A Reduction in Violation Logic

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Abstract

Using proof theoretic methods, we show that a substantial fragment of violation logic as developed by Governatori, Rotolo et al. can be translated into classical modal logic. A number of consequences of this result are discussed. Furthermore, we present a new criterion for axiomatizations of violation logic and comment on the definability of the $\otimes$-operator.

Keywords: deontic logic, contrary to duty reasoning.

1 Introduction

In a series of works [8,2,6,7,3] Governatori, Rotolo et al. introduced a family of logics intended to model contrary-to-duty reasoning. To this end they extend classical modal logic E (which features the operator O, for 'obligation') by an additional operator $\otimes$ with the intended meaning that [6]

> [t]he interpretation of a chain like $a \otimes b \otimes c$ is that a is obligatory, but if it is violated (i.e., $\neg a$ holds), then b is the new obligation (and b compensates for the violation of a); again, if the obligation of b is violated as well, then c is obligatory [...]

For these so-called $\otimes$-chains a variety of rules and axioms are proposed, resulting in a number of different systems of *violation logic*. One therefore has two levels of obligations, one stemming from the $\otimes$-chains, and the other one from the O modality of the underlying logic E. As the authors put it in [6] regarding their semantics for the $\otimes$-operator,

> We [...] split the treatment of $\otimes$-chains and obligations; the intuition is that chains are the generators of obligations and permissions [...]

In the present paper, we investigate this role of $\otimes$-chains as generators of obligations using proof theoretic methods. Our main result is that $\otimes$-chains can be replaced by formulas in the underlying logic E which generate exactly the same obligations. This yields a translation of a large fragment of violation logic into the base logic E. As a consequence, tools available for E – such

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as neighbourhood semantics on the model theoretic side, or cutfree Gentzen systems on the proof theoretic side – can be used to study violation logics. We establish coNP-completeness of the 'translatable' fragment of violation logic, and close with some remarks on the choice of axioms for $\otimes$-chains.

2 Preliminaries

Classical Modal Logic

The deontic logic underlying the treatment of $\otimes$-chains is given by axiomatic extensions of the classical non-normal modal logic E (see [4]). We have a language with a countably infinite set Var of propositional variables (denoted $a, b, c, \ldots$), a constant $\bot$ (falsum) and the following connectives:

$$\wedge, \rightarrow \text{ (binary) and } \mathrm{O} \text{ (unary)}$$

Any formula built from variables, constants and the above connectives will be called a *deontic formula* and denoted by uppercase letters A, B. Additional connectives are defined as abbreviations: $\neg A := A \rightarrow \bot$ (negation), $\top := \neg\bot$ (verum), $A \equiv B := (A \rightarrow B) \wedge (B \rightarrow A)$ (equivalence). For a set Γ of formulas, $\bigwedge \Gamma$ denotes the conjunction of all formulas in Γ, with the convention that $\bigwedge \emptyset := \top$. We call *classical* any formula not containing O, and CL denotes the sets of those classical formulas which are theorems of classical logic.

The logic E is defined to be the smallest logic of deontic formulas containing CL and closed under the rules

$$\frac{A \quad A \rightarrow B}{B} \text{ (MP)} \qquad \text{and} \qquad \frac{A \equiv B}{\mathrm{O}A \equiv \mathrm{O}B} \text{ (O-RE)}.$$

It will be convenient for our purposes to have a notion of *derivations from assumptions* in an axiomatic extension of E. Here a set Γ will play the role of *local assumptions*, whereas a set Δ plays the role of additional *axioms*.

Definition 2.1 Let $\Delta \cup \Gamma \cup \{A\}$ be a set of deontic formulas. A $(E+\Delta)$*-proof of A from* Γ is a tree of deontic formulas built from rules (MP) and (O-RE), and which is rooted in A. Its leaves are either substitution instances of formulas from $\mathrm{CL} \cup \Delta$, or formulas from Γ. The latter type of leaves are called *local assumptions.*[2] We impose the following locality condition: No instance of (O-RE) appears below a local assumption in the proof.

We write $\Gamma \vdash_{\mathrm{E}+\Delta} A$, and say that A is derivable from Γ in $\mathrm{E} + \Delta$, if there is a $(\mathrm{E} + \Delta)$-proof of A from Γ. Finally, we identify the *logic* $\mathrm{E} + \Delta$ with its derivability relation $\vdash_{\mathrm{E}+\Delta}$.

The locality condition reflects the well-known fact that modal rules such as (O-RE) should not be applied to local assumptions in modal logic, cf. the chapter on proof theory in [1]. The following Deduction Theorem holds:

Fact 2.2 (Deduction Theorem) $\Gamma \cup \{B\} \vdash_{E+\Delta} A \iff \Gamma \vdash_{E+\Delta} B \rightarrow A$.

Proof. See [1]. □

[2] More precisely, we call local assumptions only those leaves which are not at the same time instances of formulas from Δ or classical theorems.

We write $C(A)$ for a formula in which some occurences of a subformula A are distinguished, and subsequently $C(B)$ for the result of replacing in C all these distinguished occurences of A by B. Then:

Fact 2.3 (Uniform Substitution) *For any formula $C(A)$, the following rule is admissible in $E + \Delta$:*

$$\frac{A \equiv B}{C(A) \equiv C(B)} \text{ (O-RE')}$$

Proof. See [1]. □

Neighbourhood semantics We review the notion of neighbourhood models, which form the standard semantics of classical modal logics. A *neighbourhood model* $\mathcal{W} = \langle W, \mathcal{N}, V \rangle$ is composed of the following elements:

- A nonempty set W of worlds
- A neighbourhood function $\mathcal{N} : \mathcal{W} \to \mathcal{P}(\mathcal{P}(W))$
- A valuation function $V : Var \to \mathcal{P}(W)$

By abuse of notation, we write $w \in \mathcal{W}$ for worlds w instead of $w \in W$. Given e neighbourhood model $\mathcal{W}$, we can define the notion $\langle \mathcal{W}, w \rangle \models A$ of *truth at a world* $w \in \mathcal{W}$ by induction on the deontic formula A: $\langle \mathcal{W}, w \rangle \not\models \bot$, $\langle \mathcal{W}, w \rangle \models a :\Leftrightarrow w \in V(a)$, $\langle \mathcal{W}, w \rangle \models A \wedge B :\Leftrightarrow \langle \mathcal{W}, w \rangle \models A$ and $\langle \mathcal{W}, w \rangle \models B$, $\langle \mathcal{W}, w \rangle \models A \to B :\Leftrightarrow \langle \mathcal{W}, w \rangle \not\models A$ or $\langle \mathcal{W}, w \rangle \models B$, and finally

$$\langle \mathcal{W}, w \rangle \models \mathrm{O}A \quad :\Leftrightarrow \quad [A]_{\mathcal{W}} \in \mathcal{N}(w)$$

where $[A]_{\mathcal{W}} = \{w \in \mathcal{W} \mid \langle \mathcal{W}, w \rangle \models A\}$. The part $\mathcal{F}_{\mathcal{W}} = \langle W, \mathcal{N} \rangle$ of a neighbourhood model $\mathcal{W}$ is called a *neighbourhood frame*, and conversely $\langle W, \mathcal{N}, V \rangle$ is called a neighbourhood model *based on* $\mathcal{F}$. Truth on a frame is defined as follows: $\mathcal{F} \models A$ iff for all models $\mathcal{W}$ based on $\mathcal{F}$ and all worlds $w \in \mathcal{W}$, $\langle \mathcal{W}, w \rangle \models A$. For a set $\Gamma \cup \Delta \cup \{A\}$ of deontic formulas, we define the following semantic consequence relation:

$$\Gamma \models_\Delta A \quad \text{iff} \quad \text{for all neighbourhood models } \mathcal{W} \text{ and } w \in \mathcal{W}, \text{ if } \mathcal{F}_{\mathcal{W}} \models \bigwedge \Delta \text{ and } \langle \mathcal{W}, w \rangle \models \bigwedge \Gamma, \text{ then } \langle \mathcal{W}, w \rangle \models A.$$

