can make the judgment

$$b = a : P$$

which means that I have a reason, or a piece of evidence, b for P which is identical with yours. Thus identity, playing a crucial rule in reasoning, is itself thematized within the dialogical framework. Notice that this identity is not that of the propositional level: it is in the sides of reason, or of truth maker, of a judgment, and it functions in the action of reasoning not as a predicate, hence the title of the book "Immanent Reasoning or Equality in Action".

> We call our dialogues involving rational argumentation dialogues for immanent reasoning precisely because reasons backing a statement, that

> are now explicit denizens of the object language of plays, are internal to the development of the dialogical interaction itself. (p. 305)

2 A fully interpreted object language

The authors discuss that by incorporating forms of CTT, which expresses "proof-object", with its dialogical distinction between local reason and strategic reason, as well as identity functioning in the interaction of reasoning, dialogical framework puts a crucial step towards being a fully interpreted language:

> [T]he expressive power of CTT allows all these actions involved in the dialogical constitution of meaning to be incorporated as an explicit part of the object language of the dialogical framework. (p.278)

It should be said that the term "object language" in the above phrase is not quite adequate, since the distinction between object language and metalanguage belongs to the model-theoretic approach while model has no role in the dialogical framework—in contrast to some other game-theoretic frameworks. In any case, the language of DL enriched by the forms of CTT, so argue the authors, turns out to be a more powerful language able to elaborate various aspects of reasoning within itself.

3 The crucial importance of play-level

In various places of the book, the authors stress the distinction between play-level and strategy-level and show the peculiarities of the former. Such a distinction is already a feature of DL which when linked with CTT can aid the latter to avoid the problems of the sort mentioned above as Martin-Löf's circularity problem. As a result of careful examination of some formal challenges as well as philosophical debates the authors make a conclusion: "the meaning of expressions comes from the play level." (p. 289)

4 The dialogical conditions of meaningfulness: symmetry of local meaning, dialogue-definiteness

If "the play level is the level where meaning is forged" (p. 305), one may expect that some conditions for meaningfulness should be determined within this level. The authors discuss, in chapter 1 and chapter 11, two of these conditions. One is the player-independence of the meaning. If the meaning was different for the parties of the dialogue they would not speak about the same thing so that no genuine dialogue would occur. This includes the meaning of logical connectives. Thus the rules concerning them should be symmetric. The author show that by considering such a criterion the challenges such as the case of *tonk* are easily avoided (p. 286). Dialogue-definiteness is to say that in order for an expression to be meaningful the rules concerning it should be such that do not lead in endless plays.

5 A way to formalize material dialogue (or reasoning)

As opposed to formal dialogues, in material dialogues we should have rules to assert and challenge elementary propositions according to their specific contents. By the considerations listed above, the authors argue that the equipments required to formalize material dialogues have been prepared: The conditions for those rules are explained and the expressive power to deal with the specific reasons relevant to the elementary proposition is provided. Nevertheless, as the authors point out, these are only first steps and much more is needed to develop a comprehensive framework for material reasoning. However, Rahman and his collaborators provide examples of such material dialogues (sections 10.1 to 10.4).

6 The dialogical framework integrates world-directed thought and inferentialist approach

In some places of the book, including in section 10.5.1, the authors mention Sellars' idea of *space of reason* and two conflicting interpretation of it proposed by Brandom and McDowell. The authors argue that dialogical meaning explanation is not merely inferentialist, since this latter neglects the play level and consider every sequence of moves in reasoning necessarily inferential (p. 270), nor it is merely world-directed since even for material proposition there should be assigned rules of challenge and defence satisfying dialogue-definiteness:

> the dialogical framework of immanent reasoning enriched with the material level should show how to integrate world-directed thoughts (displaying empirical content) into an inferentialist approach, thereby suggesting that immanent reasoning can integrate within the same epistemological framework the two conflicting readings of the Space of Reasons brought forward by John McDowell [5, pp. 221–238] on the one hand, who insists in distinguishing world-direct thought and knowledge gathered by inference, and Robert Brandom [1] on the other hand, who interprets Sellars'

work in a more radical anti-empiricist manner. (p.233)

However, this discussion would deserve more explanation by the authors. Neither the main idea of Sellars nor the viewpoints of Brandom and McDowell were given in sufficient details and with required quotations. The reader may be convinced by the remarks of the authors but not enough space is dedicated to the presentation of the conflicting views. It could be a separate chapter.

At the end I have to emphasize that the book throughout its detailed discussions contain very stimulating ideas, besides the main one which is fully developed. The book also in each step addresses the recent critics of dialogical logic and responses them in a rather convincing way. One of the main points of the book is to contest many criticisms, such as Dutilh-Novaes [2] and Hodges [3], complaining that dialogical logic has only handled logical validity. The authors discuss that in fact such criticisms did not adequately realize that DL is a framework that can be extended and developed in several forms. The point of the chapter on material dialogues (chapter 10) is to develop a logic of content, where the authors show how to develop dialogues for natural numbers and more generally for finites sets. The authors also point out in a cursory way to some of the other recent works, e.g. [4, 6, 7], which show the fruitfulness of dialogical framework for, for example, cooperative games in legal reasoning both in classical Islamic and contemporary western jurisprudence, which may even lead to a new deontic logic.

The book is no doubt a highly valuable contribution to the studies on logic and philosophy of logic.

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