Fact 2.4 (Soundness and Completeness) $\Gamma \models_\Delta A \iff \Gamma \vdash_{E+\Delta} A$.

Proof. This follows from the strong completeness theorem for E with respect to neighbourhood models (see [4]) and the Deduction Theorem. □

Local assumptions in $\mathrm{E} + \Delta$ therefore correspond to truths at a certain world.

Violation Logics

We now discuss a family of logics which were originally introduced in [8], and then developed in a series of subsequent article (e.g., [2,6,7]). On the syntactic level, they extend classical modal logics by an operator $\otimes$, which comes in any arity $n > 0$. A formula

$$A_1 \otimes A_2 \otimes A_3 \otimes \ldots \otimes A_n$$

is meant to model a chain of obligations and corresponding compensations: A_1 is obligatory, but if A_1 is violated, then the new (secondary) obligation is A_2; The fullfillment of A_2 compensates the violation of A_1; If however A_2 is violated as well, then there is a new (ternary) obligation A_3, and so on.

Example 2.5 Consider three propositional variables w, p and f with meaning w=‘it is the weekend’, p=‘parking downtown’ and f=‘paying a fine’. Then the intended meaning of the formula

$$A_{Ex} = w \to (\neg p) \otimes f$$

taken from [7] is: *On weekends it is forbidden to park downtown; but if one does so, one has to pay a fine.* The formula A_{Ex} will serve as a running example throughout this article.

We will call various systems for logics with $\otimes$ *violation logics*, a term coined in [8]. A formula of violation logic (henceforth just called a formula) is any expression A built from $\bot, \wedge, \to, \mathrm{O}$ and $\otimes$ obeying the following *nesting condition*: No pair of operators from $\{\mathrm{O}, \otimes\}$ appears nested in A. For example, $\neg(\mathrm{O}a \wedge (b \otimes c \otimes d))$ is a formula of violation logic, whereas $\neg\mathrm{O}(a \wedge (b \otimes c))$ is not. A formula of the form $A_1 \otimes \ldots \otimes A_n$ $(n > 0)$ is called a $\otimes$-*chain*. Due to the nesting condition, every formula A_i occuring in a $\otimes$-chain is classical.

Concerning rules and axioms for $\otimes$, the literature contains a large variety of different systems, with no optimal candidate singled out. For the sake of the present article, we pick a system which is close to the one described in [6]; But we remark already here that our results apply to different systems as well, an observation which will be made precise later (Corollary 4.4). That being said, we will have the following two rules for $\otimes$:

$$\frac{A \equiv B}{\nu \otimes A \otimes \nu' \equiv \nu \otimes B \otimes \nu'} \ (\otimes\text{-RE})$$

$$\frac{A \equiv B}{\nu \otimes A \otimes \nu' \otimes B \otimes \nu'' \equiv \nu \otimes A \otimes \nu' \otimes \nu''} \ (\otimes\text{-contraction})$$

Here, a string such as $\nu \otimes A \otimes \nu'$ stands symbolically for a $\otimes$-chain containing the (classical) formula A at some position. It is allowed that ν or ν' are empty, so that A is the first or last element of the chain. The rule ($\otimes$-RE) is the generalization of (O-RE) to the language of violation logic, and ($\otimes$-contraction) is a principle of redundancy elimination.

As axioms, we take the following set Σ of formulas:

$$a_1 \otimes \ldots \otimes a_n \wedge \bigwedge_{i=1}^{k} \neg a_i \to \mathrm{O}a_{k+1} \qquad \text{(O-detachment)}$$

$$a_1 \otimes \ldots \otimes a_n \otimes a_{n+1} \to a_1 \otimes \ldots \otimes a_n \qquad (\otimes\text{-shortening})$$

$$a_1 \otimes \ldots \otimes a_{n+1} \wedge \neg a_1 \to a_2 \otimes \ldots \otimes a_{n+1} \qquad (\otimes\text{-detachment})$$

Here, $n \geq 1, 0 \leq k < n$.[3] The axiom (O-detachment) captures the intended meaning of $\otimes$-chains as descriptions of compensatory obligations: If the first k obligations expressed in a $\otimes$-chain $a_1 \otimes \ldots \otimes a_k \otimes a_{k+1} \otimes \ldots \otimes a_n$ have been violated, then the next obligation a_{k+1} comes into effect. We refer the reader to [6] for an extensive discussion of the system.

We again define a notion of derivations from assumptions.

Definition 2.6 Let $\Delta \cup \Gamma \cup \{A\}$ be a set of formulas. A *$(V_\Sigma + \Delta)$-proof of A from Γ* is a tree of formulas built from the rules (MP), (O-RE), ($\otimes$-RE) and ($\otimes$-contraction), and which is rooted in A. Its leaves are either substitution instances of formulas from $\mathrm{CL} \cup \Delta \cup \Sigma$, or formulas from Γ. The latter type of leaves are called *local assumptions.* We impose the following locality condition: No instance of (O-RE), ($\otimes$-RE) or ($\otimes$-contraction) appears below a local assumption.

We write $\Gamma \vdash_{\mathrm{V}_\Sigma + \Delta} A$, and say that A is derivable from Γ in $\mathrm{V}_\Sigma + \Delta$, if there is a $(\mathrm{V}_\Sigma + \Delta)$-proof of A from Γ. Finally, we identify the *violation logic* $\mathrm{V}_\Sigma + \Delta$ with its derivability relation $\vdash_{\mathrm{V}_\Sigma + \Delta}$.

Fact 2.7 (Deduction Theorem) $\Gamma \cup \{B\} \vdash_{V_\Sigma + \Delta} A \iff \Gamma \vdash_{V_\Sigma + \Delta} B \to A$.

Proof. By induction on the length of proofs. □

The Deduction Theorem equips us with the following mode of inference in violation logic: If we can prove A from assumption B *without using rules* (O-RE), ($\otimes$-RE) *or* ($\otimes$-contraction) *below the assumption* B, then we can infer $B \to A$.

Example 2.8 We look again at the formula A_{Ex} from Example 2.5. The following proof shows that $\{A_{Ex}, w, p\} \vdash_{\mathrm{V}_\Sigma} \mathrm{O}f$, which means that parking downtown on a weekend leads to the obligation of paying a fine:

$$\dfrac{\dfrac{\dfrac{\overset{\textit{(local assumption)}}{w} \qquad \overset{\textit{(local assumption)}}{w \to (\neg p) \otimes \mathrm{O}f}}{(\neg p) \otimes \mathrm{O}f}\,(\mathrm{MP}) \qquad \dfrac{\overset{\textit{(local assumption)}}{p}}{\neg\neg p}}{\neg\neg p \wedge ((\neg p) \otimes f)} \qquad \overset{\textit{(instance of O-detachment)}}{\neg\neg p \wedge ((\neg p) \otimes f) \to \mathrm{O}f}}{\mathrm{O}f}\,(\mathrm{MP})$$

A double line abbreviates some steps of 'classical reasoning', i.e. the use of classical theorems and (MP). Since none of the rules (O-RE), ($\otimes$-RE) or ($\otimes$-contraction) are applied in the proof above, we can also conclude, e.g., $\{A_{Ex}, w\} \vdash_{\mathrm{V}_\Sigma} p \to \mathrm{O}f$.

3 A Reduction Theorem

Throughout this section, we work in violation logics $\mathrm{V}_\Sigma + \Delta$ where Δ consists of deontic axioms only, and hence the meaning of the $\otimes$-chains is given by the axiom set Σ. The set Δ might for example consist of the single axiom

$$\mathrm{O}a \to \neg\mathrm{O}(\neg a) \tag{D}$$

[3] By the convention on empty conjunctions, it follows that $a_1 \otimes \ldots \otimes a_n \wedge \top \to \mathrm{O}a_1$ is an instance of (O-detachment).

in which case $\mathrm{V}_\Sigma + \Delta$ is the logic $D^{\otimes}$ from [6].

The technical results we are going to present apply to a fragment of violation logic that we will call the *chain negative fragment*.

Definition 3.1 (Chain Negative Fragment) An occurence of a $\otimes$-chain in a formula A is called positive if there is an even number (including zero) of implicational subformulas $B \to C$ of A such that the chain appears in B.[4] Otherwise, the occurence is called negative. We call a formula *chain negative* (resp. chain positive) if all occurences of $\otimes$-chains in it are negative (resp. positive).

For example, the chain $a \otimes b$ appears positively in the formulas $a \otimes b$, $\neg\neg(c \wedge a \otimes b)$ and $c \to a \otimes b$, and negatively in $\neg(a \otimes b)$, $(a \otimes b) \to \mathrm{O}c$ and $(a \otimes b) \wedge c \to \mathrm{O}d$.[5] The simplemost nontrivial example of a chain positive formula is a $\otimes$-chain. Intuitively, a chain negative formula is a formula in which $\otimes$-chains appear *only as assumptions, but not as conclusions.*

As our main result, we will now show that questions about the chain negative fragment of violation logic can be answered without using the machinery of violation logic, but with a suitable reduction to the underlying deontic logic $\mathrm{E} + \Delta$ instead. To this end, we first give a meaning to $\otimes$-chains as deontic formulas.

Definition 3.2 (π-translation) The translation π from $\otimes$-chains to deontic formulas is inductively defined as follows:

$$\pi(\otimes A) := \mathrm{O}A^{6}$$

$$\pi(A_1 \otimes \ldots \otimes A_n \otimes A_{n+1}) := \pi(A_1 \otimes \ldots \otimes A_n) \wedge \left((\bigwedge_{i=1}^{n} \neg A_i) \to \mathrm{O}A_{n+1} \right)$$

As an example, we have $\pi(a \otimes b \otimes c) = \mathrm{O}a \wedge (\neg a \to \mathrm{O}b) \wedge (\neg a \wedge \neg b \to \mathrm{O}c)$. In the following we will write π in closed form as

$$\pi(A_1 \otimes \ldots \otimes A_n) = \bigwedge_{i=1}^{n} \left((\bigwedge_{j=1}^{i-1} \neg A_j) \to \mathrm{O}A_i \right)$$

where by a harmless abuse of notation, we identify the conjunct $\top \to \mathrm{O}A_1$, corresponding to the index $i = 1$, with the formula $\mathrm{O}A_1$. We extend π to arbitrary formulas by letting it commute with $\wedge, \to$ and O, so that for example

$$\pi(A_{Ex}) = \pi(w \to (\neg p) \otimes f) = w \to \mathrm{O}(\neg p) \wedge (\neg\neg p \to \mathrm{O}f).$$

Given a set Γ of formulas, $\pi(\Gamma)$ denotes $\{\pi(A) \mid A \in \Gamma\}$.

[4] This is the standard notion of a positive/negative occurence of a subformula, see e.g. Definition 24.18 in [10].

[5] Recall that by definition, $\neg A = A \to \bot$.

[6] $\otimes A$ denotes a $\otimes$-chain of length 1.

We point out that the meaning given to $\otimes$-chains by the translation π is quite close to the intuitive interpretation of $\otimes$-chain from [6], which was already quoted in the introduction:

> [t]he interpretation of a chain like $a \otimes b \otimes c$ is that a is obligatory, but if it is violated (i.e., $\neg a$ holds), then b is the new obligation (and b compensates for the violation of a); again, if the obligation of b is violated as well, then c is obligatory [. . .]

As a first observation, the axioms for $\otimes$-chains remain true if translated via π:

Lemma 3.3 (Axiom Soundness) *For any axiom $A \in \Sigma$, $\vdash_E \pi(A)$.*

Proof. Below are the three axioms schemes and their respective π-translations:

(O-detachment) $a_1 \otimes \ldots \otimes a_n \wedge \bigwedge_{i=1}^{k} \neg a_i \rightarrow \mathrm{O}a_{k+1}$

$\bigwedge_{i=1}^{n} \left((\bigwedge_{j=1}^{i-1} \neg a_j) \rightarrow \mathrm{O}a_i \right) \wedge (\bigwedge_{i=1}^{k} \neg a_i) \rightarrow \mathrm{O}a_{k+1}$

($\otimes$-shortening) $a_1 \otimes \ldots \otimes a_n \otimes a_{n+1} \rightarrow a_1 \otimes \ldots \otimes a_n$

$\bigwedge_{i=1}^{n+1} \left((\bigwedge_{j=1}^{i-1} \neg a_j) \rightarrow \mathrm{O}a_i \right) \rightarrow \bigwedge_{i=1}^{n} \left((\bigwedge_{j=1}^{i-1} \neg a_j) \rightarrow \mathrm{O}a_i \right)$

($\otimes$-detachment) $a_1 \otimes \ldots \otimes a_{n+1} \wedge \neg a_1 \rightarrow a_2 \otimes \ldots \otimes a_{n+1}$

$\bigwedge_{i=1}^{n+1} \left((\bigwedge_{j=1}^{i-1} \neg a_j) \rightarrow \mathrm{O}a_i \right) \wedge \neg a_1 \rightarrow \bigwedge_{i=2}^{n+1} \left((\bigwedge_{j=2}^{i-1} \neg a_j) \rightarrow \mathrm{O}a_i \right)$

It is cumbersome but easy to check that the translations are provable in E. In fact, they are all instances of classical theorems. □

We now want to argue that in some sense, $A_1 \otimes \ldots \otimes A_n$ and its translation $\pi(A_1 \otimes \ldots \otimes A_n)$ are equivalent. One half of this claim holds in the literal sense:

Lemma 3.4 (Chain Soundness) $\vdash_{V_\Sigma} A_1 \otimes \ldots \otimes A_n \rightarrow \pi(A_1 \otimes \ldots \otimes A_n)$.

Proof. Let $1 \leq i \leq n$. From local assumptions $A_1 \otimes \ldots \otimes A_n$ and $\bigwedge_{j=1}^{i-1} \neg A_j$, we can infer $\mathrm{O}A_i$ using the axiom (O-detachment). So by the Deduction Theorem, we can infer $(\bigwedge_{j=1}^{i-1} \neg A_j) \rightarrow \mathrm{O}A_i$ for each $1 \leq i \leq n$, and by further classical reasoning we obtain

$$\bigwedge_{i=1}^{n} \left((\bigwedge_{j=1}^{i-1} \neg A_j) \rightarrow \mathrm{O}A_i \right)$$

which is precisely $\pi(A_1 \otimes \ldots \otimes A_n)$. □

Corollary 3.5 *For every chain negative formula N, $\vdash_{V_\Sigma} \pi(N) \rightarrow N$.*

Proof. By induction on the structure of N. Simultaneously, one has to prove that $\vdash_{\mathrm{V}_\Sigma} P \rightarrow \pi(P)$ for chain positive P. Both statements are trivially true when the formula does not contain $\otimes$. Furthermore, if P is a $\otimes$-chain we can use the Chain Soundness Lemma.

As an example for the inductive step, assume that a chain negative formula N is of the form $A \rightarrow B$. Then A is chain positive and B is chain negative. By induction hypothesis, we therefore have $\vdash_{\mathrm{V}_\Sigma} A \rightarrow \pi(A)$ and $\vdash_{\mathrm{V}_\Sigma} \pi(B) \rightarrow B$. From this and classical reasoning we obtain

$$\vdash_{\mathrm{V}_\Sigma} (\pi(A) \rightarrow \pi(B)) \rightarrow (A \rightarrow B)$$

which is what we need since $\pi(A \rightarrow B) = \pi(A) \rightarrow \pi(B)$.

The other cases are similar. We note that the induction step for formulas beginning with O is trivial, since by the nesting condition, such formulas do not contain the $\otimes$-operator. $\square$

Remark 3.6 The converse of Lemma 3.4 does not hold, i.e. in V_Σ we cannot prove $A_1 \otimes \ldots \otimes A_n$ from its π-translation. The intuitive reason for this is that in Σ, we do not have any axiom at hand which *creates* $\otimes$-chains from deontic formulas. For a formal argument, consider an alternative translation τ of formulas which replaces all $\otimes$-chains in a formula by $\bot$. For any axiom $A \in \Sigma$, an easy inspection shows that $\vdash_{\mathrm{V}_\Sigma} \tau(A)$. In words: The axioms of violation logic remain true if $\otimes$-chains are interpreted as contradictions.

By a simple induction on proof length it follows that $\vdash_{\mathrm{V}_\Sigma} \tau(A)$ for any theorem A of V_Σ. Hence if $\pi(A_1 \otimes \ldots \otimes A_n) \to A_1 \otimes \ldots \otimes A_n$ was provable for all $\otimes$-chains $A_1 \otimes \ldots \otimes A_n$, then so would be its τ-translation $\pi(A_1 \otimes \ldots \otimes A_n) \to \bot$, which cannot be the case.

Nevertheless, we will see that the deontic formula $\pi(A_1 \otimes \ldots \otimes A_n)$ is as strong as the $\otimes$-chain $A_1 \otimes \ldots \otimes A_n$ *when it comes to the derivation of deontic formulas*: In particular, the obligations arising from $A_1 \otimes \ldots \otimes A_n$ are exactly the obligations arising from $\pi(A_1 \otimes \ldots \otimes A_n)$.

This follows from the Reduction Theorem below, which is our main technical result in this article. We first state and prove the theorem, and then discuss its technical and conceptual consequences.

Theorem 3.7 (Reduction Theorem for the chain negative fragment) *For any chain negative formula N, the following holds:*

$$\vdash_{V_\Sigma + \Delta} N \quad \text{if and only if} \quad \vdash_{E+\Delta} \pi(N).$$

Proof. The direction from right to left is easy: If $\vdash_{\mathrm{E}+\Delta} \pi(N)$, then obviously also $\vdash_{\mathrm{V}_\Sigma+\Delta} \pi(N)$ since violation logic has all the axioms and rules of E. But then $\vdash_{\mathrm{V}_\Sigma+\Delta} N$ follows from Corollary 3.5, since N is chain negative.
For the direction from left to right, we argue by induction on the length of a proof δ witnessing $\vdash_{\mathrm{V}_\Sigma+\Delta} N$.

(i) Assume first that δ has length 1, i.e. that N is an axiom of $\mathrm{V}_\Sigma + \Delta$.
  (a) If N is a substitution instance of a classical theorem, then $\pi(N)$ is again a substitution instance of the the same classical theorem, since π commutes with boolean connectives. Hence $\vdash_{\mathrm{E}+\Delta} \pi(N)$.
  (b) Similarly, If N is a substitution instance of a formula in Δ, then $\pi(N)$ is again a substitution instance of the the same formula in Δ, since π commutes with boolean connectives and O. Hence $\vdash_{\mathrm{E}+\Delta} \pi(N)$.
  (c) If N is a substitution instance of a formula in Σ, then by the Axiom Soundness Lemma (Lemma 3.3), $\vdash_{\mathrm{E}+\Delta} \pi(N)$.

(ii) If the last step in δ is an instance of (MP) $B, A \to B/B$, then by induction hypothesis $\vdash_{\mathrm{E}+\Delta} \pi(B)$ and $\vdash_{\mathrm{E}+\Delta} \pi(A \to B)$. Since $\pi(A \to B)$ equals $\pi(A) \to \pi(B)$, we can conclude $\vdash_{\mathrm{E}+\Delta} \pi(B)$ by applying (MP) in E.

(iii) If the last step in δ is an instance of (O-RE) $A \equiv B/\mathrm{O}A \equiv \mathrm{O}B$, then by induction hypothesis $\vdash_{\mathrm{E}+\Delta} \pi(A \equiv B)$. Since $\pi(A \equiv B)$ equals $\pi(A) \equiv \pi(B)$, we can conclude $\vdash_{\mathrm{E}+\Delta} \mathrm{O}\pi(A) \equiv \mathrm{O}\pi(A)$ by applying (O-RE) in E, and $\mathrm{O}\pi(A) \equiv \mathrm{O}\pi(B)$ equals $\pi(\mathrm{O}A \equiv \mathrm{O}B)$.

(iv) Assume that the last step in δ is an inference

$$\frac{A \equiv B}{\nu \otimes A \otimes \nu' \equiv \nu \otimes B \otimes \nu'} \ (\otimes\text{-RE}).$$

By induction hypothesis $\vdash_{\mathrm{E}+\Delta} \pi(A \equiv B)$. Since A, B occur in a $\otimes$-chain, they must be classical formulas by the nesting condition, and so the premise $\pi(A \equiv B)$ equals $A \equiv B$. Now the deontic formula $\pi(\nu \otimes A \otimes \nu')$ arises from replacing some occurences of B in $\pi(\nu \otimes B \otimes \nu')$ by the formula A. Hence

$$\frac{A \equiv B}{\pi(\nu \otimes A \otimes \nu') \equiv \pi(\nu \otimes B \otimes \nu')}$$

is an instance of (O-RE') (cf. Lemma 2.3), and so $\vdash_{\mathrm{E}+\Delta} \pi(\nu \otimes A \otimes \nu' \equiv \nu \otimes B \otimes \nu')$ as desired.

(v) Assume that the last step in δ is an inference ($\otimes$-contraction). We only consider a characteristic case:

$$\frac{A \equiv B}{X \otimes A \otimes Y \otimes B \otimes Z \equiv X \otimes A \otimes Y \otimes Z} \ (\otimes\text{-contraction})$$

Again A and B must be classical, and so we have $\vdash_{\mathrm{E}+\Delta} A \equiv B$ by induction hypothesis. Now arguing in $\mathrm{E} + \Delta$, wen can use (O-RE') to derive from $A \equiv B$ the equivalence

$$\pi(X \otimes A \otimes Y \otimes B \otimes Z) \equiv \pi(X \otimes A \otimes Y \otimes A \otimes Z)$$

Written verbosely, the formula $\pi(X \otimes A \otimes Y \otimes A \otimes Z)$ equals

$$\mathrm{O}X \wedge (\neg X \to \mathrm{O}A) \wedge (\neg X \wedge \neg A \to \mathrm{O}Y) \wedge (\neg X \wedge \neg A \wedge \neg Y \to \mathrm{O}A) \\ \wedge (\neg X \wedge \neg A \wedge \neg Y \wedge \neg A \to \mathrm{O}Z).$$

By using classical reasoning we see that the fourth conjunct can be omitted since it is implied by the second conjunct. Furthermore, the second $\neg A$ in the last conjunct can be removed. The above formula is therefore equivalent to

$$\mathrm{O}X \wedge (\neg X \to \mathrm{O}A) \wedge (\neg X \wedge \neg A \to \mathrm{O}Y) \wedge (\neg X \wedge \neg A \wedge \neg Y \to \mathrm{O}Z)$$

which is precisely $\pi(X \otimes A \otimes Y \otimes Z)$. Hence we have $\vdash_{\mathrm{E}+\Delta} \pi(X \otimes A \otimes Y \otimes B \otimes Z \equiv X \otimes A \otimes Y \otimes Z)$ as desired.

This concludes the proof of the Reduction Theorem. □

It is instructive to single out a special case of Theorem 3.7.

Theorem 3.8 (Reduction Theorem, Special Case) *Let $\Gamma \cup \{D\}$ be a set of deontic formulas. Then for any chain positive formula P, the following are equivalent:*

(i) $\Gamma \cup \{P\} \vdash_{V_\Sigma+\Delta} D$

(ii) $\Gamma \cup \{\pi(P)\} \vdash_{E+\Delta} D$

(iii) $\Gamma \cup \{\pi(P)\} \vdash_{V_\Sigma+\Delta} D$

In particular, this holds if P is a $\otimes$-chain.

Proof. $\Gamma \cup \{P\} \vdash_{\mathrm{V}_\Sigma+\Delta} D$ is equivalent to $\vdash_{\mathrm{V}_\Sigma+\Delta} \bigwedge(\Gamma \cup \{P\}) \to D$ by the Deduction Theorem. Since $\bigwedge(\Gamma \cup \{P\}) \to D$ is chain negative, its provability is equivalent to $\vdash_{\mathrm{E}+\Delta} \pi(\bigwedge(\Gamma \cup \{P\}) \to D)$ by the Reduction Theorem. Now $\pi(\bigwedge(\Gamma \cup \{P\}) \to D)$ equals $\bigwedge(\Gamma \cup \{\pi(P)\}) \to D)$ since neither Γ nor D contain $\otimes$-chains by assumption. So by the Deduction Theorem, we obtain equivalence with $\Gamma \cup \{\pi(P)\} \vdash_{\mathrm{E}+\Delta} D$. We have thus established (i)↔(ii), and applying (i)↔(ii) to $\pi(P)$ instead of P yields (ii)↔(iii). □

Conceptually, of most importance is the equivalence (i)↔(iii) in the case that $P = C$ is a $\otimes$-chain, and its meaning can then be described as follows:

> *Within a context of deontic formulas, using a $\otimes$-chain C as an assumption has exactly the same effect as using its translation $\pi(C)$.*

In other words, as long as we are only interested in the role $\otimes$-chains as generators of obligations (under some circumstances described by deontic formulas), then we may as well replace all chains by their π-translations.

The questions which are not covered by the Reduction Theorem are those about the *generation of $\otimes$-chains from deontic assumptions* as well as those about *relations between different $\otimes$-chains*, such as the question when one $\otimes$-chain implies another one. We will come back to this in Section 5.

Example 3.9 Recall the formula $A_{Ex} = w \to (\neg p) \otimes f$ from Example 2.5. For any set Γ of deontic formulas, we may ask whether

$$\{A_{Ex}\} \cup \Gamma \vdash_{\mathrm{V}_\Sigma+\Delta} \mathrm{O}f$$

holds, i.e. whether under the assumption of A_{Ex}, the deontic circumstances expressed in Γ lead to the obligation of paying a fine. By the (special case of the) Reduction Theorem, this question is equivalent to asking whether

$$\{\pi(A_{Ex})\} \cup \Gamma \vdash_{\mathrm{E}+\Delta} \mathrm{O}f$$

holds, where $\pi(A_{Ex}) = w \to (\mathrm{O}(\neg p) \wedge (\neg\neg p \to \mathrm{O}f))$.

Remark 3.10 The Reduction Theorem is formulated relative to violation logics $\mathrm{V}_\Sigma + \Delta$ with a fixed axiomatization

$$\Sigma = \{(\text{O-detachment}),(\otimes\text{-contraction}),(\otimes\text{-shortening})\}$$

of $\otimes$-chains (whereas the deontic axioms Δ can be anything). Nevertheless, the proof is modular and can be adapted to violation logics $\mathrm{V}_\Pi + \Delta$ where Π is a different axiomatization of chains: We only have to check that the Axiom

Soundness Lemma (Lemma 3.3) and the Chain Soundness Lemma (Lemma 3.4) hold for the axiomatization Π, and then the proof of the Reduction Theorem goes through. Note in particular that the Chain Soundness Lemma holds for any Π which contains (O-detachment).

Remark 3.11 An easy example demonstrating that the Reduction Theorem does not hold for the full language of violation logic is the following. Consider the (chain positive!) formula $P = \pi(a \otimes b) \to (a \otimes b)$. P is not provable in V_Σ: Recall Remark 3.6, where it is argued that if P was provable in V, then so would be $\tau(P) = \pi(a \otimes b) \to \bot = \neg(\mathrm{O}a \wedge (\neg a \to \mathrm{O}b))$. But this is not a theorem of V_Σ, since it is easily seen to be falsifiable in E. On the other hand $\pi(P) = \pi(a \otimes b) \to \pi(a \otimes b)$ is obviously a theorem of E.

4 Applications of the Reduction Theorem

Throughout this section, Δ denotes a set of deontic formulas.

Corollary 4.1 *The violation logic $V_\Sigma + \Delta$ is conservative over $E + \Delta$.*

Proof. Let D be a formula without $\otimes$-chains. Then D is in the chain negative fragment and furthermore $\pi(D) = D$, and so we have $\vdash_{\mathrm{V}_\Sigma + \Delta} D$ iff $\vdash_{\mathrm{E}+\Delta} D$ by the Reduction Theorem. □

This conservativity result also follows from the *sequence semantics* for violation logic, see e.g. [6].

The main point of a reduction as expressed in Theorem 3.7 is that the logic one reduces to, i.e. $\mathrm{E} + \Delta$, is well studied, and one can transfer results about it back to the 'new' logic $\mathrm{V}_\Sigma + \Delta$. Let us see some examples.

Corollary 4.2 *The validity problem for the chain negative fragment of the violation logic V_Σ is coNP-complete.*

Proof. By the Reduction Theorem, $\vdash_{\mathrm{V}_\Sigma} D$ is equivalent to $\vdash_{\mathrm{E}} \pi(D)$ for a chain negative D, and the mapping $D \mapsto \pi(D)$ is computable in polynomial (in fact, quadratic) time. Since theoremhood in E is coNP-decidable ([11], Theorem 3.3), the same therefore holds for V_Σ. On the other hand the chain negative fragment of V_Σ is a conservative extension of CL, which is coNP-hard. □

By the same argument, complexity (or just decidability) results can be obtained for other violation logics $\mathrm{V}_\Sigma + \Delta$: We only have to know the complexity of the underlying deontic logic $\mathrm{E} + \Delta$. As far as we know, no decidability results for violation logics have been established so far.

It also follows from the Reduction Theorem that the neighbourhood semantics of classical modal logics provides a complete semantics for the chain negative fragment of violation logic. This semantics is simpler than the sequence semantics proposed in [6,7].

Corollary 4.3 *Let $\Gamma \cup \{D\}$ be a set of deontic formulas. Then for any chain positive formula P, $\Gamma \cup \{P\} \vdash_{V_\Sigma + \Delta} D$ iff for every neighbourhood model $\mathcal{W}$*

with $\mathcal{F}_\mathcal{W} \models \Delta$ the following is true: For any world $w \in \mathcal{W}$, if $\langle \mathcal{W}, w\rangle \models \bigwedge \Gamma$ and $\langle \mathcal{W}, w\rangle \models \pi(P)$, then $\langle \mathcal{W}, w\rangle \models D$.

Proof. By the Reduction Theorem, $\Gamma \cup \{P\} \vdash_{\mathrm{V}_\Sigma+\Delta} D$ is equivalent to $\Gamma \cup \{\pi(P)\} \vdash_{\mathrm{E}+\Delta} D$, which in turn is equivalent to $\Gamma \cup \{\pi(P)\} \models_\Delta D$ by Fact 2.4.□

So within a context of deontic formulas, having a $\otimes$-chain $C = a \otimes b \otimes c$ as a local assumption amounts to assuming the truth of

$$\pi(a \otimes b \otimes c) = \mathrm{O}a \wedge (\neg a \to \mathrm{O}b) \wedge (\neg a \wedge \neg b \to \mathrm{O}c)$$

at a world of a neighbourhood model $\mathcal{W}$.

Corollary 4.4 *Let $\Pi \neq \Sigma$ be any alternative axiomatization of $\otimes$-chains containing at least* (O-detachment)*, and such that $\vdash_E \pi(A)$ for every $A \in \Pi$. Then for any set of deontic formulas Δ, the chain negative fragments of $V_\Sigma + \Delta$ and $V_\Pi + \Delta$ coincide.*

Proof. By Remark 3.10, the proof of the Reduction Theorem goes through for $\mathrm{V}_\Pi + \Delta$ under the given assumptions. But then $\mathrm{V}_\Sigma + \Delta$ and $\mathrm{V}_\Pi + \Delta$ have the same characterization of their chain negative fragment (which does not depend on Σ or Π), namely

$$\vdash_{\mathrm{V}_\Sigma+\Delta} N \quad \text{iff} \quad \vdash_{\mathrm{E}+\Delta} \pi(N) \quad \text{iff} \quad \vdash_{\mathrm{V}_\Pi+\Delta} N.$$

□

An immediate consequence of Corollary 4.4 is that the axioms ($\otimes$-shortening) and ($\otimes$-detachment) are never needed for proving formulas in the chain negative fragment of $\mathrm{V}_\Sigma + \Delta$. As another consequence, consider the axiom ($\otimes$-I)

$$\left(a_1 \otimes \ldots \otimes a_n \wedge \left((\bigwedge_{i=1}^{n} \neg a_i) \to b_1 \otimes \ldots \otimes b_m \right)\right) \to a_1 \otimes \ldots \otimes a_n \otimes b_1 \otimes \ldots \otimes b_m$$

for creating $\otimes$-chains which is considered in [8,3], but not in [6,7]. It is easy to see that its π-translation is a theorem of E, and so by Corollary 4.4 its inclusion as an additional axiom has no effect on the chain negative fragment.

An axiomatization of $\otimes$-chains to which the Reduction Theorem does *not* apply is the one given in [2], where axioms such as $a \otimes (\neg a) \equiv \top$ are included. Indeed, the π-translation of the latter axiom is $\mathrm{O}a \wedge (\neg a \to \mathrm{O}\neg a) \equiv \top$, which does not hold in E.

Another consequence of the Reduction Theorem is that questions in violation logic can be tackled using the proof theory of classical modal logics. For example, [9] presents cutfree Gentzen systems for the logics

$$\mathrm{E}, \quad \mathrm{EC} = \mathrm{E} + \mathrm{O}a \wedge \mathrm{O}b \to \mathrm{O}(a \wedge b) \quad \text{and} \quad \mathrm{M} = \mathrm{E} + \mathrm{O}(a \wedge b) \to \mathrm{O}a \wedge \mathrm{O}b$$

which are called **Eseq**, **ECseq** and **Mseq** respectively.

Corollary 4.5 *Let $\Delta = \emptyset$ (resp. $\Delta = \{Oa \wedge Ob \to O(a \wedge b)\}$, resp. $\Delta = \{O(a \wedge b) \to Oa \wedge Ob\}$). Then for any chain negative formula N, $\vdash_{V_\Sigma+\Delta} N$ iff there is a cutfree proof of $\pi(N)$ in* **Eseq** *(resp.* **ECseq**, *resp.* **Mseq**).

Example 4.6 Here is a Gentzen-style proof establishing $\{A_{Ex}, w, p\} \vdash_{V_\Sigma} \mathrm{O}f$ by means of the π-translation (cf. Example 2.8):

$$\dfrac{w \Rightarrow w \quad \dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p}{p, \neg p \Rightarrow}(\neg_L)}{p \Rightarrow \neg\neg p}(\neg_R) \quad \mathrm{O}f \Rightarrow \mathrm{O}f}{\neg\neg p \to \mathrm{O}f, p \Rightarrow \mathrm{O}f}(\to_L)}{\mathrm{O}(\neg p) \wedge (\neg\neg p \to \mathrm{O}f), p \Rightarrow \mathrm{O}f}(\wedge_L)}{w \to (\mathrm{O}(\neg p) \wedge (\neg\neg p \to \mathrm{O}f)), w, p \Rightarrow \mathrm{O}f}(\to_L)$$

5 More on the interpretation of $\otimes$-chains

Arguably, the formalization of many contrary-to-duty reasoning scenarios in the framework of violation logic remains in the chain negative fragment. Recall that in particular all questions of the form

Given some (deontic) circumstances, which obligations arise from a $\otimes$-chain?

are expressible. The Reduction Theorem then suggests that in the chain negative fragment, the 'meaning' of a $\otimes$-chain can be identified with its π-translation (assuming, of course, one believes that the meaning of $\otimes$-chains is given by their proof-theoretic behaviour). Furthermore, we have seen (Corollary 4.4) that this identification is to some extent independend of the exact axiomatization Σ of $\otimes$-chains.

If we move beyond the chain negative fragment, the precise axiomatization of $\otimes$-chains matters more. So let us now consider an arbitrary violation logic $V_\Pi + \Delta$ where Π satisfies the premises of Corollary 4.4, and for which therefore the Reduction Theorem holds (Δ is again any set of deontic axioms). A typical question outside the chain negative fragment is: When does a $\otimes$-chain C imply another $\otimes$-chain C', i.e. when does $\vdash_{V_\Pi+\Delta} C \to C'$ hold? A good axiomatization Π should give a tangible meaning to the notion of implication between chains. Hence, the question we have to ask is:

When *should* a $\otimes$-chain C imply another $\otimes$-chain C'?

Here is one possible proposal. We say that a chain C *deontically subsumes* another chain C' over $V_\Pi + \Delta$ if for every deontic formula D, $\vdash_{V_\Pi+\Delta} C' \to D$ implies $\vdash_{V_{\Sigma\Pi}+\Delta} C \to D$. In words: C deontically subsumes C' if every obligation arising from C' already arises from C.

Definition 5.1 The violation logic $V_\Pi + \Delta$ is *faithful* if it proves $C \to C'$ for every pair C, C' of chains where C deontically subsumes C'.

So in a faithful violation logic, the meaning of an implication $C \to C'$ between chains is that of deontic subsumption. From the Reduction Theorem arises a simple characterization of deontic subsumption:

Lemma 5.2 *C deontically subsumes C' iff* $\vdash_{E+\Delta} \pi(C) \to \pi(C')$.

Proof. Assume that C deontically subsumes C'. Since $\vdash_{V_\Pi+\Delta} C' \to \pi(C')$ (Lemma 3.4), we also have $\vdash_{V_\Pi+\Delta} C \to \pi(C')$ by deontic subsumption. But then $\vdash_{E+\Delta} \pi(C) \to \pi(C')$ by the Reduction Theorem. Conversely, if $\vdash_{E+\Delta} \pi(C) \to \pi(C')$ and D is a deontic formula implied by C', then $\vdash_{E+\Delta} \pi(C') \to D$ by the Reduction Theorem, and so $\vdash_{E+\Delta} \pi(C) \to D$. Then again by Lemma 3.4, $\vdash_{V_\Pi+\Delta} C \to D$ follows. □

For our basic violation logic V_Σ, we can show the following:

Theorem 5.3 *V_Σ is not faithful.*

Proof. (Sketch) Let a, b be two distinct variables. The counterexample will be the two chains

$$C = a \otimes (\neg a) \quad \text{and} \quad C' = a \otimes (\neg a) \otimes b.$$

Their respective π-translations are $\pi(C) = \mathrm{O}a \wedge (\neg a \to \mathrm{O}(\neg a))$ and $\pi(C') = \mathrm{O}a \wedge (\neg a \to \mathrm{O}(\neg a)) \wedge (\neg a \wedge \neg\neg a \to \mathrm{O}b)$. Since $\pi(C)$ implies $\pi(C')$, we know by Lemma 5.2 that C deontically subsumes C'. However, while $C' \to C$ is an instance of ($\otimes$-shortening), V_Σ fails to prove $C \to C'$. We show this by providing a countermodel in the *sequence semantics* of [6]. A sequence model extends a neighbourhodel model $\mathcal{W} = \langle W, \mathcal{N}, V\rangle$ by a function $\mathcal{C}$ which maps each world w to a set $\mathcal{C}_w$ of finite nonempty sequences $\langle X_1, \ldots, X_n\rangle$ of sets of worlds, and which obeys the following closure conditions:

(i) If $\langle X_1, \ldots, X_n\rangle \in \mathcal{C}_w$ and $n > 1$, then $\langle X_1, \ldots, X_{n-1}\rangle \in \mathcal{C}_w$

(ii) Let $L \in \mathcal{C}_w$ be a list in which a set of worlds X occurs at a certain position. Then $\mathcal{C}_w$ must contain also all lists arising from removing or introducing copies of X at a later position in L.

(iii) If $\langle X_1, \ldots, X_n\rangle \in \mathcal{C}_w$ and for some $0 \leq k < n$, $w \notin X_1 \cup \ldots \cup X_k$, then $X_{k+1} \in \mathcal{N}(w)$ and $\langle X_{k+1}, \ldots, X_n\rangle \in \mathcal{C}_w$

The satisfaction clauses of the standard neighbourhood semantics are then extended by setting $\langle \mathcal{W}, \mathcal{C}, w\rangle \models A_1 \otimes \ldots \otimes A_n :\Leftrightarrow \langle [A_1]_{\mathcal{W}}, \ldots, [A_n]_{\mathcal{W}}\rangle \in \mathcal{C}_w$. It is proved in [6] that $\vdash_{V_\Sigma} A$ iff A holds in all sequence models. So our task is to construct a sequence model in which C holds, but C' fails. It will suffice to have two worlds w, v. Assume that $V(a) = \{w, v\}$ and $V(b) = \{w\}$. We let $\mathcal{N}(w) = \{\{w, v\}\}$. The value of $\mathcal{N}$ on other worlds is not relevant. Neither is the choice of C_v, which can be set to $\emptyset$ to trivially satisfy the closure conditions. We let C_w consist of all sequences of the form

$$\langle \{w, v\}, \ldots, \{w, v\}\rangle \quad \text{or} \quad \langle \{w, v\}, \emptyset, X_1, \ldots, X_n\rangle$$

where $n \geq 0$ and each X_i is either $\{w, v\}$ or $\emptyset$. Then $\mathcal{C}_w$ satisfies the closure conditions, and $\langle [a]_{\mathcal{W}}, [\neg a]_{\mathcal{W}}\rangle = \langle \{w, v\}, \emptyset\rangle \in \mathcal{C}_w$, whereas $\langle [a]_{\mathcal{W}}, [\neg a]_{\mathcal{W}}, [b]_{\mathcal{W}}\rangle = \langle \{w, v\}, \emptyset, \{w\}\rangle \notin \mathcal{C}_w$, and so $\langle W, \mathcal{N}, w\rangle \not\models a \otimes (\neg a) \to a \otimes (\neg a) \otimes b$. □

We have already seen in Remark 3.6 that $\otimes$-chains are not equivalent to their π-translation over V_Σ. From the above theorem, we can conclude that no translation with that property exists:

Corollary 5.4 (Undefinability of $\otimes$-chains over V_Σ) *There is no translation π^* from $\otimes$-chains to deontic formulas such that*

$$\vdash_{V_\Sigma} A_1 \otimes \ldots \otimes A_n \equiv \pi^*(A_1 \otimes \ldots \otimes A_n)$$

for all $\otimes$*-chains* $A_1 \otimes \ldots \otimes A_n$.

Proof. Assume that such a translation exists, and let C, C' be the two $\otimes$-chains from the proof of Theorem 5.3. Since C deontically subsumes $\mathcal{C}'$ and $\vdash_{\mathrm{V}_\Sigma} C' \to \pi^*(C')$, we have $\vdash_{\mathrm{V}_\Sigma} C \to \pi^*(C')$. Now because $\vdash_{\mathrm{V}_\Sigma} \pi^*(C') \to C'$, we can conclude $\vdash_{\mathrm{V}_\Sigma} C \to C'$, contradiction. □

From the proof of Corollary 5.4, we can extract the following observation: If in a violation logic every $\otimes$-chain is definable by a deontic formula, then the violation logic is faithful. However it is not so clear if definability of $\otimes$-chains is desirable. On a technical level, it trivializes the treatment of $\otimes$-chains, and in some sense deprives the $\otimes$-chains of their status as logical entities in their own right. If on the other hand definability does not hold, one has the burden of finding an intuition about $\otimes$-chains which is robust enough to allow for the acceptance and rejection of the principles proposed for them (such as the principle of faithfulness).

We remark that it is possible to have faithfulness without having definability of $\otimes$-chains: We obtain such a logic by formally adding to V_Σ the rule

$$\frac{\pi(C) \to \pi(C')}{C \to C'} .$$

(To show that $\otimes$-chains are not definable in the resulting logic, the argument in Remark 3.6 can be applied.)

Earlier on, we already mentioned the axiom ($\otimes$-I)

$$\left(a_1 \otimes \ldots \otimes a_n \wedge \left((\bigwedge_{i=1}^{n} \neg a_i) \to b_1 \otimes \ldots \otimes b_m \right)\right) \to a_1 \otimes \ldots \otimes a_n \otimes b_1 \otimes \ldots \otimes b_m$$

which appears in [3]. From ($\otimes$-I) we can prove $a \otimes (\neg a) \to a \otimes (\neg a) \otimes b$, the implication which was used as a counterexample to faithfulness in Theorem 5.3. This suggests the following question, to which we do not know the answer:

Is the extension of V_Σ *by* ($\otimes$-I) *a faithful violation logic?*

Finally, let us comment on the definability of $\otimes$-chains again. The easiest, but also the least illuminating way of achieving this is to add a scheme like $A_1 \otimes \ldots \otimes A_n \equiv \pi(A_1 \otimes \ldots \otimes A_n)$ to the violation logic at hand. It might also be of interest to have a 'natural' axiomatization of $\otimes$-chains which implies definability. For example, consider the following axiomatization of $\otimes$-chains:

$$\Sigma^* = (\text{O-detachment}) + (\otimes\text{-I}) \quad + \quad (\mathrm{O}\otimes)\text{: } \mathrm{O}a \to \otimes a$$

Theorem 5.5 *The violation logics* $V_{\Sigma^*} + \Delta$ *and* $V_\Sigma + \Delta$ *coincide on the chain negative fragment, and in* $V_{\Sigma^*} + \Delta$ *every* $\otimes$*-chain is definable via*

$$A_1 \otimes \ldots \otimes A_n \equiv \pi(A_1 \otimes \ldots \otimes A_n).$$

In particular, $V_{\Sigma^*} + \Delta$ *is faithful.*

Proof. $V_{\Sigma^*} + \Delta$ satisfies the premises of Corollary 4.4, and so its chain negative fragment coincides with that of $V_\Sigma + \Delta$. The Chain Soundness Lemma is satisfied in V_{Σ^*} because Σ^* contains (O-detachment). Hence for definability, it suffices to show by induction on n that

$$\vdash_{V_{\Sigma^*}+\Delta} \pi(A_1 \otimes \ldots \otimes A_n) \rightarrow A_1 \otimes \ldots \otimes A_n.$$

The base case $n = 1$ is precisely the axiom (O$\otimes$). For the induction step, we first note that the assumption $\pi(A_1 \otimes \ldots \otimes A_n \otimes A_{n+1})$ equals

$$\pi(A_1 \otimes \ldots \otimes A_n) \wedge \left((\bigwedge_{i=1}^{n} \neg A_i) \rightarrow \mathrm{O}A_{n+1} \right)$$

by the definition of π. Now by the induction hypothesis, we can replace $\pi(A_1 \otimes \ldots \otimes A_n)$ by $A_1 \otimes \ldots \otimes A_n$ and $\mathrm{O}A_{n+1}$ by $\otimes A_{n+1}$. The axiom ($\otimes$-I) then yields $A_1 \otimes \ldots \otimes A_n \otimes A_{n+1}$ as desired. □

Hence if one accepts (O-detachment) and ($\otimes$-I) as true principles for $\otimes$-chains but rejects their definability, one must argue against the validity of the axiom (O$\otimes$).

6 Conclusion

We have isolated the ‘chain negative fragment’ of violation logic, and showed how questions in this fragment can be systematically reduced to questions in the underlying classical modal logic. This made it possible to use results about classical modal logic to reason in violation logic. On top of that, we have seen that truth in the chain negative fragment is to some extent independent of the axiomatization of $\otimes$-chains. Concerning future work, we believe that the main challenge for violation logic lies in the search for intuititive, yet sufficiently formal criteria which discriminate between different possible axiomatizations of $\otimes$-chains. One such criterion called ‘faithfulness’ was suggested here.

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